On Exact Algorithms for Permutation CSP

Eun Jung Kim† and Daniel Gonçalves‡

Abstract

In the Permutation Constraint Satisfaction Problem (Permutation CSP) we are given a set of variables $V$ and a set of constraints $C$, in which constraints are tuples of elements of $V$. The goal is to find a total ordering of the variables, $\pi : V \to [1, \ldots, |V|]$, which satisfies as many constraints as possible. A constraint $(v_1, v_2, \ldots, v_k)$ is satisfied by an ordering $\pi$ when $\pi(v_1) < \pi(v_2) < \ldots < \pi(v_k)$. An instance has arity $k$ if all the constraints involve at most $k$ elements.

This problem expresses a variety of permutation problems including Feedback Arc Set and Betweenness problems. A naive algorithm, listing all the $n!$ permutations, requires $2^{O(n \log n)}$ time. Interestingly, Permutation CSP for arity 2 or 3 can be solved by Held-Karp type algorithms in time $O^*(2^n)$, but no algorithm is known for arity at least 4 with running time significantly better than $2^{O(n \log n)}$. In this paper we resolve the gap by showing that Arity 4 Permutation CSP cannot be solved in time $2^{o(n \log n)}$ unless ETH fails.

1 Introduction

Many combinatorial problems are intractable in the sense that they are unlike to have polynomial-time algorithms. One possible strategy to deal with intractability is to design moderately exponential-time algorithms. Such algorithms solve the problems optimally on any given instance. Even though exponentially many steps are required in the worst case, an exact algorithm with a slow-growing runtime function may work quite well in practice. From theoretic viewpoint, an $O^*(1.999^n)$-time algorithm is better than $O^*(2^n)$ algorithm, which is in turn better than an $O^*(n!)$-time algorithm.

The study of moderately exponential-time algorithms can be traced back to the $O^*(2^n)$-time algorithm for Hamiltonian Cycle by Held and Karp in 1962. Ever since the $O^*(2^n)$ worst-case bound seemed to be impenetrable for almost fifty years till $O^*(1.657^n)$-time (randomized) algorithm became known by Björklund in [3]. This is only a part of the success story in search for faster exact algorithms, especially in the last decade. Examples include the $O^*(2^{n/3})$-time algorithm, where $\omega$ is the matrix multiplication exponent, for MAX-2-CSP [11], a sequence of algorithmic development for COLORING culminating in $O^*(2^n)$-time algorithm [2], the very recent $O^*(c^n)$-time algorithm for SCHEDULING [5] for $c < 2$ and lots more.

The resistance of $2^n$ barrier (and its breakdown) is repeatedly observed in combinatorial problems. While remarkable algorithmic improvement has been made, for fundamental problems such as CIRCUIT SAT, TSP and COLORING, the (asymptotic) $O^*(2^n)$ runtime remains the current best. More generally, we ask what would be the lower bounds for combinatorial problems. It is widely believed that certain NP-complete problems such as 3-COLORING, INDEPENDENT SET, 3-SAT are not likely to have subexponential-time algorithms. This assumption is called Exponential Time Hypothesis (ETH) and it serves as a common ground to prove a number of hardness results.

**Exponential Time Hypothesis:** There exists an $\epsilon > 0$ such that no algorithm solves 3-SAT in time $2^{\epsilon n}$, where $n$ is the number of variables.
In this paper, we are interested in a family of problems called Permutation CSP. In the problem Permutation CSP, we are given a set of variables $V$ and a set of constraints $C$, which constraints are tuples of elements of $V$. The goal is to find a total ordering of the variables, $\pi : V \to [1, \ldots, |V|]$, which satisfies as many constraints as possible. A constraint $(v_1, v_2, \ldots, v_k)$ is satisfied by an ordering $\pi$ when $\pi(v_1) < \pi(v_2) < \ldots < \pi(v_k)$. An instance has arity $k$ if all the constraints involve at most $k$ elements. In case the arity is bounded by $k$, we call the problem Arity $k$ Permutation CSP.

The Permutation CSP is NP-complete even when restricted to instances of arity two, which is also known as Maximum Acyclic Subgraph or Feedback Arc Set. A trivial algorithm for arbitrary arity considers every possible ordering of $V$ and counts the number of satisfied constraints. This requires $O(n!|C|)$ time, which is $2^{O(n \log n)}$. However, we can do much better on instances with arity up to three using standard dynamic programming. For example, it is fairly straightforward to apply the framework of [1, 8] and obtain $2^n$-time algorithm.

Although it is not difficult to design a dynamic programming over subsets for arity up to three, it is not clear how we can proceed with arity four and so on. Is it possible to have such an algorithm, or any algorithm of runtime $O^*(c^n)$ for some constant $c$? We answer this question in the negative. The following theorem summarizes our main result stating that such improvement is impossible under ETH.

**Theorem 1.** Assuming ETH, there is no $2^{o(n \log n)}$-algorithm for Arity 4 Permutation CSP (and thus for Arity $k$ Permutation CSP, $k \geq 4$).

Our result is built on two previous results: Impagliazzo et al. [6] and Lokshtanov et al. [9]. In [6], the authors prove a computational lower bound for the $n \times n$ CLIQUE problem and transfer the lower bound to those of other natural combinatorial problems. The $n \times n$ CLIQUE is designed so that an improvement over the brute-force search would contradict ETH. It is not difficult to build a reduction from $n \times n$ CLIQUE to Permutation CSP with arity six, thus showing Theorem 1 holds for arity $k \geq 6$. The technical difficulty arises when we try to get down the arity down to four. To do this, we resort to the Sparsification Lemma of [6] and construct a variation of $n \times n$ CLIQUE with strictly constrained properties.

To be more specific, we give a sequence of reductions starting from an instance of 3-Sat in which the maximum frequency (i.e. the number of clauses containing a variable) is bounded by a fixed constant $f$. Such a 3-SAT instance is reduced to a 3-COLORING of degree bounded by $f'$. From 3-COLORING, we construct an instance of $n \times n$ CLIQUE along the line of [9]. In order to construct an $n \times n$ CLIQUE instance constrained in a subtle manner, we use Brooks’ theorem and ternary grey code. Finally from such an instance of $n \times n$ CLIQUE, we reduce to Arity 4 Permutation CSP. The whole chain of reductions are designed so that an $O^*(2^{o(n \log n)})$-time algorithm for Arity 4 Permutation CSP implies that we can solve 3-Sat instance of frequency $f$ in time $O(2^{o(n)})$ for any given $f$, which is unlikely.

An interesting dichotomy is observed as a corollary of our main result.

**Corollary 1.** The problem Arity $k$ Permutation CSP can be solved in time $O^*(2^n)$ if $k \leq 3$. Otherwise, such an algorithm is unlikely to exist under ETH.

In the next section we give some definitions and a simple proof of the fact that Arity 6 Permutation CSP has no $2^{o(n \log n)}$-algorithm, unless ETH fails. Then we prove Theorem 1 in Section 3.

## 2 Warm-up

Let us define the $n \times n$ CLIQUE Problem introduced by Lokshtanov et al. [9].

**$n \times n$ CLIQUE**

**Input:** a graph $G$ with vertex set $V(G) = [n] \times [n]$. (The vertex $(i, j)$ is said to be in row $i$ and column $j$.)

**Goal:** Determine if there exists an $n$-clique in $G$ with exactly one element from each row.
Theorem 2 (Lokshtanov et al. [9]). Assuming ETH, there is no $O(n \log n)$-time algorithm for $n \times n$ CLIQUE.

Using this result we prove the following.

Theorem 3. Assuming ETH, there is no $O(n \log n)$-algorithm for Arity 6 PERMUTATION CSP.

Proof. To prove that, we show that one could use a $O(n \log n)$-algorithm for Arity 6 PERMUTATION CSP to design a $O(n \log n)$-time algorithm for $n \times n$ CLIQUE, which does not exist if we assume ETH (by Theorem 2). This is due to an easy reduction from $n \times n$ CLIQUE to Arity 6 PERMUTATION CSP. Note that in order to achieve the desired lower bound, the reduction needs to produce an Arity 6 PERMUTATION CSP instance $(V, C)$ with $|V| = O(n)$.

Given an instance $G$ of $n \times n$ CLIQUE, we build an instance $(V, C)$ containing $4n + 1$ elements in $V$ as follows. There are (a) $n$ elements $r_i$, $i \in [n]$, corresponding to the rows of $G$, (b) $n$ elements $c_i$, $i \in [n + 1]$, corresponding to the columns of $G$ (except $c_{n+1}$ that does not exactly correspond to a column), and (c) $2n$ ‘dummy’ elements $d_i$, $i \in [2n]$. The constraint set $C$ is the union of three types of constraints, the constraints $C_G$ depending on $G$, and the structural constraints $C_1$ and $C_2$ that force the optimal orderings to be of the form $d_1d_2 \ldots d_{2n}c_1c_2c_3 \ldots c_nR_1R_2R_3 \ldots c_nR_nR_{n+1}$, where each $R_i$ is a (possibly empty) sequence of $r_i$'s.

Formally, $C_1 = \{(d_1, d_2, d_3, d_4, c_j, c_j') | \forall 1 \leq a < b < c < d \leq 2n \text{ and } \forall 1 \leq j < j' \leq n + 1\}$, $C_2 = \{(c_1, r_n, c_{n+1}) | \forall i \in [n]\}$, $C_G$ is such that for every edge $(i, j)(i', j') \in E(G)$ there is a constraint requiring $r_i$ and $r_{i'}$ to be respectively between $c_j$ and $c_{j+1}$, and between $c_{j'}$ and $c_{j'+1}$. Since we avoid an element appearing more than once in a constraint, a constraint $C_G$ can be one of three types according to the value $j' - j$: $C_G = \{(c_j, r_i, c_{j+1}, c_j', r_{i'}), r_{i'}, c_{j'+1}\} \cup \{(c_j, r_i, r_{i'}, c_{j+1}, r_{i'}), c_{j'+1}\}$ or $\{(c_j, r_i, r_{i'}, c_{j+1}, r_{i'}), c_{j'+1}\}$.

Similarly again, if in some ordering two elements $c_i$ and $c_j$ are misplaced with respect to each other, then the $\binom{2n-3}{2}n$ constraints of $C_1$ involving them are unsatisfied, and this cannot be compensated by the $|C_G| \leq \binom{n}{2}$ constraints in $C_G$ for a sufficiently large $n$.

Similarly, if in some ordering two elements $d_i$ and $d_j$ are misplaced with respect to each other, then the $\binom{2n-3}{2}n$ constraints of $C_2$ involving them are unsatisfied, and this cannot be compensated by the $|C_G| \leq \binom{n}{2}$ constraints in $C_G$ for a sufficiently large $n$.

Finally, if some $r_i$ lies before $c_1$ or after $c_{n+1}$, it cannot be part of any satisfied constraint, contrary to the case where $r_i$ lies in between $c_1$ and $c_{n+1}$. In this case, at least one satisfied constraint involves $r_i$, the one in $C_2$. Thus the optimal orderings are of the desired form.

Claim 4. Any optimal ordering is of the form $d_1d_2 \ldots d_{2n}c_1c_2c_3 \ldots c_nR_1R_2R_3 \ldots c_nR_nR_{n+1}$, where each $R_i$ is a (possibly empty) sequence of $r_i$’s. Thus in any optimal ordering all the constraints of $C_1 \cup C_2$ are satisfied.

Proof. Notice that sequences of this type exist and that they satisfy all the constraints in $C_1 \cup C_2$. Note also that since $G$ has at most $\binom{n^2}{2}$ edges, $|C_G| \leq \binom{n^2}{2}$.

If in some ordering two elements $d_i$ and $d_j$ are misplaced with respect to each other, then the $\binom{2n-3}{2}n$ constraints of $C_1$ involving them are unsatisfied, and this cannot be compensated by the $|C_G| \leq \binom{n^2}{2}$ constraints in $C_G$ (for a sufficiently large $n$).

Similarly, if in some ordering two elements $c_i$ and $c_j$ are misplaced with respect to each other, then the $\binom{2n-3}{2}n$ constraints of $C_2$ involving them are unsatisfied, and this cannot be compensated by the $|C_G| \leq \binom{n^2}{2}$ constraints in $C_G$ (for a sufficiently large $n$).

Finally, if some $r_i$ lies before $c_1$ or after $c_{n+1}$, it cannot be part of any satisfied constraint, contrary to the case where $r_i$ lies in between $c_1$ and $c_{n+1}$. In this case, at least one satisfied constraint involves $r_i$, the one in $C_2$. Thus the optimal orderings are of the desired form.

Claim 5. An optimal ordering satisfies at most $\binom{n}{2}$ constraints from $C_G$, and equality holds if and only if $G$ has an $n$-clique with one element from each row.

Proof. In an optimal ordering, for every $r_i$ there is a unique value, $\phi(i)$, such that $r_i \in R_{\phi(i)}$ (i.e. $r_i$ lies in between $c_{\phi(i)}$ and $c_{\phi(i) + 1}$). Thus, given a pair of row elements, $r_i$ and $r_{i'}$, an optimal ordering satisfies at most one constraint involving both elements (for example the constraint $(c_{\phi(i)}, r_i, c_{\phi(i) + 1}, c_{\phi(i')}, r_{i'}, c_{\phi(i') + 1})$ if
\[ \phi(i) + 2 \leq \phi(i'). \] This implies that a constraint involving \( r_i \) and \( r_{i'} \) is satisfied if and only if \( (i, \phi(i))(i', \phi(i')) \in E(G) \). Thus, \( \binom{n}{2} \) constraints from \( C_G \) are satisfied only if the vertices \( (i, \phi(i)) \) with \( i \in [n] \) form a \( n \)-clique with one vertex per row. Conversely, if \( G \) has a \( n \)-clique with one vertex per row, it is easy to order the elements in such a way that \( \binom{n}{2} \) constraints from \( C_G \) are satisfied. Note that if both \((i, j)\) and \((i', j)\), with \( i < i' \), are in the \( n \)-clique one should put \( r_i \) before \( r_{i'} \) (which are both between \( c_j \) and \( c_{j+1} \)) to satisfy the constraint \((c_j, r_{i}, r_{i'}, c_{j+1})\) associated to the edge \((i, j)(i', j)\). This concludes the proof of the claim.

Thus an optimal ordering satisfies \(|C^1_S| + |C^2_S| + \binom{n}{2}\) constraints if and only if \( G \) has an \( n \)-clique with one element from each row, and this concludes the proof of the theorem.

## 3 Main result

We have shown that a trivial enumeration algorithm for **Arity 6 Permutation CSP** cannot be significantly improved under ETH by a reduction from \( n \times n \) **CLIQUE**. In this section, we shall generalize this result to instances of arity four. For this, we successively establish lower bounds on several problems, and finally on **Arity 4 Permutation CSP** assuming ETH.

In [9], the authors prove Theorem 2 by a reduction from **3-Coloring** to \( n \times n \) **CLIQUE** such that a \( 2^{o(n \log n)} \)-algorithm for \( n \times n \) **CLIQUE** implies \( 2^{n^{o(1)}} \)-algorithm for **3-Coloring**. We follow the same line of reduction, but we need to constrain \( n \times n \) **CLIQUE** in a careful way so that we can finally reduce to **Arity 4 Permutation CSP** instances.

In the following, we present algorithmic lower bounds on \( f \)-**Sparse 3-SAT**, \( f' \)-**Sparse 3-Coloring**, \( D \)-**Degree Constrained** \( n \times n \) **CLIQUE**, and \( D \)-**Degree Constrained** \( 2n \times 2n \) **Biclique**. Then we prove Theorem 4 by an appropriate reduction of \( D \)-**Degree Constrained** \( 2n \times 2n \) **Biclique** to **Arity 4 Permutation CSP**.

### \( f \)-**Sparse 3-SAT**

**Input:** A 3CNF formula with \( n \) variables and \( m \) clauses in which each variable is contained in at most \( f \) clauses.

**Goal:** Determine if there is a satisfying assignment.

### \( f' \)-**Sparse 3-Coloring**

**Input:** A graph \( G \) on \( n \) vertices in which the maximum degree is at most \( f' \).

**Goal:** Determine if \( G \) is 3-colorable.

The Sparsification Lemma by Impagliazzo et al. states that for every \( \epsilon > 0 \), 3CNF formula on \( n \) variables can be converted as a disjunction of at most \( 2^{\epsilon n} \) 3CNF such that in each 3CNF, the frequency bounded by a function depending only on \( \epsilon \) (see [6], Corollary 1). Moreover, this disjunction can be constructed in time \( 2^{\epsilon n} \text{poly}(n) \). The following is a direct consequence of it.

**Theorem 6** (Impagliazzo et al. [6]). Assuming ETH, \( f \)-**Sparse 3-SAT** cannot be solved in \( 2^{o(n)} \)-time for every fixed \( f > 0 \).

Notice that the above theorem does not exclude the possibility of subexponential-time algorithm for some \( f \). The following theorem follows from a slight modification of the well-known reduction [10].

**Lemma 7.** Assuming ETH, \( f' \)-**Sparse 3-Coloring** cannot be solved in \( 2^{o(n)} \)-time for every fixed \( f' > 0 \).

**Proof.** We give a sketch of the polynomial-time reduction in [10] and point out how we modify the reduction in order to ensure the maximum degree. The reduction in [10] constructs a graph \( G \) from a given 3CNF formula \( \Phi \) as follows. The vertex set \( V(G) \) contains (a) three vertices \( T, F, B \) forming a triangle, (b) two
literal vertices \( v_i \) and \( \bar{v}_i \) corresponding to each variable \( x_i \) of \( \Phi \), (c) an OR-gadget \( C_j \) corresponding to \( j \)-th clause consisting of six vertices. Apart from the triangle on \( T, F, B \), edges inside OR-gadget, the edge set \( E(G) \) additionally connects (i) the pairs \( v_i \) and \( \bar{v}_i \), (ii) every literal vertex \( v_i/\bar{v}_i \) with \( N \), (iii) the literal vertex \( v_i/\bar{v}_i \) with (a vertex from) OR-gadget \( C_j \) whenever \( v_i/\bar{v}_i \) appears in \( C_j \), (iv) the vertex \( out_j \) from OR-gadget \( C_j \) with \( N \) and \( F \).

The graph \( G \) (in particular, the OR-gadget) is designed so that \( \Phi \) is satisfiable if and only if \( G \) is 3-colorable. Recall that the vertices \( T, F, N \) forms a triangle and thus has distinct colors in any 3-coloring. We say a vertex is assigned \( T (F, N \) respectively) if the vertex shares the same color with \( T (F, N \) respectively). Essentially, two properties of \( G \) ensure this if-and-only-if relation. First, in any 3-coloring of \( G \), if \( v_i \) is assigned \( T \) then \( \bar{v}_i \) is assigned \( F \) and vice versa. This is due to the connections (i), (ii). Second, the vertex \( out_j \) of OR-gadget \( C_j \) is assigned \( T \) due to (iv) and this, together with the design of OR-gadget, enforces that at least one of the literal vertices connected to \( C_j \) by (iii) is assigned \( T \).

The connection (iii) does not lead to unbounded degree of a literal vertex \( v_i/\bar{v}_i \) if we reduce from \( f\)-Sparse 3-SAT. Observe that unbounded degree may occur due to the connections (ii) and (iv). We can resolve this case by ‘expanding’ the triangle on \( T, F, B \) into a triangulated ladder as long as necessary. Now the connections in (ii) and (iv) are modified so that (ii’) every literal vertex \( v_i/\bar{v}_i \) is connected with distinct \( N \) vertex in the triangulated ladder, and (iv’) every vertex \( out_j \) from OR-gadget \( C_j \) is connected with distinct \( N \) and \( F \). Note that in the modified construction, the number of vertices created are still \( O(n + m) \). The maximum degree is now bounded by \( \max(f, 2, 5) \).

Suppose \( f'\)-Sparse 3-Coloring can be solved in time subexponential in the number of vertices for every fixed \( f' > 0 \). Then, given an instance of \( f\)-Sparse 3-SAT for any fixed \( f \), we run the presented reduction and construct \( f'\)-Sparse 3-Coloring instance with \( O(n + m) \) vertices. As the obtained instance can be solved in \( 2^{o(n+m)} \)-time and \( O(m) = O(n) \), we can solve the initial \( f\)-Sparse 3-SAT instance in time \( 2^{o(n)} + poly(n) \)-time, a contradiction to Theorem 6. Hence the statement follows.

Let us consider an instance \( G \) of \( n \times n \) Clique on the vertex set \( V = [n] \times [n] \). Let \( x \) be a vertex of \( G \) and \( X \) is a subset of \( V \). We denote the number of edges between \( x \) and \( X \) by \( \text{deg}(x, X) \). The set of vertices in the \( i \)-th row is denoted as \( R_i = \{(i, j) : j \in [n]\} \). Then given two vertex sets \( X \) and \( Y \), \( E(X, Y) \) is the set of edges with one end in \( X \) and one end in \( Y \), and \( \text{deg}(X, Y) = |E(X, Y)| \). These definition naturally extend to the case where \( X \) or \( Y \) is a single vertex.

**D-Degree Constrained \( n \times n \) Clique (D-DCnNC)**

**Input:** An instance \( G \) of \( n \times n \) Clique with two additional conditions.

(A) For every pair of rows \( i, k \in [n] \), we have \( \text{deg}((i, j), R_k) = \Delta^k \) for every \( j \in [n] \), for some constant \( \Delta^k \).

(B) For every vertex \( (i, j) \) with \( i \in [n] \), \( j \in [n-1] \), there exists a set of rows \( \exists I_{i,j} \subseteq [n] \) with \( |I_{i,j}| \geq n - D \) such that \( N(i, j) \cap R_k = N(i, j + 1) \cap R_k \) for every \( k \in I_{i,j} \).

**Goal:** Determine if there is an \( n \)-clique in \( G \) with exactly one element from each row.

**Lemma 8.** Assuming ETH, D-DCnNC cannot be solved in \( 2^{o(n \log n)} \)-time for every fixed \( D > 0 \).

**Proof.** Similarly to the proof of Theorem 2 in [9], the theorem follows from a reduction of \( f'\)-Sparse 3-Coloring to D-DCnNC. This reduction is such that an instance \( G \) of \( f'\)-Sparse 3-Coloring with \( n \) vertices is equivalent to an instance \( G' \) of D-DCnNC with \( D = f' \) and vertex set \( V' = [n'] \times [n'] \), with \( n' \log(n') = O(n) \).

Let \( x \) be the smallest integer such that \( (f'^2 + 1) + \frac{2 - f'^2 - 1}{4} \leq 3^x \). Now let \( n' = 3^x \). Note that since \( f' \) is a constant we have \( n < x 3^x < 4n \) (i.e. \( n < n' \log(n') < 4n \)) for \( n \) sufficiently large, thus \( x 3^x = O(n) \) (i.e. \( n' \log(n') = O(n) \)).
Given $G$, partition its vertices into $f^2 + 1$ parts, $V_0, \ldots, V_{f^2}$ such that any two vertices in the same part are at distance at least 3 in $G$. This is possible by Brooks’ Theorem and the fact that the maximum degree of $G^2$ is upper bounded by $f^2$. Then partition each $V_i$ into $\left\lceil \frac{|V_i|}{x} \right\rceil$ subsets $V_i^j$, such that $|V_i^j| \leq x$ and such that all the remaining sets $V_i^j$ have size exactly $x$. Here all the subsets have size $x$ except at most $f^2 + 1$ subsets of size at least 1 (and at most $x - 1$). Thus now $V$ is partitioned in at most $(f^2 + 1) + \left\lceil \frac{n - f^2 - 1}{x} \right\rceil$ (which is $\leq n'$) subsets of size at most $x$. Rename those subsets $X_1, X_2, \ldots$ and if necessary add some empty sets at the end in order to have exactly $n'$ sets $X_i$. Then add $x'n' - n$ isolated vertices in $G$ and dispatch them in the $X_i$’s in such a way that all these sets have size exactly $x$.

Now let $G'$ be the graph on $[n'] \times [n']$ such that each vertex $(i, j)$ corresponds to a 3-coloring of $X_i$. There is enough space in each row since the stable set $X_i$ has exactly $3^{|X_i|} = 3^x = n'$ 3-colorings. We enumerate these colorings in such a way that for every $i \in [n']$ and $j \in [n']$ the coloring corresponding to $(i, j)$ differs on exactly one vertex with the coloring corresponding to $(i, j + 1)$. This can be done using a ternary Gray code on $|X_i| = x$ digits, which $k^{th}$ digit corresponds to the color of the $k^{th}$ vertex in $X_i$. In $G'$ there is an edge between the vertices $(i, j)$ and $(i', j')$ if and only if the corresponding 3-colorings of $X_i$ and $X_{i'}$ are compatible, this when $i \neq i'$ and when there is no adjacent vertices $u \in X_i$ and $v \in X_{i'}$ with the same color.

It is clear that $G$ is 3-colorable if and only if $G'$ has an $n'$-clique with one element per row. So it remains now to check that $G'$ verifies conditions (A) and (B). By the construction of the sets $X_i$ (sub-partitioning the sets $V_i$), for any $i, i' \in [n]$ the induced graph $G[X_i \cup X_{i'}]$ is a matching (generally a non-perfect one). If there are $m$ edges in this matching one easily sees that any 3-coloring of $X_i$ is compatible with exactly $2^m 3^{x - m}$ 3-colorings of $X_{i'}$, and thus (A) holds. Furthermore, since two consecutive 3-colorings of $X_i$ (say the ones corresponding to $(i, j)$ and $(i, j + 1)$) differ on exactly one vertex, say $u$, of degree at most $f'$ in $G$ there are at least $n' - f' = n' - D$ sets $X_k$ without any neighbor of $u$ (eventually $X_k = X_i$). Since $u$ is isolated in $G[X_i, X_{k}]$ its color does not really matters and we clearly have that $N(i, j) \cap R_k = N(i, j + 1) \cap R_k$ (if $X_k = X_i$, then $\forall j, N(i, j) \cap R_k = \emptyset$), and thus that (B) holds. This concludes the proof of the theorem.

**D-Degree Constrained 2n × 2n Biclique (D-DCNNB)**

**Input:** A graph $G$ with vertex set $V(G) = [2n] \times [2n]$ with three additional conditions.

(A) For every edge $(i, j)(i', j') \in E(H)$ we have $1 \leq i \leq n < i' \leq 2n$ and $1 \leq j \leq n < j' \leq 2n$. Furthermore, $(i, j)(n + i', n + j') \in E(H)$ if and only if $(i', j')(n + i, n + j) \in E(H)$.

(B) For every pair of rows $i \in [n]$ and $k \in [n + 1, 2n]$, we have $\deg((i, j), R_k) = \Delta^{ik}$ for every $j \in [n]$, for some constant $\Delta^{ik}$.

(C) For every vertex $(i, j)$, with $i \in [n], j \in [n - 1]$, there exists a set of rows $I_{i,j} \subseteq [n + 1, 2n]$ with $|I_{i,j}| \geq n - D$ such that $N(i, j) \cap R_k = N(i, j + 1) \cap R_k$ for every $k \in I_{i,j}$.

**Goal:** Determine if there is $K_{n,n}$ with exactly one vertex per row.

**Lemma 9.** Assuming ETH, D-DCNNB cannot be solved in $2^{o(n \log n)}$-time for every fixed $D > 0$.

**Proof.** The theorem follows from a simple reduction of D-DCNNC to D-DCNNB. This reduction is such that an instance $G$ of D-DCNNC with vertex set $[n] \times [n]$ is equivalent to an instance $H$ of D-DCNNB with vertex set $[2n] \times [2n]$. The graph $H$ is such that $(i, j)(n + i', n + j') \in E(H)$ if and only if $(i, j)(i', j') \in E(G)$ or if $i = i'$ and $j = j'$. It is easy to see that if $H$ contains a $K_{n,n}$ with one vertex per row, the selected vertices in the first (resp. last) $n$ rows form a stable set. Furthermore for any $i \in [n]$, according to the adjacencies between $R_i$ and $R_{n+i}$, a vertex $(i, j)$ is selected if and only if $(n + i, n + j)$ is also selected. Then it is simple to conclude that $G$ has a clique with one vertex per row if and only if $H$ has a $K_{n,n}$ with one vertex per row. Thus the theorem directly follows from Lemma 9.

We can finally prove the main result.
Proof of Theorem 11 To prove the theorem we exhibit a reduction from D-DCNNB to ARITY 4 PERMUTATION CSP. This reduction associates each instance $H$ (a graph on $[2n] \times [2n]$) of D-DCNNB to an instance of $(V, C)$ of ARITY 4 PERMUTATION CSP with $|V| = (2D+4)n+1$, in such a way that $H$ is a positive instance if and only if the optimal solution to $(V, C)$ satisfies $(\begin{array}{c} 2D \\ 2 \end{array}) \binom{2n+1}{2} + (n+2) \sum_{i \in [n], j \in [n+1, 2n]} \Delta i j^\prime + n^2$ constraints. By Lemma 9, such a reduction clearly implies the theorem.

As in the proof of Theorem 3 $V$ has 3 types of elements, $2n$ elements $r_i$, with $i \in [2n]$, corresponding to the rows of $H$, $2n + 1$ elements $c_i$, with $i \in [2n + 1]$, corresponding to the columns of $H$ (except $c_{n+1}$ that does not exactly correspond to a column), and $2Dn$ dummy elements $d_i$, with $i \in [2Dn]$.

The constraint set $C$ is the union of several types of constraints, the structural constraints $C_S$ that force the shape of optimal orderings, and the constraints depending on $H$, $C_H = C_H^{crr} \cup C_H^{rcr} \cup C_H^{rcrc} \cup C_H^{ccrr}$. Formally:

- $C_S = \{(d_a, d_b, c_j, c_{j'}) \mid a < b \text{ and } j < j'\}$
- $C_H^{crr} = \{(c_j, r_i, c_{n+j}, r_{n+j'}) \mid (i, j)(n+i', n+j') \in E(H)\}$.
- $C_H^{rcr} = \{(c_j, r_i, r_{n+j'}, c_{n+j'+1}) \mid (i, j)(n+i', n+j') \in E(H)\}$.
- $C_H^{rcrc} = \{(r_i, c_{j+1}, r_{n+j'}, c_{n+j'+1}) \mid (i, j)(n+i', n+j') \in E(H)\}$.
- $C_H^{ccrr} = \{(r_i, c_{j+1}, c_{n+j'}, r_{n+j'}) \mid (i, j)(n+i', n+j') \in E(H)\}$.

There are some inconsistencies in this last case, we cannot have the same element appearing several times in some constraint. This is the case when $j = n$ and $j' = 1$ (since $c_{j+1} = c_{n+j'}$), and in such a case we replace the constraint by $(d_1, r_i, c_{n+1}, r_{n+i'}).

Claim 10. There is an optimal ordering of the form $d_1 d_2 \ldots d_{2Dn} V_{\leq n} V_{> n}$, where $V_{\leq n}$ (resp. $V_{> n}$) is a sequence of $r_i$’s and $c_j$’ with $i \leq n$ (resp. $i$ and $j > n$).

Proof. Consider any optimal ordering. Moving successively the elements $d_i$ (from $d_1$ to $d_{2Dn}$) to the $i^{th}$ position, preserves the already satisfied constraints, so after those moves the order remains optimal. Then if there are two consecutive elements $v_j$ and $v_i$ (in this order), where $v_i$ is an element $r_i$ or $c_i$ with $i \leq n$, and where $v_j$ is an element $r_j$ or $c_j$ with $j > n$, since there is no constraint involving $v_j$ and $v_i$ in this order, switching those elements preserves the already satisfied constraints. Thus there is an optimal ordering of the desired form.

Claim 11. There are optimal orderings of the form $d_1 d_2 \ldots d_{2Dn} R_0 c_1 R_1 c_2 R_2 \ldots c_{2n} R_{2n} c_{2n+1} R_{2n+1}$, where each $R_i$ with $i \leq n$ (resp. $i > n$) is a possibly empty sequence of $r_j$ with $j \leq n$ (resp. $j > n$).

Proof. By Claim 10 we can consider an optimal ordering the form $D V_{\leq n} V_{> n}$. So it remains to prove that in $V_{\leq n}$ (resp. $V_{> n}$) the $c_j$’s can be reordered increasingly. If $V_{\leq n}$ has two elements $c_{j_2}$ and $c_{j_1}$ in this order, with $1 \leq j_1 < j_2 \leq n$, separated by a (possibly empty) sequence $R$ of $r_i$’s, the following two claims show that moving $c_{j_2}$ right after $c_{j_1}$ does not decrease the number of satisfied constraints. Actually, switching $c_{j_2}$ with the $r_i$’s in $R$ decreases the number of satisfied constraints, but this is compensated by the last switch between $c_{j_2}$ and $c_{j_1}$.

Claim 12. Given any ordering with two consecutive elements $c_{j_2}$ and $r_i$ (in this order), with $j_2 \in [2, n]$ and $i \in [n]$, switching them decreases the number of satisfied constraints by at most $2Dn$.

Proof. Let us call the constraints newly satisfied after the switch activated constraints, and the newly unsatisfied constraints inactivated constraints. Note that new and inactivated constraints contain $c_{j_2}$ and $r_i$ as consecutive elements (but with opposite orders). The inactivated constraints are of the form $(c_{j_2}, r_i, c_{j'}, r_{j'}) \in C_H^{ccrr}$, for the values $i'$ and $j'$ such that $(i, j)(i', j') \in E(H)$, or of the form $(c_{j_2}, r_i, r_{i'}, c_{j'}) \in C_H^{ccrc}$.
for the values $i'$ and $j'$ such that $(i, j)(i', j' - 1) \in E(H)$. Similarly the new constraints are of the form $(r_i, c_{j2}, c_{j'}, r_{i'}) \in C_{H/crc}^r$, for the values $i'$ and $j'$ such that $(i, j - 1)(i', j') \in E(H)$, or of the form $(r_i, c_{j2}, r_{i'}, c_{j'}) \in C_{H/crc}^r$, for the values $i'$ and $j'$ such that $(i, j - 1)(i', j' - 1) \in E(H)$.

Condition (C) implies that the neighbors $N(i, j_2)$ and $N(i, j_2 - 1)$ are similar in some sense. To be precise, consider a vertex $(i', j') \in V(H)$ with $i', j' \in [n + 1, 2n]$. If the vertex is picked from row $i' \in I_{i,j_2-1}$ as defined in condition (C), then we have $(i, j_2)(i', j') \in E(H)$ if and only if $(i, j_2 - 1)(i', j') \in E(H)$. Hence it follows from the construction of $C_H$ that

$$(c_{j2}, r_i, c_{j'}, r_{i'}) \in C_{H/crc}^r$$

and

$$(c_{j2}, r_i, r_{i'}, c_{j'}) \in C_{H/crc}^r$$

Therefore, whenever the constraint $(c_{j2}, r_i, c_{j'}, r_{i'}) \in C_{H/crc}^r$ (resp. $(c_{j2}, r_i, r_{i'}, c_{j'}) \in C_{H/crc}^r$) becomes inactivated after the switch, the constraint $(r_i, c_{j2}, c_{j'}, r_{i'}) \in C_{H/crc}^r$ (resp. $(r_i, c_{j2}, r_{i'}, c_{j'}) \in C_{H/crc}^r$) becomes activated.

If the vertex $(i', j') \in V(H)$ is picked from row $i' \notin I_{i,j_2-1}$, then the inactivated constraints may not be compensated by activated constraints as in the case $i' \in I_{i,j_2-1}$. Hence the cost of the switch is bounded by $\sum_{i' \notin I_{i,j_2-1}} |N_H(i, j_2) \cap R_{i'}| \leq 2Dn$. \hfill \(\square\)

**Claim 13.** Given an ordering starting by $d_1, d_2, \ldots, d_{2Dn}$, if there are two consecutive $c_{j2}$ and $c_{j1}$ (in this order) with $j_1 < j_2$, switching them increases the number of satisfied constraints by at least $\binom{2Dn}{2}$. \hfill \(\square\)

**Proof.** Indeed, there is no constraint involving $c_{j2}$ and $c_{j1}$ in this order. Thus switching them only adds new satisfied constraints, including the $\binom{2Dn}{2}$ constraints $(d_{i1}, d_{i2}, c_{j1}, c_{j2}) \in C_S$. \hfill \(\square\)

Thus moving $c_{j2}$ along $R$ (which has length at most $n$) costs at most $n(2Dn)$ which is compensated by the benefits from the last switch between $c_{j2}$ and $c_{j1}$ ($\binom{2Dn}{2} \geq 2Dn^2$). Similar arguments hold for $V_{> n}$, and this concludes the proof of the lemma. \hfill \(\square\)

A *convenient* ordering is an orderings of the form $d_1 d_2 \ldots d_{2Dn} c_1 R_1 c_2 R_2 \ldots c_{2n} R_{2n} c_{2n+1}$, where each $R_i$ with $i \leq n$ (resp. $i > n$) is a possibly empty sequence of $r_j$ with $j \leq n$ (resp. $j > n$). Given a convenient ordering, one can define a function $\phi : [2n] \to [2n]$ such that $\forall i \in [2n], r_i \in R_{\phi(i)}$. Given such a function one can define a set of vertices with exactly one vertex per row, $V_\emptyset = \{(i, \phi(i)) \mid i \in [2n]\}$.

**Claim 14.** There are optimal orderings that are convenient. \hfill \(\square\)

**Proof.** Consider an optimal ordering as described in Claim 11. If there was a $r_i$ just before $c_1$ (resp. just after $c_{2n+1}$), switching those two elements would not contradict any satisfied constraint: there is no constraint with a $r_i$ before $c_1$ or after $c_{2n+1}$. Thus there exists an optimal orderings of the desired form. \hfill \(\square\)

In the following let us denote $\sum \Delta^{i,i'}$ the following sum $\sum_{1 \leq i \leq n < i' \leq 2n} \Delta^{i,i'}$.

**Claim 15.** The number of constraints satisfied by a convenient ordering is $\binom{2Dn}{2}(\binom{2n}{2} + (n + 2)) \sum \Delta^{i,i'} + |E(H[V_\emptyset])|$. \hfill \(\square\)

**Proof.** It is clear that a convenient ordering satisfies all the $\binom{2Dn}{2}$ constraints in $C_S$. For the satisfied constraints in $C_H$, consider any $i \in [n]$ and $i' \in [n + 1, 2n]$, and note that the constraints involving $r_i$ and $r_{i'}$ are of four types: they belong to one of $C_{H/crc}^r$, $C_{H/crc}^r$, $C_{H/crc}^r$, $C_{H/crc}^r$.

Note that the satisfied constraints from $C_{H/crc}^r$ involving $r_i$ and $r_{i'}$ are exactly the constraints of the form $(c_k, r_i, c_{k'}, r_{i'})$ with $k \in [1, \phi(i)]$ and $k' \in [n + 1, \phi(i')]$. Thus there are $\deg(R_{\phi(i)}^{c(i)}, R_{\phi(i')}^{c(i')})$ such
constraints, where \( \mathcal{R}^{\leq \phi(i)}_i \) is the set of the \( \phi(i) \) first vertices of the \( i^{th} \) row of \( H \). Similar arguments on \( \mathcal{C}^{\leq \phi(i)}_H \), \( \mathcal{C}^{\leq \phi(i)}_H \), and \( \mathcal{C}^{\leq \phi(i)}_H \) lead to the fact that the total number of satisfied constraints involving \( r_i \) and \( r'_j \) is exactly \( \text{deg}(\mathcal{R}^{\leq \phi(i)}_i, \mathcal{R}^{\leq \phi(i')}_i) + \text{deg}(\mathcal{R}^{\leq \phi(i)}_i, \mathcal{R}^{\leq \phi(i')}_i) + \text{deg}(\mathcal{R}^{\leq \phi(i)}_i, \mathcal{R}^{\leq \phi(i')}_i) \), which equals \( \text{deg}(\mathcal{R}_i, \mathcal{R}_i') + \text{deg}(\mathcal{R}_i, \phi(i)), \mathcal{R}_i + \text{deg}(\mathcal{R}_i, \phi(i)), \mathcal{R}_i + \text{deg}(\mathcal{R}_i, \phi(i)), \mathcal{R}_i ) \), and which by condition (B) equals \((n+2)^{i,i'} + \text{deg}(\mathcal{R}_i, \phi(i)), \mathcal{R}_i, \phi(i') ))\). Summing this for every \( i \in [n] \) and \( i' \in [n+1, 2n] \) leads to the mentioned value.

Thus, if an optimal ordering satisfies \((\frac{2D_H}{2})(2^{\leq n+1}) + (n+2) \sum \Delta^{i,i'} + n^2 \) constraints, there is a convenient ordering (by Claim 13) defining a function \( \phi \) such that the graph \( H[V_0] \) has \( n^2 \) edges (by Lemma 15). Since \( H \) is bipartite, this subgraph is bipartite and the number of edges implies that it is a \( K_{n,n} \) (with one vertex per row, by construction). Conversely if \( H \) has a \( K_{n,n} \) with one vertex per row, one can easily construct a convenient ordering satisfying \((\frac{2D_H}{2})(2^{\leq n+1}) + (n+2) \sum \Delta^{i,i'} + n^2 \) constraints. This concludes the proof of the theorem.

Finally, up to our best knowledge, no \( O^*(c^n) \)-time algorithm for Permutation CSP is known for \( c < 2 \) even when the arity is two. Recently, Cygan et al. proved in [4] that for a number of basic problems including Hitting Set, improving upon the \( 2^n \)-barrier contradicts Strong Exponential Time Hypothesis (SETH) [7]. Although SETH is not as widely believed as ETH, proving or disproving SETH will be a major breakthrough in the domain. We leave it as an open problem whether it is possible to break the \( 2^n \)-barrier for arity up to three.

References