

# Payoff-dependent Balancedness and Cores (revised version) <sup>\*</sup>

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## Abstract

We prove the non-emptiness of the core of an NTU game satisfying a condition of payoff-dependent balancedness, based on transfer rate mappings. We also define a new equilibrium condition on transfer rates and we prove the existence of core payoff vectors satisfying this condition. The additional requirement of transfer rate equilibrium refines the core concept and allows the selection of specific core payoff vectors. Lastly, the class of parametrized cooperative games is introduced. This new setting and its associated equilibrium-core solution extend the usual cooperative game framework and core solution to situations depending on an exogenous environment. A non-emptiness result for the equilibrium-core is also provided in the context of a parametrized cooperative game. Our proofs borrow mathematical tools and geometric constructions from general equilibrium theory with non convexities. Applications to extant results taken from game theory and economic theory are given.

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*Key words:* balancedness, cooperative game, core, parametrized game.

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## 1 Introduction

The core of an  $n$ -person cooperative game is the set of feasible outcomes that cannot be improved upon by any coalition of players. There are two well known theorems demonstrating non-emptiness of the core: the Bondareva-Shapley theorem states that the core of a TU game is non-empty if and only if the game is balanced and Scarf's theorem states that the core of a balanced NTU game is non-empty.

Since Scarf's result there have been a number of advances in the study of conditions ensuring non-emptiness of the core, but the balancedness conditions in the literature always rely on the same principle (Billera, 1970; Keiding and Thorlund-Petersen, 1987). To explain this principle, let us first consider a TU game. In such a game, within a coalition, transfers of utility can be made from one player to another at a constant one-to-one rate. In this situation, a transfer rate vector is naturally associated to each coalition. The coordinates of this vector are 1 for the members of the coalition and 0 for the other players. Given these transfer rate vectors, a family of coalitions is balanced if the transfer rate vector of the grand coalition is a positive linear combination of the transfer rate vectors of the coalitions in the family. A TU game is balanced if any feasible payoff vector for a balanced family of coalitions is feasible for the grand coalition.

For NTU games, the transfer rate vector associated to a coalition, whose coordinates are 1 for the members of the coalition and 0 for the other players, can be defined but it does not always correspond to a feasible transfer among the members of the coalition. Nevertheless the concept of balancedness has been extended to NTU games and Scarf (1967) has demonstrated the non-emptiness of the core of a balanced NTU game. Further advances introduce transfer rate vectors with unrestricted positive coordinates (Billera, 1970). In their definition of balanced games, Keiding and Thorlund-Petersen (1987) restrict Billera's requirement to specific efficient payoff vectors.

In the current paper, we extend the notion of balancedness by replacing transfer rate vectors by a transfer rate rule. The transfer rate rule associates transfer rate vectors to each efficient payoff vector of each coalition. We then introduce payoff-dependent balancedness, which is defined relative to a payoff vector. From this definition of balancedness, we deduce the notion of a payoff-dependent balanced game. Our first result, Theorem 2.1 states that any payoff-dependent balanced game has a non-empty core.

It can be shown that balanced games according to the extant literature (including the standard balancedness (Scarf, 1967);  $b$ -balancedness (Billera, 1970); balancedness for convex games (Billera, 1970);  $(b, <)$ -balancedness (Keiding

and Thorlund-Petersen, 1987)) are all payoff-dependent balanced. Hence, Theorem 2.1 generalizes the prior non-emptiness results.

In independent research, Predtetchinski and Herings (2004) introduce the condition of  $\Pi$ -balancedness, which is closely related to our condition of payoff-dependent balancedness. They prove that their condition is not only sufficient for non-emptiness of the core in NTU games but also necessary. Thus, they characterize core non-emptiness for NTU games analogous to the Bondareva-Shapley characterization for TU games.

Using the flexibility of payoff-dependent balancedness, Theorem 2.1 actually goes beyond non-emptiness of the core. Indeed, Theorem 2.1 shows the existence of a core payoff vector  $x$  satisfying an additional equilibrium condition. This equilibrium condition involves transfer rate vectors for the coalitions for which  $x$  is feasible. Then a number of results in the literature involving core payoff vectors can be deduced from Theorem 2.1.

For example, we deduce the key lemma of Reny and Wooders (1996) which exhibits a core payoff vector satisfying an equilibrium condition for credit/debit mappings. To deduce the lemma we consider a transfer rate rule that takes into account the individual contributions of the agents within the different coalitions. Consequently, the non-emptiness of the partnered core of Reny and Wooders (1996) and the existence of the average prekernel intersected with the core (Orshan et al., 2003) can be deduced from Theorem 2.1.

Lastly, we consider parametrized cooperative games. While Ichiishi (1981) does not explicitly use this abstract framework, our concept of a parametrized cooperative game is very much in the spirit of his work. The situation is the following: the payoff sets of coalitions, taken as set-valued mappings, depend on parameters which stand for an abstract environment; furthermore, an equilibrium condition on the environment is represented by a set-valued mapping depending on the environment and the payoff vector.

In the context of a parametrized game, we define an equilibrium-core vector pair, which is an environment-payoff pair where the payoff vector belongs to the core of the game associated to the environment and the environment is a fixed point of the equilibrium set-valued mapping. Under our payoff-dependent balancedness condition, we prove the existence of equilibrium-core vector pairs in Theorem 3.1. The existence of a Social Coalitional Equilibrium as stated in the work of Ichiishi (1981) is a consequence of our result.<sup>1</sup> Let us recall that the Social Equilibrium of Debreu (1952) is a particular Social Coalitional Equilibrium. In economies without ordered preferences, or in economies with increasing returns, Border (1984) or Ichiishi and Quinzii (1983) used implicitly

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<sup>1</sup> Note that we limit ourselves to a finite dimensional Euclidean space, whereas Ichiishi considers a locally convex Hausdorff topological vector space.

the parametrized framework to show non-emptiness of the core. We show how these results can be deduced from Theorem 3.1.

The geometric intuition behind our proof is borrowed from the existence of a general pricing rule equilibrium in an economy with a non-convex production sectors, see Bonnisseau and Cornet (1988, 1991) and Bonnisseau (1997). We show in the proof of Theorem 2.1 that a core payoff vector may actually be considered as an equilibrium of a two-production-set economy. Moreover, the price equilibrium condition given by the pricing rules of the economy may be restated as a transfer rate rule equilibrium condition satisfied by a core payoff vector. Finally, the analogy between core and equilibrium allocations sheds light on the close relationship between the two key assumptions of these theories: balancedness in cooperative games and survival in general economic equilibrium. Note that Vohra (1987) and Shapley and Vohra (1991) already quoted similarities between the fixed point mappings they use to show non-emptiness of the core and the fixed point mappings used in general economic equilibrium.<sup>2</sup> However these authors did not further investigate this direction.

Since our intuition comes from general economic equilibrium, the main results are naturally obtained through Kakutani's fixed-point theorem. Note that one usually associates the question of non-emptiness of cores with KKMS covering theorems or Fan's coincidence theorems, but binding the concept of core with Kakutani's theorem makes sense due to its intimate link with Walrasian economies.<sup>3</sup>

The paper is organized as follows: in Section 2, Theorem 2.1 states the existence of core payoff vectors with transfer rate rule equilibrium in payoff-dependent balanced games. Then we show how one can deduce a number of results involving balancedness. Section 3 is devoted to the parametrized model and its related topics. Under our condition of payoff-dependent balancedness, we state a result for the existence of equilibrium-core vector pair in a parametrized game, Theorem 3.1, covering Theorem 2.1. Quoted examples of applications will follow. In the body of the paper, the proofs consist mainly of geometric constructions, which are of constant use. The proofs of Theorems and technical lemmas are given in Appendix. Except for some notations and basic assumptions given below, Sections 2 and 3 can be taken independently. We discuss possible directions for future works in Section 4.

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<sup>2</sup> See Vohra (1988) to convince oneself of this fact for Shapley–Vohra mappings.

<sup>3</sup> See the discussion about these links in Ichiishi (1993, p.118-125).

## 2 Core solutions in NTU Games

*Notations.*<sup>4</sup>

$N$  is the finite set of players;

$\mathcal{N}$  is the set of non-empty subsets of  $N$ , i.e. the coalitions of players;

For each  $S \in \mathcal{N}$ ,  $L_S$  is the  $|S|$ -dimensional subspace of  $\mathbb{R}^N$  defined by

$$L_S = \{x \in \mathbb{R}^N \mid x_i = 0, \forall i \notin S\}$$

$L_{S^+}$  ( $L_{S^{++}}$ ) is the non negative orthant (positive orthant) of  $L_S$ ;

For each  $x \in \mathbb{R}^N$ ,  $x^S$  is the projection of  $x$  into  $L_S$ ;

$\mathbf{1}$  is the vector of  $\mathbb{R}^N$  whose coordinates are equal to 1;

$\mathbf{1}^\perp$  is the hyperplane  $\{s \in \mathbb{R}^N \mid \sum_{i \in N} s_i = 0\}$ ;

proj is the orthogonal projection mapping on  $\mathbf{1}^\perp$ ;<sup>5</sup>

$\Sigma_S = \text{co}\{\mathbf{1}^{\{i\}} \mid i \in S\}$ ;<sup>6</sup>

$m^S = \frac{\mathbf{1}^S}{|S|}$ ;

$\Sigma = \Sigma_N$  and  $\Sigma_{++} = \Sigma \cap \mathbb{R}_{++}^N$ .

*Game description.*

A game  $(V_S, S \in \mathcal{N})$  is a collection of subsets of  $\mathbb{R}^N$  indexed by the members of  $\mathcal{N}$ .

$x \in \mathbb{R}^N$  is a payoff vector;

$V_S \subset \mathbb{R}^N$  is the (feasible) payoff set of the coalition  $S$ ;

$\mathcal{S}(x) = \{S \in \mathcal{N} \mid x \in \partial V_S\}$  is the set of coalitions for which  $x \in \mathbb{R}^N$  is an efficient payoff vector;

$W := \cup_{S \in \mathcal{N}} V_S$  is the union of the payoff sets.

**Definition 2.1** *Let  $(V_S, S \in \mathcal{N})$  be a game. A payoff vector  $x$  is in the core of the game if  $x \in V_N \setminus \text{int } W$ .*

It is worth noting that this formulation of the core only involves two sets:  $V_N$  and  $W$ . A payoff vector for the grand coalition  $N$ , which is obviously in  $W$ , belongs to the core if it lies on the boundary of the set  $W$ . This formulation is crucial in the remainder of the paper, underpinning most of our geometric constructions. Equivalently, note also that  $x$  belongs to the core if and only if

<sup>4</sup> For any set  $Y \subset \mathbb{R}^N$ ,  $\text{co}(Y)$ ,  $\partial Y$ ,  $\text{int } Y$  and  $\text{cone} Y$ , will denote respectively its convex hull, boundary, interior and the conic hull, which is the smallest convex cone containing  $Y$ . For any set-valued mapping  $\Gamma$ ,  $\text{Gr } \Gamma$  will denote its graph.

<sup>5</sup> For each  $x \in \mathbb{R}^N$ ,  $\text{proj}(x) = x - \frac{\sum_{i \in N} x_i}{|N|} \mathbf{1}$ .

<sup>6</sup> Given the definition of  $x^S$ ,  $\mathbf{1}^S$  is the vector with coordinates equal to 1 in  $S$  and equal to 0 outside  $S$ .

$x \in \partial W$  and  $N \in \mathcal{S}(x)$ .<sup>7</sup>

We now posit two basic assumptions on the game. Roughly, (A.1) states that the payoff set of any coalition  $S$  satisfies free disposal and is represented as a cylinder of  $\mathbb{R}^N$  where the coordinates associated with non members of  $S$  are unconstrained. (A.2) states that the individually rational payoff set of any coalition is bounded above.<sup>8</sup>

**(A.1)** (i)  $V_{\{i\}}$ ,  $i \in N$ , and  $V_N$  are non-empty.

(ii) For each  $S \in \mathcal{N}$ ,  $V_S$  is closed,  $V_S - \mathbb{R}_+^N = V_S$ ,  $V_S \neq \mathbb{R}^N$  and, for all  $(x, x') \in (\mathbb{R}^N)^2$ , if  $x \in V_S$  and  $x^S = x'^S$ , then  $x' \in V_S$ .

**(A.2)** There exists  $m \in \mathbb{R}$  such that, for each  $S \in \mathcal{N}$ , for each  $x \in V_S$ , if  $x \notin \text{int } V_{\{i\}}$  for all  $i \in S$ , then  $x_j \leq m$  for all  $j \in S$ .

## 2.1 The main theorem

Before stating the main result of this section, we note that, under (A.1),  $V_N$  and  $W$  satisfy the assumptions of Bonnisseau and Cornet (1988, Lemma 5.1. p.139). Therefore, there exist continuous mappings  $p_N$  from  $\mathbb{R}^N$  to  $\partial V_N$  and  $\lambda_N$  from  $\mathbf{1}^\perp$  to  $\mathbb{R}$  such that, for all  $x \in \mathbb{R}^N$ ,  $p_N(x) = \text{proj}(x) - \lambda_N(\text{proj}(x))\mathbf{1}$ . The projection mapping  $p_N$  is illustrated in Figure 1.

In the remainder we will also need a second projection mapping  $p_W$ . The only difference between  $p_W$  and  $p_N$  is that  $p_W$  is defined as into  $\partial W$  instead of  $\partial V_N$ .

We are now in position to define the notions of transfer rate rule and of payoff-dependent balanced game.

**Definition 2.2** *Let  $(V_S, S \in \mathcal{N})$  be a game satisfying (A.1)*

*(i) A transfer rate rule is a collection of set-valued mappings  $((\varphi_S)_{S \in \mathcal{N}}, \psi)$  such that: for each  $S \in \mathcal{N}$ ,  $\varphi_S : \partial V_S \rightarrow \Sigma_S$  is upper semi-continuous*

<sup>7</sup> Obviously, Definition 2.1 can be adapted to analyze situations where the grand coalition is not the only candidate for the stability concept. For instance, if  $x \in \partial W$  and  $\{S_1, \dots, S_K\} \subset \mathcal{S}(x)$  where the collection  $\{S_1, \dots, S_K\}$  is a partition of  $N$  then  $x$  belongs to the coalition structure core as named by Greenberg (1992, Definition 6.1, p. 1316), see also the end of Sub-section 3.2.1.

<sup>8</sup> Assumption (A.1) implies that there exists, for each  $i \in N$ ,  $v_i \in \mathbb{R}$  such that  $V_{\{i\}} = \{x \in \mathbb{R}^N \mid x_i \leq v_i\}$ . Note that we do not require  $V_S$  to be non-empty for  $S \neq N$  and  $S \neq \{i\}$ ,  $i \in N$ . But, note that the core would be unchanged if we replace the empty sets  $V_S$  by  $V_S := \{x \in \mathbb{R}^N \mid x_i \leq v_i, \forall i \in S\}$ . We do not normalize the game as usually done. For instance, in Shapley and Vohra (1991), the game is normalized without loss of generality by imposing  $v_i > 0$  for each  $i \in N$ .

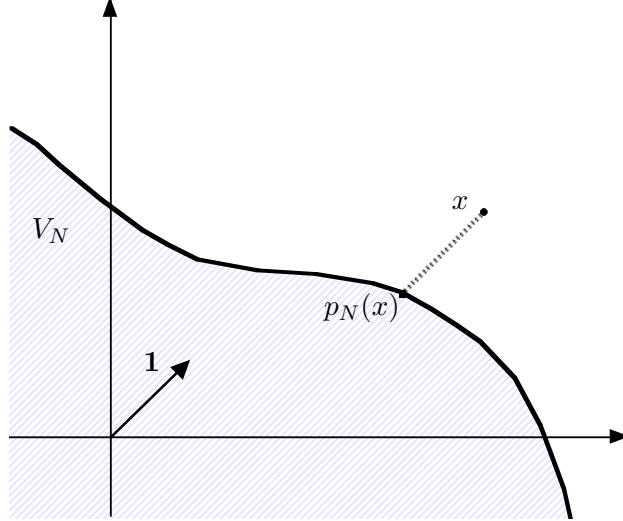


Fig. 1. The projection mapping  $p_N$

with non-empty compact and convex values;  $\psi : \partial V_N \rightarrow \Sigma$  is upper semi-continuous with non-empty compact and convex values.

(ii) The game  $(V_S, S \in \mathcal{N})$  is payoff-dependent balanced if there exists a transfer rate rule  $((\varphi_S)_{S \in \mathcal{N}}, \psi)$  such that, for each  $x \in \partial W$ ,

$$\text{if } \text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset, \text{ then } x \in V_N.$$

The usual transfer rate rule is constant and single-valued, for instance in Scarf (1967),  $\varphi_S = m^S$  and in Billera (1970),  $\varphi_S = b_S$  for some  $b_S \in L_{S^+} \setminus \{0\}$ . Our innovation comes from the fact that we allow the transfer rates to depend on the payoff vector  $x$ . An example of such transfer rate rule is given in Figure 2. Let  $\bar{x} \in \partial V_N$ . For each  $x \in \partial V_N$ , the mapping  $\varphi_N$  is defined as:<sup>9</sup>

$$\varphi_N(x) = \{t \in \Sigma \mid t \cdot x = t \cdot \bar{x}\}$$

To give some intuition on the balancedness condition, let us first consider the case where  $\psi = \varphi_N$ . For any given efficient payoff vector  $x \in \partial W$ , for any coalition  $S \in \mathcal{S}(x)$ ,  $\varphi_S(x)$  defines a set of admissible transfer rates between agents within the coalition. Let  $b_S \in \varphi_S(x)$  for each  $S \in \mathcal{S}(x)$  and  $b_N \in \varphi_N(p_N(x))$ . Thus, the game is payoff-dependent balanced at  $x$  if  $x \in V_N$  whenever the family  $\mathcal{S}(x)$  is balanced in the standard sense, that is,  $b_N$  belongs to the convex hull of the family  $(b_S)_{S \in \mathcal{S}(x)}$ .

The class of payoff-dependent balanced games includes all concepts of balanced

<sup>9</sup> This specific transfer rate rule plays a crucial role in the contribution of Predtetchinski and Herings (2004), as demonstrated by the proof of their main result.

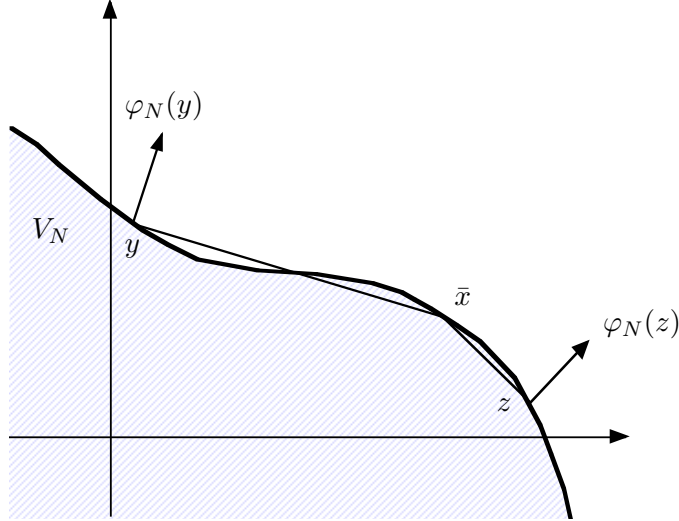


Fig. 2. Example of transfer rate rule

games in the literature (Scarf, 1967; Billera, 1970; Keiding and Thorlund-Petersen, 1987) and convex games à la Billera (1970) (see Sub-section 2.3). In addition, the introduction of the set-valued mapping  $\psi$  allows us to get some refinements to the core, whenever  $\psi$  differs from  $\varphi_N$  (see Sub-section 2.4).

We also point out that the balancedness condition holds only on  $\partial W$ , the (*weakly*) efficient frontier of the game.

The following theorem is the first result of the paper. Its proof is given in Appendix.

**Theorem 2.1** *Let  $(V_S, S \in \mathcal{N})$  be a game satisfying (A.1) and (A.2). If the game is payoff-dependent balanced with respect to the transfer rate rule  $((\varphi_S)_{S \in \mathcal{N}}, \psi)$ , then there exists a core payoff vector  $x$  satisfying:*

$$\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset.$$

Predtetchinski and Herings (2004) do not obtain the second part of the conclusion of Theorem 2.1, that is, the fact that the family  $\mathcal{S}(x)$  is balanced with respect to the transfer rate vectors given by the transfer rate rule. Note also that their boundedness assumption is weaker than ours (A.2) but their notion of payoff-dependent balancedness is slightly stronger. Actually, Theorem 2.1 brings to mind numerous contributions (Ichiishi and Idzik, 2002; Kannai and Wooders, 2000; Orshan et al., 2003; Reny and Wooders, 1996, 1998) that show that balancedness implies existence of an element  $x$  in the core satisfying the condition that  $\mathcal{S}(x)$  is balanced (in Scarf's sense). The relationship with the contributions of Orshan et al. (2003) and Reny and Wooders (1996) is illustrated in Sub-section 2.4. Since these authors are dealing with extensions



of KKMS Theorems, the link with the results of Ichiishi and Idzik (2002), Kannai and Wooders (2000) and Reny and Wooders (1998) is beyond the scope of our paper.

## 2.2 About the proof

To emphasize the geometric intuition that leads our reasonings, we provide a brief outline of the proof. We will use a weak version of an abstract result of Bonnisseau and Cornet (1991, Theorem 1 p.67), to obtain Theorem 2.1 as a corollary of existence of general equilibrium in an economy with non-convex production sets.

Before recalling the abstract result, we first posit some notations. Let  $C$  be a closed, convex cone included in  $\mathbb{R}_{++}^N \cup \{0\}$  and containing  $\mathbf{1}$  in its interior. Let  $\Delta = \{x \in -C^\circ \mid x \cdot \mathbf{1} = 1\}$ , where  $C^\circ$  is the negative polar cone of  $C$ , i.e.  $C^\circ = \{t \in \mathbb{R}^N \mid t \cdot x \leq 0, \text{ for all } x \in C\}$ . Note that  $\Sigma \subset \text{int } \Delta$ .

**Theorem 2.2 (Bonnisseau–Cornet (1991))** *Let  $Y_1$  and  $Y_2$  two subsets of  $\mathbb{R}^N$ . For each  $j = 1; 2$ , let  $\tilde{\varphi}_j$  be a set-valued mapping from  $\partial Y_j$  to  $\Delta$ . Assume the following:*

- (P) For  $j = 1; 2$ ,  $Y_j$  is closed, non-empty and  $Y_j - C = Y_j$  (free disposal);  $\tilde{\varphi}_j$  is upper semi-continuous with non-empty convex values; there exists  $\alpha_j \in \mathbb{R}$  such that for all  $y_j \in \partial Y_j$  and, for all  $p \in \tilde{\varphi}_j(y_j)$ , it holds that  $p \cdot y_j \geq \alpha_j$  (bounded losses).
- (B) For each  $t \geq 0$ ,  $A_t = \{(y_1, y_2) \in Y_1 \times Y_2 \mid y_1 + y_2 + t\mathbf{1} \in C\}$  is bounded.
- (S) For each  $t > 0$ , for each  $(p, y_1, y_2) \in \Delta \times \partial Y_1 \times \partial Y_2$ , if  $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$  and  $y_1 + y_2 + t\mathbf{1} \in C$ , then  $p \cdot (y_1 + y_2 + t\mathbf{1}) > 0$ .

*Then there exists  $(y_1, y_2, p) \in \partial Y_1 \times \partial Y_2 \times \Delta$  such that  $y_1 + y_2 \in C$  and  $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$ .*

To make the link with Bonnisseau and Cornet (1991, Theorem 1 p.67) precise, the reader must set  $C = X = \mathbb{R}_+^N$ . It is an easy matter to check that the proof works with a general convex cone  $C$ .<sup>10</sup> The use of the cone  $C$  is necessary to show that (S) holds true, which is not true for the cone  $\mathbb{R}_+^N$ , see Appendix, proof of Lemma 4.4.

The proof of Theorem 2.1 relies on the construction of a fictitious economy with two (*non convex*) production sets built upon the payoff sets of the coalitions and with pricing rules derived from the transfer rate rule. Then, one obtains the existence of a core payoff vector from an equilibrium,

<sup>10</sup> See Bonnisseau and Jamin (2003).

which exists thanks to Theorem 2.2. The last conclusion of Theorem 2.1,  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset$ , comes from the equilibrium condition  $p \in \tilde{\varphi}(y_1) \cap \tilde{\varphi}(y_2)$ .

Our balancedness assumption is used only once in the proof of Theorem 2.1, namely in Lemma 4.4 (Claim 3), where we show that the fictitious economy satisfies (S). Surprisingly (or not?), the argument binds intimately the most questionable assumptions of general equilibrium theory and cooperative game theory, respectively survival and balancedness.

### 2.3 About the balancedness condition

In this sub-section, we focus on the particular case of Theorem 2.1 where  $\psi = \varphi_N$ . We show how Theorem 2.1 generalizes the existing results on core non-emptiness. The proofs are based on the definition of the "right" transfer rate rule  $((\varphi_S)_{S \in \mathcal{N}}, \psi)$ . Motivated by a result of Keiding and Thorlund-Petersen (1987), we also provide another necessary and sufficient for non-emptiness of the core. Next, an example of a game that is not balanced according to the prior definitions is given. Nevertheless, the game satisfies our condition of payoff-dependent balancedness and its core is non-empty. Finally, the particular case of convex NTU games is also briefly described.

There are three dominant versions of balancedness in the NTU game literature due to Scarf (1967), Billera (1970) and Keiding and Thorlund-Petersen (1987). In Scarf (1967), a family of coalitions  $\mathcal{B} \subset \mathcal{N}$  is balanced if for each  $S \in \mathcal{B}$ , there exists  $\lambda_S \in \mathbb{R}_+$  such that  $\sum_{S \in \mathcal{B}} \lambda_S \mathbf{1}^S = \mathbf{1}$ . A game  $(V_S, S \in \mathcal{N})$  is balanced if for any balanced family of coalitions  $\mathcal{B} \subset \mathcal{N}$ ,  $\cap_{S \in \mathcal{B}} V_S \subset V_N$ . In Billera (1970), a transfer rate vector  $b_S \in L_{S^+} \setminus \{0\}$  is associated to each coalition  $S \neq N$  and for the grand coalition, the transfer rate vector is given by  $b_N \in L_{N^{++}}$ . A family of coalitions  $\mathcal{B} \subset \mathcal{N}$  is  $b$ -balanced if for each  $S \in \mathcal{B}$ , there exists  $\lambda_S \in \mathbb{R}_+$  such that  $\sum_{S \in \mathcal{B}} \lambda_S b_S = b_N$ . The game is  $b$ -balanced if for any  $b$ -balanced family of coalition  $\mathcal{B} \subset \mathcal{N}$ ,  $\cap_{S \in \mathcal{B}} V_S \subset V_N$ . Keiding and Thorlund-Petersen (1987) define yet another notion of balanced family, called a  $(b, <)$ -balanced family. Since their definition is complicated, we refer the reader to their paper. The game is  $(b, <)$ -balanced if for any  $(b, <)$ -balanced family of coalition  $\mathcal{B} \subset \mathcal{N}$ ,  $\cap_{S \in \mathcal{B}} V_S \subset V_N$ .

The prior results can be obtained from ours, let us first define  $(b, \partial)$ -balanced games. A game  $(V_S, S \in \mathcal{N})$  is  $(b, \partial)$ -balanced if for any  $b$ -balanced family of coalitions  $\mathcal{B} \subset \mathcal{N}$ ,  $\partial W \cap (\cap_{S \in \mathcal{B}} V_S) \subset V_N$ . Note that this notion of a balanced game is the same as Billera's except that we include  $\partial W$  in the intersection condition.

**Corollary 2.1** *Let  $(V_S, S \in \mathcal{N})$  be a game satisfying (A.1) and (A.2). For*

each  $S \in \mathcal{N}$ , let  $b_S \in L_{S^+} \setminus \{0\}$ . If the game is  $(b, \partial)$ -balanced then it admits a core payoff vector.

**Proof of Corollary 2.1.** We show that the game is payoff-dependent balanced. For each  $S \in \mathcal{N}$ , let  $\varphi_S$  be the constant mapping which associates  $\frac{1}{\sum_{i \in S} b_{Si}} b_S$  to each  $x \in \partial V_S$  and let  $\psi = \varphi_N$ . Let  $x \in \partial W$  be such that  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(x) \neq \emptyset$ . This means that there exist  $\lambda_S \in \mathbb{R}_+$ ,  $S \in \mathcal{S}(x)$ , which satisfy

$$\sum_{S \in \mathcal{S}(x)} \lambda_S \frac{1}{\sum_{i \in S} b_{Si}} b_S = \frac{1}{\sum_{i \in N} b_{Ni}} b^N$$

Consequently,

$$\sum_{S \in \mathcal{S}(x)} \lambda_S \frac{\sum_{i \in N} b_{Ni}}{\sum_{i \in S} b_{Si}} b_S = b_N,$$

implying that the family  $\mathcal{S}(x)$  is  $b$ -balanced. Then,  $(b, \partial)$ -balancedness implies that  $x \in V_N$ .

Applying Theorem 2.1, the game has a non-empty core.  $\square$

The result of Billera (1970) is immediately deduced from Corollary 2.1. In Scarf (1967), the author proves that the core of a balanced game is non-empty under (A.1) and

**(A. W2)** There exists  $m \in \mathbb{R}$  such that, for each  $x \in V_N$ , if  $x \notin \text{int } V_{\{i\}}$  for all  $i \in N$ , then  $x \leq m\mathbf{1}$ .

Under (A.1) and (A.W2), we remark that (A.2) also holds true if the game is balanced, thus Scarf's result is a consequence of Corollary 2.1. Indeed, given  $S \in \mathcal{N}$ , we remark that  $\{S, (\{i\})_{i \notin S}\}$  is a balanced family. Now let  $x \in V_S$  satisfies  $x_i \geq v_i$ ,  $i \in S$ . Let  $x'$  be defined by  $x'_i = x_i$ ,  $i \in S$  and  $x'_i = v_i$ ,  $i \notin S$ . From (A.1),  $x' \in V_S \cap (\cap_{i \notin S} V_{\{i\}})$ . From the balancedness of the game,  $x' \in V_N$  and clearly,  $x'_i \geq v_i$ ,  $i \in N$ . Consequently, from (A.W2),  $x' \leq m\mathbf{1}$ , which implies  $x_i \leq m$ ,  $i \in S$ . Thus, Scarf's result is obtained as a corollary of Theorem 2.1.

We leave it to the reader to check that  $(b, \partial)$ -balancedness is weaker than the  $(b, <)$ -balancedness, so that the non-emptiness result given by Keiding and Thorlund-Petersen (1987, Theorem 2.1 p.277) is also a consequence of Corollary 2.1.

To motivate our next result, recall that Keiding and Thorlund-Petersen (1987) characterize a class of games with non-empty cores, called *weakly*  $(b, <)$ -balanced games. Defining a member of this class of games involves approximation of a game by a sequence of games. As we show now, one can simplify

the statement of Keiding and Thorlund-Petersen result by replacing the notion of  $(b, <)$ -balancedness by  $(b, \partial)$ -balancedness. For each coalition  $S$ , let  $V_{S^*} = \{x \in V_S \mid x \notin \text{int } V_{\{i\}}, \forall i \in S\}$  be the set of individually rational payoff vectors. A game is weakly  $(b, \partial)$ -balanced if there exists a sequence  $\{V^\tau\}_{\tau=1}^\infty$  of  $(b, \partial)$ -balanced games such that:  $V_N = V_N^\tau$  for all  $\tau$  and, for all  $S \in \mathcal{N}$ , the sequence  $\{V_{S^*}^\tau\}_{\tau=1}^\infty$  converges to the set  $V_{S^*}$  for the Hausdorff topology on the non-empty compact sets of  $\mathbb{R}^N$ . Using the arguments of Keiding and Thorlund-Petersen (1987, Proof of Theorem 5.1. p.286), one can show the following result:

**Proposition 2.1** *Let  $(V_S, S \in \mathcal{N})$  be a game satisfying (A.1) and (A.2). The game has a non-empty core if and only if there exists a weakly  $(b, \partial)$ -balanced game  $(V', N)$  such that  $V_N = V'_N$  and  $V_S \subset V'_S, S \in \mathcal{N}$ .*

### 2.3.1 An example

The following 3-player game with a non-empty core is not  $(b, \partial)$ -balanced. Let  $N = \{1, 2, 3\}$  and define:

$$V_{\{i\}} = \{x \in \mathbb{R}^3 \mid x_i \leq 1\} \text{ for all } i = 1, 2, 3;$$

$$V_{\{ij\}i \neq j} = (\{x \in \mathbb{R}^3 \mid x_i \leq 1\} \cup \{x \in \mathbb{R}^3 \mid x_j \leq 1\}) \cap \{x \in \mathbb{R}^3 \mid x_i \leq 2; x_j \leq 2\};$$

$$V_{\{123\}} = \{x \in \mathbb{R}^3 \mid \sum_{i=1}^3 x_i \leq 3\}.$$

The game satisfies (A.1) and (A.2) (consider  $m = 2$ ), and, the core is non-empty and reduced to the element  $(1, 1, 1)$ ,

**Proposition 2.2** *The game is not  $(b, \partial)$ -balanced.*

**Proof of Proposition 2.2.** Consider the two points  $(1, 2, 1)$  and  $(1, 1, 2)$ . They lie outside  $V_N$  and they belong to  $\partial W$ . Note that

$$\mathcal{S}((1, 2, 1)) = \{\{1\}, \{3\}, \{12\}, \{13\}, \{23\}\},$$

$$\mathcal{S}((1, 1, 2)) = \{\{1\}, \{2\}, \{12\}, \{13\}, \{23\}\}.$$

Note that either  $C_1 = \{\{1\}, \{3\}, \{23\}\} \subset \mathcal{S}((1, 2, 1))$  or  $C_2 = \{\{1\}, \{2\}, \{23\}\} \subset \mathcal{S}((1, 1, 2))$  must be  $b$ -balanced. Indeed, let  $(b_S, S \in \mathcal{N})$  be a family of transfer rate vectors. Since  $b_{\{2,3\}} \in L_{\{2,3\}} \setminus \{0\}$ , one easily checks that the simplex  $\Sigma$  is the union of the convex hulls of  $\{b'_{\{1\}} = (1, 0, 0), b'_{\{3\}} = (0, 0, 1), b'_{\{2,3\}}\}$  and  $\{b'_{\{1\}} = (1, 0, 0), b'_{\{2\}} = (0, 1, 0), b'_{\{2,3\}}\}$ , where  $b'_S$  is equal to  $(1/\sum_{i=1}^3 b_S^i)b_S$ . Consequently,  $b'_{\{1,2,3\}}$  belongs to at least one of these convex hulls and the

game cannot be  $(b, \partial)$ -balanced since  $(1, 2, 1)$  and  $(1, 1, 2)$  do not belong to  $V_N$ .  $\square$

The game is not  $(b, \partial)$ -balanced, hence, neither  $b$ -balanced nor balanced. But, we obtain the following result.

**Proposition 2.3** *For all  $S \in \mathcal{N}$ , let  $\varphi_S$  be defined on  $\partial V_S$  as follows:*

$$\varphi_S(x) = \{t_S \in \Sigma_S \mid t_S \cdot x = 1\}$$

*Then the game is payoff-dependent balanced with respect to  $(\varphi_S)_{S \in \mathcal{N}}$  and  $\psi = \varphi_N$ .*

**Proof of Proposition 2.3.** Since  $(1, 1, 1) \in \partial V_S$ , for each  $S \in \mathcal{N}$ , and from (A.1)(ii) the set-valued mappings  $\varphi_S$ ,  $S \in \mathcal{N}$ , have convex values which are non-empty. Furthermore, it is routine to check that these set-valued mappings are upper semi-continuous.<sup>11</sup>

Suppose that  $x \in \partial W$  is such that  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(p_N(x)) \neq \emptyset$ . Then there exist  $\lambda_S \in \mathbb{R}_+$  and  $b_S \in \varphi_S(x)$  for each  $S \in \mathcal{S}(x)$ , and  $b_N \in \varphi_N(p_N(x))$ , such that  $\sum_{S \in \mathcal{S}(x)} \lambda_S = 1$  and  $\sum_{S \in \mathcal{S}(x)} \lambda_S b_S = b_N$ . Suppose that  $N \notin \mathcal{S}(x)$ , this implies that  $x = p_N(x) + \alpha \mathbf{1}$  with  $\alpha > 0$ . Therefore  $b_N \cdot x > b_N \cdot p_N(x)$ . But  $b_N \cdot x = (\sum_{S \in \mathcal{S}(x)} \lambda_S b_S) \cdot x = \sum_{S \in \mathcal{S}(x)} \lambda_S b_S \cdot x = \sum_{S \in \mathcal{S}(x)} \lambda_S = 1$  and  $b_N \cdot p_N(x) = 1$ , which is a contradiction.  $\square$

So far, it is worth noticing that the transfer rate rules  $(\varphi_S)_{S \in \mathcal{N}}$  have been taken as constant in the proof of Corollary 2.1. In this example, the transfer rate rules must depend on the payoff vectors in order to have balancedness. For example,  $\varphi_{\{2,3\}}(1, 2, 1) = \{(0, 0, 1)\}$ ,  $\varphi_{\{1,2,3\}}(p_N(1, 2, 1)) = \{(t_1, t_2, t_3) \in \Sigma \mid t_2 = 1/3\}$ ,  $\varphi_{\{2,3\}}(1, 1, 2) = \{(0, 1, 0)\}$  and  $\varphi_{\{1,2,3\}}(p_N(1, 1, 2)) = \{(t_1, t_2, t_3) \in \Sigma \mid t_3 = 1/3\}$ .

### 2.3.2 Convex games

To conclude our discussion on balancedness, we recall Billera's contribution (1970) for convex games, i.e. games with convex payoff sets. Let us stress that his contribution clearly anticipates our notion of payoff-dependent balancedness. Roughly, Billera implicitly uses a transfer rate rule depending on payoff vectors to describe the possible transfers among agents. In the situation of convex games, it can be shown that the transfer rate rule coincides, quite naturally, with the normal cone of convex analysis (see the proof below).

<sup>11</sup> We remark that the mappings  $(\varphi_S)_{S \in \mathcal{N}}$  can be seen as the exact analogues of average cost pricing rules up to a translation (we consider the point  $(1, 1, 1)$  instead of  $(0, 0, 0)$ ). An example of such rules is given in Bonnisseau and Cornet (1988, Corollary 3.3 p.130).

Consider the case involving convexity in payoff sets for which Billera (1970) gives a necessary and sufficient condition for non-emptiness of the core. He uses the notion of the support function. For all  $S \in \mathcal{N}$ ,  $\sigma_S$  denotes the support function of  $V_S$ , that is, the mapping from  $\mathbb{R}^N$  to  $\mathbb{R} \cup \{+\infty\}$  defined by  $\sigma_S(p) = \sup\{p \cdot v \mid v \in V_S\}$ . We show that Billera's result is a corollary of our main theorem.

**Corollary 2.2 (Billera (1970))** *The core of a game  $(V_S, S \in \mathcal{N})$  is non-empty if (A.1) and (A.2) are satisfied, if  $V_N$  is convex and if*

**(B.1)** *For all  $S \in \mathcal{N} \setminus \{N\}$ , there exists  $b_S \in \mathbb{R}^N \setminus \{0\}$  such that  $\sigma_S(b_S)$  is finite and, for all  $b \in \text{cone}\{b_S \mid S \in \mathcal{N} \setminus N\}$ , it holds that  $\sigma_N(b) \geq \max\{\sum_{S \in \mathcal{N} \setminus \{N\}} \lambda_S \sigma_S(b_S) \mid \forall S \in \mathcal{N} \setminus \{N\}, \lambda_S \geq 0, \sum_{S \in \mathcal{N} \setminus \{N\}} \lambda_S b_S = b\}$ .*

**Remark 2.1** *Under (A.1) and (A.2), if all payoff sets are convex then (B.1) is also necessary for non-emptiness of the core. Note also that a TU game is a convex game à la Billera.<sup>12</sup> In TU games,  $\sigma_S(b_S)$  is finite if and only if  $b_S$  is positively proportional to  $\mathbf{1}^S$ . This implies that for TU games, (B.1) is equivalent to the notion of balanced game of Bondareva (1963) and Shapley (1967).*

**Proof of Corollary 2.2.** It suffices to prove that the game is payoff-dependent balanced. For each  $S \in \mathcal{N} \setminus \{N\}$ , we let  $\varphi_S$  be the constant mapping which associates  $\frac{1}{\sum_{i \in N} b_{Si}}$  to each  $x \in \partial V_S$  and we let  $\varphi_N(x) = N_{V_N}(x) \cap \Sigma$ , where  $N_{V_N}(x)$  is the normal cone of convex analysis to  $V_N$  at  $x$ .<sup>13</sup> From the convexity of  $V_N$  and (A.1), the set-valued mapping  $\varphi_N$  has convex values and it is upper semi-continuous. Since  $\sigma_S(b_S)$  is finite,  $V_S$  is a cylinder and from (A.1) it holds that  $b_S \in L_{S+}$ ,  $S \in \mathcal{N}$ .

Let  $x \in \partial W$  satisfy  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(p_N(x)) \neq \emptyset$  and suppose that  $N \notin \mathcal{S}(x)$ . From the definition of  $\varphi_S$ , this implies that there exists  $b \in N_{V_N}(p_N(x)) \cap \text{cone}\{b_S \mid S \in \mathcal{S}(x)\}$ . Note that  $b \cdot p_N(x) = \sigma_N(b)$ . Remark also that, if  $x$  does not belong to  $V_N$  then  $x = p_N(x) + \alpha \mathbf{1}$  with  $\alpha > 0$ . Consequently,  $b \cdot x > b \cdot p_N(x) = \sigma_N(b)$ . On the other hand, for all  $S \in \mathcal{S}(x)$ ,  $b_S \cdot x \leq \sigma_S(b_S)$ . Since  $b \in \text{cone}\{b_S \mid S \in \mathcal{S}(x)\}$ , there exists  $\lambda_S \geq 0$ ,  $S \in \mathcal{S}(x)$ , such that  $b = \sum_{S \in \mathcal{S}(x)} \lambda_S b_S$ . From our assumption, one has  $\sigma_N(b) \geq \sum_{S \in \mathcal{S}(x)} \lambda_S \sigma_S(b_S) \geq (\sum_{S \in \mathcal{S}(x)} \lambda_S b_S) \cdot x = b \cdot x$ . This yields a contradiction.  $\square$

<sup>12</sup> In the TU case, there exists a payment  $v_S \in \mathbb{R}$  for each coalition, in other terms  $V_S = \{x \in \mathbb{R}^N \mid \sum_{i \in S} x_i \leq v_S\}$ .

<sup>13</sup> Let  $K \subset \mathbb{R}^N$  be a convex set, the normal cone to  $K$  at  $x \in K$  is  $N_K(x) = \{p \in \mathbb{R}^N \mid p \cdot x \geq p \cdot y, \forall y \in K\}$ .

## 2.4 Core payoff vector with transfer rate rule equilibrium

We now consider the case where the mapping  $\psi$  can differ from the mapping  $\varphi_N$ . In this case, the statement of Theorem 2.1 allows us to pick a particular element of the core satisfying the transfer rate rule equilibrium condition.

The following result is due to Reny and Wooders (1996). We deduce it from Theorem 2.1 by constructing a well-chosen transfer rate rule. In particular,  $\psi$  will depend on the cooperative commitments of each player in all the coalitions. We then apply this corollary to demonstrate the existence of solutions concepts closely related to fair division schemes, namely the partnered core and the core intersected with the average prekernel.

**Corollary 2.3 (Reny-Wooders (1996))** *Let  $(V_S, S \in \mathcal{N})$  be a  $\partial$ -balanced game satisfying (A.1) and (A.2).<sup>14</sup> Suppose that for each pair of players  $i$  and  $j$ , there is a continuous mapping  $c_{ij}: \partial W \rightarrow \mathbb{R}_+$  such that  $c_{ij}$  is zero on  $V(S) \cap \partial W$  whenever  $i \notin S$  and  $j \in S$ . Then there exists a core payoff vector  $x$  such that, for each  $i \in N$ ,  $\eta_i(x) := \sum_{j \in N} (c_{ij}(x) - c_{ji}(x)) = 0$ .*

It makes sense to interpret the mappings  $c_{ij}$  as credit/debit mappings. Then, one can see  $\eta_i(x)$  as the measure of the grand coalition's net indebtedness to  $i$  or as  $i$ 's net credit against the grand coalition. Letting  $c_{ij} = 0$  for each  $i, j \in N$  yields Scarf's result. We provide a direct and intuitive proof for Corollary 2.3. The set-valued mapping  $\psi$  will take into account individual contributions in the payoff vectors of the grand coalition; then  $\psi^i$  stands for the cooperation index of the agent  $i$ . This leads to show the existence of a constant index among the agents.

**Proof of Corollary 2.3.** Firstly, notice that, for each  $x \in \partial W$ ,  $\eta(x) \in \mathbf{1}^\perp$ . For each  $x \in \partial W$ , define  $\eta_*(x) = \max_{i \in N} \{|\eta_i(x)|\}$  and let  $\tilde{\eta}_i$ ,  $i \in N$ , be mappings from  $\partial W$  to  $\mathbb{R}_+$ , where  $\tilde{\eta}_i(x) := \frac{1}{\eta_*(x)+1} \frac{\eta_i(x)}{|N|}$ . The idea of the proof is to suitably incorporate the net credit and normalized mapping  $\tilde{\eta}$  into  $\psi$ . Define for each  $x \in \partial V_S$ ,  $S \in \mathcal{N}$ , and each  $x' \in \partial V_N$ :

$$\varphi_S(x) = m^S \text{ and } \psi(x') = m^N - \tilde{\eta}(p_W(x'))$$

It appears clearly that, for each  $S \in \mathcal{N}$ ,  $\varphi_S$  has values in  $\Sigma_S$  and  $\psi$  has values in  $\Sigma$ . Furthermore, the mappings are all upper semi-continuous with non-empty compact and convex values. This result stems from the continuity of the mappings  $c_{ij}$  and the normalization.

<sup>14</sup>The original result is stated for balanced games, it is slightly improved by considering  $\partial$ -balanced games. A game is  $\partial$ -balanced if for any balanced family of coalitions  $\mathcal{B}$ ,  $\partial W \cap (\cap_{S \in \mathcal{B}} V_S) \subset V_N$ .

**Lemma 2.1** *For each  $x \in \partial W$ , if  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$ , then  $\psi(p_N(x)) = m^N$ .*

In Appendix, we provide the detailed proof of Lemma 2.1. Let  $x \in \partial W$ ; then from Lemma 2.1 the condition  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$  says that the family  $\mathcal{S}(x)$  is balanced and since the game is  $\partial$ -balanced, one deduces that the game is payoff-dependent balanced. Now, applying Theorem 2.1, there exists  $x$  in the core of the game such that  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$ . Noticing that  $x = p_N(x) = p_W(x)$  and using once again Lemma 2.1, it follows that  $\tilde{\eta}(x) = 0$ , that is  $\eta(x) = 0$ , so Corollary 2.3 is proved.  $\square$

We briefly recall two applications of Corollary 2.3. We refer the reader respectively to the works of Reny and Wooders (1996) and Orshan et al. (2003) for a complete presentation.

The analysis of core payoff vectors of NTU games with partnerships properties was initiated in Reny and Wooders (1996) and furthered in other fields by Page and Wooders (1996); Reny and Wooders (1998). A payoff vector  $x \in \partial W$  is said to be partnered if the family  $\mathcal{S}(x)$  satisfies, for all  $i, j \in N$ ,  $\mathcal{S}_i(x) \subset \mathcal{S}_j(x) \Rightarrow \mathcal{S}_j(x) \subset \mathcal{S}_i(x)$ , where  $\mathcal{S}_i(x) = \{S \in \mathcal{S}(x) \mid i \in S\}$ . The partnered core is the core intersected with the set of partnered payoff vectors. Reny and Wooders (1996) apply Corollary 2.3 to suitable mappings  $c_{ij}$  to prove the existence of an element in the partnered core.<sup>15</sup> The mappings are defined as follows:  $c_{ij}(x) = \min\{\text{dist}(x_S, V(S)) \mid S \in \mathcal{N} \text{ and } i \notin S \ni j\}$  for each  $x \in \partial W$  where  $\text{dist}$  is the Euclidean distance.

As a second direct application, one can prove the existence of an element lying in the core intersected with the average prekernel (also called bilateral consistent prekernel) as defined in Orshan and Zarzuelo (2000), see also Serrano and Shimomura (2006).<sup>16 17</sup> To define the average prekernel, we need to introduce two additional assumptions on the game, non-levelness and smoothness. The average prekernel is the consistent extension of the usual prekernel in TU games. It also generalizes the Nash bargaining solution. The existence result can be deduced from Corollary 2.3 by considering suitable credit mappings. Indeed, the average prekernel may be rewritten as the set of elements  $x \in \partial V_N$

<sup>15</sup> The core of the game is not tight if there exists a payoff vector  $x$  in the core which does not belong to  $V_S$  for each  $S \neq N$ . In that case, every element satisfying this property has the partnership property. Indeed since  $x \notin V_S$  for each  $S \in \mathcal{N} \setminus N$ , then  $\mathcal{S}_i(x) = \mathcal{S}_j(x) = \{N\}$  for each  $i, j \in N$ . The statement of Theorem 2.1 is in the same spirit since in the case of non tight core one gets  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} = \varphi_N(x)$  and thus  $\varphi_N(x) \cap \psi(x) \neq \emptyset$ , that is, for the transfer rate rules in the proof of Corollary 2.3,  $m^N = \varphi_N(x) = \psi(x) = m^N - \tilde{\eta}(x)$ , then  $\eta(x) = 0$ .

<sup>16</sup> The kernel originated in Maschler, Peleg and Shapley (1979), where partnered (or separating) collections of sets were also introduced.

<sup>17</sup> The proof can be found in Jehlé (2004).



such that  $\sum_{j \in N} (c_{ij}(x) - c_{ji}(x)) = 0$  for some credit mappings satisfying the requirements of Corollary 2.3. In this context,  $c_{ij}(x)$  can be seen as the weighted surplus of agent  $i$  with respect to agent  $j$  at the point  $x$ . Orshan et al. (2003) show non-emptiness of the core intersected with the average prekernel in  $\partial$ -separating games, here this result can be improved by considering the larger class of  $\partial$ -balanced games.<sup>18</sup>

### 3 Parametrized cooperative game

#### 3.1 Existence of equilibrium-core vector pairs

This section is intended to unify that part of the literature which uses explicitly or implicitly parametrized games. To take into account the environment and the possible interactions between players' payoff vectors, we introduce a canonical version of a parametrized game and an associated solution concept, called equilibrium-core vector pair.

The environment set is  $\Theta$  and a game is associated to each  $\theta \in \Theta$ . Formally, we define a set-valued mapping  $V_S$  from  $\Theta$  to  $\mathbb{R}^N$  for each coalition  $S$ . To encompass several results in the literature, we add a set-valued mapping  $V$  from  $\Theta$  to  $\mathbb{R}^N$ , which possibly differs from  $V_N$ . In Border (1984),  $V$  is the set of attainable utilities of the so-called whole economy whereas  $V_N$  is the set of attainable utilities of the grand coalition. In Boehm (1974),  $V$  possibly differs from  $V_N$  due to costs of forming a coalition. In Ichiishi (1981),  $V$  is the set of attainable utilities when coalition structures are allowed whereas  $V_N$  is the set of attainable utilities of the grand coalition. Finally, the equilibrium condition on the environment is represented by a set-valued mapping  $G$  from  $\Theta \times \mathbb{R}^N$  to  $\Theta$ . We denote by  $W(\theta)$  the union of the payoff sets  $\cup_{S \in \mathcal{N}} V_S(\theta)$ .

**Definition 3.1** *An equilibrium-core vector pair is a pair  $(\theta^*, x^*) \in \Theta \times \mathbb{R}^N$  such that:*

$$x^* \in \partial V(\theta^*) \setminus \text{int } W(\theta^*) \text{ and } \theta^* \in G(\theta^*, x^*).$$

The model is closely linked up to the model of Ichiishi (1981), who defines a Social Coalitional Equilibrium. We show that existence of Ichiishi's Equilibrium is a corollary of our main result. We also show non-emptiness of the core of an economy without ordered preferences, previously demonstrated by Border (1984). Furthermore, our general framework allows to investigate some other topics of economic theory. For instance, Ichiishi and Idzik (1996) have

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<sup>18</sup>Note that a  $\partial$ -separating game is  $\partial$ -balanced.

shown non-emptiness of the incentive compatible core in incomplete information framework using the existence of Ichiishi's Social Coalitional Equilibrium. Their result can also be obtained from our theorem.

**Remark 3.1** *Obviously, the parametrized framework encompasses the case of constant payoff sets with respect to the environment. Thus, the results of Section 2 are all covered by Theorem 3.1 presented below.*

The assumptions on the game are the following (we just add continuous dependencies with respect to the environment to (A.1) and (A.2) of Section 2):

(A.4)  $\Theta$  is a non-empty, convex, compact subset of an Euclidean space.  $G$  is an upper semi-continuous set-valued mapping with non-empty and convex values.

(A.5) (i) The set-valued mappings  $V_{\{i\}}$ ,  $i \in N$ , and  $V$  are non-empty valued.  $V_S$ ,  $S \in \mathcal{N}$ , and  $V$  are lower semi-continuous set-valued mappings with closed graph.

(ii) For each  $\theta \in \Theta$ ,  $V_S(\theta)$ ,  $S \in \mathcal{N}$ , and  $V(\theta)$  satisfy (A.1)(ii).

(A.6) For each  $\theta \in \Theta$ , there exists  $m(\theta) \in \mathbb{R}$  such that for each  $S \in \mathcal{N}$  and for each  $x \in V_S(\theta)$ , if  $x \notin \text{int } V_{\{i\}}(\theta)$  for each  $i \in S$  then  $x_j \leq m(\theta)$  for each  $j \in S$ . For all  $\theta \in \Theta$ , for each  $x \in V(\theta)$ , if  $x_i \notin \text{int } V_{\{i\}}(\theta)$  for each  $i \in N$ , then  $x \leq m(\theta)\mathbf{1}$ .

(A.5) implies that there exist functions  $v_i$ ,  $i \in N$ , from  $\Theta$  to  $\mathbb{R}$ , such that, for each  $\theta \in \Theta$ ,  $V_{\{i\}}(\theta) = \{z \in \mathbb{R}^N \mid z_i \leq v_i(\theta)\}$ . Bonnisseau (1997, Lemma 3.1. p.217) states that if a set-valued mapping  $M$  from  $\Theta$  to  $\mathbb{R}^N$  is lower semi-continuous with non-empty values, has a closed graph and satisfies (A.1)(ii) for all  $\theta \in \Theta$ , then there exists a continuous mapping  $\lambda$  from  $\Theta \times \mathbf{1}^\perp$  to  $\mathbb{R}$  such that, for all  $(\theta, s) \in \Theta \times \mathbf{1}^\perp$ , it holds that  $s - \lambda(\theta, s)\mathbf{1} \in \partial M(\theta)$ . Let  $p_V$  and  $\lambda_V$  be continuous mappings defined respectively on  $\Theta \times \mathbb{R}^N$  and  $\Theta \times \mathbf{1}^\perp$  such that:  $p_V(\theta, x) = \text{proj}(x) - \lambda_V(\theta, \text{proj}(x))\mathbf{1} \in \partial V_N(\theta)$ . We define similarly mappings  $p_W$  and  $\lambda_W$  associated to  $W$ .

We can extend the notion of a transfer rate rule  $((\varphi_S)_{S \in \mathcal{N}}, \psi)$  to a parametrized game. The unique difference is that the set-valued mappings are now defined on the graphs  $\text{Gr } \partial V_S$  or  $\text{Gr } \partial V$ .

**Definition 3.2** *Let  $(V, (V_S)_{S \in \mathcal{N}}, \Theta)$  be a parametrized game satisfying (A.5). The game is payoff-dependent balanced if there exists a transfer rate rule  $((\varphi_S)_{S \in \mathcal{N}}, \psi)$  such that, for each  $(\theta, x) \in \text{Gr } \partial W$ ,*

$$\text{if } \text{co}\{\varphi_S(\theta, x) \mid S \in \mathcal{S}_\theta(x)\} \cap \psi(\theta, p_V(\theta, x)) \neq \emptyset, \text{ then } x \in V(\theta).$$

The second result of the paper is the following.

**Theorem 3.1** *Let  $(V, (V_S)_{S \in \mathcal{N}}, \Theta)$  be a parametrized game satisfying (A.4), (A.5) and (A.6). If the game is payoff-dependent balanced with respect to the transfer rate rule  $((\varphi_S)_{S \in \mathcal{N}}, \psi)$ , then there exists an equilibrium-core vector pair  $(\theta^*, x^*)$  such that:*

$$\text{co} \{ \varphi_S(\theta^*, p_W(\theta^*, x^*)) \mid S \in \mathcal{S}_{\theta^*}(p_W(\theta^*, x^*)) \} \cap \psi(\theta^*, x^*) \neq \emptyset.$$

The proof, given in Appendix, follows the geometric construction given in the proof of Theorem 2.1. But note that, contrary to the proof of Theorem 2.1, the proof is self contained in the sense that we do not appeal to an existence result of general economic equilibrium.

Let us consider the parametrized game  $(V_N, (V_S)_{S \in \mathcal{N}}, \Theta)$ . For all  $\theta \in \Theta$ ,  $(V(\theta), V_S(\theta), S \in \mathcal{N})$  is the game defined in Sub-section 2.3.1. If  $\Theta$  and  $G$  satisfy (A.4), then there exists an equilibrium-core vector pair,  $(\theta^*, (1, 1, 1))$  where  $\theta^*$  is a fixed-point of  $G(\cdot, (1, 1, 1))$ . Nevertheless,  $(V(\theta), V_S(\theta), S \in \mathcal{N})$  is not  $(b, \partial)$ -balanced for any  $\theta \in \Theta$ . Once again, we need to consider a non-constant transfer rate rule to deduce the existence of an equilibrium-core vector pair.

### 3.2 Two applications of Theorem 3.1

In both applications below, we do not make full use of Theorem 3.1 since for any given environment  $\theta$ , the game  $(V(\theta), V_S(\theta), S \in \mathcal{N})$  will be balanced with a constant transfer rate rule. So we mostly focus on the role of the parametrization in these models. Both applications will clarify the usefulness of the mapping  $G$ , which is explicitly given in each case. Thanks to Theorem 3.1, the proofs are elementary.

#### 3.2.1 Ichiishi's Social Coalitional Equilibrium

The Social Coalitional Equilibrium of Ichiishi (1981) is a pioneering work in the framework of parametrized games. Furthermore, the general formulation of the Equilibrium, where the agents can realize a coalition structure, encompasses Social Equilibrium of Debreu (1952) and the usual core as special cases. A number of applications of this seminal result are reviewed in Ichiishi (1993).

A coalition structure is a partition of  $N$ . Let  $\mathcal{P}$  be a non-empty collection of coalition structure and denote a member of  $\mathcal{P}$  by  $P$ . Each player has an environment set  $\Theta_i$  ( $\Theta_S = \prod_{i \in S} \Theta_i$ ,  $\Theta = \Theta_N$ ). For each  $S \in \mathcal{N}$ , let  $F^S$  be a mapping from  $\Theta$  into  $\Theta_S$ . A preference relation of each player  $i$  in a coalition  $S$  is represented by a utility function;  $v_S^i : \text{Gr } F^S \rightarrow \mathbb{R}$ .

A Social Coalitional Equilibrium of a society is a pair consisting of an environment  $\theta^* \in \Theta$  and an admissible coalition structure  $P^* \in \mathcal{P}$ , such that: (i) For each  $D \in P^*$ ,  $\theta^{*D} \in F^D(\theta^*)$ . (ii) It is not true that there exists  $S \in \mathcal{N}$  and  $\theta' \in F^S(\theta^*)$  such that  $v_S^i(\theta^*, \theta') > v_{D(i)}^i(\theta^*, \theta^{*D(i)})$  for every  $i \in S$ , where  $D(i) \in P^*$  and  $i \in D(i)$ .

**Corollary 3.1 (Ichiishi (1981))** *A Social Coalitional Equilibrium exists if:*  
(1) For every  $i \in N$ ,  $\Theta_i$  is a non-empty, convex compact subset of an Euclidean space. (2) For every  $S \in \mathcal{N}$ ,  $F^S$  is a lower and upper semi-continuous set-valued mapping with non-empty values. (3) For every  $S \in \mathcal{N}$ ,  $v_S^i$  is continuous on  $\text{Gr } F^S$ . (4) For every  $\theta \in \Theta$  and every  $v \in \mathbb{R}^N$ , if there exists a balanced family  $\mathcal{B}$  such that for each  $S \in \mathcal{B}$  there exists  $\theta(S) \in F^S(\theta)$  for which  $v_i \leq u_S^i(\theta, \theta(S))$  for each  $i \in S$ , then there exist  $P \in \mathcal{P}$  and  $\theta^{D(i)} \in F^D(\theta)$  for every  $D \in P$  such that  $v_i \leq u_D^i(\theta, \theta^{D(i)})$  for all  $i \in D$ . (5) For every  $\theta \in \Theta$  and for every  $v \in \mathbb{R}^N$ , the set

$$\bigcup_{P \in \mathcal{P}} \left\{ \theta' \in \Theta \mid \forall D \in P, \theta^{D(i)} \in F^D(\theta) \text{ and } v \leq (v_{D(i)}^i(\theta, \theta^{D(i)}))_{i \in N} \right\}$$

is convex.

Although in the original paper of Ichiishi, the strategy sets are taken as Hausdorff topological vector spaces, here we limit the framework to Euclidean spaces.

**Proof of Corollary 3.1.** For each  $S \in \mathcal{N}$ , let us define:

$$V_S(\theta) = \left\{ u \in \mathbb{R}^N \mid \exists \theta' \in F^S(\theta), u_i \leq v_S^i(\theta, \theta'), i \in S \right\}$$

$$\tilde{V}_S(\theta) = \left\{ u \in V_S(\theta) \mid \forall i \in N \setminus S, u_i = 0 \right\}$$

$$V(\theta) = \bigcup_{P \in \mathcal{P}} \sum_{D \in P} \tilde{V}_D(\theta)$$

And let  $G(\theta, x)$  be equal to

$$\bigcup_{P \in \mathcal{P}} \left\{ \theta' \in \Theta \mid \forall D \in P, \theta^{D(i)} \in F^D(\theta) \text{ and } p_V(\theta, x) \leq (v_{D(i)}^i(\theta, \theta^{D(i)}))_{i \in N} \right\}.$$

We show that the parametrized game defined above meets the requirements of Theorem 3.1. We begin with the condition of balancedness. For each  $S \in \mathcal{N}$ , let  $\varphi_S$  be the constant mapping equals to  $m^S$  and  $\psi = \varphi_N$ . Let  $(\theta, x)$  be in  $\text{Gr } \partial W$  and satisfies  $\text{co}\{\varphi_S(\theta, x) \mid S \in \mathcal{S}_\theta(x)\} \cap \psi(\theta, p_V(\theta, x)) \neq \emptyset$ . As seen before, one easily checks that the family  $\mathcal{S}_\theta(x)$  is balanced. For each  $S \in \mathcal{S}_\theta(x)$ , there exists  $\theta(S) \in F^S(\theta)$  for which  $x_i \leq u_S^i(\theta, \theta(S))$ ,  $i \in S$ . Then, from (4), there exist  $P \in \mathcal{P}$  and  $\theta^{D(i)} \in F^D(\theta)$  for every  $D \in P$  such that  $x_i \leq u_D^i(\theta, \theta^{D(i)})$ ,  $i \in D \in P$ . This implies that  $x \in \sum_{D \in P} \tilde{V}_D(\theta) \in V(\theta)$ .

Ichiishi proved the continuity of the set-valued mappings  $V_S$  and  $V$  (Ichiishi, 1981, Proof of Lemma, step 1, p.372), so (A.5) clearly holds true. From the definitions of  $p_V$  and  $V$ ,  $G$  is non-empty and convex valued from (5). It suffices to prove that  $G$  has a closed graph to imply that  $G$  is upper semi-continuous. This is straightforward from the finiteness of  $\mathcal{P}$  and (2), (3), so (A.4) holds. The continuity of  $v_S^i$  and (1) guarantee that (A.6) is satisfied.

From Theorem 3.1 there exists  $(\theta^*, x^*)$  such that  $\theta^* \in G(\theta^*, x^*)$  and  $x^* \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$ . Hence, there exists  $P \in \mathcal{P}$  such that for each  $D \in P$ ,  $\theta^{*D} \in F^D(\theta^*)$  and  $p_V(\theta^*, x^*) \leq (v_{D(i)}^i(\theta^*, \theta^{*D(i)}))_{i \in N}$ ; furthermore  $x^* = p_V(\theta^*, x^*)$ . Necessarily  $(v_{D(i)}^i(\theta^*, \theta^{*D(i)}))_{i \in N} \in \partial V(\theta^*) \setminus \text{int } \cup_{S \in \mathcal{N}} V_S(\theta^*)$ ; thus the pair  $(\theta^*, P)$  satisfies the requirement of Social Coalitional Equilibrium.  $\square$

Ichiishi and Quinzii (1983) use a variant of Corollary 3.1 to prove the non-emptiness of the core of an economy with increasing returns. The authors split the environment set into an abstract environment set and action sets for each individual. In addition in their model, the set  $\mathcal{P}$  is restricted to  $N$ . Therefore,  $V = V_N$  and the feasibility condition in the definition of Social Coalitional Equilibrium must hold only on the coarsest coalition  $N$ . Using our own materials, one can prove directly the result of Ichiishi and Quinzii (1983, Lemma A p.406).

One can combine results from Sub-section 2.4 with the solution concept of Social Coalitional Equilibrium. For instance, by considering the equilibrium condition on the transfer rate rule  $((\varphi_S)_{S \in \mathcal{N}}, \psi)$  and under assumptions given in Ichiishi and Quinzii (1983, Lemma A p.406), we can obtain a Social Coalitional Equilibrium with partnered payoff vectors. Though the concept of partnered structure is much weaker than the concept of partition structure, it avoids the use of assumption (4) in Corollary 3.1. Then, the emerging structure is obtained as the outcome of an endogenous process. To the best of our knowledge, such solutions have not been explored any further in this parametrized framework.

We remark also that if we come back to NTU cooperative games without parameters, Theorem 3.1 provides also a sufficient condition for non-emptiness of the coalition structure core by considering  $V = \cup_{P \in \mathcal{P}} \cap_{S \in P} V_S$  (see footnote 7).

### 3.2.2 Core allocations for non-ordered preferences

Border proves the non-emptiness of the core of an economy with non-ordered preferences. We recover Border's result (1984) from Theorem 3.1 by defining a suitable parametrized game. We also show how Kajii (1992)'s generalization of Border's result can be recovered from our result.

Let  $\Xi_i, i \in N$ , be the payoff set of agent  $i$ ,  $\Xi_S = \prod_{i \in S} \Xi_i$  and  $\Xi = \Xi_N$ . For each  $S \in \mathcal{N}$ , let  $F^S$  be the feasibility mapping from  $\Xi$  into  $\Xi_S$  and let  $\Theta \subset \Xi$  be the set of all jointly feasible allocations. A preference relation for each player is represented by a set-valued mapping  $P_i$  from  $\Xi_i$  into  $\Xi_i$ .

An element  $\xi \in \Xi$  is said to be in the core if : (i)  $\xi \in \Theta$ . (ii) There is no  $S \in \mathcal{N}$  and  $\xi' \in F^S(\xi)$  satisfying  $\xi'_i \in P_i(\xi_i)$  for all  $i \in S$ .

**Corollary 3.2 (Border (1984))** *The core is non-empty if: (1) For each  $i$ ,  $\Xi_i$  is a non-empty convex subset of an Euclidean space. (2) For each  $S \in \mathcal{N}$ ,  $F^S : \Xi \rightarrow \Xi_S$  is a lower and upper semi-continuous set-valued mapping with compact values and  $F^i, i \in N$ , is non-empty valued. (3)  $\Theta$  is compact and convex. (4) For each  $i$ ,  $P_i$  has an open graph in  $\Xi_i \times \Xi_i$ , for all  $\xi_i \in \Xi_i$ ,  $P_i(\xi_i)$  is convex and  $\xi_i \notin P_i(\xi_i)$ . (5) The game is balanced: for all  $\xi' \in \Xi$ , for any balanced family  $\beta$  with the balancing weights  $(\lambda_B)_{B \in \beta}$ , if there exist  $(\xi^B)_{B \in \beta}$  such that  $\xi^B \in F^B(\xi')$ ,  $B \in \beta$ , then  $\xi \in \Theta$  where  $\xi_i = \sum_{B \in \beta, i \in B} \lambda_B \xi_i^B$ .*

**Proof of Corollary 3.2.** We define pseudo-utility functions  $v_i : \Xi_i \times \Xi_i \rightarrow \mathbb{R}$ ,  $i \in N$ , as follows:  $v_i(\xi'_i, \xi_i) = \text{dist}[(\xi_i, \xi'_i), (\text{Gr } P_i)^c]$ . The convexity of  $P_i(\xi_i)$  implies that  $v_i$  is quasi-concave in its first argument (see Border's Appendix).

The parametrized game is defined on the compact set  $\Theta$ . For each  $\xi \in \Theta$ , for each  $S \in \mathcal{N}$ , let  $V_S(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in F^S(\xi), u_i \leq v_i(\xi'_i, \xi_i), i \in S\}$  and let  $V(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in \Theta, u_i \leq v_i(\xi'_i, \xi_i), i \in N\}$ . Let  $W(\xi) = \cup_{S \in \mathcal{N}} V_S(\xi)$ . Remark that  $\xi \in \Theta$  is in the core if and only if  $0 \in V(\xi) \setminus \text{int } W(\xi)$ . Let

$$G(\xi, x) = \{\xi' \in \Theta \mid p_V(\xi, x) \leq (v_i(\xi'_i, \xi_i))_{i \in N}\}.$$

Consider the game above. We provide in Appendix the detailed and rather technical proof that the assumptions of Theorem 3.1 are all fulfilled. Then, there exists a bundle  $(\theta^*, x^*)$  such that  $x^* \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$  and  $\theta^* \in G(\theta^*, x^*)$ . Hence,  $x^* = p_V(\theta^*, x^*) \leq (v_i(\theta_i^*, \theta_i^*))_i = 0$ . From the free-disposal assumption, since  $\theta^* \in \Theta$ ,  $0 \in V(\theta^*)$  and  $x^* \in \partial V(\theta^*)$ ,  $x^* \leq 0$  implies that  $0 \in \partial V(\theta^*)$ . Furthermore,  $x^* \notin \text{int } W(\theta^*)$  and  $x^* \leq 0$  implies that  $0 \notin \text{int } W(\theta^*)$ . That is to say that  $0 \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$  and  $\theta^* \in \Theta$ , so the core is non-empty.  $\square$

The analysis of nonordered preferences has been carried out more generally. Kajii (1992) proposes a generalization of both Border's result and of Scarf's  $\alpha$ -core non-emptiness result (Scarf, 1971). The same construction as in the previous proof can be applied to obtain Kajii's result. The difference between Border's result and Kajii's comes from the fact that Kajii considers preferences with interdependencies, that is, the mappings  $P^i$  are defined from  $\Xi$  into  $\Xi$ . Consequently the pseudo utility mappings are defined on  $\Xi \times \Xi$  but still verify quasi-concavity in their first variables (Kajii, 1992, p.196).

In Kajii's setting, a coalition  $S$  blocks a feasible allocation  $\xi \in \Theta$  if there exists  $\xi' \in F^S(\xi)$  such that for all  $\xi''$  with  $\xi''_i = \xi'_i$  all  $i \in S$ , it holds that  $\xi'' \in P_i(\xi)$ . Payoff sets are naturally defined as:  $V_S(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in F^S(\xi) \text{ such that for all } \xi''_i = \xi'_i, i \in S, u_i \leq v_i(\xi'', \xi), i \in S\}$ , for all  $S \in \mathcal{N}$  and  $V(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in \Theta u_i \leq v_i(\xi', \xi), i \in N\}$ . These set-valued mappings satisfy the expected properties of continuity as in Border's setting. We obtain the result of Kajii (1992, Corollary p.201) (he additionally assumes that  $F^N(\xi) = \Theta$  and  $\Xi_i = \Theta_i$ ) if we posit:

$$G(\xi, x) = \bigcap_{i \in N} \left\{ \xi' \in \Theta \mid p_V^i(\xi, x) \leq v_i(\xi', \xi) \right\}.$$

As a finite intersection of upper semi-continuous set-valued mappings with convex values, the mapping  $G$  is an upper semi-continuous set-valued mapping with convex values.  $G$  has non-empty values since, for each  $(\theta, x) \in \Theta \times \mathbb{R}^N$ ,  $p_V(\theta, x) \in V(\theta)$  by definition. Therefore the parametrized game meets the requirements of Theorem 3.1.

## 4 Further developments

We point out three directions for research that can be initiated from concepts and results of our paper: core of a possibly non-convex economy, core with asymmetric information and core selections.

In a non-convex economy, the payoff-dependent balancedness enlarges the geometric possibilities to get a non-empty core (see Sub-section 2.3.1). The negative result (Scarf, 1986, Theorem 5 p.426) delimits however the range of new results. In a convex economy with production, Florenzano (1989) uses a direct proof to show the core non-emptiness under a balancedness assumption which holds on the production sets of the economy. In her setting where no cooperative game structure is defined, one should restate payoff-dependent balancedness by defining transfer rates directly on the fundamentals of the economy.

In the setting of parametrized games, Theorem 3.1 allows us to show the non-emptiness of the incentive cores with asymmetric information (Ichiishi and Idzik, 1996; Ichiishi and Radner, 1999) and the non-emptiness of the  $\alpha$ -core (Kajii, 1992; Scarf, 1971). To show these results, one makes use of the standard balancedness. In future research, one could exploit the flexibility of the payoff-dependent balancedness.

Our first result, Theorem 2.1, sheds light on the possibility for selecting core payoff vectors using the transfer rate rule (see Sub-section 2.4). Other con-

tributions in various fields deal with core selections, see for instance Herings et al. (2007); Ichiishi and Idzik (2002); Kannai and Wooders (2000); Page and Wooders (1996); Reny and Wooders (1998). The incorporation of appropriate transfer rate rules might lead to a unified treatment to describe core selections (e.g. Iehlé, 2004).

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## Appendix

### *Proof of Theorem 2.1*

Let  $Y_2 = -(\text{int } W)^c$  where  $(\text{int } W)^c$  denotes the complement of the interior of the set  $W$ . Note that  $Y_2$  is bounded above by  $-v$ . Let  $\tilde{\varphi}_2$  be the set-valued mapping from  $\partial Y_2$  to  $\Sigma$  defined by

$$\tilde{\varphi}_2(y_2) = \text{co} \{ \varphi_S(-y_2) \mid S \in \mathcal{S}(-y_2) \}.$$

**Lemma 4.1** *For all  $(y_2, p) \in \text{Gr } \tilde{\varphi}_2$  such that  $y_{2i} < -m$  for some  $i \in N$ , it holds that  $p_i = 0$ .*

**Proof of Lemma 4.1.** Let  $y_2 \in \partial Y_2$  be such that  $y_2 \notin \{-m\mathbf{1}\} + \mathbb{R}_+^N$ . Let  $i \in N$  be such that  $y_{2i} < -m$ . Then, for all  $S \in \mathcal{S}(-y_2)$  we show that  $i \notin S$ . Indeed, recalling that  $-y_2 \geq v$ , (A.2) states that if  $-y_2 \in V_S$ , then  $-y_{2j} \leq m$  for all  $j \in S$ . Thus,  $i \notin S$ . Consequently, for all  $S \in \mathcal{S}(-y_2)$ , for all  $p \in \varphi_S(-y_2)$ ,  $p_i = 0$  since  $\varphi_S$  takes its values in  $\Sigma_S$ . Hence, for all  $p \in \tilde{\varphi}_2(y_2)$ ,  $p_i = 0$ .  $\square$

Let  $m$  be the upper bound given in (A.2). Since  $Y_2$  is bounded above by  $-v$  and from (A.2), the set  $Y_2 \cap (\{-m\mathbf{1}\} + \mathbb{R}_+^N)$  is compact, therefore there exists  $\rho > 0$  such that  $\text{proj}(y) \in B_{\mathbf{1}^\perp}(0, \rho)$  for all  $y \in Y_2 \cap (\{-m\mathbf{1}\} + \mathbb{R}_+^N)$ .



Define  $Y_1 = \{p_N(s) \mid s \in \bar{B}_{1^\perp}(0, \rho)\} - \mathbb{R}_+^N$ . We remark that, for all  $y_1 \in \partial Y_1$ , if  $\text{proj}(y_1) \in \bar{B}_{1^\perp}(0, \rho)$  then  $y_1 \in \partial V_N$ . Let us choose  $\underline{y} \in \text{int } Y_1$ .

**Lemma 4.2** *There exists a continuous mapping  $c$  from  $\partial Y_1$  to  $\Sigma_{++}$  such that  $c(y_1) \cdot (y_1 - \underline{y}) \geq 0$  for all  $y_1 \in \partial Y_1$ .*

**Proof of Lemma 4.2.** Since  $Y_1$  satisfies the free disposal condition and  $\underline{y} \in \text{int } Y_1$ , for all  $y_1 \in \partial Y_1$ , there exists a vector  $p \in \Sigma_{++}$  such that  $p \cdot (y_1 - \underline{y}) > 0$ . Define the set valued mapping  $\Gamma$  from  $\partial Y_1$  to  $\Sigma_{++}$  as  $\Gamma(y_1) = \{p \in \Sigma_{++} \mid p \cdot (y_1 - \underline{y}) > 0\}$ ; from the argument above,  $\Gamma$  has non-empty values. It is an easy matter to check that it has open graph and convex values. By applying a weak version of Michael's selection theorem (Florenzano, 2003, Proposition 1.5.1., p.29), one gets the existence of a continuous selection of  $\Gamma$ .  $\square$

Let  $\tilde{\varphi}_1$  be the set-valued mapping from  $\partial Y_1$  to  $\Sigma$  defined by:

$$\tilde{\varphi}_1(y_1) = \begin{cases} \psi(y_1) & \text{if } \|\text{proj}(y_1)\| < \rho \\ \text{co}\{\psi(y_1), c(y_1)\} & \text{if } \|\text{proj}(y_1)\| = \rho \\ c(y_1) & \text{if } \|\text{proj}(y_1)\| > \rho \end{cases}$$

**Lemma 4.3** *For all  $(y_1, y_2) \in \partial Y_1 \times \partial Y_2$  and  $p \in \Sigma$  such that  $\text{proj}(y_1) = -\text{proj}(y_2)$  and  $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$ , it holds that  $p \in \psi(y_1)$  and  $y_1 \in \partial V_N$ .*

**Proof of Lemma 4.3.** We first prove that  $\|\text{proj}(y_1)\| \leq \rho$ . Indeed, if it is not true, then  $\varphi_1(y_1) = c(y_1) \in \Sigma_{++}$ . From Lemma 4.1 and the choice of  $\rho$ , since  $\|\text{proj}(y_2)\| = \|\text{proj}(y_1)\| > \rho$  it holds that  $\varphi_2(y_2) \notin \Sigma_{++}$ . But, this contradicts  $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$ .

Now, the construction of  $\rho$  and  $\|\text{proj}(y_1)\| \leq \rho$  imply that  $y_1 \in \partial V_N$ .

If  $p \notin \psi(y_1)$ , then  $\|\text{proj}(y_1)\| = \rho$  and  $p \in \Sigma_{++}$ . The same argument as in the first paragraph leads again to a contradiction.  $\square$

Given these materials, one can state the following technical lemma. The proof consists of the verification of the assumptions of Theorem 2.2 with respect to the construction above.

**Lemma 4.4** *For any closed convex cone  $C$  included in  $\mathbb{R}_{++}^N \cup \{0\}$  such that  $\mathbf{1} \in \text{int } C$  it holds that  $Y_1, Y_2$  and  $\tilde{\varphi}_1, \tilde{\varphi}_2$  satisfy the requirements of Theorem 2.2.*

**Proof of Lemma 4.4.** We check with the three following claims that the assumptions of Theorem 2.2 hold true for the sets  $Y_1$  and  $Y_2$  and for the mappings  $\tilde{\varphi}_1$  and  $\tilde{\varphi}_2$ .

**Claim 4.1**  $Y_1$  and  $\tilde{\varphi}_1$  satisfies (P).

**Proof of Claim 4.1.**  $Y_1$  and  $\tilde{\varphi}_1$  clearly satisfies the first properties in (P) the definition of the set-valued mapping and the continuity of the function  $c$ .

Bounded losses assumption in (P) also holds true. Indeed, there exists  $\alpha \in \mathbb{R}$  such that for all  $s \in \bar{B}(0, \rho)$  and, for all  $p \in \Sigma$ ,  $p \cdot (s - \lambda_N(s)\mathbf{1}) \geq \alpha$  and  $p \cdot \underline{y} \geq \alpha$ . Thus one gets that  $p \cdot y_1 \geq \alpha$  for all  $(y_1, p) \in \text{Gr } \tilde{\varphi}_1$ , using Lemma 4.2.  $\square$

**Claim 4.2**  $Y_2$  and  $\tilde{\varphi}_2$  satisfies (P).

**Proof of Claim 4.2.** One easily checks that  $Y_2$  is closed and that it satisfies the free-disposal assumption in (P). From the assumptions on the transfer rate rule given in Definition 2.2,  $\tilde{\varphi}_2$  has obviously convex compact values. Since  $y_2 \in \partial Y_2$  implies that  $-y_2 \in \partial V_S$  for at least one  $S \in \mathcal{N}$ ,  $\tilde{\varphi}_2$  has non-empty values. In order to prove that  $\tilde{\varphi}_2$  is upper semi-continuous, since  $\Sigma$  is compact, it suffices to show that the set-valued mapping  $\tilde{\psi}_2$  defined by  $\tilde{\psi}_2(y_2) = \cup_{S \in \mathcal{S}(-y_2)} \varphi_S(-y_2)$  has a closed graph. Let  $(y_2^\nu, p^\nu)$  be a sequence in  $\partial Y_2 \times \Sigma$  which converges to  $(y_2, p)$  and such that  $p^\nu \in \tilde{\psi}_2(y_2^\nu)$  for all  $\nu$ . From the definition of  $\mathcal{S}$ , for  $\nu$  sufficiently large,  $\mathcal{S}(-y_2^\nu) \subset \mathcal{S}(-y_2)$ . Consequently, for all  $\nu$  sufficiently large, there exists  $S^\nu \in \mathcal{S}(-y_2)$  such that  $p^\nu \in \varphi_{S^\nu}(-y_2^\nu)$ . Since  $\mathcal{S}(-y_2)$  is a finite set, there exists a subsequence such that  $S^\nu$  is constant and equal to  $S$ . Since  $\varphi_S$  is upper semi-continuous, this implies that  $p \in \varphi_S(-y_2) \subset \tilde{\psi}_2(y_2)$  and concludes the proof.

Bounded losses assumption in (P) also holds true. Indeed, for all  $(y_2, p) \in \text{Gr } \tilde{\varphi}_2$ , from Lemma 4.1, one has:

$$p \cdot y_2 \geq \sum_{i \in N, y_{2i} \geq -m} p_i y_{2i} \geq -m \sum_{i \in N, y_{2i} \geq -m} p_i = -m \sum_{i \in N} p_i = -m. \quad \square$$

**Claim 4.3** (B) and (S) are satisfied.

**Proof of Claim 4.3.** (B) is satisfied since  $Y_2$  is bounded above by  $-v$  and  $Y_1$  is also bounded above since  $\{p_N(s) \mid s \in \bar{B}_{1^\perp}(0, \rho)\}$  is a compact set.

(S) holds true. If it is not the case, there exists  $t > 0$ ,  $(y_1, y_2) \in \partial Y_1 \times \partial Y_2$  and  $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$  such that  $y_1 + y_2 + t\mathbf{1} \in C$  and  $p \cdot (y_1 + y_2 + t\mathbf{1}) = 0$ . Since  $p \in \mathbb{R}_+^N \setminus \{0\}$  and  $C \subset \mathbb{R}_{++}^N \cup \{0\}$ , one deduces that  $y_1 + y_2 + t\mathbf{1} = 0$ . Let  $s_1 = \text{proj}(y_1)$  and  $s_2 = \text{proj}(y_2)$ . Clearly,  $s_1 = -s_2$ . Then one can apply Lemma 4.3 which states that  $y_1 \in \partial V_N$  and  $p \in \psi(y_1)$ .

Let  $x = -y_2$ , thus  $x \in \partial W$ .  $s_1 = -s_2$  implies that  $p_N(x) = y_1$  consequently  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$ . Thus, since the game is payoff-dependent balanced, one has  $x \in V_N$ . But,  $y_1 = x - t\mathbf{1}$  contradicts the fact that  $y_1 \in \partial V_N$  from the free disposal property of  $V_N$ .  $\square$

From the previous claims, the conclusion of Lemma 4.4 is satisfied.  $\square$

Theorem 2.2 implies that there exists a vector  $(y_1, y_2, p) \in \partial Y_1 \times \partial Y_2 \times \Sigma$  such that:  $y_1 + y_2 \in C \subset \{0\} \cup \mathbb{R}_{++}^N$  and  $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$ .

We first show that  $y_1 + y_2 = 0$ . If it is not true,  $-y_2 \ll y_1$ . But  $y_1 \in Y_1 \subset V_N$  and thus  $-y_2 \in \text{int } V_N \subset \text{int } W$ , which contradicts that  $-y_2 \in (\text{int } W)^c$ . Since  $\text{proj}(y_1) = -\text{proj}(y_2)$ , applying Lemma 4.3, we get  $p \in \text{co}\{\varphi_S(y_1) \mid S \in \mathcal{S}(y_1)\} \cap \psi(y_1) \neq \emptyset$  and  $y_1 \in \partial V_N$ . Since  $y_1 = -y_2 \in \partial W$ ,  $y_1 \notin \text{int } V_S$  for all  $S \in \mathcal{N}$ . As was to be proved,  $y_1$  satisfies the conclusion of Theorem 2.1.  $\square$

*Proof of Lemma 2.1.*

Suppose for some  $x \in \partial W$  that  $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$ . Then there exists some non-negative  $\lambda_S$  for each  $S \in \mathcal{S}(x)$  such that:

$$\sum_{S \in \mathcal{S}(x)} \lambda_S = 1, x = p_W(p_N(x)) \text{ and } m^N - \sum_{S \in \mathcal{S}(x)} \lambda_S m_S = \tilde{\eta}(p_W(p_N(x))) \quad (*).$$

To end the proof, we show that  $\eta^* := \tilde{\eta}(x) = 0$ . Putting  $M := \{m \in N \mid \eta_m^* = \max_{i \in N} \eta_i^*\}$ , we only need to show that  $\sum_{i \in M} \eta_i^* \leq 0$  since  $\eta^* = \tilde{\eta}(x) \in \mathbf{1}_\perp$ . If  $i \in M$  and  $j \in N \setminus M$ , that is  $\eta_j^* < \eta_i^*$ , then, from (\*), there exists  $R \in \mathcal{S}(x)$  such that  $j \in R$  and  $i \notin R$ . Therefore,  $c_{ij}(x) = 0$  from the definition of the mapping  $c_{ij}$ . Using the following argument extracted from Ichiishi and Idzik (2002), which was also exhibiting the partnership property, we get, denoting by  $t(x) = 1/(|N|(\eta_*(x) + 1))$ :<sup>19</sup>

$$\begin{aligned} \sum_{i \in M} \eta_i^* &= \sum_{i \in M} \tilde{\eta}_i(x) = t(x) \sum_{i \in M} \sum_{j \in N} (c_{ij}(x) - c_{ji}(x)) \\ &= t(x) (\sum_{i \in M} \sum_{j \in M} (c_{ij}(x) - c_{ji}(x)) + \sum_{i \in M} \sum_{j \in N \setminus M} (c_{ij}(x) - c_{ji}(x))) \\ &= 0 + t(x) \sum_{i \in M} \sum_{j \in N \setminus M} (-c_{ji}(x)) \leq 0. \end{aligned}$$

Consequently  $\eta^* = 0$ .  $\square$

*Proof of Theorem 3.1*

We first introduce some uniform bounds with respect to the environment set. From the lower semi-continuity and closed graph assumptions of the set valued  $V_{\{i\}}$ , it is immediate to see that the mappings  $v_i$ ,  $i \in N$ , are continuous. Let us denote  $v = \min\{v_i(\theta) \mid \theta \in \Theta, i \in N\} \mathbf{1}$ . The bound  $m(\theta)$  given in (A.6)

<sup>19</sup>They provide a new proof of an extension of the KKMS theorem proposed by Reny and Wooders (1998).

can also be chosen continuously since the set-valued mapping  $V_S$ ,  $S \in \mathcal{N}$ , are lower semi-continuous with closed graph. Let  $m = \max\{m(\theta) \mid \theta \in \Theta\}$ . These elements exist since  $\Theta$  is compact.

We define the set-valued mapping  $Y_2$  from  $\Theta$  into  $\mathbb{R}^N$  by:

$$Y_2(\theta) = -\text{int } (W(\theta)^c).$$

Note that  $Y_2$  is lower semi-continuous with a closed graph and, for all  $\theta \in \Theta$ ,  $Y_2(\theta) - \mathbb{R}_+^N = Y_2(\theta)$  and  $Y_2(\theta) \neq \mathbb{R}^N$ . Let  $\tilde{\varphi}_2$  be the set-valued mapping from  $\text{Gr } \partial Y_2$  into  $\Sigma$  defined by:

$$\tilde{\varphi}_2(\theta, y_2) = \text{co } \{\varphi_S(\theta, -y_2) \mid S \in \mathcal{S}_\theta(-y_2)\}.$$

**Lemma 4.5** *Let  $(\theta, y_2) \in \text{Gr } \partial Y_2$ , if  $y_{2i} < -m$  for some  $i \in N$ , then  $p_i = 0$  for all  $p \in \tilde{\varphi}_2(\theta, y_2)$ .*

**Proof of Lemma 4.5.** We apply Lemma 4.1 to the set-valued mappings  $\tilde{\varphi}_2(\theta, \cdot)$  and the set  $Y_2(\theta)$ .  $\square$

Since  $Y_2(\theta)$  is uniformly bounded above by  $-v$ , there exists  $\rho$  such that  $\text{proj}(\theta, y_2) \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)$  for all  $(\theta, y_2) \in \text{Gr } \partial Y_2$  such that  $y_2 \in (\{-me\} + \mathbb{R}_+^N)$ . Let us define the set-valued mapping  $Y_1$  from  $\Theta$  into  $\mathbb{R}^N$  by:

$$Y_1(\theta) = \left\{ p_V(s, \theta) \mid s \in \bar{B}_{\mathbf{1}^\perp}(0, \rho) \right\} - \mathbb{R}_+^N.$$

Since  $p_V$  is continuous, note that  $Y_1$  is lower semi-continuous with a closed graph and, for all  $\theta \in \Theta$ ,  $Y_1(\theta) - \mathbb{R}_+^N = Y_1(\theta)$  and  $Y_1(\theta) \neq \mathbb{R}^N$ . Then, the compactness of  $\Theta$  implies the existence of two real numbers  $\alpha_1$  and  $\beta_1$  such that for all  $y_1 \in \{z_1 \in \partial Y_1(\theta) \mid \|\text{proj}(z_1)\| \leq \rho, \theta \in \Theta\}$ ,  $\alpha_1 \mathbf{1} \leq y_1 \leq \beta_1 \mathbf{1}$ . Note also that, for all  $\theta \in \Theta$ , for all  $y_1 \in \partial Y_1(\theta)$ , if  $\|\text{proj}(y_1)\| \leq \rho$ , then  $y_1 \in \partial V(\theta)$ . Let us choose  $\underline{y}' \in \text{int } Y_1(\theta)$  for all  $\theta \in \Theta$ . Such element exists since every element strictly inferior to  $\alpha_1 \mathbf{1}$  satisfies this condition.

**Lemma 4.6** *There exists a continuous mapping  $c$  from  $\text{Gr } \partial Y_1$  to  $\Sigma_{++}$  such that  $c(\theta, y_1) \cdot (y_1 - \underline{y}') \geq 0$  for all  $(\theta, y_1) \in \text{Gr } \partial Y_1$ .*

**Proof of Lemma 4.6.** Define a mapping  $\Gamma'$  on  $\text{Gr } \partial Y_1$  such that  $\Gamma'(\theta, y_1) = \{p \in \Sigma_{++} \mid p \cdot (y_1 - \underline{y}') > 0\}$  and use the arguments given in the proof of Lemma 4.2.  $\square$

Let  $\tilde{\varphi}_1$  be the set-valued mapping from  $\text{Gr } \partial Y_1$  into  $\Sigma$  defined by:

$$\tilde{\varphi}_1(\theta, y_1) = \begin{cases} \psi(\theta, y_1) & \text{if } \|\text{proj}(y_1)\| < \rho \\ \text{co}\{\psi(\theta, y_1), c(\theta, y_1)\} & \text{if } \|\text{proj}(y_1)\| = \rho \\ c(\theta, y_1) & \text{if } \|\text{proj}(y_1)\| > \rho \end{cases}$$

**Lemma 4.7** *There exists  $\alpha \in \mathbb{R}$  such that, for all  $(\theta, y_1, y_2) \in \text{Gr } (\partial Y_1 \times \partial Y_2)$ ,  $(p_1, p_2) \in \tilde{\varphi}_1(\theta, y_1) \times \tilde{\varphi}_2(\theta, y_2)$ , one has  $p_1 \cdot y_1 + p_2 \cdot y_2 \geq \alpha$ .*

**Proof of Lemma 4.7.** For all  $(\theta, y_2) \in \text{Gr } \partial Y_2$ , for all  $p \in \tilde{\varphi}_2(\theta, y_2)$ , from Lemma 4.5, one has  $p \cdot y_2 \geq \sum_{i \in N, y_{2i} \geq -m} p_i y_{2i} \geq -m \sum_{i \in N, y_{2i} \geq -m} p_i = -m \sum_{i \in N} p_i = -m$  since  $p \in \Sigma$ . Secondly, for all  $(\theta, y_1) \in \text{Gr } \partial Y_1$ ,  $p \in \tilde{\varphi}_1(\theta, y_1)$ , if  $\|\text{proj}(y_1)\| \leq \rho$  then  $p \cdot y_1 \geq p \cdot \alpha_1 \mathbf{1} = \alpha_1$ ; if  $\|\text{proj}(y_1)\| > \rho$ , from Lemma 4.6,  $p \cdot y_1 = c(\theta, y_1) \cdot y_1 \geq c(\theta, y_1) \cdot \underline{y}' \geq \min\{q \cdot \underline{y}' \mid q \in \Sigma\}$ . Hence  $p \cdot y_1$  is bounded below, which proves the result.  $\square$

Since the values of  $Y_1$  and  $Y_2$  are respectively uniformly bounded above by  $\beta_1 \mathbf{1}$  and  $-v$ , there exists a convex and compact set  $\bar{B} \in (\mathbf{1}^\perp)^2$  such that:  $B(0, \rho) \times B(0, \rho) \subset \bar{B}$  and, for all  $(\theta, y_1, y_2) \in \text{Gr } (\partial Y_1 \times \partial Y_2)$  such that  $y_1 + y_2 - \alpha \mathbf{1} \in \mathbb{R}_{++}^N \cup \{0\}$ ,  $(\text{proj}(y_1), \text{proj}(y_2)) \in \text{int } \bar{B}$ .

Finally, using again (Bonnisseau, 1997, Lemma 3.1 p.217), one introduces the continuous mappings  $\lambda_1$  and  $\lambda_2$  from  $\Theta \times \mathbf{1}^\perp$  to  $\mathbb{R}$  associated to  $Y_1$  and  $Y_2$ . We fix  $\eta > 0$  arbitrary, let  $\Sigma_\eta$  be the set  $\{p \in \mathbb{R}^N \mid \sum_{i \in N} p_i = 1; p_i \geq -\eta, i \in N\}$ .

Let  $F$  be the set-valued mapping from  $\Theta \times B \times \Sigma_\eta \times \Sigma^2$  into itself.  $F = \prod_{j=1}^4 F_j$ .

$$F_1(\theta, (s_1, s_2), p, (p_1, p_2)) = G(\theta, y_1)$$

$$F_2(\theta, (s_1, s_2), p, (p_1, p_2)) = \{\sigma \in B \mid \sum_{i=1}^2 (p-p_i) \cdot \sigma_i \geq \sum_{i=1}^2 (p-p_i) \cdot \sigma'_i, \forall \sigma' \in B\}$$

$$F_3(\theta, (s_1, s_2), p, (p_1, p_2)) = \{q \in \Sigma_\eta \mid (q - q') \cdot (y_1 + y_2) \leq 0, \forall q' \in \Sigma_\eta\}$$

$$F_4(\theta, (s_1, s_2), p, (p_1, p_2)) = (\tilde{\varphi}_1(\theta, y_1), \tilde{\varphi}_2(\theta, y_2))$$

where for  $i = 1; 2$ ,  $y_i = s_i - \lambda_i(\theta, s_i) \mathbf{1}$ .

**Lemma 4.8** *The mapping  $F$  satisfies Kakutani's fixed point theorem conditions.*

**Proof of Lemma 4.8.**  $F$  is a set valued mapping from a non-empty, convex, compact set into itself. Actually, it suffices to verify for  $F_4$  that the assumptions of Kakutani's fixed point theorem are satisfied since the others components obviously meet the assumptions.

By construction,  $\tilde{\varphi}_1$  has non-empty and convex values and  $\tilde{\varphi}_1$  is upper semi-

continuous. Since  $(\theta, y_2) \in \text{Gr } \partial Y_2$  implies that  $-y_2 \in \partial V_S(\theta)$  for at least one  $S \in \mathcal{N}$ ,  $\tilde{\varphi}_2$  has obviously convex and non-empty values. Since  $\Sigma$  is compact, it suffices to show that the set-valued mapping  $\tilde{\psi}_2$  defined by  $\tilde{\psi}_2(\theta, y_2) = \cup_{S \in \mathcal{S}_\theta(-y_2)} \varphi_S(\theta, -y_2)$  has a closed graph in order to prove that  $\tilde{\varphi}_2$  is upper semi-continuous.

Let  $(\theta^\nu, y^\nu, p^\nu)$  be a sequence of  $\text{Gr } \partial Y_2 \times \Sigma$  which converges to  $(\theta, y, p)$  and such that  $p^\nu \in \tilde{\psi}_2(\theta^\nu, y^\nu)$  for all  $\nu$ . From the definition of  $\mathcal{S}$ , for  $\nu$  large enough,  $\mathcal{S}_{\theta^\nu}(-y_2^\nu) \subset \mathcal{S}_\theta(-y_2)$ . Indeed, it is not true, since  $\mathcal{N}$  is a finite set, there exists  $S \in \mathcal{N}$  and a subsequence  $(\theta^\nu, y^\nu)$  such that for  $\nu$  large enough  $-y_2^\nu \in \partial V_S(\theta^\nu)$  and  $-y_2 \notin \partial V_S(\theta)$ . Since  $V_S$  is a lower semi-continuous set-valued mapping with a closed graph and  $V_S(\theta) - \mathbb{R}_+^N = V_S(\theta)$ , the set-valued mapping  $\theta \rightarrow \partial V_S(\theta)$  has a closed graph. Since  $-(y_2^\nu)$  converges to  $-y_2$ , one gets a contradiction.

Consequently, for all  $\nu$  large enough, there exists  $S^\nu \in \mathcal{S}_\theta(-y_2)$  such that  $p^\nu \in \varphi_{S^\nu}(\theta^\nu, -y_2^\nu)$ . Since  $\mathcal{S}_\theta(-y_2)$  is a finite set, there exists a subsequence such that  $S^\nu$  is constant and equal to  $S$ . Since  $\varphi_S$  is upper semi-continuous, this implies that  $p \in \varphi_S(\theta, -y_2)$ , which is included in  $\tilde{\psi}_2(\theta, y_2)$  since  $S \in \mathcal{S}_\theta(-y_2)$ .  $\square$

From the previous lemma, there exists  $(\theta^*, (s_1^*, s_2^*), p^*, (p_1^*, p_2^*))$  such that, if, for  $i = 1, 2$ ,  $y_i^* = s_i^* - \lambda_i(\theta^*, s_i^*)\mathbf{1}$ :

$$\theta^* \in G(\theta^*, y_1^*) \quad (1)$$

$$(s_1^*, s_2^*) = (\text{proj}(y_1^*), \text{proj}(y_2^*)) \text{ and } (y_1^*, y_2^*) \in \partial Y_1(\theta^*) \times \partial Y_2(\theta^*) \quad (2)$$

$$\sum_{i=1}^2 (p^* - p_i^*) \cdot s_i^* \geq \sum_{i=1}^2 (p^* - p_i^*) \cdot \sigma'_i \text{ for each } \sigma' \in B \quad (3)$$

$$(p^* - q') \cdot (y_1^* + y_2^*) \leq 0 \text{ for each } q' \in \Sigma_\eta \quad (4)$$

$$(p_1^*, p_2^*) \in (\tilde{\varphi}_1(\theta^*, y_1^*), \tilde{\varphi}_2(\theta^*, y_2^*)) \quad (5)$$

We now exhibit from the above equations an element satisfying the conclusion of Theorem 3.1.<sup>20</sup> Let  $\gamma^* = -p^* \cdot (y_1^* + y_2^*)$ , remark that  $p^* \cdot (y_1^* + y_2^* + \gamma^*e) = 0$  and  $\gamma^* \leq -\alpha$ . Indeed,  $p^* \cdot \sum_{i=1}^2 y_i^* = p^* \cdot (\sum_{i=1}^2 s_i^* - \lambda_i(\theta^*, s_i^*)e)$ . From (3) with  $\sigma' = 0$ , one gets:  $p^* \cdot \sum_{i=1}^2 y_i^* \geq \sum_{i=1}^2 p_i^* \cdot s_i^* - \lambda_i(\theta^*, s_i^*) = \sum_{i=1}^2 p_i^* \cdot y_i^* \geq \alpha$  from Lemma 4.7.

From (4), for each  $q' \in S_\eta$ ,  $q' \cdot (y_1^* + y_2^* + \gamma^*\mathbf{1}) = q' \cdot (y_1^* + y_2^*) + \gamma^* \geq p^* \cdot (y_1^* + y_2^*) + \gamma^* = 0$ . Therefore,  $y_1^* + y_2^* + \gamma^*\mathbf{1} \in \{0\} \cup \mathbb{R}_{++}^N$  and it follows that  $(s_j^*) \in \text{int } \bar{B}$  by construction of the set  $\bar{B}$ . Then  $p^* = p_1^* = p_2^* \in \Sigma$ , from (3), since the maximum of a linear function is interior only if it is a null mapping.

<sup>20</sup> Bonnisseau (1997) used a similar argument to show the existence of a general equilibrium with externalities.

$p^* \in \Sigma$  implies  $y_1^* + y_2^* + \gamma^* \mathbf{1} = 0$ . From (2) that means  $s_1^* = \text{proj}(y_1^*) = -\text{proj}(y_2^*) = -s_2^*$ .

It remains to show that  $y_1^* \in \partial V_N(\theta^*)$  and  $p^* \in \psi(\theta^*, y_1^*)$ . The argument is exactly the same as the one in the proof of Lemma 4.3.

Let  $\xi^* = -y_2^*$  and  $x^* = y_1^*$ . We deduce that  $\xi^* \in \partial W(\theta^*)$  and therefore, from  $x^* - \xi^* + \gamma^* \mathbf{1} = 0$ , it follows that  $p_W(\theta^*, x^*) = \xi^*$ , or, equivalently,  $p_V(\theta^*, \xi^*) = x^*$ . From (5),  $p^* \in \psi(\theta^*, p_V(\theta^*, \xi^*)) \cap \text{co}\{\varphi_S(\theta^*, \xi^*) \mid S \in \mathcal{S}_{\theta^*}(\xi^*)\}$ . So we deduce from the condition of payoff-dependent balancedness that  $\xi^* \in V(\theta^*)$ . Since  $\xi^* \in \partial W(\theta^*) \cap V(\theta^*)$ ,  $x^* = p_V(\theta^*, \xi^*) \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$ . Using (1), one can say that  $(\theta^*, x^*)$  is an equilibrium-core vector pair and  $p^* \in \psi(\theta^*, x^*) \cap \text{co}\{\varphi_S(\theta^*, p_W(\theta^*, x^*)) \mid S \in \mathcal{S}_{\theta^*}(p_W(\theta^*, x^*))\}$  as was to be proved.  $\square$

### *Proof of Corollary 3.2.*

(A.4):  $G$  has convex values from the quasi-concavity of  $v_i$  with respect to the first variable.  $G$  has non-empty values since, from the definition of  $V_S$ , for all  $\theta \in \Theta$ , for all  $x \in \partial V_N(\theta)$ , there exists  $\theta' \in \Theta$  such that  $x \leq (v_i(\theta'_i, \theta_i))_i$ . From the continuity of the mappings  $v_i$  and  $p_N$ ,  $G$  is clearly an upper semi-continuous set-valued mapping.

(A.5): Since  $F^i$  has non-empty values for all  $i \in N$  and from the balancedness assumption, taking the balanced family  $(\{i\}, i \in N)$  one can prove the non-emptiness of  $\Theta$ . Now, the lower-semi continuity and closed graph assumption of the set-valued mappings  $V_S$ ,  $S \in \mathcal{N}$ , are proved.

(*lower semi-continuity*) For all  $\theta^\nu \in \Theta$  a sequence converging to  $\theta \in \Theta$ , we show that for each  $x \in V_S(\theta)$ , there exists a sequence  $(x^\nu)$  converging to  $x$  with  $x^\nu \in V_S(\theta^\nu)$  for  $\nu$  large enough. Since  $x \in V_S(\theta)$ , there exists  $\theta' \in F^S(\theta)$  such that  $x_i \leq v_i(\theta'_i, \theta_i)$ ,  $i \in S$ . Since  $F^S$  is lower semi-continuous, there exists a sequence  $(\theta'^\nu)$  converging to  $\theta'$  with  $\theta'^\nu \in F^S(\theta^\nu)$ . Then, from the continuity of the mapping  $v_i$ ,  $v_i(\theta'^\nu_i, \theta_i)$  tends to  $v_i(\theta'_i, \theta_i)$ ,  $i \in S$ . Let  $T$  be a subset of  $S$  such that, for each  $i \in T$ ,  $x_i = v_i(\theta'_i, \theta_i)$ . Now, it suffices to take  $x_i^\nu = v_i(\theta'^\nu_i, \theta_i)$ ,  $i \in T$ , and  $x_i^\nu = x_i$ ,  $i \in S \setminus T$  to conclude the proof.

(*closed graph*) Let  $(\theta^\nu)$  be a sequence converging to  $\theta$ , we show that if  $x^\nu \in V_S(\theta^\nu)$  converges to  $x \in \mathbb{R}^N$ , then  $x \in V_S(\theta)$ . For all  $\nu \geq 0$ , there exists  $\theta'^\nu \in F^S(\theta^\nu)$  such that  $x_i^\nu \leq v_i(\theta'^\nu_i, \theta_i)$ . Since  $F^S$  is upper-semi continuous with compact values,  $F^S(\Theta)$  is compact. Then, the sequence  $(\theta'^\nu)$  remains in a compact set. So taking a subsequence if we need to, one can say that  $(\theta'^\nu)$  tends to an element  $\theta' \in F^S(\theta)$ . Taking the limit and from the continuity of the mappings  $v_i$ , one gets  $\theta' \in F^S(\theta)$  such that  $x_i \leq v_i(\theta'_i, \theta)$  for all  $i \in S$ , that

is to say that  $x \in V_S(\theta)$ , as was to be proved.

Remark now, that from a well known argument relying on the quasi-concavity of the functions  $v_i(\cdot, \xi_i)$ , the balancedness condition given in (5) is equivalent to the balancedness of the game  $(V(\theta), V_S(\theta)_{S \in \mathcal{N}})$  for each  $\theta \in \Theta$ , that is, for each balanced family  $\mathcal{B}$ ,  $\cap_{S \in \mathcal{B}} V_S(\theta) \subset V(\theta)$ . This fact is used in the two paragraphs below.

We check that (A.6) holds true. Let  $S \in \mathcal{N}$ . The family  $\{S, (\{i\})_{i \notin S}\}$  is a balanced family. Let  $(\theta, x) \in \text{Gr } V_S$ , since  $F^i(\theta)$  is non-empty, there exist  $\xi^i \in F^i(\theta)$ ,  $i \notin S$ . Let  $x'$  be defined by  $x'_i = x_i$ ,  $i \in S$  and  $x'_i = v_i(\xi^i, \theta_i)$ ,  $i \notin S$ . Clearly,  $x' \in V_S(\theta) \cap (\cap_{i \notin S} V_{\{i\}}(\theta))$ . From the balancedness of the game,  $x' \in V(\theta)$ . Consequently, from the compactness of  $\Theta$  and the continuity of  $v_i$ ,  $i \in N$ , there exists  $m(\theta)$  such that  $x' \leq m(\theta)\mathbf{1}$ , hence  $x_i \leq m(\theta)$ ,  $i \in S$ .

Finally, the parametrized game is payoff-dependent balanced since the game is balanced (see the argument in the proof of Corollary 2.1).  $\square$

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