Economic Theory manuscript No.
(will be inserted by the editor)

Properties and applications of dual reduction

Yannick Viossat

Received: date / Accepted: date

Abstract The dual reduction process, introduced by Myerson, allows a finite game to be reduced to a smaller-dimensional game such that any correlated equilibrium of the reduced game is an equilibrium of the original game. We study the properties and applications of this process. It is shown that generic two-player normal form games have a unique full dual reduction (a known refinement of dual reduction) and that all strategies that have probability zero in all correlated equilibria are eliminated in all full dual reductions. Among other applications, we give a linear programming proof of the fact that a unique correlated equilibrium is a Nash equilibrium, and improve on a result due to Nau, Gomez-Canovas and Hansen on the geometry of Nash equilibria and correlated equilibria.

Keywords Correlated equilibrium · Nash equilibrium · Dual reduction · Linear duality

JEL Classification C72

1 Introduction

Dual reduction (Myerson 1997) is a reduction process for finite normal form games which, in a sense, generalizes elimination of dominated strategies. Its roots lie in the proofs of existence of correlated equilibria of Hart and Schmeidler (1989) and Nau and McCardle (1990). The reduction consists in eliminating some pure strategies or merging several pure strategies into a single mixed strategy that serves as a new pure strategy. The main property is that any Nash or correlated equilibrium of the reduced game is an equilibrium of

Y. Viossat
CEREMADE, Université Paris-Dauphine, Place du maréchal de Lattre de Tassigny, F-75016 Paris, France
E-mail: viossat@ceremade.dauphine.fr

the original game. Moreover, by iterative reduction, any finite game can be reduced to an elementary game, that is, to a game in which all incentive constraints defining correlated equilibria may be satisfied as strict inequalities in a correlated equilibrium. Myerson (1997) also showed that, while some games can be reduced in several ways, this ambiguity is alleviated if we focus on a specific class of dual reductions, called full dual reductions.

A first aim of this article is to better understand the properties of dual reduction: in particular, which strategies and equilibria are eliminated, and whether focusing on full dual reduction allows us to get a uniquely defined reduction process. Our main results are that: (i) full dual reductions need not eliminate weakly dominated strategies; however, they eliminate all strategies which have probability zero in all correlated equilibria; (ii) generic two-player normal form games have a unique full dual reduction. The main step in proving (ii) is to understand that the ways in which dual reduction merges pure strategies is linked to the Nash equilibria of the games obtained by omitting certain pure strategies of the original game.

The second aim of this article is to show that dual reduction is a useful tool to study properties of Nash and correlated equilibria. We give three examples: (i) a linear programming proof of the fact that a unique correlated equilibrium is a Nash equilibrium; (ii) a proof that any tight game (Nitzan 2005) has a completely mixed Nash equilibrium; (iii) a refinement of a result of Nau et al. (2004) on the geometry of Nash equilibria and correlated equilibria. Other applications are discussed.

The remainder of the article is organized as follows: the next section sums up known results, with some new insights. In Section 3, we try to understand which strategies and equilibria are eliminated by dual reduction. Section 4 gives examples of applications of dual reduction. Finally, the Appendix shows that almost all two-player games have a unique full dual reduction. All references to Myerson are to Myerson (1997).

Notation. As Myerson, we denote a finite game in strategic form by

$$\Gamma = (N, (C_i)_{i \in N}, (U_i)_{i \in N})$$

where N is the finite set of players, C_i the finite set of pure strategies of player i, and $U_i : \times_{j \in N} C_j \to \mathbb{R}$ the utility function of player i. The set of pure strategy profiles is denoted by $C = \times_{j \in N} C_j$. For each player i, we let $C_{-i} = \times_{j \in N \setminus \{i\}} C_j$. If $c = (c_j)_{j \in N}$ is a pure strategy profile and d_i a pure strategy of player i, we let (c_{-i}, d_i) denote the pure strategy profile that differs from c only in that the i-component is d_i.

For any finite set S, we let $|S|$ denote its number of elements and let $\Delta(S)$ be the set of probability distributions over S. We identify the element s of S with the corresponding vertex of $\Delta(S)$. A correlated strategy of the players in N is an element of $\Delta(C)$. Thus, $\mu = (\mu(c))_{c \in C}$ is a correlated strategy if and only if $\mu(c) \geq 0$ for any c in C and $\sum_{c \in C} \mu(c) = 1$.

Let μ be a correlated strategy. Assume that before play, a mediator draws a pure strategy profile c with probability $\mu(c)$ and then privately recommends c_i

to player i, for every i in N. The correlated strategy μ is a *correlated equilibrium* (Aumann 1974; 1987) if no player has an incentive to deviate unilaterally from these recommendations. That is, μ is a correlated equilibrium if and only if it satisfies the following incentive constraints:

$$\sum_{c_{-i} \in C_{-i}} \mu(c) \left[U_i(c_{-i}, d_i) - U_i(c)\right] \leq 0 \qquad \forall i \in N, \forall c_i \in C_i, \forall d_i \in C_i \qquad (1)$$

For any mapping $\alpha_i : C_i \to \Delta(C_i)$ and any c_i, d_i in C_i, we may write $\alpha_i * c_i$ instead of $\alpha_i(c_i)$ to denote the image of c_i by this mapping, and $\alpha_i(d_i|c_i)$ instead of $(\alpha_i * c_i)(d_i)$ to denote the probability of d_i in the mixed strategy $\alpha_i * c_i$. This is for consistency with Myerson.

2 Basics of dual reduction

Unless stated otherwise, all results of this section are due to Myerson.

Dual vectors. Assume that before play a mediator privately recommends a pure strategy to each player, who can either obey or deviate from this recommendation. The behavior of player i can then be described by a mapping $\alpha_i : C_i \to \Delta(C_i)$, which associates to every pure strategy c_i the randomized strategy $\alpha_i * c_i$ that she will play if recommended c_i. The mapping α_i may be called a *deviation plan* for player i. If a mediator tries to implement a pure strategy profile c, then player i's gain from deviating unilaterally according to α_i instead of following the mediator's recommendation is

$$D_i(c, \alpha_i) := U_i(c_{-i}, \alpha_i * c_i) - U_i(c)$$

Let $\alpha = (\alpha_i)_{i \in N}$ be a profile of deviation plans and let $D(c, \alpha)$ denote the sum of the above gains over the set of players:

$$D(c, \alpha) := \sum_{i \in N} D_i(c, \alpha_i) = \sum_{i \in N} \left[U_i(c_{-i}, \alpha_i * c_i) - U_i(c)\right]$$

Definition. A *dual vector* is a profile of deviation plans $\alpha = (\alpha_i)_{i \in N}$ such that $D(c, \alpha) \geq 0$ for every pure strategy profile c in C.

Any game has at least one dual vector. Indeed, letting $\alpha_i * c_i = c_i$ for all i in N and all c_i in C_i defines a dual vector. We call it the *trivial dual vector*.

Duality and existence of correlated equilibria. Dual vectors naturally arise from the linear programming proofs of existence of correlated equilibria (Hart and Schmeidler 1989; Nau and McCardle 1990). The term dual vector comes from the following duality theorem: For any c in C, the problem of finding a correlated equilibrium μ such that $\mu(c) > 0$ is a linear program whose dual is equivalent to the problem of showing that there is no dual vector α such that $D(c, \alpha) > 0$ (Nau and McCardle 1990, Proposition 2 [1]).

[1] Up to normalization of α, the quantity $D(c, \alpha)$ here corresponds to $-A(s)\alpha$ there.

In light of this result, to prove existence of correlated equilibria, it suffices to show that there is at least one c for which there is no dual vector α^c such that $D(c, \alpha^c) > 0$. To prove this by contradiction, assume that for every c there is some such dual vector. Then any interior combination of these dual vectors yields a dual vector α such that $D(c, \alpha) > 0$ for all c. But for every mixed strategy profile σ and every j in N,

$$\sum_{c \in C} \sigma(c) D_j(c, \alpha_j) = U_j(\sigma_{-j}, \alpha_j * \sigma_j) - U_j(\sigma) \tag{2}$$

where

$$\alpha_j * \sigma_j = \sum_{c_j \in C_j} \sigma_j(c_j)(\alpha_j * c_j)$$

Indeed, both sides of (2) represent the expected gain for player j from deviating according to α_j, when all other players obey a mediator who is trying to implement σ. Summing over the set of players N, we get

$$\sum_{c \in C} \sigma(c) D(c, \alpha) = \sum_{j \in N} [U_j(\sigma_{-j}, \alpha_j * \sigma_j) - U_j(\sigma)] \tag{3}$$

Now the mapping $\alpha_j : C_j \to \Delta(C_j)$ may be seen as a transition probability on C_j and as such induces a Markov chain on C_j. By basic properties of Markov chains, every player j has a mixed strategy σ_j which is α_j*-stationary*, that is, such that $\alpha_j * \sigma_j = \sigma_j$. For such an α_j-stationary strategy σ_j and any σ_{-j}, $U_j(\sigma_{-j}, \alpha_j * \sigma_j) = U_j(\sigma)$ so that $\sum_{c \in C} \sigma(c) D_j(c, \alpha_j) = 0$ by (2). If σ_j is α_j-stationary for all j in N, then $\sum_{c \in C} \sigma(c) D(c, \alpha) = 0$ by (3), contradicting the assumption that $D(c, \alpha) > 0$ for all c.

A variant of this proof of existence of correlated equilibria has recently been used by Papadimitriou and Roughgarden (2008) to develop algorithms for finding correlated equilibria in a broad class of succinctly representable multiplayer games, in a time which is polynomial in the succinct representation of the game. For more on (mostly Nash) equilibrium computation, see the excellent survey by Roughgarden (2010) and the other articles in the same special issue of Economic Theory.

How to reduce a game using a dual vector. Let α be a dual vector. As noted above, the mapping $\alpha_i : C_i \to \Delta(C_i)$ induces a Markov chain on C_i. This Markov chain partitions C_i into a set of transient states and disjoint minimal absorbing sets.[2] From the basic theory of Markov chains, it follows that for every minimal absorbing set $B_i \subset C_i$, there is a unique α_i-stationary mixed strategy with support B_i. Let C_i/α_i denote the set of such α_i-stationary strategies with support equal to a minimal absorbing set. The set C_i/α_i is thus a subset of the set of mixed strategies of player i in Γ, and, because the minimal absorbing sets are disjoints, it has as most as many elements as C_i. The α-reduced game, denoted by Γ/α, is the game with the same set of players and

[2] A nonempty subset B_i of C_i is a *minimal absorbing set* for the Markov chain induced by α_i if: (i) for every c_i in B_i, $\alpha_i * c_i$ has support in B_i and (ii) it contains no nonempty proper subset satisfying (i).

the same utility functions as in Γ, but in which, for every i in N, the pure strategy set of player i is C_i/α_i:

$$\Gamma/\alpha = \{N, (C_i/\alpha_i)_{i\in N}, (U_i)_{i\in N}\}$$

The reduction thus operates by eliminating some strategies and merging other strategies. The pure strategies of Γ/α are mixed strategies of Γ. If a pure strategy $c_i \in C_i$ is a transient state for the Markov chain induced by α_i, then c_i is *eliminated*. It is not a pure strategy and does not take part in any pure strategy of the reduced game Γ/α: for all σ_i in C_i/α_i, seen as elements of $\Delta(C_i)$, we have $\sigma_i(c_i) = 0$. If c_i is α_i-stationary (i.e. $\alpha_i * c_i = c_i$), then c_i is *kept* as a pure strategy of the reduced game: $c_i \in C_i/\alpha_i$. Finally, if c_i is neither transient nor stationary, then the strategies in the minimal absorbing set of c_i are *merged* (Myerson says "*consolidated*") into an equivalence class, which is represented by the unique α_i-stationary strategy with support equal to this minimal absorbing set. Thus, c_i is not in C_i/α_i, but there exists σ_i in C_i/α_i such that $\sigma_i(c_i) > 0$ (where σ_i is seen as an element of $\Delta(C_i)$).

Furthermore, due to properties of Markov chains, a mixed strategy σ_i in $\Delta(C_i)$ is α_i-stationary if and only if it is a convex combination of the α_i-stationary strategies with support equal to some minimal absorbing set, that is, if and only if it belongs to $\Delta(C_i/\alpha_i)$. So the set of mixed strategies of player i in Γ/α corresponds to his set of α_i-stationary mixed strategies in Γ.

Definition. A *dual reduction* of Γ is any α-reduced game Γ/α where α is a dual vector. An *iterative dual reduction* of Γ is any game $\Gamma/\alpha^1/\alpha^2/\ldots$ where each α^k is a dual vector for $\Gamma/\alpha^1/\alpha^2/\ldots/\alpha^{k-1}$. (The expression "dual reduction" may refer either to a reduced game or to the reduction technique.)

Dual reduction and equilibria. Let $C/\alpha = \times_{i\in N} C_i/\alpha_i$ denote the set of pure strategy profiles of the reduced game Γ/α. So the set of correlated strategies of the reduced game is $\Delta(C/\alpha)$. Any correlated strategy μ in C/α can be mapped back to a Γ-equivalent correlated strategy $\bar{\mu}$ in the natural way:

$$\bar{\mu}(c) = \sum_{\sigma\in C/\alpha} \mu(\sigma)\sigma(c) \quad \forall c \in C$$

(the mapping $\mu \mapsto \bar{\mu}$ can be shown to be injective). Myerson's main result is that if μ is a correlated equilibrium of a dual reduction of Γ, then $\bar{\mu}$ is a correlated equilibrium of Γ. The same result holds for Nash equilibrium.

The main step of the proof is to show that, if σ is a mixed strategy profile such that σ_j is α_j-stationary for all $j \neq i$, then $U_i(\sigma) \leq U_i(\sigma_{-i}, \alpha_i * \sigma_i)$. To see this, note that if $\alpha_j * \sigma_j = \sigma_j$ for all $j \neq i$, then (3) is reduced to

$$\sum_{c\in C} \sigma(c) D(c,\alpha) = U_i(\sigma_{-i}, \alpha_i * \sigma_i) - U_i(\sigma) \tag{4}$$

Because α is a dual vector, $D(c,\alpha) \geq 0$ for all c, hence the left hand side of (4) is nonnegative. Thus, (4) implies $U_i(\sigma) \leq U_i(\sigma_{-i}, \alpha_i * \sigma_i)$.

From this, it can be seen that if σ_j is α_j-stationary for all $j \neq i$, then player i has an α_i-stationary best-response to σ_{-i} (for instance, the Cesàro limit - that is, the limit of the arithmetic mean - of the sequence (σ_i^k) defined inductively by $\sigma_i^0 = \sigma_i$ and $\sigma_i^{k+1} = \alpha_i * \sigma_i^k$, where σ_i is any best-response to σ_{-i}; the sequence (σ_i^k) need not converge, but its arithmetic mean does). It is then relatively straightforward to show that any equilibrium of Γ/α is an equilibrium of Γ.

Payoff rescaling. Linear transformations of the utility functions do not affect the players' preferences. So, for dual reduction not to depend on specific representations of these preferences, such transformations ought not affect the ways in which a game may be reduced. We show that this is indeed the case, although Myerson does not mention it.

For instance, consider the following games Γ and Γ', where Γ' is obtained from Γ by multiplying player 1's payoffs by 1/2:

Γ :

	x_2	y_2
x_1	2, 0	0, 2
y_1	0, 2	2, 0

Γ' :

	x_2	y_2
x_1	1, 0	0, 2
y_1	0, 2	1, 0

Define α by $\alpha_i * x_i = y_i$ and $\alpha_i * y_i = x_i$ for every i in $\{1, 2\}$ (recall that pure strategies are identified with the vertices of the simplex). Then α is a dual vector of Γ but not of Γ', because $D(\alpha, c) < 0$ for $c = (x_1, y_2)$. However, α can be rescaled into a dual vector of Γ' inducing the same reduction. The idea is to replace α_1 by a deviation plan α'_1 which deviates similarly but twice as much. However, $\alpha'_1(y_1|x_1)$, for instance, would then be greater than 1. So instead, we replace α_1 and α_2 by deviations plans which deviate $2K$ and K times as much, respectively, with $2K \leq 1$. This doubles the impact of player 1's deviations compared to player 2's, hence compensates for the fact that player 1's payoffs have been multiplied by 1/2, while ensuring that the new deviation plans are well defined.

Formally, let $\lambda_1 = 1/2$ and $\lambda_2 = 1$, and let $0 < K \leq \min_{i \in N} \lambda_i = 1/2$. Let

$$\alpha'_i * c_i = \left(1 - \frac{K}{\lambda_i}\right) c_i + \left(\frac{K}{\lambda_i}\right) [\alpha_1 * c_i]$$

for all c_i in C_i. Then for every σ_i in $\Delta(C_i)$, $[\alpha'_i * \sigma_i] - \sigma_i = (K/\lambda_i)[\alpha_i * \sigma_i]$. It follows that α' is a dual vector of Γ' and that the same mixed strategies are stationary under α_i and α'_i, so that α and α' induce the same reduced game: $\Gamma'/\alpha' = \Gamma'/\alpha$.

More generally, let Γ and Γ' be two games with the same sets of players and strategies. Let us say that Γ' is a *rescaling* of Γ if for every player i in N, there exists a positive constant λ_i and a function $f_i : C_{-i} \to \mathbb{R}$ such that $U'_i(c) = \lambda_i . U_i(c) + f_i(c_{-i})$ for all c in C. Drawing on the above construction, it is easy to show that if Γ' is a rescaling of Γ then for any dual vector α of Γ there is a dual vector α' of Γ' such that $\Gamma'/\alpha' = \Gamma'/\alpha$.

Symmetric games. Let Γ be a two-player symmetric game. That is, $C_1 = C_2 = \{1, 2, ..., m\}$ and for all (k, l) in $\{1, 2, ..., m\}^2$, $U_1(k, l) = U_2(l, k)$. If

$\alpha = (\alpha_1, \alpha_2)$ is a dual vector, then so are $\alpha' = (\alpha_2, \alpha_1)$ and $\bar{\alpha} = (\alpha + \alpha')/2$. Moreover, $\Gamma/\bar{\alpha}$ is symmetric. More generally, if there are some symmetries in the roles of the players (e.g. if within a subset of the set of players, the roles of the players are cyclically symmetric), then the game can be reduced in a way that respects these symmetries (Viossat 2008a, Proposition 21).

Jeopardization and reduction to elementary games. A game is *elementary* (Myerson) if it has a correlated equilibrium μ which satisfies all incentive constraints with strict inequality:

$$\sum_{c \in C} \mu(c) \left[U_i(c_{-i}, d_i) - U_i(c)\right] > 0 \quad \forall i \in N, \forall c_i \neq d_i \tag{5}$$

If μ satisfies (5), then for any correlated strategy μ' with full support and for any ϵ small enough, $(1-\epsilon)\mu + \epsilon\mu'$ is a strict correlated equilibrium with full support; conversely, any strict correlated equilibrium with full support satisfies (5). So, elementary games can also be defined as those games having a strict correlated equilibrium with full support.[3]

Myerson shows that a game may be strictly reduced if and only if it is not elementary; this implies that, though many games are not elementary (e.g., Matching Pennies), any game may be reduced to an elementary game by iterative dual reduction. The proof is based on the concept of jeopardization and Proposition 1 below which we will use extensively.

Definition. Let c_i and d_i be pure strategies of player i. The strategy d_i *jeopardizes* c_i if in every correlated equilibrium μ, the incentive constraint stipulating that player i should not gain by deviating from c_i to d_i is tight. That is,

$$\sum_{c_{-i} \in C_{-i}} \mu(c) \left[U_i(c_{-i}, d_i) - U_i(c)\right] = 0$$

Proposition 1 (Myerson) *There exists a dual vector α such that $\alpha_i(d_i|c_i) > 0$ if and only if d_i jeopardizes c_i.*

Proof This is a special case of strong complementary slackness in linear programming. See Myerson for details. □

Full dual reduction Let us say that a dual vector is *full* (or has full support) if it is positive in every component that is positive in at least one dual vector. By Proposition 1, for any full dual vector α and any pure strategies c_i and d_i in C_i, $\alpha_i(d_i|c_i) > 0$ if and only if d_i jeopardizes c_i. Existence of full dual vectors follows from the convexity of the set of dual vectors. A *full dual reduction* of Γ is any reduced game Γ/α where α is a full dual vector.

[3] A correlated equilibrium μ has *full support* if $\mu(c) > 0$ for all c in C. It is *strict* if the incentive constraint (1) is satisfied with strict inequality, for any i in N, any c_i that has positive marginal probability in μ, and any $d_i \neq c_i$.

Full dual reduction significantly refines dual reduction. Indeed, we show in the Appendix that almost all two-player games have a unique full dual reduction. This is not true of general dual reductions (for instance, any game with several weakly dominated strategies has several nontrivial dual reductions).

The interest of full dual reductions also lies in the following properties, which remain implicit in Myerson: all full dual vectors, having the same positive components, induce the same minimal absorbing sets. Therefore, in all full dual reductions, the pure strategies that are eliminated (resp. kept as pure strategies, merged together) are the same. Moreover, in any full dual reduction, there are weakly fewer pure strategies than in any other dual reduction of the same game. That is, if α is a full dual vector and β is any dual vector, then $|C_i/\alpha_i| \leq |C_i/\beta_i|$ for all i. This follows from the fact that any component that is positive in β_i is also positive in α_i.

3 Which strategies and equilibria are eliminated?

We say that the pure strategy c_i is *eliminated* in the dual reduction Γ/α if $\sigma_i(c_i) = 0$ for all σ_i in C_i/α_i. A pure strategy profile c is eliminated if $\sigma(c) = 0$ for all σ in C/α; that is, if c_i is eliminated for some i in N. Finally, a correlated equilibrium μ is eliminated if there is no correlated strategy of Γ/α which induces μ in Γ. In this section, we try to understand which kind of strategies and equilibria are eliminated in dual reductions.

Strategies. As shown by Myerson, if a pure strategy is weakly dominated then there is a dual reduction which exactly eliminates this strategy. Moreover, iterative dual reduction always leads to a game that has no weakly dominated strategies. Finally, if a dual reduction consists in eliminating a pure strategy, say c_i, then this strategy is weakly dominated, or gives exactly the same payoffs as another mixed strategy. Indeed, for all $j \neq i$, every pure strategy in C_j must be stationary under the associated dual vector α. Due to (4), this implies that $U_i(c_{-i}, \alpha_i * c_i) \geq U_i(c)$ for all c_{-i} in C_{-i}.

In the sense that these properties hold, dual reduction generalizes elimination of dominated strategies. However, full dual reductions need not eliminate weakly dominated strategies:

Example 1

	x_2	y_2
x_1	$1,1$	$1,0$
y_1	$1,0$	$0,0$

In this game, μ is a correlated equilibrium if and only if y_2 is not played in μ. That is, $\mu(x_1, y_2) = \mu(y_1, y_2) = 0$. Therefore x_1 and y_1 jeopardize each other. It follows that, in all full dual reductions, x_1 and y_1 are merged, hence y_1 is not eliminated.

By contrast, in full dual reductions, strategies that have marginal probability zero in all correlated equilibria are eliminated, and a similar result holds for strategy profiles.

Proposition 2 *Let $c \in C$ (resp. $c_i \in C_i$) be a pure strategy profile (resp. pure strategy) that has probability zero in all correlated equilibria. Then c (resp. c_i) is eliminated in all full dual reductions and in all elementary iterative dual reductions.*

To prove Proposition 2, we introduce a new class of dual vectors.

Definition. A dual vector is *strong* if for every pure strategy profile c in C that has probability zero in all correlated equilibria

$$D(c, \alpha) = \sum_{i \in N} [U_i(c_{-i}, \alpha_i * c_i) - U_i(c)] > 0.$$

(the first equality simply recalls the definition of $D(c, \alpha)$). Existence of strong dual vectors follows from Nau and McCardle's (1990) proof of existence of correlated equilibria. Due to the linearity of the conditions defining dual vectors and their refinements, any interior convex combination of a full dual vector and of a strong dual vector is a dual vector which is both strong and full. This implies that there is at least one strong and full dual vector. It is actually easy to show that a dual vector is both strong and full if and only if it belongs to the relative interior of the (convex) set of dual vectors.

Proof of Proposition 2. First consider elementary iterative dual reductions: if Γ^e is an elementary iterative dual reduction of Γ, then it has a correlated equilibrium with full support, which induces a correlated equilibrium of Γ. Therefore all pure strategy profiles and pure strategies of Γ that have not been eliminated in Γ^e have positive probability in some correlated equilibrium.[4]

Now consider full dual reductions. Let α be a dual vector. Let $\sigma \in C/\alpha$. By definition of C/α, $\alpha_i * \sigma_i = \sigma_i$ for all i in N. Therefore, it follows from (3) that $\sum_{c \in C} \sigma(c) D(c, \alpha) = 0$. Since by definition of dual vectors, $D(c, \alpha) \geq 0$ for all c in C, this implies that

$$D(c, \alpha) > 0 \Rightarrow \sigma(c) = 0. \tag{6}$$

Assume now that α is strong and full, and let c be a pure strategy profile that has probability zero in all correlated equilibria. Since α is strong, by definition, $D(c, \alpha) > 0$, hence $\sigma(c) = 0$ by (6). Therefore, c is eliminated in the dual reduction Γ/α. Furthermore, α is a full dual vector and in all full dual reductions, the same strategies, hence the same strategy profiles are eliminated. It follows that c is eliminated in all full dual reductions.

Finally, let c_i be a pure strategy profile with marginal probability zero in all correlated equilibria. For all c_{-i} in C_{-i}, the profile $c = (c_{-i}, c_i)$ has probability zero in all correlated equilibria. Therefore, in a full dual reduction, (c_{-i}, c_i) is eliminated for all c_{-i} in C_{-i}, hence c_i itself is eliminated. This completes the proof. □

[4] I owe my understanding of this point to Roger Myerson.

Equilibria. We now turn to equilibria. Let us say that in a dual reduction, a correlated equilibrium μ is eliminated if there is no correlated strategy of the reduced game which induces μ in the original game.

Proposition 3 *Strict correlated equilibria cannot be eliminated, not even in an iterative dual reduction.*

Proof If μ is a strict correlated equilibrium, then a strategy that has positive marginal probability in μ cannot be jeopardized by another strategy. It follows that in any dual reduction of Γ, all pure strategies used in μ remain as pure strategies. Furthermore, as the player's options for deviating are more limited in the reduced game than in Γ, the distribution μ is a strict correlated equilibrium of the reduced game. Inductively, in any iterative dual reduction of Γ, all strategies used in μ are available and μ is a strict correlated equilibrium.[5] $\square$

Recall that a Nash equilibrium σ is *quasi-strict* if for all i in N, any pure best response to σ_{-i} belongs to the support of σ_i. Contrary to strict equilibria, quasi-strict equilibria may be eliminated by dual reduction as follows from Example 1. Actually, in the following game :

	x_2	y_2
x_1	$1,1$	$1,1$
y_1	$1,1$	$0,0$

the unique full dual reduction consists in eliminating y_1 and y_2; thus, in this game, every full dual reduction eliminates all quasi-strict equilibria (this also happens in the game of Myerson's figure 6).

Moreover, the following example shows that completely mixed Nash equilibria may be eliminated in all nontrivial dual reductions. In the left game, playing each strategy with equal probability is a completely mixed Nash equilibrium:

	x_2	y_2	z_2
x_1	$3,1$	$2,2$	$0,0$
y_1	$3,1$	$0,0$	$2,2$

	y_2	z_2
y_1	$2,2$	$0,0$
z_1	$0,0$	$2,2$

However, the unique nontrivial dual reduction is the game on the right, in which x_2 and thus all completely mixed Nash equilibria of the original game have been eliminated. (To see that the only nontrivial dual reduction consists in eliminating x_2, note that, for player 2, x_2 is equivalent to $\frac{1}{2}y_2 + \frac{1}{2}z_2$; this implies that y_2 and z_2 jeopardize x_2. Furthermore, for any i in $\{1,2\}$, the strategies y_i and z_i must be stationary under any dual vector because they have positive probability in a strict correlated equilibrium. The result follows.)

In this example, the reduced game is obtained by eliminating a pure strategy of player 2 which is redundant in the sense that it gives her the same payoffs as another mixed strategy. More generally, let $\Gamma' = \{N, (C_i')_{i \in N}, (U_i)_{i \in N}\}$ be

[5] This proof shows that a pure strategy that has positive marginal probability in some strict correlated equilibrium can never be eliminated nor merged with other strategies. This generalizes the fact that elementary games cannot be reduced.

a game obtained from Γ by omitting some pure strategies (thus $C'_i \subset C_i$ for all i). Assume that the omitted strategies are redundant in the sense that for every omitted pure strategy c_i, there is a mixed strategy in $\Delta(C'_i)$ that gives the same payoffs to player i as c_i:

$$\forall i \in N, \forall c_i \in C_i \backslash C'_i, \exists \sigma_i \in \Delta(C'_i), \forall c_{-i} \in C_{-i}, U_i(c_{-i}, c_i) = U_i(c_{-i}, \sigma_i) \quad (7)$$

Then Γ' is a dual reduction of Γ. Indeed, for all i in N, let $\alpha_i * c_i = c_i$ if $c_i \in C_i$ and $\alpha_i * c_i = \sigma_i$ (defined in (7)) otherwise; this defines a dual vector α such that $\Gamma/\alpha = \Gamma'$. In particular, dual reduction allows to reduce any game to its reduced normal form (Kohlberg and Mertens 1986).

4 Some applications of dual reduction

This section aims at showing that dual reduction is a useful tool to study properties of Nash equilibria and correlated equilibria. We focus on three applications. First, we prove by linear programming that a unique correlated equilibrium is a Nash equilibrium. Second, we show that tight games (Nitzan 2005) have a completely mixed Nash equilibrium. Third, we refine a result of Nau et al. (2004) on the geometry of Nash equilibria and correlated equilibria.

A unique correlated equilibrium is a Nash equilibrium: an elementary proof. Consider the fact that, if a game has a unique correlated equilibrium, then this correlated equilibrium is a Nash equilibrium. Of course, this follows from the existence of a Nash equilibrium and the fact that any Nash equilibrium is a correlated equilibrium. However, as for existence of correlated equilibria (Hart and Schmeidler 1989; Nau and McCardle 1990), it would be nice to find a more direct proof, relying solely on linear programming. Dual reduction provides such a proof.

Indeed, let Γ be a game with a unique correlated equilibrium. By iterative dual reduction, Γ may be reduced to an elementary game Γ^e. Since Γ^e is elementary (i.e. has a strict correlated equilibrium with full support), it follows that either Γ^e has an infinity of correlated equilibria, or Γ^e has a unique strategy profile. Since Γ has a unique correlated equilibrium and since different correlated equilibria of Γ^e induce different correlated equilibria of Γ, the first case is ruled out. Therefore, Γ^e has a unique strategy profile, hence trivially a Nash equilibrium. This implies that Γ has a Nash equilibrium. Hence that the unique correlated equilibrium of Γ is a Nash equilibrium.

This proof relies on: a) the definition of dual reduction, which requires the Minimax theorem and existence of stationary distributions for finite Markov chains; and b) the fact that any game may be reduced to an elementary game, which Myerson proved through the strong complementary property of linear programs. Since the existence of stationary distributions for finite Markov chains can be deduced from the Minimax theorem, the above proof relies solely on linear duality. In particular no fixed point theorem is used.[6]

[6] The fact that the existence of stationary distributions for finite Markov chains can be deduced from the Minimax theorem is mentioned by Mertens et al. (1994, ex. 9, p.41). Here

Tight games. Consider the class of tight games (Nitzan 2005):

Definition. A game is *tight* if in every correlated equilibrium μ, all incentive constraints are tight:

$$\sum_{c_{-i}\in C_{-i}} \mu(c)[U_i(c_{-i},d_i)-U_i(c)] = 0 \qquad \forall i\in N, \forall c_i \in C_i, \forall d_i \in C_i \tag{8}$$

Nitzan (2005) shows that, for any positive integer n, there is an open set of tight games within the set of $n \times n$ two-player games.

Proposition 4 *Any tight game has a completely mixed Nash equilibrium*

Proof The definition of tight games may be rephrased as follows: a game is tight if d_i jeopardizes c_i, for every player i and every pair of pure strategies (c_i, d_i) in $C_i \times C_i$. Therefore, if α is a full dual vector of a tight game Γ, then for every pair of pure strategies (c_i, d_i) in $C_i \times C_i$, we have $\alpha_i(d_i|c_i) > 0$. Thus, for the Markov chain induced by α_i, there is a unique minimal absorbing set, namely the whole of C_i. Therefore, in the full dual reduction Γ/α, all strategies of player i are merged into a single representative, and this holds for every i in N. It follows that Γ/α has a unique strategy profile σ, which is a completely mixed strategy profile of Γ; furthermore, since σ is trivially a Nash equilibrium of Γ/α, it is a Nash equilibrium of Γ. □

Pretight games Let C_i^c denote the set of pure strategies of player i that have positive marginal probability in at least one correlated equilibrium. Let us say that a game is *pretight* if, in every correlated equilibrium μ, every incentive constraint (1) with c_i and d_i in C_i^c is tight. That is,

$$\sum_{c_{-i}\in C_{-i}} \mu(c)[U_i(c_{-i},d_i)-U_i(c)] = 0 \quad \forall i\in N, \forall c_i \in C_i^c, \forall d_i \in C_i^c \tag{9}$$

This condition, which is weaker than (8), has been introduced by Nau et al. (2004). Recall that the inequalities defining the set of correlated equilibria are linear in μ, so that the set of correlated equilibria is a convex polytope. Nau et al (2004, Proposition 2) showed that if there exists a Nash equilibrium in the relative interior of the correlated equilibrium polytope, then (a) the game is pretight, and (b) there exists a Nash equilibrium with support $\times_{i\in N} C_i^c$. Dual reduction allows us to show that (a) implies (b).

Proposition 5 *Any pretight game has a quasi-strict Nash equilibrium with support* $\times_{i\in N} C_i^c$.

is a proof: let $M = (m_{ij})$ denote a stochastic matrix (that is, the m_{ij} are nonnegative and each column sums to unity). Applying the lemma of Hart and Schmeidler (1989, p.19) with $a_{jk} = m_{kj}$ and u a basis vector, we get that there exists a probability vector x such that $Mx = x$. Since Hart and Schmeidler prove this lemma via the Minimax theorem, this shows that existence of stationary distributions for finite Markov chains can indeed be deduced from the Minimax theorem.

Proof Let Γ be pretight. Let α be a strong and full dual vector. Let $i \in N$. By Proposition 2, in any full dual reduction, all strategies in $C_i \backslash C_i^c$ are eliminated. Moreover, by the definition of pretight games, all strategies of C_i^c jeopardize each other. Therefore, in a full dual reduction, they are either all eliminated or all merged together. The first case is ruled out because all other pure strategies of player i are eliminated. It follows that C_i/α_i consists of a unique mixed strategy σ_i, with support C_i^c. Hence, in Γ/α there is a unique pure strategy profile σ, which has support $\times_{i\in N} C_i^c$, and which is a Nash equilibrium of Γ/α, hence of Γ.

We now show that σ is quasi-strict. Let $c_i \in C_i \backslash C_i^c$ and let $\tau = (\sigma_{-i}, c_i)$. Since c_i has marginal probability zero in all correlated equilibria, it follows that every strategy profile c with $\tau(c) > 0$ has probability zero in all correlated equilibria. By definition of strong dual vectors, this implies that $D(c, \alpha) > 0$. Therefore

$$\sum_{c\in C} \tau(c) D(c, \alpha) > 0.$$

However, because for every $j \neq i$ the strategy τ_j is equal to σ_j hence α_j-stationary, it follows that equation (4) is satisfied (with τ replacing σ). Therefore

$$0 < \sum_{c\in C} \tau(c) D(c, \alpha) = U_i(\tau_{-i}, \alpha_i * \tau_i) - U_i(\tau) = U_i(\sigma_{-i}, \alpha_i * c_i) - U_i(\sigma_{-i}, c_i)$$

which implies that c_i is not a best reply to σ_{-i}, so σ is quasi-strict. □

Proposition 5 does not only show that Nau et al.'s condition (a) implies their condition (b), but also that any pretight game has a quasi-strict Nash equilibrium: a nontrivial fact, since for $n \geq 3$, not all n-player games have a quasi-strict Nash equilibrium (van Damme 1991). Moreover, from Proposition 5 and a few basic arguments given in Viossat (2006), a converse of Nau et al.'s result can be obtained. Namely, if a game is pretight, then there is a Nash equilibrium in the relative interior of the correlated equilibrium polytope.

Remark: it follows from the proof of Proposition 5 that in any full dual reduction of a pretight game, for every player, all pure strategies that have positive marginal probability in at least one correlated equilibrium are merged together, and all others are eliminated. Since two-player zero-sum games and games with a unique correlated equilibrium are pretight (Viossat 2006), this result also holds in these classes of games. This generalizes examples given by Myerson (Figures 3 and 5).

Other applications. Dual reduction is used in Viossat (2008b) to show that a unique correlated equilibrium is a quasi-strict Nash equilibrium. A generalization, still based on dual reduction, is given in Viossat (2008a). Namely, every finite game has a Nash equilibrium σ such that, for all i in N, all pure best-responses to σ_{-i} have positive probability in some correlated equilibrium. Dual reduction can also be used to show that, generically, there are certain dimensions that the correlated equilibrium polytope cannot have (Viossat, 2008a) or to check that a game has a unique correlated equilibrium.[7] Finally, dual reduction is used in Viossat (2005, Chapter 9B) to show that, in all 3×3 symmetric games, from any interior initial condition and under any two-population convex monotonic dynamics (Hofbauer and Weibull 1996), all strategies that have marginal probability zero in all correlated equilibria are eliminated.

Appendix: Uniqueness of the reduction process

In this appendix, we show that generic two-player normal form games have a unique full dual reduction. We first show that this is not true of all games. Let Γ denote the rather trivial game:

	x_2	y_2
x_1	1, 1	0, 1

Let $0 < \epsilon < 1$. Define the full dual vector α^ϵ by $\alpha_2^\epsilon * x_2 = \alpha_2^\epsilon * y_2 = \epsilon x_2 + (1-\epsilon) y_2$. In the full dual reduction Γ/α_ϵ, there is a unique pure strategy profile whose payoffs $(\epsilon, 1)$ depend on ϵ. Thus, even if only full dual reductions are used, there might still be multiple ways to reduce a game. Other examples (omitted) suggest however that multiplicity of full dual reductions typically arises when a player is indifferent between some of his strategies, or becomes so after elimination of strategies of the other players. Such indifference is a non-generic phenomenon in the normal form payoff space, and we show below that almost all two-player normal form games have a unique full dual reduction. We first show that there are severe restrictions on the ways strategies may be merged.

Notation: for all i in N, let $B_i \subset C_i$ and let $B = \times_{i \in N} B_i$. We denote by $\Gamma_B = (N, (B_i)_{i \in N}, (U_i)_{i \in N})$ the game obtained from Γ by reducing the pure strategy set of player i to B_i, for all i in N.

Proposition 6 *Let α be a dual vector. For each i in N, let $B_i \subset C_i$ denote a minimal α_i-absorbing set and $B = \times_{i \in N} B_i$. Let σ_{B_i} denote the unique α_i-stationary strategy of player i with support B_i and $\sigma_B = (\sigma_{B_i})_{i \in N}$. Then σ_B is a completely mixed Nash equilibrium of Γ_B.*

Proof The proof draws on the remark made by Myerson at the end of the proof of his lemma 2. Since B_i is α_i-absorbing, we may define $\alpha'_i : B_i \to \Delta(B_i)$ by $\alpha'_i * c_i = \alpha_i * c_i$ for all c_i in B_i. Since α is a dual vector of Γ, it follows that α' is a dual vector of Γ_B. Moreover, $B/\alpha' = \{\sigma_B\}$, hence σ_B is a Nash equilibrium of Γ_B/α'. This implies that σ_B is a Nash equilibrium of Γ_B. Finally, σ_B is completely mixed because σ_{B_i} has support B_i. □

Corollary 1 *Assume that for every product $B = \times_{i \in N} B_i$ of subsets B_i of C_i, the game Γ_B has at most one completely mixed Nash equilibrium. Then Γ has a unique full dual reduction.*

Proof Let α and α' be two full dual vectors. Let $\sigma \in C/\alpha$. Let B_i denote the support of σ_i (seen as an element of $\Delta(C_i)$) and let $B = \times_{i \in N} B_i$. Note that, as the support of an α_i-stationary strategy, B_i is a minimal α_i-absorbing set. Since full dual vectors have the same minimal absorbing sets, it follows that B_i is also a minimal α'_i-absorbing set. Therefore, there exists τ in C/α' such that τ_i has support B_i, for all i in N. By Proposition 6, both σ and τ are completely mixed Nash equilibria of Γ_B. By assumption, this implies $\sigma = \tau$, hence $\sigma \in C/\alpha'$. Therefore $C/\alpha \subset C/\alpha'$ with equality by symmetry. □

[7] As shown in Viossat (2006): (i) a game has a unique correlated equilibrium if and only it is pretight and has a unique Nash equilibrium; (ii) a game is pretight if and only if it has some specific dual vectors. So if it is known that a game has a unique Nash equilibrium, it can be shown that it has a unique correlated equilibrium by exhibiting appropriate dual vectors, without computing the correlated equilibria.

Proposition 7 *Let Γ be a two-player game. Assume that for any game Γ_B obtained from Γ by omitting some pure strategies, and any Nash equilibrium σ of Γ_B, the supports of σ_1 and σ_2 have the same number of elements. Then Γ has a unique full dual reduction.*[8]

Proof It suffices to show that the assumption of corollary 1 is met. The proof is by contradiction. Assume that there exists $B = B_1 \times B_2 \subset C_1 \times C_2$ such that Γ_B has two distinct completely mixed Nash equilibria σ and τ. Without loss of generality, assume $\sigma_1 \neq \tau_1$. There exists λ in $\mathbb{R}$ such that $\sigma_1^\lambda := \lambda\sigma_1 + (1-\lambda)\tau_1$ is in $\Delta(C_1)$ but its support is a strict subset of the support of σ_1. Since Γ_B is a two-player game and σ and τ are completely mixed (in Γ_B), it follows that $(\sigma_1^\lambda, \sigma_2)$ is a Nash equilibrium of Γ_B. But so is σ, and for at least one of these equilibria, the supports of the strategies of the players do not have the same number of elements. Therefore the assumption of Proposition 7 is not satisfied. □

Additional arguments can be used to show that almost all two-player normal form games have a unique sequence of iterative full dual reductions. See (Viossat 2008a) for details.

Acknowledgements This article originated in my Ph.D. thesis, written at the laboratoire d'économétrie de l'Ecole polytechnique, under the supervision of Sylvain Sorin. I am deeply grateful to Bernhard von Stengel and to Françoise Forges, Ehud Lehrer, Roger Myerson, seminar audiences, and several anonymous referees whose constructive comments helped me to improve the presentation of this article. All shortcomings are mine. The author gratefully acknowledges the support of the ANR, project "Croyances" and of the Fondation du Risque, Chaire Groupama, "Les particuliers face au risque".

References

1. Aumann, R. (1974), "Subjectivity and Correlation in Randomized Strategies", *Journal of Mathematical Economics* **1**, 67–96
2. Aumann, R.J. (1987), "Correlated Equilibria as an Expression of Bayesian Rationality", *Econometrica* **55**, 1–18
3. Hart, S. and D. Schmeidler (1989), "Existence of Correlated Equilibria", *Mathematics of Operations Research* **14**, 18–25
4. Hofbauer, J. and J.W. Weibull (1996), "Evolutionary Selection against Dominated Strategies", *Journal of Economic Theory* **71**, 558–573.
5. Kohlberg, E. and J.F. Mertens (1986), "On the Strategic Stability of Equilibria", *Econometrica* **54**, 1003–1038
6. Mertens, J.F., S. Sorin and S. Zamir (1994) "Repeated games, Part A, Background material", CORE discussion paper 9402, Université Catholique de Louvain
7. Myerson, R.B. (1997), "Dual Reduction and Elementary Games", *Games and Economic Behavior* **21**, 183–202
8. Nau R.F., S. Gomez Canovas, and P. Hansen (2004), "On the Geometry of Nash Equilibria and Correlated Equilibria", *International Journal of Game Theory* **32**, 443–453
9. Nau, R.F. and K.F. McCardle (1990), "Coherent Behavior in Noncooperative Games", *Journal of Economic Theory* **50**, 424–444
10. Nitzan, N. (2005), "Tight Correlated Equilibrium", discussion paper #394, Center for the Study of Rationality, the Hebrew University of Jerusalem
11. Papadimitriou C.H. and T. Roughgarden (2008), "Computing Correlated Equilibria in Multi-Player games", *Journal of the ACM* **55**, Article 14
12. Roughgarden, T. (2010), "Computing Equilibria: a computational complexity perspective", to appear in *Economic Theory* **42**, doi:10.1007/s00199-009-0448-y
13. van Damme, E. (1991), *Stability and Perfection of Nash Equilibria*, Springer-Verlag

[8] This assumption is met by almost all two-player games; indeed, by all two-player games Γ such that every game Γ_B obtained by omitting pure strategies from Γ is nondegenerate in the sense of von Stengel (2002, Def. 2.6 and Thm 2.10).

14. Viossat, Y. (2005), *Correlated Equilibria, Evolutionary Games and Polutation Dynamics*, Ph.D. dissertation, Ecole polytechnique, Paris
15. Viossat, Y. (2006), "The Geometry of Nash Equilibria and Correlated Equilibria and a Generalization of Zero-Sum Games", S-WoPEc working paper 641, Stockholm School of Economics, Stockholm
16. Viossat, Y. (2008a), "Properties and Applications of Dual Reduction", Cahier du Ceremade, Université Paris-Dauphine, hal-00264051[2]
17. Viossat, Y. (2008b), "Is Having a Unique Correlated Equilibrium Robust?", *Journal of Mathematical Economics* **44**, 1152–1160
18. von Stengel, B. (2002), "Computing Equilibria for Two-Person Games", in *Handbook of Game Theory*, R.J. Aumann and S. Hart Eds, Elsevier Science Publishers (North Holland), vol. 3, chap. 45, 1723–1759