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Abstract

This PhD dissertation presents three independent research topics in the field of stochastic target and optimal control problems with applications to financial mathematics.

In a first part, we provide a PDE characterization of the super hedging price of an American option of barrier types in a Markovian model of financial market. This extends to the American case a recent works of Bouchard and Benatjar [9], who considered European barrier options, and Karatzas and Wang [29], who discussed the case of perpetual American barrier options in a Black and Scholes type model. Contrary to [9], we do not use the usual dual formulation, which allows to reduce to a standard control problem, but instead prove and appeal to an American version of the geometric dynamic programming principle for stochastic targets of Soner and Touzi [47]. This allows us to avoid the non-degeneracy assumption on the volatility coefficients of [9], and therefore extends their results to possibly degenerate cases which typically appear when the market is not complete. As a by-product, we provide an extension to the case of American type targets of the result of [47], which is of own interest.

In the second part, within a Brownian diffusion Markovian framework, we provide a direct PDE characterization of the minimal initial endowment required so that the terminal wealth of a financial agent (possibly diminished by the payoff of a random claim) can match a set of constraints in probability. Such constraints should be interpreted as a rough description of a targeted profit and loss (P&L) distribution. This allows to give a price to options under a P&L constraint, or to provide a description of the discrete P&L profiles that can be achieved given an initial capital. This approach provides an alternative to the standard utility indifference (or marginal) pricing rules which is better adapted to market practices. From the mathematical point of view, this is an extension of the stochastic target problem under controlled loss, studied in Bouchard, Elie and Touzi [12], to the case of multiple constraints. Although the associated Hamilton-Jacobi-Bellman operator is fully discontinuous, and the terminal condition is irregular, we are able to construct a numerical scheme that converges at any continuity points of the pricing function.

The last part of this thesis is concerned with the extension of the optimal control
of direction of reflection problem introduced in Bouchard [8] to the jump diffusion case. In a Brownian diffusion framework with jumps, the controlled process is defined as the solution of a stochastic differential equation reflected at the boundary of a domain along oblique directions of reflection which are controlled by a predictable process which may have jumps. We also provide a version of the weak dynamic programming principle of Bouchard and Touzi [14] adapted to our context and which is sufficient to provide a viscosity characterization of the associated value function without requiring the usual heavy measurable selection arguments nor the a-priori continuity of the value function.
Introduction générale

Dans la littérature, un problème de cible stochastique est souvent étudié en utilisant des arguments de dualité qui permet de se ramener à un problème écrit sous une forme standard de contrôle stochastique, voir par exemple Cvitanić et Karatzas [19], Föllmer et Kramkov [22], et, Karatzas et Shreve [30]. Cependant, cette approche ne requiert pas seulement la preuve d’une formulation duale, mais aussi ne s’appliquent qu’aux dynamiques linéaires. Afin d’éviter ces difficultés, on se base sur les étapes introduites par Soner et Touzi [46] et [47], qui ont proposé un nouveau principe de la programmation dynamique, appelé géométrique. Cela ouvre les portes pour un nombre vaste d’applications, notamment pour le contrôle des risques en finance et assurance. Dans un cadre markovien, il permet de dériver les équations aux dérivées partielles associées au problème de cible stochastique de la manière plus directe, à l’exception de la formulation standard de dualité, voir [51] et [44].

L’objectif principal de cette thèse est d’étendre leurs résultats à des applications importantes dans le contrôle des risques en finance et en assurance. En particulier, nous proposons une extension du principe de la programmation dynamique géométrique, qui est associée à une option barrière de type américain avec contraintes. Nous étudions également une classe de problèmes de cible stochastique avec multiples contraintes au sens de la probabilité. Ceci caractérise le prix de sur-réplication sur une forme de P&L comme une unique solution de viscosité d’une équation aux dérivées partielles. Finalement, nous considérons une version faible du principe classique de la programmation dynamique introduite par Bouchara et Touzi [14] et l’appliquons à une nouvelle classe de problèmes de contrôle stochastique associées à des diffusions mixtes. En laquelle le processus contrôlé est défini comme la solution d’une équation différentielle stochastique, qui est rejetée à la frontière d’un domaine borné $\mathcal{O}$ et ses directions de réflexion peuvent être contrôlées. Cette thèse est organisée selon les trois parties suivant.

Dans la première partie, notre objectif est de caractériser les options barrières des type américain dans un modèle markovien du marché financier à l’aide des solutions de viscosité. Récemment, ce problème a été étudié par Bouchard et Bentahar [9] qui ont considéré la couverture sous contraintes avec une barrière dans un
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contexte européen, ainsi qu’une extension du résultat précédent de Shreve, Schmock et Wystup [41]. Karatzas et Wang [29] ont traté de une façon similaire pour les options barrières de type américain dans un modèle de Black-Scholes. La difficulté principale provient de la condition au bord \( \partial O \) avant une maturité. Au cas d’option vanille, il faut envisager une version ‘face-lifted’ du pay-off comme une condition au bord. Cependant, cette condition est insuffisante pour traiter une option barrière plus générale et doit être considérée au sens faible, voir [41]. Ceci conduit à une mélange de la condition au bord de type Neumann et Dirichlet. Au cas américain, un phénomène similaire apparaît, mais nous devons également faire attention au fait que l’option peut être exercée à tout moment avant une maturité.

Au contraire de Bouchard et Bentahar [9], nous n’utilisons pas la formulation usuelle de dualité, au lieu d’appliquer le principe de la programmation dynamique géométrique (PPDG) au problème de cible stochastique. On commence par étendre le travail de Soner et Touzi [47] sur une cible de type américain, appelée une version ‘obstacle’. En général, un problème de cible américaine stochastique est introduit suivant : trouver l’ensemble \( V(0) \) des conditions initiales \( z \) pour \( Z_{t,0}^\nu, z(t) \in G(t) \) \( \forall t \in [0, T] \) \( \mathbb{P} \)-a.s. pour un contrôle \( \nu \). Ici, \( t \mapsto G(t) \) est une fonction en prenant des valeurs dans la collection de sous-ensembles borélienne de \( \mathbb{R}^d \). Dans le Chapitre 1, nous proposons une version obstacle du PPDG écrit sous la forme :

\[
V(t) = \{ z \in \mathbb{R}^d : \exists \nu \in A \text{ s.t. } Z_{t,z}^\nu(\tau \wedge \theta) \in G(t) \quad \forall \tau \in T(t,T) \}, \tag{0.1}
\]

où

\[
G(t) := G(\tau) 1_{\tau \leq \theta} + V(\theta) 1_{\tau > \theta}
\]

et \( T(t,T) \) désignant l’ensemble des temps d’arrêt à valeurs dans \( [t, T] \). Nous concluons cette partie à traiter l’application de la version obstacle au-dessus à l’option barrière américaine avec contraintes dans le Chapitre 2. Soient un modèle markovien du marché financier composé d’un actif non risqué et d’une ensemble des \( d \) actifs risqués \( X = (X^1, \ldots, X^d) \), et un processus de la richesse \( Y_{t,x,y}^\pi \) associé à un capital initial \( y \in \mathbb{R}_+ \) et une stratégie financière \( \pi \). Une option barrière de type américaine est exprimé par une fonction \( g \) définie sur \( [0, T] \times \mathbb{R}_+^d \) et par un domaine ouvert \( O \) de \( \mathbb{R}_+^d \) pour que l’acheteur reçoive le paiement \( g(\theta, X_{t,x}(\theta)) \) quand il exerce l’option au moment de \( \theta \leq \tau_{t,x} \), où \( \tau_{t,x} \) est le temps de la première sortie du processus de prix \( X_{t,x} \) en dehors d’un domaine ouvert \( O \). Dans ce cas, l’ensemble des dotations associées à notre problème est donné par

\[
V(t, x) := \{ y \geq \mathbb{R}_+ : \exists \pi \text{ t.q. } Z_{t,x,y}^\pi(\tau \wedge \tau_{t,x}) \in G(\tau \wedge \tau_{t,x}) \quad \forall \tau \in T(t,T) \},
\]

avec \( Z_{t,x,y}^\pi := (X_{t,x}, Y_{t,x,y}^\pi) \) et \( G(t) := \{(x, y) \in \bar{O} \times \mathbb{R}_+ : y \geq g(t, x)\} \). Ensuite, la fonction valeur \( v(t, x) := \inf V(t, x) \) caractérise complètement l’infini de l’ensemble \( V(t, x) \) et une version du PPDG peut être déclarée en termes de la fonction
valeur $v$ :

(\textbf{DP1}) Si $y > v(t, x)$, alors il existe une stratégie financière $\pi$ telle que

$$Y_\pi^{t,x,y}(\theta) \geq v(\theta, X_{t,x}(\theta)) \text{ pour tout } \theta \in \mathcal{T}_{t,x}.$$ 

(\textbf{DP2}) Si $y < v(t, x)$, alors il existe $\tau \in \mathcal{T}_{t,x}$ tel que

$$\mathbb{P} \left[ Y_\pi^{t,x,y}(\theta \wedge \tau) \geq v(\theta, X_{t,x,y}(\theta)) + g(\tau, X_{t,x,y}(\tau)) \mathbb{1}_{\theta < \tau} + \mathbb{1}_{\theta \geq \tau} \right] < 1$$

pour toute stratégie financière $\pi$ et $\theta \in \mathcal{T}_{t,x} := \{ \tau \in \mathcal{T}_{t,T} : \tau \leq \tau_{t,x} \}$.

L'analyse ci-dessus peut être comparée à Soner et Touzi [49], auquel l’auteurs ont considéré une option de type européen. Et la preuve est vraiment très proches lorsque la programmation dynamique (DP2) est remplacée par une forme correspondante. Les différences principales proviennent du fait que nous n’imposons pas l’hypothèse de non-dégénérescence sur les coefficients des actifs financiers sous-jacents. Cette hypothèse apparaissent dans les marchés financiers complets et est aussi nécessaire à la formulation de dualité, voir [9], [13], [19], [28] et [29]. Nous prenons cette occasion pour expliquer comment ce problème peut être traité et être transposé à l’option américaine sans barrière dont nous parlerons plus tard à la fin de ce chapitre. Afin de séparer les difficultés, on se restreint au cas où les stratégies de $\pi$ sont à valeurs dans un ensemble compact convexe donné. Le cas du contrôle non borné peut être manipulé à l’aide de la technique développée par Bouchard, Elie et Touzi [12]. De même, les sauts pourrait être ajoutée à la dynamique sans aucunes difficultés majeures, voir Bouchard [7].

Dans la deuxième partie, nous étudions la plus petite richesse initiale telle que la richesse terminale d’un agent financier peut être satisfaite d’un ensemble de contraintes en probabilité. En pratique, cet ensemble de contraintes doit être considéré comme une description sommaire d’une distribution ciblée de P&L. Plus précisément, nous supposons que la valeur liquidative de l’option s’écrit $g(X_{t,x}(T))$, où $X_{t,x}$ sont les actifs risqués. Le processus de la richesse $Y^{\nu}_{t,x,y}$ est associé à partir d'un capital initial $y \in \mathbb{R}_+$ et une stratégie financière $\nu$. Pour une collection des seuils $\gamma := (\gamma^i)_{i \leq \kappa} \in \mathbb{R}^\kappa$ et des probabilités $p := (p^i)_{i \leq \kappa} \in [0, 1]^\kappa$, le prix de l’option est défini comme le plus petit capital initial $y$ tel que il existe une stratégie financière $\pi$ pour quelle le filet de la perte ne dépasse pas $-\gamma^i$ avec une probabilité plus de $p^i$ pour tout $i \in \{1, \ldots, \kappa\}$. Cela conduit à une contrainte pour la distribution de P&L qui est imposée par un histogramme discret associé à $(\gamma, p)$. En évitant la grandé négativité du processus de richesse ( même avec une probabilité faible), nous également imposons que $Y^{\nu}_{t,x,y}(T) \geq \ell$ pour certains $\ell \in \mathbb{R}_-$. Le prix est alors écrit.
sous la forme suivante :

\[ v(t, x, p) := \inf \{ y : \exists \nu \text{ s.t. } Y_{t,x,y}^{\nu}(T) \geq \ell, \ P \{ Y_{t,x,y}^{\nu}(T) - g(X_{t,x}(T)) \geq -\gamma^i \} \geq p^i \ \forall \ i \leq \kappa \}, \]

où on suppose que \( \gamma^{\kappa} \geq \cdots \geq \gamma^2 \geq \gamma^1 \geq 0 \).

Car \( \gamma^i \geq \gamma^j \) pour \( i \leq j \), notre problème naturellement se limite aux cas \( p^i \leq p^j \) pour \( i \leq j \). Ceci conduit à l’introduction de conditions aux bords sur les plans où \( p^i = p^j \) pour certains \( i \neq j \). Mais cette restriction n’est pas nécessaire dans notre approche. Nous alors ne l’utilisons pas et nous concentrons sur les points de la frontière \((t, x, p)\) tels que \( p_i = 0 \) ou \( p_i = 1 \). Du point de vue numérique, on pourrait cependant utiliser le fait que \( v(\cdot, p) = v(\cdot, \hat{p}) \), où \( \hat{p} \) est défini par \( \hat{p}^j = \max_{i \leq j} p^j \) pour \( i \leq \kappa \).

Au cas \( \kappa = 1 \), un tel problème est appelé le “quantile hedging problem”. Il a été largement étudié par Föllmer and Leukert [23] qui ont donné une description explicite de la richesse terminale optimale de \( Y_{t,x,y}^{\nu}(T) \) quand le sous-jacent du marché financier est complet. Ce résultat est dérivé d’une utilisation intelligente du Lemme de Neyman-Pearson en statistiques mathématiques et s’applique aux cadres non-Markoviens. Ensuite, une approche directe, basée sur la notion de problèmes cibles stochastiques, a été proposée par Bouchard, Elie et Touzi [12]. Cela nous permet de caractériser la fonction valeur même dans les marchés incomplets ou dans les cas où le processus de prix \( X_{t,x} \) peut être influencé par la stratégie de négociation \( \nu \), voir par exemple Bouchard et Dang [10]. Le problème (0.2) est une généralisation de ce travail au cas de multiples contraintes en probabilité.

Comme dans [12], la première étape consiste à récrire (0.2) comme un problème standard de cible stochastique :

\[ v(t, x, p) = \inf \{ y : \exists (\nu, \alpha) \text{ t.q. } Y_{t,x,y}^{\nu}(T) \geq \ell, \ \min_{i \leq \kappa} (1_{Y_{t,x,y}^{\nu}(T) - g(X_{t,x}(T)) \geq -\gamma^i - P_{t,p}^{\alpha,i}(T)}) \geq 0 \}. \]

La réduction ci-dessus est obtenue par ajouter une famille de martingales bornées \( d \)-dimensionnelles \( P_{t,p}^{\alpha,i} \) dont \( i \)-ème composante \( P_{t,p}^{\alpha,i} \) est définie par la représentation des martingales pour \( 1_{Y_{t,x,y}^{\nu}(T) - g(X_{t,x}(T)) \geq -\gamma^i} \). Il conduit à un principe de la programmation dynamique géométrique, qui nous permet de caractériser la fonction valeur comme une solution discontinue de viscosité d’une certaine équation aux dérivées partielles. Dans cette thèse, on étudie des telles équations dans le cadre de marchés complets pour deux cas suivants.

Au premier cas, le montant d’argent investi dans les actifs risqués est nonborné, voir l’exemple d’une option d’achat dans le modèle de Black-Scholes dans Föllmer and Leukert [23]. Un opérateur associé de Hamilton-Jacobi-Bellman est indiqué dans 8
la Subsection 4 du Chapitre 3 avec une condition terminale :

\[ v^*(T, x, p) = v_\nu(T, x, p) = l(1 - \max_{i \leq \kappa} p^i) + \sum_{i=0}^{\kappa-1} (g(x) - \gamma^i)(p^{i+1} - \max_{k \leq i} p^k)_+. \] (0.3)

Nous nous concentrerons sur le deuxième cas où le montant d’argent investi dans les actifs risqués est borné. C’est-à-dire que les stratégies financières \( \nu \) appartiennent à l’ensemble \( \mathcal{U} \) de processus progressivement mesurable prenant des valeurs dans un sous-ensemble compact \( \mathbb{R}^d \). Dans ce cas, la condition terminale de \( v \) n’est pas une forme naturelle de (0.3) à cause de l’absence de convexité par rapport à la variable \( p \). De plus, l’apparition de la contrainte de gradient "\( \text{diag}[x] D_x v \in U \)" conduit à la version "facelifted " de \( g \), appliquée \( \hat{g} \), voir par exemple Cvitanić, Pham et Touzi [20]. En suite, la condition terminale peut être naturellement formulée au termes faible \( v_\nu(T, \cdot) \geq \hat{G}_\nu \) and \( v^*(T, \cdot) \leq \hat{G}^* \), où

\[ \hat{G}(x, p) := \max_{i \in \mathcal{K}} \left( \ell \mathbf{1}_{p^i = 0} + (g(x) - \gamma^i)\mathbf{1}_{0 < p^i < 1} + (\hat{g}(x) - \gamma^i)\mathbf{1}_{p^i = 1} \right). \]

Il est clair que \( \hat{G}_\nu < \hat{G}^* \) pour certain \( p \in \partial[0, 1]^\kappa \) et l’opérateur associé de Hamilton-Jacobi-Bellman est également discontinu. Ceci est impossible d’établir un résultat de comparaison et aussi de construire un schéma numérique basé sur ce opérateur. Afin de résoudre ces difficultés, nous introduisons dans le Chapitre 4 une séquence de problèmes approximatifs qui sont plus réguliers et pour lesquels une convergence des schémas numériques associés est obtenue. En particulier, nous montrerons que une telle séquence permet de préciser une estimation de les problèmes suivantes :

\[ \underline{v}(t, x, p) = \inf \left\{ y : \forall \varepsilon > 0 \exists \nu^\varepsilon \text{ t.q. } Y^\nu^\varepsilon_{t,x,y}(T) \geq \ell, \quad \mathbb{P}\left[ Y^\nu^\varepsilon_{t,x,y}(T) - g(X^\nu_{t,x}(T)) \geq -\gamma^i \right] \geq p^i - \varepsilon \forall i \right\} \]

et

\[ \bar{v}(t, x, p) = \inf \left\{ y : \exists \nu \text{ t.q. } Y^\nu_{t,x,y}(T) \geq \ell, \quad \mathbb{P}\left[ Y^\nu_{t,x,y}(T) - g(X^\nu_{t,x}(T)) \geq -\gamma^i \right] > p^i \forall i \leq \kappa \right\}. \]

On montre effectivement que la première fonction value \( \underline{v} \) est la limite à gauche en \( p \) de \( v \) tandis que la deuxième \( \bar{v} \) est la limite à droid en \( p \) de \( v \). Aux cas où \( v \) is continu, en suite \( \bar{v} = v = \underline{v} \) et nos schémas convergent vers la fonction valeur originale. Cependant, il semble que cette continuité est difficile à être démontrée par les absences de la convexité et de la monotonie stricte de la fonction indicatrice. Pourtant, l’un des deux approximations ci-dessus peut être choisi pour résoudre des problèmes pratiques.

Dans la dernière partie, nous considérons un problème de contrôle optimal stochastique pour les processus de réflexion dans un cadre de la diffusion Brownienne.
avec des sauts. L’originalité de ce travail provient du fait que le processus contrôlé est défini comme la solution d’une équation différentielle stochastique qui se reflète à la frontière d’un domaine $O$. La direction de réflexion oblique $\gamma$ est contrôlée par un processus prévisible qui peut avoir des sauts.

Lorsque la direction de réflexion $\gamma$ n’est pas contrôlée, on obtient une classe des équations différentielles stochastiques plus simples avec réflexion, dans laquelle le processus de réflexion est défini sur $\mathbb{R}_+$, voir Skorokhod [42]. Tanaka [50] a montré l’existence d’une solution de telle équation dans des domaines convexes, voir El Karoui et Marchan [17], et, Ikeda et Watanabe [26]. Plus tard, Lions et Sznitman [35] ont étudié le cas où le domaine $O$ satisfait une condition de la sphère extérieure uniforme, qui signifie que il existe $r > 0$ tel que pour tout $x \in \partial O$ il y a $x_1 \in \mathbb{R}^d$ satisfaisant $B(x_1, r) \cap \bar{O} = \{x\}$. En général, l’existence et l’unicité des solutions au problème de Skorokhod avec réflexion oblique régulière sont encore des questions ouvertes, seulement quelques cas particuliers ont été traités. En particulier, Lions et Sznitman [35] ont montré que l’unicité quand soit le domaine $O$ est régulier, soit il satisfait une condition de la sphère homogène extérieure uniforme mais le cône de réflexion oblique doit être transformé en la normale multipliée par une fonction matricielle régulière. Quand la direction de réflexion $\gamma$ n’est pas contrôlée, c’est à dire que $\gamma$ est une fonction régulière de $\mathbb{R}^d$ à $\mathbb{R}^d$ qui ne dépend pas de $\varepsilon$ et satisfait $|\gamma| = 1$, Dupuis et Ishii [21] ont démontré l’existence et l’unicité forte des solutions pour deux cas. Au premier cas, la direction de réflexion $\gamma$ à chaque point de la frontière est une valeur singulière et varie doucement, même si le domaine $O$ peut être non régulier, telle que

$$\bigcup_{0 \leq \lambda \leq r} B(x - \lambda \gamma(x), \lambda r) \subset O^c, \quad \forall x \in \partial O,$$

pour quelque $r \in (0,1)$. Au second cas, on suppose que le domaine $O$ est l’intersection d’un nombre fini de domaines à frontière régulière. Dans Bouchard [8],
l’existence est obtenue lorsque $\mathcal{O}$ est un ensemble ouvert borné, $\gamma$ est une fonction régulière de $\mathbb{R}^d \times E$ à $\mathbb{R}^d$ satisfaisant $|\gamma| = 1$, et il existe un $r \in (0, 1)$ tel que
\[
\bigcup_{0 \leq \lambda \leq r} B(x - \lambda \gamma(x, e), \lambda r) \subset \mathcal{O}^c, \quad \forall (x, e) \in \partial \mathcal{O} \times E, \tag{0.5}
\]
qui impose uniformément la condition (0.10) dans la variable de contrôle. Cela permet de montrer l’existence forte d’une solution au problème de Skorokhod dans la famille de fonctions continues quand $\varepsilon$ est une fonction continue à variation bornée. Nous faisons une extension de ce résultat au problème de Skorokhod pour la famille des fonctions càdlàg ayant un nombre fini de points de discontinuité. La difficulté vient de la manière de définir la solution aux temps de sauts. Dans cette thèse, nous étudions une classe particulière de solutions, qui est paramétrée par le choix d’un opérateur de projection $\pi$. Si la valeur de $\phi(s-) + \Delta \psi(s)$ sors de la clôture du domaine à un temps de saut, nous projetons cette valeur sur la frontière $\partial \mathcal{O}$ du domaine selon la direction $\gamma$. La valeur après le saut de $\phi$ est choisie égale à $\pi(\phi(s-) + \Delta \psi(s), \varepsilon(s))$, où la projection $\pi$ selon la direction oblique $\gamma$ satisfait
\[
y = \pi(y, e) - l(y, e)\gamma(\pi(y, e), e), \quad \text{pour tous } y \notin \bar{\mathcal{O}} \text{ et } e \in E,
\]
pour une telle fonction positive $l$. Quand la réflexion n’est pas oblique et le domaine $\mathcal{O}$ est convexe, la fonction $\pi$ est exactement la projection usuelle et $l(y)$ coïncide avec la distance à la clôture du domaine $\bar{\mathcal{O}}$. Nous savons déjà que l’existence et l’unicité à un tel problème entre les temps de saut sont garanties, voir Dupuis et Ishii [21]. Cela conduit à une existence sur l’intervalle de $[0, T]$, ainsi que les solutions aux temps de sauts suivent la règle ci-dessus, quand $\psi$ et $\varepsilon$ ont seulement un nombre fini de discontinuités.

Par la suite, nous démontrons l’existence d’une paire unique formée par un processus avec réflexion $X^\varepsilon$ et par un processus non décroissant $L^\varepsilon$ qui satisfont
\[
\begin{cases}
X(r) = x + \int_t^r F(X(s-))dZ_s + \int_t^r \gamma(X(s), \varepsilon(s))1_{X(s)\in\partial\mathcal{O}}dL(s), \\
X(r) \in \bar{\mathcal{O}}, \quad \text{for all } r \in [t, T]
\end{cases} \tag{0.6}
\]
ou $Z$ est la somme d’un drift, une intégrale stochastique par rapport au mouvement Brownien et un processus de Poisson composé, et le processus de contrôle $\varepsilon$ appartient à la classe $\mathcal{E}$ de $E$-valeur processus prévisibles càdlàg à variation bornée et activité infinie. Comme le cas déterministe à préciser, nous ne étudions qu’une classe particulière de solutions paramétrée par $\pi$. C’est à dire que $X$ est projeté sur la frontière selon l’opérateur de projection $\pi$ chaque fois qu’il sors du domaine en raison d’un saut. La valeur après le saut de $X$ est choisie égal à $\pi(X(s-) + F_s(X(s-))\Delta Z_s, \varepsilon(s))$. L’objectif principal du Chapitre 6 est d’étendre le résultat de Bouchard [8] à un processus de la diffusion avec sauts. Ce processus contrôlé est défini par la solution de l’équations différentielles stochastiques avec la réflexion (1.2). Nous introduisons
ensuite un problème de contrôle optimal \( v(t, x) := \sup_{\varepsilon \in \mathcal{E}} J(t, x; \varepsilon) \), où la fonction de coût \( J(t, x; \varepsilon) \) est définie par

\[
J(t, x; \varepsilon) := \mathbb{E} \left[ \beta_{t,x}^{\varepsilon} (T) g(X_{t,x}^{\varepsilon}(T)) + \int_t^T \beta_{t,x}^{\varepsilon}(s) f(X_{t,x}^{\varepsilon}(s)) ds \right]
\]

avec \( \beta_{t,x}^{\varepsilon}(s) = e^{-\int_t^s \rho(X_{t,x}^{\varepsilon}(r)) dL_{t,x}^{\varepsilon}(r)} \), l’indice ‘\( t, x \)’ est la condition initiale de (1.2) et \( f, g, \rho \) sont des fonctions données. De façon classique, la technique clé permettant de dériver les équations aux dérivées partielles associées est le principe de la programmation dynamique, voir Bouchard [8], Fleming et Soner [24] et Lions [34]. Bouchard et Touzi [14] ont récemment discuté une version faible de celui, qui est suffisante pour caractériser la fonction valeur associée au sens de viscosité sans avoir besoin de l’argument usuel de sélection mesurable et de la continuité a priori de la fonction valeur. Dans ce travail, nous appliquons leur résultat à donner par

\[
v(t, x) \leq \sup_{\varepsilon \in \mathcal{E}} \mathbb{E} \left[ \beta_{t,x}^{\varepsilon}(\tau) \left[ v^*, g(\tau, X_{t,x}^{\varepsilon}(\tau)) \right] + \int_t^\tau \beta_{t,x}^{\varepsilon}(s) f(X_{t,x}^{\varepsilon}(s)) ds \right],
\]

et, pour toute fonction semi-continue supérieurement \( \varphi \) telle que \( \varphi \leq v^* \),

\[
v(t, x) \geq \sup_{\varepsilon \in \mathcal{E}} \mathbb{E} \left[ \beta_{t,x}^{\varepsilon}(\tau) \left[ \varphi, g(\tau, X_{t,x}^{\varepsilon}(\tau)) \right] + \int_t^\tau \beta_{t,x}^{\varepsilon}(s) f(X_{t,x}^{\varepsilon}(s)) ds \right],
\]

où \( v^* \) (resp. \( v_* \)) sont l’enveloppe semicontinue supérieurement (resp. semi-continue inférieurement) de \( v \), et \([w, g](s, x) := w(s, x)1_{s<T} + g(x)1_{s=T}\) pour toute fonction \( w \) définie sur \([0, T] \times \mathbb{R}^d\). Cela nous permet de caractériser la fonction valeur de \( v \) au sens de viscosité. Finalement, nous étendons le principe de comparaison de Bouchard [8] à notre contexte.
In the mathematical finance literature, the problem of super-hedging financial options has usually been treated via the so-called dual formulation approach, see e.g. Cvitanić and Karatzas [19], Föllmer and Kramkov [22], and, Karatzas and Shreve [30]. This allows us to convert this problem into a standard stochastic control formulation. However, this not only requires to first prove a suitable duality result but also essentially only applies to linear dynamics, and is therefore very restrictive. In this work, we follow the steps of Soner and Touzi [46] and [47], who proposed to appeal to a new dynamic programming principle, called geometric. Opening the doors to a wide range of applications, particularly in risk control in finance and insurance, this provides a direct PDE characterization of super-hedging type problems in very general Markovian setting, without appealing the standard dual formulation from mathematical finance, see e.g. [51] and [44].

The main aim of this thesis is to extend their results to some non-trivial applications in risk control in finance and assurance. In particular, we prove an extension of the geometric dynamic programming principle, which allows us to study the pricing of American option under constraints neither with a barrier or without a barrier. We also study a class of the stochastic target problems with multiple constraints in probability, which allows to provide a PDE characterization of P&L based option price in models with constraints on the hedging strategy. Finally, we consider a weak version of the classical dynamic programming principle introduced in Bouchard and Touzi [14] and apply it to study a new class of optimal control problem for jumps-diffusion processes. In which, the controlled process is defined as the solution of a stochastic differential equation rejected on the boundary of a bounded domain \( \mathcal{O} \), and whose direction of reflection can be controlled. The rest of thesis is organized as follows.

In the first part of this thesis, our goal is to provide a PDE characterization for the super hedging price of an American option of barrier types in a Markovian model of financial market. Recently, this problem has been further studied by Bouchard and Bentahar [9] who considered the super-hedging under constraints for barrier type options in a European context, thus extending previous results of Shreve, Schmock and Wystup [41]. A similar analysis has been carried out by Karatzas and Wang...
[29] for perpetual American barrier options within a Black and Scholes type financial model. The main difficulty comes from the boundary condition on $\partial O$ before maturity. As in the vanilla option case, they have to consider a 'face-lifted' version of the pay-off as a boundary condition. However, it turns out not to be sufficient and this boundary condition has to be considered in a weak sense, see also Shreve, Schmock and Wystup [41]. This gives a rise to a mixed Neumann/Dirichlet type boundary condition. In the American case, a similar phenomenon appears but we also have to take into account the fact that the option may be exercised at any time. This introduces an additional free boundary feature inside the domain.

In [9], the derivation of the associated PDE relies on the dual formulation of Cvitanić and Karatzas [19]. Contrary to this, we do not use the usual dual formulation, which allows to reduce to a standard control problem, but instead appeal to the geometric dynamic programming principle for stochastic targets of Soner and Touzi [47], call 'an obstacle version'.

In the Chapter 1, we start with an extension to the case of American type targets of the result of Soner and Touzi [47], which is of own interest. It can be formalized as follows. Let us consider a family of $d$-dimensional controlled processes $Z_{t,z}$ with initial condition $Z_{t,z} = z$, depending on a control $\nu \in \mathcal{A}$. As an obstacle version, we mean the following problem: find the set $V(0)$ of initial conditions $z$ for $Z_{0,z}$ such that $Z_{t,z}(t) \in G(t) \forall t \in [0,T]$ $\mathbb{P}$-a.s. for some $\nu \in \mathcal{A}$. Here, $t \mapsto G(t)$ is now a set valued map taking values in the collection of Borel subsets of $\mathbb{R}^d$, which corresponds to an American target stochastic problem. The main aim of this chapter is to provide an obstacle version of the geometric dynamic programming principle to this setting:

$$V(t) = \{z \in \mathbb{R}^d : \exists \nu \in \mathcal{A} \text{ s.t } Z_{t,z}^{\nu}(\tau \land \theta) \in G^\tau \bigoplus V \mathbb{P} \text{ - a.s } \forall \tau \in \mathcal{T}_{[t,T]} \}, \quad (0.7)$$

where

$$G^\tau \bigoplus V := G(\tau) 1_{\tau \leq \theta} + V(\theta) 1_{\tau > \theta},$$

where $\mathcal{T}_{[t,T]}$ denotes the set of stopping times taking values in $[t, T]$.

The Chapter 2 of this dissertation deals with the application of the above obstacle version to the super-hedging of an American option of barrier types, in a Markovian model of financial market. We consider a financial market composed of a non-risky asset, price process is normalized to unity, and $d$ risky assets $X = (X^1, \ldots, X^d)$. To an initial capital $y \in \mathbb{R}_+$ and a financial strategy $\pi$, we associate the induced wealth process $Y_{t,x,y}^{\pi}$. The American barrier option is described by a map $g$ defined on $[0, T] \times \mathbb{R}^d_+$ and an open domain $O$ of $\mathbb{R}^d_+$ so that the buyer receives the payoff $g(\theta, X_{t,x}(\theta))$ when he/she exercises the option at time $\theta \leq \tau_{t,x}$, where $\tau_{t,x}$ is the first time when $X_{t,x}$ exits $O$. In this case, the set of initial endowments that allow one to
hedge the claim is given by

\[ V(t, x) := \{ y \geq \mathbb{R}_+ : \exists \pi \text{ s.t. } Z^\pi_{t,x,y}(\tau \wedge \tau_t) \in G(\tau \wedge \tau_t) \forall \tau \in \mathcal{T}_{t,T} \}, \]

with \( Z^\pi_{t,x,y} := (X_{t,x}, Y^\pi_{t,x,y}) \) and \( G(t) := \{ (x, y) \in \mathcal{O} \times \mathbb{R}_+ : y \geq g(t, x) \} \).

Then the associated value function \( v(t, x) := \inf V(t, x) \) completely characterizes the (interior of the) set \( V(t, x) \) and a version of geometric dynamic programming principle can be stated in terms of the value function \( v \):

**(DP1).** If \( y > v(t, x) \), then there exists a financial strategy \( \pi \) such that

\[ Y^\pi_{t,x,y}(\theta) \geq v(\theta, X_{t,x}(\theta)) \text{ for all } \theta \in \mathcal{T}_{t,x}. \]

**(DP2)** If \( y < v(t, x) \), then there exists \( \tau \in \mathcal{T}_{t,x} \) such that

\[ \mathbb{P} \left[ Y^\pi_{t,x,y}(\theta \wedge \tau) > v(\theta, X_{t,x,y}(\theta))1_{\theta < \tau} + g(\tau, X_{t,x,y}(\tau))1_{\theta \geq \tau} \right] < 1 \]

for all financial strategy \( \pi \) and \( \theta \in \mathcal{T}_{t,x} := \{ \tau \in \mathcal{T}_{t,T} : \tau \leq \tau_t \} \).

The above analysis can be compared to Soner and Touzi [49] who considered European type options, and the proofs are actually very close once the dynamic programming (DP2) is replaced by a corresponding one. The main differences come from the fact that we do not impose a non-degeneracy assumption on the volatility coefficients of the underlying financial assets, which is also required in a dual formulation, see [9], [13], [19], [28] and [29]. We therefore take this opportunity to explain how it can be treated, which is of own interest and could be transposed to the American option without barrier, which we discuss later at the end of this chapter.

In order to separate the difficulties, we shall restrict to the case where the strategies \( \pi \) take values in a given convex compact set. The case of unbounded control sets can be handled by using the technology developed in Bouchard, Elie and Touzi [12]. Similarly, jumps could be added to the dynamics without major difficulties, see Bouchard [7].

In the second part, we study a direct PDE characterization of the minimal initial endowment required so that the terminal wealth of a financial agent (possibly diminished by the payoff of a random claim) can match a set of constraints in probability. In practice, this set of constraints has to be viewed as a rough description of a targeted P&L distribution. To be more precise, let us consider the problem of a trader who would like to hedge a European claim of the form \( g(X_{t,x}(T)) \), where \( X_{t,x} \) models the evolution of some risky assets, assuming that their value is \( x \) at time \( t \). Here, the price is chosen so that the net wealth \( Y^\nu_{t,x,y}(T) - g(X_{t,x}(T)) \) satisfies a P&L constraint. Namely, given a collection of thresholds \( \gamma := (\gamma^i)_{i \leq \kappa} \in \mathbb{R}^\kappa \) and of
probabilities \((p^i)_{i \leq \kappa} \in [0,1]^\kappa\), for some \(\kappa \geq 1\), the price of the option is defined as the minimal initial wealth \(y\) such that there exists a strategy \(\nu\) satisfying the net hedging loss should not exceed \(-\gamma^i\) with probability more than \(p^i\). This coincides with a constraint on the distribution of the P&L of the trader, in the sense that it should match the constraints imposed by the discrete histogram associated to \((\gamma,p)\). In order to avoid that the wealth process goes too negative, even with small probability, we further impose that \(Y_{t,x,y}^\nu(T) \geq \ell\) for some \(\ell \in \mathbb{R}_-\). The minimal initial endowment required to achieve the above constraints is given by:

\[
v(t,x,p) := \inf \{ y : \exists \nu \text{ s.t. } Y_{t,x,y}^\nu(T) \geq \ell, \mathbb{P} \left[ Y_{t,x,y}^\nu(T) - g(X_{t,x}(T)) \geq -\gamma^i \right] \geq p^i \quad \forall \ i \leq \kappa \},
\]

where we suppose that \(\gamma^\kappa \geq \cdots \geq \gamma^2 \geq \gamma^1 \geq 0\).

Since \(\gamma^i \geq \gamma^j\) for \(i \leq j\), it would be natural to restrict to the case where \(p^i \leq p^j\) for \(i \leq j\). From the PDE point of view, this would lead to the introduction of boundary conditions on the planes for which \(p^i = p^j\) for some \(i \neq j\). Since this restriction does not appear to be necessary in our approach, we deliberately do not use this formulation and focus on the points of the boundary \((t,x,p)\) satisfying \(p_i = 0\) or \(p_i = 1\) for some \(i \leq k\). From the pure numerical point of view, one could however use the fact that \(v(\cdot,p) = v(\cdot,\hat{p})\) where \(\hat{p}\) is defined by \(\hat{p}^i = \max_{i \leq j} p^j\) for \(i \leq \kappa\).

In the case \(\kappa = 1\), such a problem is referred to as the “quantile hedging problem”. It has been widely studied by Föllmer and Leukert [23] who provided an explicit description of the optimal terminal wealth \(Y_{t,x,y}^\nu(T)\) in the case where the underlying financial market is complete. This result is derived from a clever use of the Neyman-Pearson Lemma in mathematical statistics and applies to non-Markovian frameworks. A direct approach, based on the notion of stochastic target problems, has then been proposed by Bouchard, Elie and Touzi [12]. It allows to provide a PDE characterization of the pricing function \(v\), even in incomplete markets or in cases where the stock price process \(X_{t,x}\) can be influenced by the trading strategy \(\nu\), see e.g. Bouchard and Dang [10]. The problem (0.8) is a generalization of this work to the case of multiple constraints in probability.

As in Bouchard, Elie and Touzi [12], the first step consists in rewriting the stochastic target problem with multiple constraints in probability (0.8) as a stochastic target problem in the \(\mathbb{P}\)-a.s. sense. This is achieved by introducing a suitable family of \(d\)-dimensional bounded martingales \(\{P_{t,p}^\alpha, \alpha\}\) and by re-writing \(v\) as

\[
v(t,x,p) = \inf \{ y : \exists (\nu,\alpha) \text{ s.t. } Y_{t,x,y}^\nu(T) \geq \ell, \min_{i \leq \kappa} \left( 1_{Y_{t,x,y}^\nu(T) - g(X_{t,x}(T)) \geq -\gamma^i} - P_{t,p}^{\alpha,i}(T) \right) \geq 0 \},
\]

where \(P_{t,p}^{\alpha,i}\) denotes the \(i\)-th component of \(P_{t,p}^\alpha\). The above reduction leads to the Geometric dynamic programming principle of Soner and Touzi [47], which allows us to provide the PDE characterization in the sense of viscosity.
In this paper, we restrict to the case where the market is complete but the amount of money that can be invested in the risky assets is bounded. The incomplete market case could be discussed by following the lines of the paper, but will add extra complexity. Since the proofs below are already complex, we decided to restrict to the complete market case. The fact that the amount of money that can be invested in the risky assets is bounded could also be considered. In this case, the associated Hamilton-Jacobi-Bellman operator is stated in the Proposition 4 of Chapter 3 with a suitable terminal condition:

$$v^*(T, x, p) = v_*(T, x, p) = l(1 - \max_{i \leq \kappa} p^i) + \sum_{i=0}^{\kappa-1} (g(x) - \gamma^{i+1})(p^{i+1} - \max_{k \leq i} p^k)_+$$ \hspace{1cm} (0.9)

It does not really simplify the arguments. On the other hand, it is well-known that quantile hedging type strategies can lead to the explosion of the number of risky assets to hold in the portfolio near the maturity. This is due to the fact that it typically leads to hedging discontinuous payoffs, see the example of a call option in the Black-and-Scholes model in Föllmer and Leukert [23]. In our multiple constraint case, we expect to obtain a similar behavior. The constraint on the portfolio is therefore imposed to avoid this explosion, which leads to strategies that can not be implemented in practice. We then could suppose that a financial strategy is described by an element $\nu$ of the set $U$ of progressively measurable processes taking values in a given compact subset $U \subset \mathbb{R}^d$. In this case, the terminal condition for $v$ is not the natural one (0.9) because of the lack of convexity with respect to variable $p$. Moreover, the associated gradient constraint that $\text{diag}[x] D_x v \in U$ leads to the “face-lifted” version of $g$, see e.g. Cvitanić, Pham and Touzi [20]. The terminal boundary condition can be naturally stated in terms of $v_*(T, \cdot) \geq \hat{G}_*$ and $v^*(T, \cdot) \leq \hat{G}^*$, where

$$\hat{G}(x, p) := \max_{i \in K} \left( \ell 1_{p^i=0} + g^i(x) 1_{0 < p^i < 1} + \hat{g}^i(x) 1_{p^i=1} \right).$$

It is clear that $\hat{G}_* < \hat{G}^*$ for some $p \in \partial[0,1]^\kappa$ and the associated Hamilton-Jacobi-Bellman operator is also discontinuous, which leave little hope to be able to establish a comparison result, and therefore build a convergent numerical scheme directly based on this PDE characterization. In order to surround this difficulty, we shall introduce in the Chapter 4 a sequence of convergent approximating problems which are more regular and for which convergent schemes can be constructed. In particular, we will show that it allows to approximate point-wise the relaxed problems:

$$\underline{v}(t, x, p) = \inf \left\{ y : \forall \varepsilon > 0 \exists \nu^\varepsilon \text{ s.t.} \quad Y_{t,x,y}^\nu(T) \geq \ell, \quad \mathbb{P} \left[ Y_{t,x,y}^\nu(T) - g(X_{t,x}(T)) \geq -\gamma^i \right] \geq p^i - \varepsilon \quad \forall i \right\}$$

and

$$\overline{v}(t, x, p) = \inf \left\{ y : \exists \nu \text{ s.t.} \quad Y_{t,x,y}^\nu(T) \geq \ell, \quad \mathbb{P} \left[ Y_{t,x,y}^\nu(T) - g(X_{t,x}(T)) \geq -\gamma^i \right] > p^i \quad \forall i \leq \kappa \right\}.$$
The first value function \( v \) is indeed shown to be the left-limit in \( p \) of \( v \), while \( \tilde{v} \) is the right-limit in \( p \) of \( v \). In cases where \( v \) is continuous, then \( \tilde{v} = v = v \) and our schemes converge to the original value function. However the continuity of \( v \) in its \( p \)-variable seems are a-priori difficult to prove by lack of convexity and strict monotonicity of the indicator function, and may fail in general. Still, one of the two approximations can be chosen to solve practical problems.

In the last part, we consider an optimal control problem for reflected processes within a Brownian diffusion framework with jumps. The originality of this work comes from the fact that the controlled process is defined as the solution of a stochastic differential equation (SDE) reflected at the boundary of a domain \( O \), along oblique directions of reflection \( \gamma \) which are controlled by a predictable process which may have jumps.

When the direction of reflection \( \gamma \) is not controlled, a simple class of reflected SDEs was introduced by Skorokhod [42], in which the reflecting diffusion process defines on \( \mathbb{R}^+ \). Tanaka [50] proved the existence of a solution to such equation in convex domains, see also El Karoui and Marchan [17], and, Ikeda and Watanabe [26]. Later Lions and Sznitman [35] studied the case when the domain \( O \) satisfies a uniform exterior sphere condition, which means that we can choose \( r > 0 \) such that for all \( x \in \partial O \) there exists \( x_1 \in \mathbb{R}^d \) satisfies \( \bar{B}(x_1, r) \cap \bar{O} = \{x\} \). Motivated by applications in financial mathematics, Bouchard [8] then proved the existence of a solution to a class of reflected SDEs, in which the oblique direction of reflection is controlled. This result is restricted to Brownian SDEs and to the case where the control is a deterministic combination of an Itô process and a continuous process with bounded variation.

In the Chapter 5, we extend Bouchard’s result to the case of jump diffusion and allow the control to have discontinuous paths. As a first step, we start with an associated deterministic Skorokhod problem:

\[
\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s), \varepsilon(s))1_{\phi(s) \in \partial O} d\eta(s), \quad \phi(t) \in O,
\]

where \( \eta \) is a non decreasing function and \( \gamma \) is controlled by a control process \( \varepsilon \) taking values in a given compact set \( E \) of \( \mathbb{R}^l \).

In general, the existence and uniqueness of solutions to the Skorokhod problem with smoothly varying oblique reflection is still an open question and has been settled only in some specific cases. In particular, Lions and Sznitman [35] proved the such uniqueness when either the domain \( O \) is smooth, or it may satisfy an uniform exterior sphere condition but the oblique reflection cone has to be transformed into the normal one by multiplication of a smooth matrix function. In the case where the direction of reflection \( \gamma \) is not controlled, i.e. \( \gamma \) is a smooth function from \( \mathbb{R}^d \)
to $\mathbb{R}^d$ satisfying $|\gamma| = 1$ which does not depend on $\varepsilon$, Dupuis and Ishii [21] proved the strong existence and uniqueness of solutions in two cases. In the first case, the direction of reflection $\gamma$ at each point of the boundary is single valued and varies smoothly, even if the domain $O$ may be non smooth, so that

$$\bigcup_{0 \leq \lambda \leq r} B(x - \lambda\gamma(x), \lambda r) \subset O^c, \ \forall x \in \partial O, \quad (0.10)$$

for some $r \in (0,1)$. In the second case, the domain $O$ is the intersection of a finite number of domains with relatively smooth boundaries. In Bouchard [8], the existence holds whenever $O$ is a bounded open set, $\gamma$ is a smooth function from $\mathbb{R}^d \times E$ to $\mathbb{R}^d$ satisfying $|\gamma| = 1$, and there exists some $r \in (0,1)$ such that

$$\bigcup_{0 \leq \lambda \leq r} B(x - \lambda\gamma(x,e), \lambda r) \subset O^c, \ \forall (x,e) \in \partial O \times E, \quad (0.11)$$

which imposes the condition (0.10) uniformly in the control variable. This allows to prove the strong existence of a solution for the Skorokhod problems in the family of continuous functions when $\varepsilon$ is a continuous function with bounded variation.

Extending this result, we consider the Skorokhod problem in the family of càdlàg functions with finite number of points of discontinuity. The difficulty comes from the way the solution map is defined at the jump times. In this thesis, we will investigate on a particular class of solutions, which is parameterized through the choice of a projection operator $\pi$. If the value $\phi(s-) + \Delta \psi(s)$ is out of the closure of the domain at a jump time $s$, we simply project this value on the boundary $\partial O$ of the domain along the direction $\gamma$. The value after the jump of $\phi$ is chosen as $\pi(\phi(s-) + \Delta \psi(s), \varepsilon(s))$, where the projection $\pi$ along the oblique direction $\gamma$ satisfies

$$y = \pi(y,e) - l(y,e)\gamma(\pi(y,e),e), \ \text{for all } y \notin \bar{O} \text{ and } e \in E,$$

for some suitable positive function $l$. When the direction of reflection is not oblique and the domain $O$ is convex, the function $\pi$ is just the usual projection operator and $l(y)$ coincides with the distance to the closure of the domain $\bar{O}$.

We already know that the existence and the uniqueness between the jump times is guaranteed, recall Dupuis and Ishii [21]. By pasting together the solutions at the jumps times according to the above rule, we clearly obtain an existence on the whole time interval $[0, T]$ when $\psi$ and $\varepsilon$ have only a finite number of discontinuous points.

To conclude this chapter, we prove the existence of an unique pair formed by a reflected process $X^\varepsilon$ and a non decreasing process $L^\varepsilon$ satisfying

$$\left\{ \begin{array}{l} X(r) = x + \int_t^r F(X(s-))dZ_s + \int_t^r \gamma(X(s), \varepsilon(s))1_{X(s) \in \partial O}dL(s), \\
X(r) \in \bar{O}, \ \text{for all } r \in [t, T] \end{array} \right. \quad (0.12)$$

where $Z$ is the sum of a drift term, a Brownian stochastic integral and an adapted compound Poisson process, and the control process $\varepsilon$ belongs to the class $\mathcal{E}$ of
E-valued càdlàg predictable processes with bounded variation and finite activity. As in the deterministic case, we only study a particular class of solutions, which is parameterized by $\pi$. Namely, $X$ is projected on the boundary $\partial O$ through the projection operator $\pi$ whenever it is out of the domain because of a jump. The value after the jump of $X$ is chosen as $\pi(X(s-)+F_s(X(s-))(\Delta Z_s,\varepsilon(s)))$.

The main aim of the Chapter 6 is to extend the framework of Bouchard [8] to the jump diffusion case, in which the controlled process is defined as the solution of the reflected SDE (0.12). We then introduce an optimal control problem, $v(t,x) = \sup_{\varepsilon \in \mathcal{E}} J(t,x;\varepsilon)$ where the cost function $J(t,x;\varepsilon)$ is defined as

$$E \left[ \beta^{\varepsilon}_{t,x}(T)g(X^{\varepsilon}_{t,x}(T)) + \int_{t}^{T} \beta^{\varepsilon}_{t,x}(s)f(X^{\varepsilon}_{t,x}(s))ds \right]$$

with $\beta^{\varepsilon}_{t,x}(s) = e^{-\int_{t}^{s} \rho(X^{\varepsilon}_{t,x}(r-))dL^{\varepsilon}_{t,x}(r)}$, $f$, $g$, $\rho$ are some given functions, and the subscript $t,x$ means that the solution of (0.12) is considered from time $t$ with the initial condition $x$.

As usual, the technical key for deriving the associated PDEs is the dynamic programming principle (DPP), see Bouchard [8], Fleming and Soner [24] and Lions [34]. Bouchard and Touzi [14] recently discussed a weaker version of the classical DPP, which is sufficient to provide a viscosity characterization of the associated value function, without requiring the usual heavy measurable selection argument nor the a priori continuity on the associated value function. In this paper, we apply their result to our context:

$$v(t,x) \leq \sup_{\varepsilon \in \mathcal{E}} E \left[ \beta^{\varepsilon}_{t,x}(\tau)[v^{*},g](\tau,X^{\varepsilon}_{t,x}(\tau)) + \int_{t}^{\tau} \beta^{\varepsilon}_{t,x}(s)f(X^{\varepsilon}_{t,x}(s))ds \right]$$

and, for every upper semi-continuous function $\varphi$ such that $\varphi \leq v^{*}$,

$$v(t,x) \geq \sup_{\varepsilon \in \mathcal{E}} E \left[ \beta^{\varepsilon}_{t,x}(\tau)[\varphi,g](\tau,X^{\varepsilon}_{t,x}(\tau)) + \int_{t}^{\tau} \beta^{\varepsilon}_{t,x}(s)f(X^{\varepsilon}_{t,x}(s))ds \right],$$

where $v^{*}$ (resp. $v_{*}$) is the upper (resp. lower) semi-continuous envelope of $v$, and $[w,g](s,x) := w(s,x)1_{s<T} + g(x)1_{s=T}$ for any map $w$ define on $[0,T] \times \bar{O}$. This allows us to provide a PDE characterization of the value function $v$ in the viscosity sense. We finally extend the comparison principle of Bouchard [8] to our context.
Notation

Following are some notations that will be used throughout this thesis:

We consider the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) as being the space of continuous functions \(C([0, T], \mathbb{R}^d)\) equipped with the product measure \(\mathbb{P}\) induced by the Wiener measure and the \(\mathbb{P}\)-complete filtration \(\mathcal{F} := \\{\mathcal{F}_t\}_{0 \leq t \leq T}\) generated by a standard \(d\)-dimensional Brownian motion \(W\). We denote by \(\omega\) or \(\tilde{\omega}\) a generic point.

Given \(0 \leq s \leq t \leq T\), we denote by \(\mathcal{T}_{[s,t]}\) the set of \([s,t]\)-valued stopping times. We simply write \(\mathcal{T}\) for \(\mathcal{T}_{[0,T]}\). For any \(\theta \in \mathcal{T}\), we let \(L^p_\theta\) be the set of all \(p\)-integrable \(\mathbb{R}^d\)-valued random variables which are measurable with respect to \(\mathcal{F}_\theta\).

For ease of notations, we set \(\rho := (\mathcal{T}, \mathcal{F})\) equipped with the product measure \(\mathbb{P} \times \mathbb{P}\)-a.e., where \(\mathbb{P}\) is the Lebesgue measure on \([0, T]\).

For \(T > 0\) and a Borel set \(K \subset \mathbb{R}^d\), \(D^f([0, T], K)\) is the set of càdlàg functions from \([0, T]\) into \(K\) with a finite number of discontinuous points, and \(BV^f([0, T], K)\) is the subset of elements in \(D^f([0, T], K)\) with bounded variation. For \(\varepsilon \in BV^f([0, T], K)\), we set \(|\varepsilon| := \sum_{i \leq n} |e_i|\), where \(|e_i|\) is the total variation of \(e_i\). We denote by \(N^\varepsilon_{[t, T]}\) the number of jump times of \(\varepsilon\) on the interval \([t, T]\).

In the space \(\mathbb{R}^d\), we denote by \(\langle \cdot, \cdot \rangle\) natural scalar product and by \(|\cdot|\) the associated norm. Any element \(x\) of \(\mathbb{R}^d\) is identified to a column vector whose \(i\)-th component is denoted by \(x^i\) and \(x_I := (x_i)_{i \in I}\) for \(I \subset \{1, \ldots, d\}\). We write \(\text{diag}\ [x]\) to denote the diagonal matrix of \(\mathbb{M}^d\) who \(i\)-th diagonal element is \(x^i\). For \(x, y \in \mathbb{R}^d\), we write \(x \geq y\) for \(x^i \geq y^i\) for all \(i \leq d\). Given \(x, \rho \in \mathbb{R}^d\) we set \(x\rho := (x^i\rho^i)_{i \leq d}\), \(e^\rho := (e^i\rho^i)_{i \leq d}\) and \(x^\rho := \prod_{i=1}^d(x^i)^\rho^i\), whenever the operations are well defined. We denote by \(B(x, r)\) the open ball of radius \(r > 0\) and center \(x\).

We denote by \(\mathbb{M}^{n,d}\) the set of \(n \times d\) matrices, \(\text{Trace} [M]\) the trace of \(M \in \mathbb{M}^{d,d}\) =: \(\mathbb{M}^d\) and \(M^\top\) its transposition. The \(i\)-th line of \(M \in \mathbb{M}^{n,d}\) is denoted by \(M^i\). We identify \(\mathbb{R}^d\) to \(\mathbb{M}^{d,1}\).

For a set \(A \subset \mathbb{R}^d\), we note \(\text{int}(A)\) its interior, \(A^c\) its complement, \(\partial A\) its boundary and \(\partial_T A := \{x \in \mathbb{R}^{d-1} : \langle T, x \rangle \in \partial A\}\).

Given a smooth map \(\varphi\) on \([0, T] \times \mathbb{R}^d\), we denote by \(\partial_x \varphi\) its partial derivatives
with respect to its first variable, and by $D\varphi$ and $D^2\varphi$ is partial gradient and Hessian matrix with respect to its second variable. For a locally bounded map $v$ on $[0, T] \times (0, \infty)^d$, we set

$$v_*(t, x) := \liminf_{(s, z) \to (t, x)} v(s, z) \quad \text{and} \quad v^*(t, x) := \limsup_{(s, z) \to (t, x)} v(s, z)$$

for $(t, x) \in [0, T] \times (0, \infty)^d$.

If nothing else is specified, identities involving random variables have to be taken in the a.s. sense.
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Première partie

American geometric dynamic programming principle and applications
Chapitre 1

American geometric dynamic programming principle

1 Introduction

A stochastic target problem can be described as follows. Given a set of controls $\mathcal{A}$, a family of $d$-dimensional controlled processes $Z_{0,z}^\nu$ with initial condition $Z_{0,z}^\nu(0) = z$ and a Borel subset $G$ of $\mathbb{R}^d$, find the set $V(0)$ of initial conditions $z$ for $Z_{0,z}^\nu$ such that $Z_{0,z}^\nu(T) \in G$ $\mathbb{P}$-a.s. for some $\nu \in \mathcal{A}$. Such a problem appears naturally in mathematical finance. In such applications, $Z_{0,x,y}^\nu = (X_{0,x}^\nu, Y_{0,x,y}^\nu)$ typically takes values in $\mathbb{R}^{d-1} \times \mathbb{R}$, $Y_{0,x,y}^\nu$ stands for the wealth process, $\nu$ corresponds to the financial strategy, $X_{0,x}^\nu$ stands for the price process of some underlying financial assets, and $G$ is the epigraph $\mathcal{E}(g)$ of some Borel map $g : \mathbb{R}^{d-1} \to \mathbb{R}$. In this case, $V(0) = \{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \exists \nu \in \mathcal{A} \text{ s.t. } Y_{0,x,y}^\nu(T) \geq g(X_{0,x}^\nu(T)) \mathbb{P}$-a.s.\}$, which, for fixed values of $x$, corresponds to the set of initial endowments from which the European claim of payoff $g(X_{0,x}^\nu(T))$ paid at time $T$ can be super-replicated.

In the mathematical finance literature, this kind of problems has usually been treated via the so-called dual formulation approach, which allows to reduce this non-standard control problem into a stochastic control problem in standard form, see e.g. Karatzas and Shreve [31], and, El Karoui and Quenez [33]. However, such a dual formulation is not always available. This is the case when the dynamics of the underlying financial assets $X^\nu$ depends on the control $\nu$ in a non-trivial way. This is also the case for certain kind of models with portfolio constraints, see e.g. Soner and Touzi [45] for Gamma constraints. Moreover, it applies only in mathematical finance where the duality between super-hedgeable claims and martingale measures can be used, see Karatzas and Shreve [31]. In particular, it does not apply in most problems coming from the insurance literature, see Bouchard [7] for an example.

In [46] and [47], the authors propose to appeal to a new dynamic programming principle, called geometric, which is directly written on the stochastic target pro-
CHAPITRE 1. AMERICAN GEOMETRIC DYNAMIC PROGRAMMING

PRINCIPLE

blem :

\[ V(t) = \{ z \in \mathbb{R}^d : \exists \nu \in \mathcal{A} \text{ s.t. } Z^\nu_{0,z}(\theta) \in V(\theta) \ \mathbb{P}-\text{a.s.} \} \text{ for all } \theta \in \mathcal{T}_{[t,T]}, \]  

where \( V(\theta) \) extends the definition of \( V(0) \) to the non trivial time origin \( \theta \) and \( \mathcal{T}_{[t,T]} \) denotes the set of stopping times taking values in \([t,T]\). Since then, this principle has been widely used in finance and insurance to provide PDE characterizations to super-hedging type problems, see also Soner and Touzi [48] for an application to mean curvature flows. Recent results of Bouchard, Elie and Touzi [12] also allowed to extend this approach to the case where the condition \( \mathbb{P} \left[ Z^\nu_{0,z}(T) \in G \right] = 1 \) is replaced by the weaker one \( \mathbb{P} \left[ Z^\nu_{0,z}(T) \in G \right] \geq p \), for some fixed \( p \in [0,1] \). Similar technologies are used in Bouchard, Elie and Imber [11] to study optimal control problems under stochastic target or moment constraints.

Surprisingly, it seems that no American version of this geometric dynamic programming principle has been studied so far. By American version, we mean the following problem : find the set \( V(0) \) of initial conditions \( z \) for \( Z^\nu_{0,z} \) such that \( Z^\nu_{0,z}(t) \in G(t) \ \forall \ t \in [0,T] \ \mathbb{P}-\text{a.s.} \) for some \( \nu \in \mathcal{A} \). Here, \( t \mapsto G(t) \) is now a set valued map taking values in the collection of Borel subsets of \( \mathbb{R}^d \). The main aim of this chapter is to extend the geometric dynamic programming principle to this setting. We shall show in Section 2 that the counterpart of (1.1) for American targets is given by, for all \( \theta \in \mathcal{T}_{[t,T]} \), :

\[ V(t) = \{ z \in \mathbb{R}^d : \exists \nu \in \mathcal{A} \text{ s.t. } Z^\nu_{0,z}(\tau \wedge \theta) \in G^{\tau,\theta} V \ \mathbb{P}-\text{a.s.} \ \forall \ \tau \in \mathcal{T}_{[t,T]} \}, \]  

where

\[ G^{\tau,\theta} V := G(\tau) \ 1_{\tau \leq \theta} + V(\theta) \ 1_{\tau > \theta}. \]

2 The American geometric dynamic programming principle

Given a Borel subset \( U \) of an Euclidean space, we denote by \( \mathcal{A} \) the set of all progressively measurable processes \( \nu \) in \( L^2([0,T] \times \Omega) \) taking values in \( U \). Easily check that \( \mathcal{A} \) satisfies the following two conditions, which are natural in optimal control (see e.g. [32]), and already appear in Soner and Touzi [47], :

A1. Stability under concatenation at stopping times : For all \( \nu_1, \nu_2 \in \mathcal{A} \) and \( \theta \in \mathcal{T}_{[t,T]} \),

\[ \nu_1 1_{[0,\theta]} + \nu_2 1_{[\theta,T]} \in \mathcal{A}. \]
A2. Stability under measurable selection: For any \( \theta \in T_{[t,T]} \) and any measurable map \( \phi: (\Omega, \mathcal{F}(\theta)) \to (\mathcal{A}, \mathcal{B}_\mathcal{A}) \), there exists \( \nu \in \mathcal{A} \) such that

\[
\nu = \phi \text{ on } [\theta, T] \times \Omega, \text{ Leb } \times \mathbb{P} \text{ - a.e.,}
\]

where \( \mathcal{B}_\mathcal{A} \) is the set of all Borel subsets of \( \mathcal{A} \).

For a technical key appearing in the proof of the geometric dynamic programming principle (see Lemma 2.3), we restrict at the time set \( T \) on the controls following the target principle. To ensure some new assumption which will allow us to extend their result to our framework:

- Stability under measurable selection:
- Right-continuity assumption on the target \( G_1 \)
- Some square integrable assumptions on the controls following the target \( G_2 \).

For any \( \theta \in T \), we let \( L_d^p(\theta) \) be the set of all \( p \)-integrable \( \mathbb{R}^d \)-valued random variables which are measurable with respect to \( \mathcal{F}_\theta \). For ease of notations, we set \( L_d^0 := L_d^0(\theta) \). We denote by \( \mathcal{S} \) the set of \( (\theta, \xi) \in T \times L_d^0 \) such that \( \xi \in L_d^0(\theta) \). Observe that \( \mathcal{S} := [0, T] \times \mathbb{R}^d \subset \mathcal{S} \).

On the state process:

The controlled state process is a map from \( \mathcal{S} \times \mathcal{A} \) into a subset \( \mathcal{Z} \) of \( \mathbb{H}_d^0 \)

\[
(\theta, \xi, \nu) \in \mathcal{S} \times \mathcal{A} \mapsto Z_{\theta, \xi}^\nu \in \mathcal{Z}.
\]

As in [47], except for \( Z_2 \) which is stated in a stronger form, the state process is assumed to satisfy the following conditions, for all \( (\theta, \xi) \in \mathcal{S} \) and \( \nu \in \mathcal{A} \):

- \( Z_{\theta, \xi}^\nu \) is \( \mathbb{H}_d^0 \) valued
- Consistency with deterministic initial data: For all \( (t, z) \in \mathcal{S} \), \( s > t, \nu \in \mathcal{A} \) and bounded Borel function \( f \), there exists \( \nu_\omega \in \mathcal{A}^{\theta(\omega)} \) such that

\[
\mathbb{E} \left[ f(Z_{t,z}^\nu(s \lor \theta)) \mid \mathcal{F}_\theta \right] (\omega) = \int f(Z_{t,z}^\nu(\tilde{\omega}) \mid \mathcal{F}_\theta)(s \lor \theta(\omega))(\tilde{\omega})) d\mathbb{P}(\tilde{\omega}).
\]
Z3. Flow property: For \( \tau \in T \) such that \( \theta \leq \tau \) \( \mathbb{P} \)-a.s.:

\[
Z^\nu_{\theta, \xi} = Z^\nu_{\tau, \mu} \text{ on } [\tau, T], \quad \text{where } \mu := Z^\nu_{\theta, \xi}(\tau).
\]

Z4. Causality: Let \( \tau \) be defined as in Z3 and fix \( \nu_1, \nu_2 \in \mathcal{A} \). If \( \nu_1 = \nu_2 \) on \( [\theta, \tau] \), then

\[
Z^\nu_{\theta, \xi} = Z^\nu_{\theta, \xi} \text{ on } [\theta, \tau].
\]

Z5. Measurability: For any \( u \leq T \), the map

\[
(t, z, \nu) \in S \times \mathcal{A} \mapsto Z^\nu_{t, z}(u)
\]

is Borel measurable.

On the target:

The map \( G \) is a measurable set-valued map from \([0, T]\) to the set \( \mathcal{B}_{\mathbb{R}^d} \) of Borel sets of \( \mathbb{R}^d \). It is assumed to be right-continuous in the following sense:

G1. Right-continuity of the target: For all sequence \((t_n, z_n)_n\) of \([0, T] \times \mathbb{R}^d\) such that \((t_n, z_n) \to (t, z)\), we have

\[
t_n \geq t_{n+1} \text{ and } z_n \in G(t_n) \forall n \geq 1 \implies z \in G(t).
\]

We now provide an additional assumption, which is required in the proof of Lemma 2.2 later.

G2. The square integration on the controls following the target: For all \((t, z) \in [0, T] \times \mathbb{R}^d\), if a \( \mathbb{F} \)-predictable process \( \nu \) satisfies

\[
\nu_s \in U \text{ for } s \in [t, T] \text{ and } Z^\nu_{t, z}(T) \in G(T) \mathbb{P} - \text{a.e}
\]

then

\[
\nu \in L^2([0, T] \times \Omega).
\]

Remark 2.1 Note that the assumption G2 is trivially satisfied when \( U \) is a compact set. A case of unbounded control sets will be discussed later in Example 2.2, see also Section 4 of Chapter 3.

2.1 The American Geometric Dynamic Programming Principle

We can now state the main result of this section which extends Theorem 3.1. in Soner and Touzi [47].
Theorem 2.1 Suppose that the assumptions \(Z1 - Z5\) and \(G1, G2\) are satisfied. Fix \(t \leq T\) and \(\theta \in \mathcal{T}_{[t,T]}^t\). Then

\[
V(t) = \{z \in \mathbb{R}^d : \exists \nu \in \mathcal{A}\ s.t. \ Z_{t,x}^\nu(\tau \wedge \theta) \in G \bigoplus_{\tau} V \ \forall \tau \in \mathcal{T}_{[t,T]}^t\},
\]

where

\[
G \bigoplus_{\tau} V := G(\tau) 1_{\tau \leq \theta} + V(\theta) 1_{\tau > \theta}.
\]

The above result states that not only \(Z_{t,x}^\nu(\tau) \in G(\tau)\) for all \(\tau \in \mathcal{T}_{[t,T]}^t\) but also that \(Z_{t,x}^\nu(\theta)\) should belong to \(V(\theta)\), the set of initial data \(\xi\) such that \(G(\theta, \xi) \neq \emptyset\), where

\[
G(t, z) := \{\nu \in \mathcal{A} : Z_{t,x}^\nu(\tau) \in G(\tau) \text{ for all } \tau \in \mathcal{T}_{[t,T]}^t\}, \ (t, z) \in S.
\]

To conclude this subsection, let us discuss some particular cases. In the discussions below, we assume that the conditions \(Z1 - Z5\) and \(G1\) of Theorem 2.1 hold and only focus on the condition \(G2\). The first example corresponds to the case of bounded control set. In the second one, the square integration of control \(\nu\) comes from the almost everywhere boundedness of \(Y_{t,x,y}^\nu\).

Example 2.1 (One sided constraint) Let \(Z_{t,x,y}^\nu\) be of the form \((X_{t,x,y}^\nu, Y_{t,x,y}^\nu)\) where \(X_{t,x,y}^\nu\) takes values in \(\mathbb{R}^{d-1}\) and \(Y_{t,x,y}^\nu\) takes values in \(\mathbb{R}\). Assume further that:

- \(U\) is a compact set.
- \(G(t) := \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : y \geq g(t, x)\}\), for some Borel measurable map \(g : [0, T] \times \mathbb{R}^{d-1} \to \mathbb{R}\).
- The sets \(\Gamma(t, x) := \{y \in \mathbb{R} : (x, y) \in V(t)\}\) are half spaces, i.e.: \(y_1 \geq y_2\) and \(y_2 \in \Gamma(t, x) \Rightarrow y_1 \in \Gamma(t, x)\), for any \((t, x) \in [0, T] \times \mathbb{R}^{d-1}\).

Note that the last condition is satisfied if \(X_{t,x,y}^\nu\) does not depend on the initial condition \(y\) and when \(y \mapsto Y_{t,x,y}^\nu\) is non-decreasing. In this case, the associated value function

\[
v(t, x) := \inf \Gamma(t, x) \text{, \ } (t, x) \in [0, T] \times \mathbb{R}^{d-1}
\]

completely characterizes the (interior of the) set \(V(t)\) and a version of Theorem 2.1 can be stated in terms of the value function \(v:\)

\((DP1)\) If \(y > v(t, x)\), then there exists \(\nu \in \mathcal{A}\) such that

\[
Y_{t,x,y}^\nu(\theta) \geq v(\theta, X_{t,x,y}^\nu(\theta)) \text{ for all } \theta \in \mathcal{T}_{[t,T]}^t.
\]

\((DP2)\) If \(y < v(t, x)\), then \(\exists \tau \in \mathcal{T}_{[t,T]}^t\) such that

\[
P\left(Y_{t,x,y}^\nu(\theta \wedge \tau) > v(\theta, X_{t,x,y}^\nu(\theta))1_{\theta < \tau} + g(\tau, X_{t,x,y}^\nu(\tau))1_{\theta \geq \tau}\right) < 1
\]

for all \(\theta \in \mathcal{T}_{[t,T]}^t\) and \(\nu \in \mathcal{A}\).

Note that, by definition of \(v\) we have \(v \geq g\), which explains why the constraints \(Y_{t,x,y}^\nu(\tau) \geq g(\tau, X_{t,x,y}^\nu(\tau))\) needs not to appear in \((DP1)\).
Example 2.2 (Two sided constraint) Let \( Z_{i,x,y}^\nu \) be as in Example 2.1 such that
\[
dY_{i,y}^\nu (s) = y + \int_t^s \nu_r dW_r.
\]
Assume that \( G \) satisfies :
- \( G(t) := \{ (x,y) \in \mathbb{R}^{d-1} \times \mathbb{R} : g(t,x) \leq y \leq \bar{g}(t,x) \} \), for some Borel measurable maps \( g, \bar{g} : [0,T] \times \mathbb{R}^{d-1} \mapsto \mathbb{R} \) satisfying
  \[
  \ell < g \leq \bar{g} < L \text{ on } [0,T] \times \mathbb{R}^{d-1},
  \]
for some constants \( \ell \) and \( L \).
- The sets \( \Gamma(t,x) := \{ y \in \mathbb{R} : (x,y) \in V(t) \} \) are convex, i.e. : \( \lambda \in [0,1] \) and \( y_1, y_2 \in \Gamma(t,x) \Rightarrow \lambda y_1 + (1-\lambda) y_2 \in \Gamma(t,x) \), for any \( (t,x) \in [0,T] \times \mathbb{R}^{d-1} \).
In this case, the associated value functions
\[
\underline{v}(t,x) := \inf \Gamma(t,x) \quad \text{and} \quad \bar{v}(t,x) := \sup \Gamma(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^{d-1}
\]
completely characterize the (interior of the) set \( V(t) \).

Note that if \( Z_{i,x,y}^\nu (T) \in G(T) \) then \( Y_{i,y}^\nu (T) \in [\ell, L] \) \( \mathbb{P} \)-a.e, which together with the dynamic form of \( Y_{i,y}^\nu \) implies that \( \nu \in L^2([0,T] \times \Omega) \). Therefore, a version of Theorem 2.1 can be stated in terms of these value functions :

\begin{align*}
(DP1) \quad & \text{If } y \in (\underline{v}(t,x), \bar{v}(t,x)), \text{ then there exists } \nu \in \mathcal{A}^t \text{ such that } \\
& \underline{v}(\theta, X_{t,x,y}^\nu (\theta)) \leq Y_{i,x,y}^\nu (\theta) \leq \bar{v}(\theta, X_{t,x,y}^\nu (\theta)) \text{ for all } \theta \in \mathcal{T}_{[t,T]}^t. \\

(DP2) \quad & \text{If } y \notin (\underline{v}(t,x), \bar{v}(t,x)), \text{ then } \exists \tau \in \mathcal{T}_{[t,T]}^t \text{ such that } \\
& \mathbb{P} \left[ Y_{i,x,y}^\nu (\theta \land \tau) \in (\underline{v}, \bar{v}) (\theta, X_{t,x,y}^\nu (\theta))1_{\theta < \tau} + (g, \bar{g}) (\tau, X_{t,x,y}^\nu (\tau))1_{\theta \geq \tau} \right] < 1,
\end{align*}
for all \( \theta \in \mathcal{T}_{[t,T]}^t \) and \( \nu \in \mathcal{A}^t \).

2.2 Proof of Theorem 2.1

The proof follows from similar argument as the one used in Soner and Touzi [47], which we adapt to our context.

Given \( t \in [0,T] \), we have to show that \( V(t) = \bar{V}(t) \) where
\[
\bar{V}(t) := \{ z \in \mathbb{R}^d : \exists \nu \in \mathcal{A}^t \text{ s.t } Z_{i,x}^\nu (\tau \land \theta) \in G \bigoplus_{\tau,\theta}^\tau V \text{ for all } \tau \in \mathcal{T}_{[t,T]}^t \},
\]
for some \( \theta \in \mathcal{T}_{[t,T]}^t \).

We split the proof in several Lemmas. From now on, we assume that the conditions of Theorem 2.1 hold.
Lemma 2.1 \( V(t) \subset \hat{V}(t) \).

**Proof.** Fix \( z \in V(t) \) and \( \nu \in \mathcal{G}(t, z) \), i.e. such that \( Z^\nu_{t,z}(\tau) \in G(\tau) \) for all \( \tau \in \mathcal{T}^t_{[t,T]} \). It follows from Z3 that \( Z^\nu_{\theta,\xi}(\vartheta \lor \theta) \in G(\vartheta \lor \theta) \) for all \( \vartheta \in \mathcal{T}^t_{[t,T]} \), where \( \xi := Z^\nu_{t,z}(\theta) \). Hence,

\[
1 = \mathbb{P} \left[ Z^\nu_{\theta,\xi}(s \lor \theta) \in G(s \lor \theta) \big| \mathcal{F}_\theta \right] (\omega) \quad \text{for} \quad s \geq t.
\]

It follows from Z2 that there exists \( \nu_\omega \in \mathcal{A}^{\theta(\omega)} \) satisfying

\[
1 = \mathbb{E} \left[ 1\{Z^\nu_{t,z}(s \lor \theta) \in G(s \lor \theta)\} \big| \mathcal{F}_\theta \right] (\omega)
\]

\[
= \int 1\{Z^\nu_{\theta(\omega),\xi}(s \lor \theta(\omega)) \in G(s \lor \theta(\omega))\} d\mathbb{P}(\omega) \quad \text{for} \quad s \geq t,
\]

which implies that

\[
Z^\nu_{\theta(\omega),\xi}(s \lor \theta(\omega)) \in G(s \lor \theta(\omega)) \quad \text{for} \quad s \geq t.
\]

Therefore, \( Z^\nu_{t,z}(\theta) \in V(\theta) \, \mathbb{P} - \text{a.s.} \) Since we already know that \( Z^\nu_{t,z}(\tau) \in G(\tau) \) for all \( \tau \in \mathcal{T}^t_{[t,T]} \), this shows that \( z \in \hat{V}(t) \).

It remains to prove the opposite inclusion.

**Lemma 2.2** \( \hat{V}(t) \subset V(t) \).

**Proof.** We now fix \( z \in \hat{V}(t) \) and \( \nu \in \mathcal{A}^t \) such that

\[
Z^\nu_{t,z}(\theta \land \tau) \in \bigoplus_{\tau,\theta} V \, \mathbb{P} - \text{a.s.} \quad \text{for all} \quad \tau \in \mathcal{T}^t_{[t,T]}.
\]

**1.** We first work on the event set \( \{\theta < \tau\} \). On this set, we have \( Z^\nu_{t,z}(\theta) \in V(\theta) \) and therefore

\[
(\theta, Z^\nu_{t,z}(\theta)) \in D := \{(t, z) \in S : z \in V(t)\}.
\]

Let \( \mathcal{B}_D \) denote the collection of Borel subsets of \( D \). Applying Lemma 2.3 below to the measure induced by \( (\theta, Z^\nu_{t,z}(\theta)) \) on \( S \), we can construct a measurable map \( \phi : (D, \mathcal{B}_D) \to (\mathcal{A}, \mathcal{B}_A) \) such that

\[
Z^{\phi(\theta, Z^\nu_{t,z}(\theta))}_{\theta,\xi}(\vartheta) \in G(\vartheta) \quad \text{for all} \quad \vartheta \in \mathcal{T}^\theta_{[\theta,T]}.
\]

We can then find a \( \mathbb{F}^t \)-predictable process \( \nu_1 \) such that \( \nu_1 = \phi(\theta, Z^\nu_{t,z}(\theta)) \) on \( [\theta, T] \, \text{Leb} \times \mathbb{P} \)-a.e and is independent on \( \mathcal{F}_\theta \). Hence, \( \hat{\nu} := \nu_1|_{[0,\theta]} + \nu_1|_{[\theta,T]} \) is \( \mathbb{F}^t \)-predictable and satisfies

\[
Z^\nu_{\theta,\xi}(T) \in G(T).
\]

It follows from G2 that \( \hat{\nu} \) is a square integrable process and then belongs to \( \mathcal{A}^t \). Moreover, according to Z3 and Z4, we have

\[
Z^\hat{\nu}_{t,z}(\tau) = Z^{\phi(\theta, Z^\nu_{t,z}(\theta))}_{\theta,\xi}(\tau) \in G(\tau) \quad \text{on} \quad \{\theta < \tau\}.
\]
2. Let \( \hat{\nu} \) be defined as above and note that, by (2.1), we also have

\[
Z_{t,z}^\hat{\nu}(\tau) = Z_{t,z}^\nu(\tau) \in G \bigoplus V = G(\tau) \text{ on } \{ \tau \leq \theta \}.
\]

3. Combining the two above steps shows that \( \hat{\nu} \in G(t, z) \) and therefore \( z \in V(t) \).

It remains to prove the following result which was used in the previous proof.

**Lemma 2.3** For any probability measure \( \mu \) on \( S \), there exists a Borel measurable function \( \phi : (D, \mathcal{B}_D) \to (A, \mathcal{B}_A) \) such that

\[
\phi(t, z) \in G(t, z) \text{ for } \mu - \text{a.e. } (t, z) \in D.
\]

**Proof.** We need prove that \( B := \{(t, z, \nu) \in S \times A : \nu \in G(t, z)\} \) is a Borel set and therefore an analytic subset in \( L^2([0, T] \times \mathbb{R}^d) \). This allows to apply the Jankov-von Neumann Theorem (see [5] Proposition 7.49), and then deduce that there exists an analytically measurable function \( \phi : D \to A \) such that

\[
\phi(t, z) \in G(t, z) \text{ for all } (t, z) \in D.
\]

Since an analytically measurable map is also universally measurable, the required result follows from Lemma 7.27 in [5].

1. We first show that

\[
B_1 := \{(t, z, \nu) \in S \times A : \nu \in \mathcal{A}^t\}
\]

is closed in \( L^2([0, T] \times \mathbb{R}^d) \). Let \( (t_n, \nu_n) \in [0, T] \times \mathcal{A} \) such that

\[
\nu_n \in \mathcal{A}^{t_n} \text{ for } n \geq 1
\]

and

\[
(t_n, \nu_n) \to (t, \nu) \text{ in } L^2([0, T] \times \mathbb{R}^d).
\]

Fixing \( s \geq t \) and a closed subset \( U \) in \( \mathbb{R}^d \), we have to prove that

\[
\{\omega \in \Omega : \nu(s) \in U\} =: U_\infty \in \mathcal{F}^t.
\]

Note that \( U_\infty = \bigcup_{n \geq 1} U_n \), where \( U_n := \{\omega \in \Omega : \nu_k(s) \in U \text{ for } k \geq n\} \). This, together with the fact that \( \mathbb{F}^r \) is decreasing in \( r \in [0, s] \) and the fact that \( \nu_n(s) \) is \( \mathcal{F}^{t_n}_s \)-measurable, implies that \( U_n \in \mathcal{F}^{\min \{t, n\}}_s \subset \mathcal{F}^t_s \) for all \( n \geq 1 \), therefore leads to the required result in the case where \( t_n \geq t \), after possibly passing to a subsequence.

In the case that \( \{t_n\}_{n \geq 1} \) is an increasing sequence satisfying \( t_n < t \) for \( n \geq 1 \), the required result follows the left-continuous property of \( \mathcal{F}^t_s \) with respect to \( t \) in \([0, s] \), i.e.,

\[
\bigcap_{n \geq 1} \mathcal{F}^{t_n}_s =: \mathcal{F}^t_{s-} = \mathcal{F}^t_s.
\]
3. CONCLUSION AND DISCUSSION

which is obtained by a similar argument as in the proof of the right-continuity of
the complete filtration generated by a standard Brownian motion, see Theorem 31
Protter [38].

2. It remains to show that

\[ B_2 := \{(t, z, \nu) \in S \times A : Z_{t, z}^\nu(\tau) \in G(\tau) \text{ for all } \tau \in T_{[t, T]}\} \]

is a Borel set in \( L^2([0, T] \times \mathbb{R}^d) \), and therefore conclude that \( B \) is the intersection of
two Borel sets \( B_1 \) and \( B_2 \).

It follows from Z5 that the map \((t, s, \nu) \in S \times A \rightarrow Z_{t, z}^\nu(r) \) is Borel measurable,
for any \( r \leq T \). Then, for any bounded continuous function \( f \), the map \( \psi_f^r : (t, s, \nu) \in S \times A \rightarrow \mathbb{E}[f(Z_{t, z}^\nu(r))] \) is Borel measurable. Since \( G(r) \) is a Borel set, the map
\( 1_{G(r)} \) is the limit of a sequence of bounded continuous functions \((f^n)_{n \geq 1} \).
Therefore, \( \psi_{1_{G(r)}} = \lim_{n \rightarrow \infty} \psi_{f^n}^r \) is a Borel function. This implies that \( B^r \) is a Borel set, where,
for \( \theta \in T_{[0, T]} \),

\[ B^\theta := \{(t, s, \nu) \in S \times A : \psi_{1_{G(t)}}^\theta(t, z, \nu) \geq 1\} = \{(t, s, \nu) \in S \times A : Z_{t, z}^\nu(\theta \lor t) \in G(\theta \lor t)\} . \]

Since \( B_2 = \bigcap_{\theta \in T_{[0, T]}} B^\theta \), appealing to the right-continuous assumption \( G \) and the
right-continuity of \( Z_{t, z}^\nu \), we deduce that \( B_2 = \bigcap_{r \leq T, \; r \in \mathbb{Q}} B^r \). This shows that \( B_2 \) is a
Borel set.

3 Conclusion and discussion

In the next chapter, we explain how the American geometric dynamic program-
ing principle of Theorem 2.1 can be used to relate the super-hedging price of an
American option under constraints. The problem of super-hedging financial options
under constraints has been widely studied since the seminal works of Cvitanić and
Karatzas [19] and Broadie et al. [13]. From a practical point of view, it is motivated
by two important considerations:

1. constraints may be imposed by a regulator,

2. imposing constraints avoids the well-know phenomenon of explosion of the
hedging strategy near the maturity when the payoff is discontinuous, see e.g.
Shreve, Schmock and Wystup [41].
Recently, this problem has been further studied by Bouchard and Bentahar [9] who
considered the problem of super-hedging under constraints for barrier type options
in a general Markovian setting, thus extending previous results of [41]. Moreover,
the results of [9] are proved under a non-degeneracy assumption on the volatility
coefficients of the underlying financial assets, which was also crucial in [9], [13], [19],
[28], [29] and [49]. A similar analysis has been carried out by [28] for American options and by [29] for perpetual American barrier options both within a Black and Scholes type financial model. In the next chapter, we show that this assumption can be relaxed. To achieve this we use a different approach. Instead of appealing to the dual formulation as in [9], we make use of an American version of the geometric dynamic programming principle. Our analysis can be compared to [49] who considered European type options, and the proofs are actually very close once the dynamic programming (1.1) is replaced by (1.2). This introduces new technical difficulties which we tackle in this particular context but could be transported without major difficulties to the cases discussed in the above quoted papers.

An application of European version will be discussed later in Chapter 3, where we consider a stochastic target approach for P&L matching problems under two cases:

a. either the amount of money that can be invested in the risky assets is bounded,

b. or the terminal value of the hedging portfolio is bounded.

Those cases allows us to assure the satisfaction of $G_2$, recall Remark 2.1.
Chapitre 2

Application to American option pricing under constraints

1 Introduction

The aim of this chapter is to provide a PDE characterization of the super hedging price of an American option of barrier types in a Markovian model of financial market. This extends to the American case a recent works of [9], who considered European barrier options, and [29], who discussed the case of perpetual American barrier options in a Black and Scholes type model. Namely, the American payoff is assumed to be of the form $(g(t \land \tau,X_{t\land\tau}))_{t\leq T}$ where $\tau$ is the first exit time of a $d$-dimensional price process $X$ from a given domain $O$. It is payed to the buyer whenever he/she decides to exercise the option at any time before the final maturity $T$. In [9], the derivation of the associated PDE relies on the dual formulation of Cvitanić and Karatzas [19]. The main difficulty comes from the boundary condition on $\partial O$ before maturity. As in the vanilla option case, they have to consider a 'face-lifted' version of the pay-off as a boundary condition. However, it turns out not to be sufficient and this boundary condition has to be considered in a weak sense, see also [41]. This gives rise to a mixed Neumann/Dirichlet type boundary condition. In the American case, a similar phenomenon appears but we also have to take into account the fact that the option may be exercised at any time. This introduces an additional free boundary feature inside the domain.

Contrary to [9], we do not use the usual dual formulation, which allows to reduce to a standard control problem, but instead appeal to the American version of the geometric dynamic programming principle proved in the previous chapter. This allows us to avoid the non-degeneracy assumption on the volatility coefficients of [9], and therefore extends their results to possibly degenerate cases which typically appear when the market is not complete.

In order to separate the difficulties, we shall restrict to the case where the controls
take values in a given convex compact set. The case of unbounded control sets can be handled by using the technology developed in [12].

2 The barrier option hedging

2.1 The financial model

We consider a Markovian model of financial market composed of a non-risky asset, which price process is normalized to unity, and \( d \) risky assets \( X = (X^1, ..., X^d) \) whose dynamics are given by the stochastic differential equation

\[
dX(s) = \text{diag}[X(s)] \mu(X(s)) ds + \text{diag}[X(s)] \sigma(X(s)) dW_s. \tag{2.1}
\]

Here \( \mu : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{M}_d \), the set of \( d \times d \)-matrices. Time could be introduced in the coefficients without difficulties, under some additional regularity assumptions. We deliberately choose a time homogeneous dynamics to alleviate the notations.

Given \( x \in (0, \infty)^d \), we denote by \( X_{t,x} \) the solution of the above equation on \([t, T]\) satisfying \( X_{t,x}(t) = x \). In order to guarantee the existence and uniqueness of a strong solution to (2.1), we assume that

\[
(\mu, \sigma) \text{ is bounded}
\]

and

\[
x \in (0, \infty)^d_+ \implies (\text{diag}[x] \sigma(x), \text{diag}[x] \mu(x)) \text{ is Lipschitz continuous}. \tag{2.2}
\]

Importantly, we do not assume that \( \sigma \) is uniformly elliptic nor invertible, as in e.g. [4], [9] or [49].

A financial strategy is described by a \( d \)-dimensional predictable process \( \pi = (\pi^1, ..., \pi^d) \). Each component stands for the proportion of the wealth invested in the corresponding risky asset. In this paper, we restrict to the case where the portfolio strategies are constrained to take values in a given compact convex set \( K : \pi \in K \text{ Leb} \times d\mathbb{P} \text{ a.e }, \text{ with } 0 \in K \subset \mathbb{R}^d \text{ convex and compact}. \)

We denote by \( \mathcal{A} \) the set of such strategies and \( \mathcal{A}^t \) the set of strategies which are independent on \( \mathcal{F}^t \).

To an initial capital \( y \in \mathbb{R}_+ \) and a financial strategy \( \pi \), we associate the induced wealth process \( Y_{t,x,y}^{\pi} \) defined as the solution on \([t, T] \) of

\[
dY(s) = Y(s)\pi_s^\top \text{diag}[X_{t,x}(s)]^{-1} dX_{t,x}(s) \tag{2.3}
\]

\[
= Y(s)\pi_s^\top \mu(X_{t,x}(s)) ds + Y(s)\pi_s^\top \sigma(X_{t,x}(s)) dW_s \tag{2.4}
\]

with \( Y(t) = y \).
2.2 The barrier option hedging and the dynamic programming principle

The option is described by a locally bounded map \( g \) defined on \([0, T] \times (0, \infty)^d\) and an open domain \( \mathcal{O} \subset (0, \infty)^d\) satisfying

\[
g \geq 0 \text{ on } [0, T] \times \mathcal{O} \quad \text{and} \quad g = 0 \text{ on } [0, T] \times \mathcal{O}^c. \quad (2.5)
\]

We assume that the function \( g \) is continuous on \([0, T] \times \mathcal{O}\). The buyer receives the payoff \( g(\tau, X_{t,x}(\tau)) \) when he/she exercises the option at time \( \tau \leq \tau_{t,x} \), where \( \tau_{t,x} \) is the first time when \( X_{t,x} \) exists \( \mathcal{O} \):

\[
\tau_{t,x} = \inf \{ s \geq t : X_{t,x}(s) \notin \mathcal{O} \} \wedge T.
\]

The super-hedging price is thus defined as

\[
v(t, x) := \inf \{ y \in \mathbb{R}_+ : \exists \pi \in \mathcal{A}^t \text{ s.t. } Y_{t,x,y}^\pi(\tau \wedge \tau_{t,x}) \geq g(\cdot, X_{t,x}(\cdot))(\tau \wedge \tau_{t,x}) \forall \tau \in \mathcal{T}_{[t,T]} \}. \quad (2.6)
\]

**Remark 2.1** Note that \( v(t, x) \) coincides with the lower bound of the set \( \{ y \in \mathbb{R}_+ : (x, y) \in V(t) \} \) where

\[
V(t) := \{(x, y) \in \mathcal{O} \times \mathbb{R}_+ : \exists \pi \in \mathcal{A}^t \text{ s.t. } Z_{t,x,y}^\pi(\tau \wedge \tau_{t,x}) \in G(\tau \wedge \tau_{t,x}) \forall \tau \in \mathcal{T}_{[t,T]} \},
\]

with \( Z_{t,x,y}^\pi \) and \( G \) defined as

\[
Z_{t,x,y}^\pi := (X_{t,x}, Y_{t,x,y}^\pi) \quad \text{and} \quad G(t) := \{(x, y) \in \mathcal{O} \times \mathbb{R}_+ : y \geq g(t, x) \}.
\]

**Remark 2.2** Also notice that \( Z1 \) and \( Z3 - Z5 \) in the section 2 of the previous chapter hold for \( Z_{t,x,y}^\pi \) and the continuity assumption (2.5) implies that \( G(t) \) satisfies the right-continuity condition \( G1 \). To see that \( Z2 \) holds, we now view \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})\) as the \(d\)-dimensional canonical filtered space equipped with the Weiner measure.

We denote by \( \omega \) or \( \tilde{\omega} \) a generic point. The Brownian motion is thus defined as \( W(\omega) = (\omega_t)_{t \geq 0} \). For \( \omega \in \Omega \) and \( r \geq 0 \), we denote \( \omega^r := \omega_{\cdot r} \) and \( T_r(\omega) := \omega_{r r} - \omega_r \). For \((\theta, \xi) \in \mathcal{S}, \pi \in \mathcal{A}, s \in [0, T]\) and a bounded Borel function, we have

\[
\mathbb{E} \left[ f(s \vee \theta, Z_{t,x}^\pi(s \vee \theta)) | \mathcal{F}_t \right](\omega) = \mathbb{E} \left[ f(s \vee \theta(\omega), Z_{t,x}^{\pi(\theta(\omega))}(s \vee \theta(\omega))(\tilde{\omega})) d\mathbb{P}(\tilde{\omega}) \right] = \mathbb{E} \left[ f(s \vee \theta(\omega), Z_{t,x}^{\pi(\theta(\omega))}(s \vee \theta(\omega))(\tilde{\omega})) \right],
\]

where, for fixed \( \omega \in \Omega \), \( \tilde{\pi}_\omega : \tilde{\omega} \mapsto \pi(\theta(\omega) + T_{\theta(\omega)}(\omega)) \) can be identified to an element of \( \mathcal{A}^{\theta(\omega)} \).
From now on, we denote \( \mathcal{T}_{t,x} := \{ \tau \in \mathcal{T}_{t,T} : \tau \leq \tau_{t,x} \} \). It follows from the above Remark that the American geometric dynamic programming principle of Theorem 2.1 of Chapter 1 applies to \( v \), compare with Example 2.1 of that chapter.

**Corollary 2.1** Fix \((t, x, y) \in [0, T] \times \bar{O} \times \mathbb{R}_+ \).

**\( \text{(DP1)} \)** If \( y > v(t, x) \), then there exists \( \pi \in \mathcal{A}^t \) such that
\[
Y_{t,x,y}^\pi(\theta) \geq v(\theta, X_{t,x}(\theta)) \text{ for all } \theta \in \mathcal{T}_{t,x}^t.
\]

**\( \text{(DP2)} \)** If \( y < v(t, x) \), then there exists \( \tau \in \mathcal{T}_{t,x}^t \) such that
\[
\mathbb{P} \left[ Y_{t,x,y}^\pi(\theta \land \tau) > v(\theta, X_{t-x,y}(\theta))1_{\theta < \tau} + g(\tau, X_{t-x,y}(\tau))1_{\theta \geq \tau} \right] < 1
\]
for all \( \theta \in \mathcal{T}_{t,x}^t \) and \( \pi \in \mathcal{A}^t \).

### 2.3 PDE characterization

In this section, we show how the dynamic programming principle of Corollary 2.1 allows to provide a PDE characterization of \( v \). We start with a formal argument. The first claim **(DP1)** of the geometric dynamic programming principle can be formally interpreted as follow. Set \( y := v(t, x) \) and assume that \( v \) is smooth. Assuming that **(DP1)** of Corollary 2.1 holds, we must then have, at least at a formal level, \( dY_{t,x,y}^\pi(t) \geq dv(t, X_{t,x}(t)) \), which can be achieved only if \( \pi^\top \mu(x)v(t, x) - \mathcal{L}v(t, x) \geq 0 \) and \( v(t, x)\pi^\top \sigma(x) = (Dv)^\top(t, x)\text{diag}[x]\sigma(x) \), where
\[
\mathcal{L}v(t, x) := \partial_t v(t, x) + (Dv)^\top(t, x)\text{diag}[x]\mu(x) + \frac{1}{2}\text{Trace} \left[ \text{diag}[x]\sigma(x)\sigma^\top(x)\text{diag}[x]D^2v(t, x) \right].
\]
Moreover, we have by definition \( v \geq g \) on \([0, T] \times \bar{O}\). Thus, \( v \) should be a supersolution, on \( D := [0, T] \times \bar{O} \), of

\[
\mathcal{H}\varphi(t, x) := \min\{ \sup_{\pi \in \mathcal{N}\varphi(t, x)} (\pi^\top \mu(x)\varphi(t, x) - \mathcal{L}\varphi(t, x)) , \varphi - g \} = 0 , \tag{2.7}
\]
where, for a smooth function \( \varphi \), we set \( \mathcal{N}\varphi(t, x) := N(x, \varphi(t, x), D\varphi(t, x)) \) with
\[
N(x, y, p) := \{ \pi \in K : y\pi^\top \sigma(x) = p^\top \text{diag}[x]\sigma(x) \} , \text{ for } (x, y, p) \in (0, \infty)^d \times \mathbb{R}_+ \times \mathbb{R}^d ,
\]
and we use the usual convention \( \sup \emptyset = -\infty \).

Note that the supersolution property implies that \( \mathcal{N}v \neq \emptyset \), in the viscosity sense. We shall show in Lemma 3.2 below that, for \((x, y, p) \in (0, \infty)^d \times \mathbb{R}_+ \times \mathbb{R}^d \),
\[
N(x, y, p) \neq \emptyset \iff M(x, y, p) \geq 0 , \tag{2.8}
\]
where
\[
M(x, y, p) := \inf_{\rho_0 \in K_x} \{ \delta_x(\rho_0)y - \rho_0^\top \text{diag}[x]p \}.
\]
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with

\[ \delta_x(\rho_0) := \sup \{ \tilde{\pi}^\top \rho_0, \tilde{\pi} \in K_x \}, \]

\[ K_x := \{ \tilde{\pi} \in \mathbb{R}^d : \tilde{\pi}^\top \sigma(x) = \pi^\top \sigma(x) \text{ for some } \pi \in K \}, \]

and

\[ \tilde{K}_x := \{ \rho_0 \in \mathbb{R}^d : |\rho_0| = 1 \text{ and } \delta_x(\rho_0) < \infty \}. \]

Hence, \( v \) should be a supersolution of

\[ \min \{ \mathcal{H} \varphi, \mathcal{M} \varphi \} = 0 \text{ on } D, \quad (2.9) \]

where \( \mathcal{M} \varphi(t, x) := M(x, \varphi(t, x), D\varphi(t, x)) \), the possible identity

\[ M(x, v(t, x), Dv(t, x)) = 0 \]

which is equivalent to \( v(t, x)^{-1} \text{diag } [x] Dv(t, x) \in K_x \), see [39], reflecting the fact that the constraint is binding.

**Remark 2.3** Note that when \( \sigma(x) \) is invertible

\[ K = K_x \quad \text{and} \quad \delta_x(\rho_0) = \delta(\rho_0) \quad \text{where} \quad \delta(\rho_0) := \sup \{ \pi^\top \rho_0 : \pi \in K \}. \quad (2.10) \]

Since \( \mathcal{N} \varphi(t, x) \) is a singleton, when \( \sigma \) is invertible, we then retrieve a formulation similar to [9] and [46].

Moreover, the minimality condition in the definition of \( v \) should imply that \( v \) actually solves (in some sense) the partial differential equation (2.9), with the usual convention \( \sup \emptyset = -\infty \).

We shall first prove that \( v \) is actually a viscosity solution of (2.9) in the sense of discontinuous viscosity solutions. In order to prove the subsolution property, we shall appeal to the additional regularity assumption:

**Assumption 2.1** Fix \( (x_0, y_0, p_0) \in \bar{O} \times (0, \infty) \times \mathbb{R}^d \) such that \( y_0^{-1} \text{diag } [x_0] p_0 \in \text{int}(K_{x_0}) \). Set \( \pi_0 \in \mathcal{N}(x_0, y_0, p_0) \). Then, for all \( \varepsilon > 0 \), there exists an open neighborhood \( \bar{B} \) of \( (x_0, y_0, p_0) \) and a locally Lipschitz map \( \hat{\pi} \) such that

\[ |\hat{\pi}(x_0, y_0, p_0) - \pi_0| \leq \varepsilon \]

and

\[ \hat{\pi}(x, y, p) \in \mathcal{N}(x, y, p) \text{ on } \bar{B} \cap (\bar{O} \times (0, \infty) \times \mathbb{R}^d). \]

**Remark 2.4** In the case where \( \sigma \) is invertible, it corresponds to Assumption 2.1 in [12].
CHAPITRE 2. APPLICATION TO AMERICAN OPTION PRICING UNDER CONSTRAINTS

**Theorem 2.1** Assume that $v$ is locally bounded. Then, $v_*$ is a viscosity supersolution of (2.9) on $D$. If moreover Assumption 2.1 holds, then $v^*$ is a viscosity subsolution of (2.9) on $D$.

It remains to provide a boundary condition of $v$. Not surprisingly, we could expect that the constraint $Mv \geq 0$ should propagate to the boundary point which implies formally implies that $v$ should be a viscosity solution of

$$B_{\varphi} = 0,$$

where

$$B_{\varphi} := \begin{cases} \min\{H_{\varphi}, M_{\varphi}\} & \text{on } D, \\ \min \{\varphi(t,x) - g(t,x), M_{\varphi}(t,x)\} & \text{on } \partial D. \end{cases}$$

However, we shall see Remark 3.1 below that the above equation has to be corrected in order to admit a viscosity supersolution and therefore makes sense. When $\sigma$ is convertible, such a boundary condition was already observed by [9] in the context of European option pricing, where they proved that, under a suitable condition, $v_*$ is a supersolution of

$$\min\{v(t,x) - g(t,x), \hat{M}v(t,x)\} \geq 0 \text{ on } \partial D,$$

and $v^*$ is a subsolution of

$$\min\{v(t,x) - g(t,x), Mv(t,x)\} \leq 0 \text{ on } \partial D,$$

where, for a smooth function $\varphi$, $\hat{M}\varphi$ is defined as $M\varphi$ but with

$$\tilde{K}(x,\tilde{O}) := \{\rho_0 \in \mathbb{R}^d : \exists \tau_0 > 0 \text{ s.t. } xe^{\tau\rho_0} \in \tilde{O} \text{ for all } \tau \leq \tau_0\}$$

instead of $K_x$. In our general context, where $\sigma$ is not assumed to be invertible anymore, we face a more complex structure. We proceed as follows. Let $L^1(\text{Leb})$ be the set of Lebesgue-integrable functions equipped with the strong measure topology, and define $L^1_{\text{Leb}}$ as the subset of elements $\rho \in L^1(\text{Leb})$ such that $\rho = (\rho(s))_{s \geq 0}$ on $\mathbb{R}_+$ satisfying $|\rho(s)| = 1$ for Lebesgue almost every $s \geq 0$. For $\rho \in L^1_{\text{Leb}}$ and $x \in \tilde{O}$, define the process $\chi_{\rho}^t$ as the solution of

$$\chi(t) = x + \int_0^t \text{diag} [\chi(s)] \rho(s) ds, \quad t \geq 0.$$ 

For $x \in E \subset \tilde{O}$, we denote

$$S_0(x,E) := \{\rho_0 \in \tilde{K}_x : \exists (\rho,\tau) \in S(x,E) \text{ s.t. } \rho(0) = \rho_0 \text{ and } \tau > 0\},$$

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where
\[
S(x, E) := \{ (\rho, \tau) \in L^0_0(\text{Leb}) \times \mathbb{R}^+ : \chi_{\rho}^x(s) \in E, \rho(s) \in \tilde{K}_{\chi^x(s)} \text{ Leb. a.e. on } [0, \tau] \}.
\]

In Lemma 3.5 below, we will show that \( v^* \) is a supersolution of
\[
\mathcal{M}_d v(t, x) := \inf_{\rho_0 \in S_0(x, \bar{\mathcal{O}})} \{ \delta_x(\rho_0) v - \rho_0^\top \text{diag} \ [x] \ Dv \} = 0 \text{ on } \partial D,
\]
under the following additional assumptions:

**Assumption 2.2**

(i) For all \( x \in \mathcal{O} \), the closure of \( S_0(x, \mathcal{O}) \) is equal to \( S_0(x, \bar{\mathcal{O}}) \).

(ii) \((x_n)_{n \geq 1}\) is a sequence in \( \mathcal{O} \) such that
\[
x_n \to x \in \partial \mathcal{O} \text{ and } (\rho, \tau) \in S(x, \bar{\mathcal{O}}),
\]
there exists a sequence \((\rho_n, \tau_n) \to (\rho, \tau)\) such that, up to a subsequence,
\[
(\rho_n, \tau_n) \in S(x_n, \bar{\mathcal{O}}) \text{ for } n \geq 1,
\]
and
\[
\int_0^{\tau_n} \delta_{\chi_{\rho_n}^x}(\rho_n(s)) ds \to \int_0^{\tau} \delta_{\chi_{\rho}^x}(\rho(s)) ds.
\]

**Assumption 2.3** There exists a map \( d : (0, \infty)^d \to \mathbb{R} \) such that

(i) \( \{ x \in (0, \infty)^d : d(x) > 0 \} = \mathcal{O} \).

(ii) \( \{ x \in (0, \infty)^d : d(x) = 0 \} = \partial \mathcal{O} \).

(iii) For all \( x \in \partial \mathcal{O} \), there exists \( r > 0 \) such that \( d \in C^2(B_r(x)) \).

(iv) For all \( x \in \partial \mathcal{O} \), \( \sigma(x)^\top \text{diag} \ [x] \ Dd(x) \neq 0 \).

**Remark 2.5** The conditions (i), (ii) and (iii) of Assumption 2.3 are automatically satisfied when \( \mathcal{O} \) has a smooth enough boundary, see [25]. The condition (iv) is not required in the proof of the viscosity property stated in Theorem 2.2 below, but will be used to prove the comparison result of Theorem 2.3.

**Remark 2.6** We discuss here the relationship between the operators \( \mathcal{M}, \tilde{\mathcal{M}} \) and \( \mathcal{M}_d \):

1. In the case where \( \sigma \) is invertible, (2.10) implies that \( S_0(x, \bar{\mathcal{O}}) = \tilde{K}(x, \bar{\mathcal{O}}) \), which therefore leads to the coincidence of \( \mathcal{M}_d v(t, x) \) and \( \tilde{\mathcal{M}} v(t, x) \).

2. Note that, if \( x \in \mathcal{O} \), then there exists \( r_x > 0 \) such that \( B_{r_x}(x) \subset \mathcal{O} \). Then
\[
S_0(x, \bar{\mathcal{O}}) \supset S_0(x, B_{r_x}(x)) = \tilde{K}_x \text{ for all } x \in \mathcal{O}.
\]
Moreover, \( \mathcal{M}_d v(t, x) = \mathcal{M} v(t, x) \) on \( \{ T \} \times \mathcal{O} \). We therefore only realize the difference between \( \mathcal{M}_d \) and \( \mathcal{M} \) as \( x \in \partial \mathcal{O} \).
The above remark leads to the separation of the boundary into the two sub-domains
$\partial_T D := \{ T \} \times O$ and $\partial_x D := [0,T] \times \partial O$ :

**Definition 2.1** A lower semicontinuous (resp. upper semicontinuous) function $w$ on $[0,T] \times \tilde{O}$ is a viscosity supersolution (resp. subsolution) of

\[
\mathcal{B}_d \phi = 0,
\]

if, for any test function $\phi \in C^{1,2}([0,T] \times \tilde{O})$ and $(t_0, x_0) \in [0,T] \times \tilde{O}$ that achieves a local minimum (resp. maximum) of $w - \phi$ so that $(w - \phi)(t_0, x_0) = 0$, we have

\[
\mathcal{B}_d \phi(t_0, x_0) \geq 0 \text{ (resp. } \mathcal{B}_d \phi(t_0, x_0) \leq 0),
\]

where

\[
\mathcal{B}_d \phi := \begin{cases} 
\mathcal{B} \phi & \text{on } D \cup \partial_T D, \\
\min \{ \mathcal{B} \phi, g(t, x), \mathcal{M}_d \phi(t, x) \} & \text{on } \partial_x D.
\end{cases}
\]

A locally bounded function $w$ is a discontinuous viscosity solution of (2.15) if $w_*$ (resp. $w^*$) is a supersolution (resp. subsolution) of (2.15).

**Theorem 2.2** Let Assumptions 2.1, 2.3 and 2.2 hold. Then, $v$ is a discontinuous viscosity solution of (2.15).

We conclude by establishing a comparison result for (2.15) which implies that $v$ is continuous on $[0,T] \times O$, with a continuous extension to $[0,T] \times \tilde{O}$, and is the unique viscosity solution of (2.15) in a suitable class of functions. To this purpose, we shall need the following additional assumptions:

**Assumption 2.4**

(i) There exists $\bar{\gamma} \in K \cap [0, \infty)^d$ and $\lambda > 1$ such that $\lambda \bar{\gamma} \in K_x$ for all $x \in \tilde{O}$.

(ii) There exists a constant $C > 0$ such that $|g(t, x)| \leq C (1 + x^\gamma)$ for all $(t, x) \in [0,T] \times \tilde{O}$.

(iii) There exists $c_K > 0$ such that $\delta_x(\rho_0) \geq c_K$ for all $x \in \tilde{O}$ and $\rho_0 \in \tilde{K}_x$.

(iv) There exists $C > 0$ such that, for all $x, y \in \tilde{O}$ and $\rho_0 \in \tilde{K}_x$, we can find $\tilde{\rho} \in \tilde{K}_y$ satisfying $|\rho - \tilde{\rho}| \leq C |x - y|$ and $\delta_y(\tilde{\rho}) - \delta_x(\rho_0) \leq \epsilon(x, y)$, where $\epsilon$ is a continuous map satisfying $\epsilon(z, z) = 0$ for all $z \in (0, \infty)^d$.

(v) Either

(v.a.) There exists $L > 0$ such that, for all $(x, \bar{x}, y, \bar{y}, p, \bar{p}) \in (0, \infty)^d \times \mathbb{R}_+^2 \times \mathbb{R}^d$ : $\pi \in N(x, y, p) \neq \emptyset$, $N(\bar{x}, \bar{y}, \bar{p}) \neq \emptyset$ and $|x - \bar{x}| \leq L^{-1}$ implies $\exists \pi \in N(\bar{x}, \bar{y}, \bar{p})$ s.t. $|y \pi^\top \mu(x) - \bar{y} \pi^\top \mu(\bar{x})| \leq L|p^\top \text{diag } [x] \mu(x) - \bar{p}^\top \text{diag } [\bar{x}] \mu(\bar{x})|.$

or

(v.b.) For all $p, q \in \mathbb{R}^d$ and $x \in (0, \infty)^d : p^\top \sigma(x) = q^\top \sigma(x) \Rightarrow p^\top \mu(x) = q^\top \mu(x).$
Remark 2.7 1. The first condition holds whenever $\lambda \bar{\gamma} \in K$.
2. The condition (ii) implies that

$$\exists C > 0 \text{ s.t. } |v(t, x)| \leq C (1 + x^{\bar{\gamma}}) \quad \forall (t, x) \in [0, T] \times (0, \infty)^d.$$  (2.16)

Indeed, let $\pi \in \mathcal{A}$ be defined by $\pi_s = \bar{\gamma}$ for all $t \leq s \leq T$. Since $\sigma$ is bounded, one easily checks from the dynamics of the processes $X_{t,x}$ and $Y_{t,x,1}^\pi$ that

$$1 + \prod_{i=1}^d (X_{t,x}^i(u))^{\bar{\gamma}_i} \leq C \left(1 + \prod_{i=1}^d (x^i)^{\bar{\gamma}_i}\right) Y_{t,x,1}^\pi(u) \quad \text{for all } u \in [t, T],$$

where $C > 0$ depends only on $|\bar{\gamma}|$ and the bound on $|\sigma|$. Then, after possibly changing the value of the constant $C$, (ii) of Assumption 2.4 implies

$$g(u, X_{t,x}(u)) \leq C (1 + x^{\bar{\gamma}}) Y_{t,x,1}^\pi(u) \quad \text{for all } u \in [0, T].$$

Since $yY_{t,x,1}^\pi = Y_{t,x,y}^\pi$ for $y > 0$, we deduce (2.16) from the last inequality.

3. The condition (iii) is implied by $0 \in \text{int}(K)$. Indeed, if $0 \in \text{int}(K)$, then $\delta_x \geq \delta$ where the later is uniformly strictly positive, see [39].

4. The condition (iv) is trivially satisfied if $\delta_x = \delta$ for all $x \in \bar{O}$, which is the case when $\sigma$ is invertible.

5. The condition (v) is trivially satisfied when $\sigma$ is invertible. The condition (v.b.) is natural in the case $0 \in \text{int}(K)$ as, in this case, it is equivalent to $\pi^\top \sigma(x) = 0 \Rightarrow \pi^\top \mu(x) \leq 0$ for all $x \in K$, which is intimately related to the minimal no-arbitrage condition: $\pi \in \mathcal{A}$ and $Y_{t,x,y}^\pi \sigma(X_{t,x}) = 0$ Leb$\times$P-a.e. on $[t, T] \Rightarrow Y_{t,x,y}^\pi \mu(X_{t,x}) \leq 0$ Leb$\times$P-a.e. on $[t, T]$.  

Theorem 2.3 Let Assumptions 2.1, 2.2, 2.3 and 2.4 hold. Then, $v^* = v_*$ is continuous on $[0, T] \times \bar{O}$ and is the unique viscosity solution of (2.15) in the class of non-negative functions satisfying the growth condition (ii) of Assumption 2.4.

2.4 The “face-lift” phenomenon

When $\sigma$ is invertible, it can be shown under mild assumptions that the unique viscosity solution of (2.12)-(2.13) in a suitable class, is given by $\hat{g}(T, \cdot)$ where

$$\hat{g}(t, x) := \sup_{\rho_0 \in K(x, \bar{O})} e^{-R(\rho_0)} g(t, x e^{\rho_0}).$$

A standard comparison theorem, see Section 3.4 below, then implies that the boundary condition can actually be written in $v_*(T, \cdot) = v^*(T, \cdot) = \hat{g}(T, \cdot)$. This is the so-called “face-lift” procedure which was already observed by [13] in the context of European option pricing, see also [4], [7], [9] or [20].
In our general context, where \( \sigma \) is not assumed to be invertible anymore, standard optimal control arguments actually show that it should be related to the deterministic control problem

\[
\bar{g}(t_0, x) := \sup_{(\rho, \tau) \in \mathcal{D}(x, \partial)} e^{-\int_{t_0}^{\tau} \delta(\rho(s)) ds} g(t_0, \chi_{\rho}^x(\tau)),
\]

which is well defined on \( \partial \) by (2.14).

Note that, in the case where \( \sigma \) is invertible, (2.10) implies that it can be rewritten in

\[
\sup_{(\rho, \tau) \in \mathcal{D}(x, \partial)} e^{-\int_{t_0}^{\tau} \delta(\rho(s)) ds} g(t_0, \chi_{\rho}^x(\tau))
\]

whose value function is easily seen to coincide with \( \hat{g}(t_0, \cdot) \) by using the fact that \( \delta \) is convex and homogeneous of degree one, and \( g \geq 0 \).

We now make precise the above discussion in the two following Corollaries. The first one actually states that \( g \) can be replaced by \( \bar{g} \) in the definition of \( H \) and in the terminal condition.

**Corollary 2.2** Let Assumption 2.1 and (i)-(iv) of Assumption 2.4 hold. Assume further that:

(i) For each \( x \in \partial \), the map \( t \in [0, T] \mapsto \bar{g}(t, x) \) is continuous.

(ii) The map \( x \in \partial \mapsto \sup \{ \delta_x(\rho_0), \rho_0 \in \tilde{K}_x \} \) is locally bounded.

Then, \( v \) is a discontinuous viscosity solution of (2.15) and satisfies the boundary condition

\[
v_*(T, \cdot) = v^*(T, \cdot) = \hat{g}(T, \cdot) \quad \text{on} \quad \partial.
\]

If moreover, (v) of Assumption 2.4 hold, then it is the unique viscosity solution of (2.15)-(2.17), in the class of non-negative functions satisfying the growth condition (2.16).

3 Proof of the PDE characterization

From now on, we assume that \( v \) is locally bounded.

3.1 The supersolution property

We start with the supersolution property of Theorem 2.2.

**Lemma 3.1** The map \( v_* \) is a viscosity supersolution of \( H \varphi = 0 \) on \( [0, T] \times \partial \).  

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Proof. Note that \( v \geq g \) by definition. Since \( g \) is continuous, this implies that \( v \leq g \). It thus suffices to show that \( v \) is a supersolution of

\[
\sup_{\pi \in \mathcal{N}_\varphi(t,x)} (\pi^\top \mu(x) \varphi(t,x) - \mathcal{L}\varphi(t,x)) = 0 \text{ on } D.
\]

The proof follows from similar arguments as in [46], the main difference comes from the fact that \( \sigma \) is not assumed to be non-degenerate which only modifies the terminal argument of the above paper. We therefore only sketch the proof and focus on the main differences. Fix \((t_0, x_0) \in [0, T) \times \mathcal{O}\) and let \( \varphi \) be a smooth function such that \((t_0, x_0)\) achieves a strict minimum of \( v - \varphi \) on \([0, T] \times \mathcal{O}\) satisfying \((v - \varphi) (t_0, x_0) = 0\).

Let \((t_n, x_n)_{n \geq 1}\) be a sequence in \([0, T) \times \mathcal{O}\) that converges to \((t_0, x_0)\) so that

\[
v(t_n, x_n) \to v(t_0, x_0) =: y_0 \text{ as } n \to \infty.
\]

Set

\[
y_n := v(t_n, x_n) + \frac{1}{n}.
\]

Since \( y_n > v(t_n, x_n), \) it follows from (DP1) of Corollary 2.1 that we can find \( \pi_n \in \mathcal{A} \) such that, for any stopping time \( \tau_n < \tau_{n,x,n} \), we have

\[
Y_{t_n,x_n,y_n}^{\pi_n}(\tau_n) \geq v(\tau_n, X_{t_n,x_n}(\tau_n)) .
\]

Since \( v \geq v \geq \varphi, \) it follows that

\[
Y_{t_n,x_n,y_n}^{\pi_n}(\tau_n) \geq v(\tau_n, X_{t_n,x_n}(\tau_n)) \geq \varphi(\tau_n, X_{t_n,x_n}(\tau_n)).
\]

Set \( Y_n := Y_{t_n,x_n,y_n}^{\pi_n}, X_n := X_{t_n,x_n}. \) It follows from the previous inequality and Itô’s Lemma that

\[
0 \leq y_n + \int_{t_n}^{\tau_n} Y_n(s) \pi_n^\top(s) \sigma(X_n(s))dW_s + \int_{t_n}^{\tau_n} Y_n(s) \pi_n^\top(s) \mu(X_n(s))ds - \varphi(t_n, x_n)
\]

\[
- \int_{t_n}^{\tau_n} \mathcal{L}\varphi(s, X_n(s))ds - \int_{t_n}^{\tau_n} (D\varphi)^\top(s, X_n(s)) \text{diag } [X_n(s)] \sigma(X_n(s))dW_s
\]

which can be written as

\[
0 \leq \beta_n + \int_{t_n}^{\tau_n} Y_n(s) \pi_n^\top(s) \mu(X_n(s)) - \mathcal{L}\varphi(s, X_n(s))ds + \int_{t_n}^{\tau_n} \psi(s, X_n(s), Y_n(s), \pi_n(s))dW_s,
\]

where \( \beta_n := y_n - \varphi(t_n, x_n) \) and

\[
\psi : (s, x, y, \pi) \to (y \pi^\top - (D\varphi)^\top(s, x) \text{diag } [x]) \sigma(x).
\]

By choosing a suitable sequence of stopping times \((\tau_n)_n\), introducing a well-chosen sequence of change of measures as in Section 4.1 of [46] and using the Lipschitz continuity assumption (2.2) and the fact that \( K \) is convex and compact, we deduce
we finally obtain \( \kappa > 0 \),

\[
0 \leq \sup_{\pi \in K} (\varphi(t_0, x_0) \pi^T \mu(x_0) - \mathcal{L}\varphi(t_0, x_0) + \kappa |\psi(t_0, x_0, y_0, \pi)|^2).
\]

Recalling that \( K \) is compact and \( \psi \) is continuous, we obtain by sending \( \kappa \) to \( \infty \) that

\[
\varphi(t_0, x_0) \pi^T \mu(x_0) - \mathcal{L}\varphi(t_0, x_0) \geq 0 \quad \text{and} \quad |\psi(t_0, x_0, y_0, \pi)|^2 = 0 \quad \text{for some} \quad \pi \in K.
\]

Noting that

\[
0 = \psi(t_0, x_0, y_0, \pi) = (y_0 \pi - (D\varphi)^T(t_0, x_0) \text{diag} [x_0]) \sigma(x_0) \Rightarrow \pi \in N\varphi(t_0, x_0),
\]

we finally obtain

\[
\sup_{\pi \in N\varphi(t_0, x_0)} (\varphi(t_0, x_0) \pi^T \mu(x_0) - \mathcal{L}\varphi(t_0, x_0)) \geq 0.
\]

\[\square\]

As explained in Section 2.3, we now use the fact that \( N\varphi(t, x) \neq \emptyset \) if and only if \( M\varphi(t, x) \geq 0 \).

**Lemma 3.2** Fix \((x, y, p) \in \bar{O} \times \mathbb{R}_+ \times \mathbb{R}^d\). Then, \( N(x, y, p) \neq \emptyset \) if and only if \( M(x, y, p) \geq 0 \). If moreover \( y > 0 \), then \( y^{-1}\text{diag} [x] p \in \text{int}(K_x) \) if and only if \( M(x, y, p) > 0 \).

**Proof.** For \( y > 0 \), \( N(x, y, p) \neq \emptyset \iff y^{-1}\text{diag} [x] p \in K_x \iff M(x, y, p) \geq 0 \) since \( K_x \) is a closed convex set, see [39], and similarly, for \( y > 0 \), \( y^{-1}\text{diag} [x] p \in \text{int}(K_x) \) if and only if \( M(x, y, p) > 0 \). We now consider the case \( y = 0 \). Since \( 0 \in K \subset K_x \), we have \( \delta_x \geq 0 \). Hence, \( N(x, 0, p) \neq \emptyset \iff 0 \sigma(x) = p^T\text{diag} [x] \sigma(x) \iff \varepsilon^{-1}p^T\text{diag} [x] \in K_x \) for each \( \varepsilon > 0 \iff M(x, \varepsilon, p) \geq 0 \) for each \( \varepsilon > 0 \iff M(x, 0, p) \geq 0 \).

\[\square\]

As a corollary of Lemma 3.1 and the previous Lemma, we obtain :

**Corollary 3.1** The map \( v_* \) is a viscosity supersolution of \( M\varphi = 0 \) on \([0, T) \times O\).

We now turn to the boundary condition at \( t = T \).

**Lemma 3.3** The map \( v_*(T, \cdot) \) is a viscosity supersolution of

\[
\min\{\phi_*(T, \cdot) - g(T, \cdot), M\phi\} = 0 \quad \text{on} \quad O.
\]

**Proof.** The fact that \( v_*(T, \cdot) \geq g(T, \cdot) \) follows from the continuity of \( g \) and the fact that \( v \geq g \) on \([0, T) \times \bar{O}\) by definition. Let \( \phi \) be a smooth function and \( x_0 \in O \) be

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such that \( x_0 \) achieves a strict minimum of \( v_s(T, \cdot) - \phi \) and \( v_s(T, x_0) - \phi(x_0) = 0 \). Let \((s_n, \xi_n)_n\) be a sequence in \([0, T) \times \mathcal{O}\) satisfying
\[
(s_n, \xi_n) \rightarrow (T, x_0) , \ s_n < T \quad \text{and} \quad v_s(s_n, \xi_n) \rightarrow v_s(T, x_0) .
\]
For all \( n \in \mathbb{N} \) and \( k > 0 \), we define
\[
\varphi^k_n(t, x) := \phi(x) - \frac{k}{2} |x - x_0|^2 + k \frac{T - t}{T - s_n}.
\]
Notice that \( 0 \leq (T - t)(T - s_n)^{-1} \leq 1 \) for \( t \in [s_n, T] \), and therefore
\[
\lim_{k \to 0} \limsup_{n \to \infty} \sup_{(t, x) \in [s_n, T] \times B_r(x_0)} |\varphi^k_n(t, x) - \phi(x)| = 0 , \tag{3.2}
\]
where \( r > 0 \) is such that \( B_r(x_0) \subset \bar{\mathcal{O}} \). Next, let \((t^k_n, x^k_n)\) be a sequence of local minimizers of \( v_s - \varphi^k_n \) on \([s_n, T] \times B_r(x_0)\) and set \( \epsilon^k_n := (v_s - \varphi^k_n)(t^k_n, x^k_n) \). Following line by line the arguments of the proof of Lemma 20 in \([7]\), one easily checks that, after possibly passing to a subsequence,
\[
\begin{align*}
\text{for all} \quad k > 0 , & \quad (t^k_n, x^k_n) \rightarrow (T, x_0) , \tag{3.3} \\
\text{for all} \quad k > 0 , & \quad t^k_n < T \quad \text{for sufficiently large} \ n , \tag{3.4} \\
\varphi^k_n(t^k_n, x^k_n) \rightarrow & \varphi^k_n(T, x_0) = \phi(x_0) \quad \text{as} \ n \to \infty \quad \text{and} \ k \to 0 . \tag{3.5}
\end{align*}
\]
Notice that (3.3) and a standard localization argument implies that we may assume that \( x^k_n \in B_r(x_0) \) for all \( n \geq 1 \) and \( k > 0 \). It then follows from (3.4) that, for all \( k > 0 \), \((t^k_n, x^k_n)\) is a sequence of local minimizers of \( v_s - \varphi^k_n \) on \([s_n, T] \times B_r(x_0)\). Also, notice that (3.2), (3.3) and (3.5) imply
\[
\begin{align*}
\text{for all} \quad k > 0 , & \quad D\varphi^k_n(t^k_n, x^k_n) = D\phi(x^k_n) - k(x^k_n - x_0) \to D\phi(x_0) , \tag{3.6} \\
\text{and} \quad \lim_{k \to 0} \lim_{n \to \infty} \epsilon^k_n = & \ 0 . \tag{3.7}
\end{align*}
\]
It then follows from Theorem 2.2, recall the convention \( \sup \emptyset = -\infty \), (3.4) and the fact that \((t^k_n, x^k_n)\) is a local minimizer for \( v_s - \varphi^k_n \) that, for sufficiently large \( n \), we can find \( \pi^k_n \in K \) such that
\[
v_s(t^k_n, x^k_n)(\pi^k_n)^\top \sigma(x^k_n) = D\varphi^k_n(t^k_n, x^k_n)^\top \text{diag} \[ x^k_n \] \sigma(x^k_n) .
\]
Since \( K \) is compact, we can assume that \( \pi^k_n \to \pi \in K \) as \( n \to \infty \) and then \( k \to 0 \). Taking the limit as \( n \to \infty \) and then as \( k \to 0 \) in the previous inequality, and using (3.3), (3.5), (3.6), as well as the continuity of \( x \mapsto \text{diag} \[ x \] \sigma(x) \) thus implies that
\[
v_s(T, x_0)\pi^\top \sigma(x_0) = D\phi(x_0)^\top \text{diag} \[ x_0 \] \sigma(x_0) .
\]
Appealing to Lemma 3.2 then implies the required result. \( \square \)

In order to prove the boundary condition on \( \partial_x D \), we need following Lemma :
Lemma 3.4 Let \((t, x) \in \bar{D}\). Then
\[
v_*(t, x) = \sup_{(\rho, \tau) \in \mathcal{S}(x, \bar{O})} e^{-\int_0^\tau \delta_\chi(t)(\rho(s))ds} v_*(t, \chi^\rho_x(\tau)). \tag{3.8}
\]

Proof. We start with the case \((t, x) \in D\). It follows from Assumption 2.2 and the lower semicontinuity of \(v_*\) that
\[
\sup_{(\rho, \tau) \in \mathcal{S}(x, \bar{O})} e^{-\int_0^\tau \delta_\chi(t)(\rho(s))ds} v_*(t, \chi^\rho_x(\tau)) = \sup_{(\rho, \tau) \in \mathcal{S}(x, O)} e^{-\int_0^\tau \delta_\chi(t)(\rho(s))ds} v_*(t, \chi^\rho_x(\tau)).
\]
It suffices to show that
\[
v_*(t, x) = \sup_{(\rho, \tau) \in \mathcal{S}(x, O)} e^{-\int_0^\tau \delta_\chi(t)(\rho(s))ds} v_*(t, \chi^\rho_x(\tau)).
\]
We fix \((\rho, \tau) \in \mathcal{S}(x, O)\) so that \(\tau > 0\), denote the map
\[
w : r \to w(r) = e^{-\int_0^\tau \delta_\chi(t)(\rho(s))ds} v_*(t, \chi^\rho_x(r))
\]
defined on \([0, \tau]\). We shall prove that \(w(0) \geq w(\tau)\).

The fact that \(\chi^\rho_x(s) \in \mathcal{O}\) for all \(s \leq \tau\), together with Corollary 3.1, implies that \(v_*\) is a supersolution of
\[
\inf_{\rho_0 \in \mathcal{K}_x} \{\delta_x(\rho_0)\varphi - \rho_0^\top \text{diag}[x] D\varphi\} \geq 0 \text{ on } \{t\} \times I^\rho_x,
\]
where \(I^\rho_x := \{\chi^\rho_x(r) : r \in [0, \tau]\}\).

The map \(w\) therefore is a supersolution of
\[-Dw \geq 0 \text{ on } [0, \tau].\]

This allows us to prove the required result by following the steps below:

1. For any \(r_0 \in (0, \tau]\), there exist \(a(r_0) \geq 0\) satisfying
   \[
   0 \leq a(r_0) < r_0
   \]
   and\[
w(r_0) \leq w(a) \text{ for } a \in [a(r_0), r_0].\]

Given \(\epsilon > 0\), we define \(w_\epsilon(r) := w(r) - \epsilon r\) for \(r \in [0, \tau]\). Then \(w_\epsilon\) is a supersolution of \(-Dw_\epsilon \geq \epsilon\) on \([0, \tau]\). This means that for a smooth function \(\varphi\) such that \(r_0\) achieves a strict minimum of \(w_\epsilon - \varphi\) on \([0, \tau]\) satisfying \((w_\epsilon - \varphi)(r_0) = 0\), we have \(D\varphi(r_0) \leq -\epsilon\). This implies that there exists \(a(r_0) \geq 0\) such that \(a(r_0) < r_0\) and \(\varphi\) is strictly decreasing on \([a(r_0), r_0]\). We intend to prove that\[
w_\epsilon(a) > w_\epsilon(r_0) \text{ for all } a \in [a(r_0), r_0].\]
Assume to the contrary that there exists \( a \in [a(r_0), r_0) \) such that
\[
\omega_\varepsilon(a) \leq \omega_\varepsilon(r_0). \tag{3.9}
\]
Note that the map \( \omega_\varepsilon^a(r) \equiv \omega_\varepsilon(a) \) solves the equation \( D\varphi = 0 \) on \([a, r_0]\) with the boundary conditions \( \omega_\varepsilon^a(a) = \omega_\varepsilon^a(r_0) = \omega_\varepsilon(a) \). It follows from a standard comparison argument and (3.9) that
\[
\sup_{[a,r_0]} (\omega_\varepsilon^a - \omega_\varepsilon) \leq \max\{\omega_\varepsilon^a(a) - \omega_\varepsilon(a), \omega_\varepsilon^a(r_0) - \omega_\varepsilon(r_0)\} \leq 0.
\]
Then,
\[
\omega_\varepsilon(a) = \omega_\varepsilon^a(r_0) \leq \omega_\varepsilon(r_0).
\]
Recalling that the test function \( \varphi \) is strictly decreasing on \([a, r_0]\), we have
\[
\omega_\varepsilon(a) - \varphi(a) < \omega_\varepsilon(r_0) - \varphi(r_0),
\]
which leads to a contradiction of the strict minimum property of \( \omega_\varepsilon - \varphi \) at \( r_0 \).

2. We now prove that
\[
\omega_\varepsilon(r_0) \geq \lim\inf_{b \to r_0, b > r_0} \omega_\varepsilon(b) \text{ for } r_0 \in [0, \tau).
\]
Assume to the contradiction that there exists \( b(r_0) > r_0 \) such that
\[
\omega_\varepsilon(r_0) < \omega_\varepsilon(b) \text{ for all } b \in (r_0, b(r_0)]. \tag{3.10}
\]
In view of step 1., there exists \( a(r_0) < r_0 \) such that
\[
\omega_\varepsilon(a) \geq \omega_\varepsilon(r_0) \text{ for all } a \in [a(r_0), r_0].
\]
This, together with (3.10), implies that a test function
\[
\phi(r) := \omega_\varepsilon(r_0)
\]
satisfies
\[
\min_{[a(r_0), b(r_0)]} (\omega_\varepsilon - \phi) = \omega_\varepsilon(r_0) - \phi(r_0) = 0,
\]
but \( D\phi(r_0) = 0 \), which leads to a contradiction of the supersolution property of \( \omega_\varepsilon \) stated in step 1.

3. In order to complete the proof, we set
\[
\tau_0 = \inf\{s \leq \tau : \omega(r) \geq w(\tau) \text{ for all } r \in [s, \tau]\}, \tag{3.11}
\]
and show that \( \tau_0 = 0 \). Note that \( \tau_0 \leq a(\tau) < \tau \), where the function \( a(\cdot) \) is defined in step 1. Assume that \( \tau_0 > 0 \).

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It then follows from step 1 and step 2, applied \( r_0 = \tau_0 \), that there exists \( a < \tau_0 \) such that
\[
w(r) \geq w(\tau_0) \text{ for all } r \in [a, \tau_0),
\]
\[
w(\tau_0) \geq \liminf_{b \to \tau_0, b > \tau_0} w(b) \geq w(\tau),
\]
which leads to a contradiction of the definition of \( \tau_0 \). The required result follows from step 2, with \( r_0 = 0 \).

We have just proved the equality (3.8) when \((t, x) \in D\). We could show that this equality is also valued when \((t, x) \in \partial_T D\), by taking limits as \( t \to T \). It remains to consider the case where \((t, x) \in \partial_x D\). Let \((t_n, x_n)_{n \geq 1} \) be a sequence in \( D \) such that \((t_n, x_n) \to (t, x)\) and \( v(t_n, x_n) \to v_*(t, x)\). Fix \((\rho, \tau) \in \mathcal{S}(x, \bar{O})\). It follows from the condition (ii) of Assumption 2.2 that there exists \((\rho_n, \tau_n) \in \mathcal{S}(x_n, \bar{O})\) for all \( n \geq 1 \) satisfying \((\rho_n, \tau_n) \to (\rho, \tau)\). Then,
\[
v_*(t_n, x_n) \geq e^{-\int_0^{\tau_n} \delta\chi_{\rho_n}(r) \, ds} v_*(t_n, \chi_{\rho_n}^x(\tau_n)),
\]
which leads to the required result by sending \( n \to \infty \). \( \square \)

We now provide a boundary condition on \( \partial_x D \):

**Lemma 3.5** The map \( v_* \) is a viscosity supersolution of
\[
\mathcal{M}_d \varphi(t, x) = 0 \text{ on } D.
\]

**Proof.** Let \((t, x) \in \bar{D} \) and \( \varphi \) be a \( C^2 \) function such that
\[
0 = (v_* - \varphi)(t, x) = \min_{\bar{D}} (v_* - \varphi).
\]
We fix \((\rho, \tau) \in \mathcal{S}(x, \bar{O})\). It follows from (3.8) that
\[
\varphi(t, x) = v_*(t, x) \geq e^{-\int_0^\tau \delta\chi_{\rho}(r) \, ds} v_*(t, \chi_{\rho}^x(r)) \geq e^{-\int_0^\tau \delta\chi_{\rho}(r) \, ds} \varphi(t, \chi_{\rho}^x(r))
\]
for all \( r \leq \tau \). Dividing by \( r \) and sending \( r \to 0 \), leads to the required result. \( \square \)

**Remark 3.1** Note that, under Assumption 2.3 the map \( v_* \) can not be a supersolution of \( \mathcal{M} \varphi(t, x) = 0 \) on \( \partial_x D \). Assume to the contradiction that for all \((t_0, x_0) \in \partial_x D \) and a smooth function \( \varphi \) such that \( \min_D (v_* - \varphi) = (v_* - \varphi)(t_0, x_0) = 0 \) then
\[
\mathcal{M} \varphi(t_0, x_0) \geq 0.
\]
Let \((t_0, x_0) \in \partial_x D \) and a smooth function \( \varphi \) such that \((t_0, x_0)\) is a minimum of \( v_* - \varphi \). In view of Assumption 2.3, \((t_0, x_0)\) is also a minimum of \( v_* - \varphi^\varepsilon \), where \( \varphi^\varepsilon(t, x) := \varphi(t, x) - \varepsilon^{-1}d(x) \) for \( \varepsilon > 0 \). This implies that
\[
\inf_{\rho_0 \in K_{r_0}} \{ \delta_{x_0}(\rho)\varphi(t_0, x_0) - \rho_0^T \text{diag} [x_0] (D\varphi(t_0, x_0) - \varepsilon^{-1}d(x_0)) \} \geq 0. \quad (3.12)
\]
Let \( \rho_0 \in \tilde{K}_{r_0} \setminus \mathcal{S}(x_0, \bar{O}) \), then there exist a sequence \( \tau_n \to 0 \) and \( \rho \in L^1(\text{Leb}) \) such that \( d(\chi_{x_0}^\varepsilon(\tau_n)) < 0 \) and \( \rho(0) = \rho_0 \), recall Assumption 2.3. This implies that \( \rho_0^T \text{diag} [x_0] Dd(x_0) < 0 \), which leads to a contradiction of (3.12) when \( \varepsilon \) comes to 0.

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3.2 The subsolution property

We now turn to the subsolution property of Theorem 2.2.

Lemma 3.6 Under Assumption 2.1, the map $v^*$ is a viscosity subsolution of

$$\min \{ \mathcal{H} \varphi, \mathcal{M} \varphi \} = 0 \text{ on } [0, T) \times \mathcal{O}.$$ 

Proof. Fix $(t_0, x_0) \in [0, T) \times \mathcal{O}$ and let $\varphi$ be a smooth function such that $(t_0, x_0)$ achieves a strict maximum of $v^* - \varphi$ on $[0, T] \times \mathcal{O}$ satisfying $(v^* - \varphi)(t_0, x_0) = 0$.

We assume that

$$\min \{ \mathcal{H} \varphi(t_0, x_0), \mathcal{M} \varphi(t_0, x_0) \} := 2 \varepsilon > 0,$$

and work towards a contradiction. Note that $v^*(t_0, x_0) = \varphi(t_0, x_0) > 0$ since $g \geq 0$.

In view of Lemma 3.2, this implies that $\varphi(t_0, x_0)^{-1} \text{diag } [x_0] D \varphi(t_0, x_0) \in \text{int}(K_{x_0})$. It then follows from Assumption 2.1 that we can find $r > 0$ and a Lipschitz continuous map $\hat{\pi}$ such that

$$\min \left\{ y \hat{\pi}(x, y, D \varphi(t, x))^\top \mu(x) - \mathcal{L} \varphi(t, x), \ y - g(t, x) \right\} > \varepsilon$$

and, for $(t, x, y) \in B_r(t_0, x_0) \times B_r(\varphi(t, x)) \subset D \times (0, \infty)$,

$$\hat{\pi}(x, y, D \varphi(t, x)) \in N(x, y, D \varphi(t, x)). \quad (3.13)$$

Let $(t_n, x_n)_n$ be a sequence in $B_r(t_0, x_0)$ such that $v(t_n, x_n) \to v^*(t_0, x_0)$ and set $y_n := v(t_n, x_n) - n^{-1}$ so that $y_n > 0$ for $n$ large enough. Without loss of generality, we can assume that $y_n \in B_r(\varphi(t_n, x_n))$ for each $n$. Let $(X^n, Y^n)$ denote the solution of (2.1) and (2.3) associated to the Markovian control $\hat{\pi}(X^n, Y^n, D \varphi(\cdot, X^n))$ and the initial conditions $(X^n(t_0), Y^n(t_0)) = (x_n, y_n)$. Note that these processes are well defined on $[t_n, \tau_n]$ where

$$\tau_n := \inf \left\{ t \geq t_n : \ (t, X^n(t)) \notin B_r(t_0, x_0) \text{ or } |Y^n(t) - \varphi(t, X^n(t))| \geq r \right\}.$$

Moreover, it follows from the definition of $(t_0, x_0)$ as a strict maximum point of $v^* - \varphi$ that

$$(v^* - \varphi)(\tau_n, X^n(\tau_n)) < -\zeta \quad \text{or} \quad |Y^n(\tau_n) - \varphi(\tau_n, X^n(\tau_n))| \geq r \quad (3.14)$$

for some $\zeta > 0$. Since $v^* \leq \varphi$, applying Itô’s Lemma to $Y^n - \varphi(\cdot, X^n)$, recalling (3.13) and using a standard comparison Theorem for stochastic differential equations shows that

$$Y^n(\tau_n) - v(\tau_n, X^n(\tau_n)) \geq Y^n(\tau_n) - \varphi(\tau_n, X^n(\tau_n)) \geq r$$

on $\{|Y^n(\tau_n) - \varphi(\tau_n, X^n(\tau_n))| \geq r\}$.
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The same arguments combined with (3.14) also implies that

\[ Y^n(t_n) - v(t_n, X^n(t_n)) \geq y_n - \varphi(t_n, x_n) + \zeta \quad \text{on} \quad \{ |Y^n(t_n) - \varphi(t_n, X^n(t_n))| < r \} . \]

Since \( y_n - \varphi(t_n, x_n) \to 0 \), combining the two last assertions shows that \( Y^n(t_n) - v(t_n, X^n(t_n)) > 0 \) for \( n \) large enough. Moreover, it follows from (3.13) that \( Y^n > g(\cdot, X^n) \) on \([t_n, \tau_n] \). Since \( y_n < v(t_n, x_n) \), this contradicts \((\text{DP}2)\) of Corollary 2.1. \( \square \)

Lemma 3.7 Let Assumption 2.1, 2.2 and 2.3 hold. Fix \((t_0, x_0) \in \partial T \cup \partial_x D\) and assume that there exists a smooth function \( \phi \) such that \( M\phi(t_0, x_0) > 0 \) and \((t_0, x_0)\) achieves a strict local maximum of \( v^* - \phi \). Then,

\[ v^*(t_0, x_0) \leq g(t_0, x_0) . \]

Proof. 1. We first prove the required result when \((t_0, x_0) \in \partial_x D\). Since \( g \) is continuous and \( M\phi(t_0, x_0) > 0 \), it follows from Lemma 3.2, and Assumption 2.1, 2.3 that we can find \( r, \eta > 0 \) and a Lipschitz continuous map \( \hat{\pi} \) such that

\[ d(x_0) = 0, \quad 0 \leq d(x) < \epsilon \lambda \text{ on } B_r(x_0) \cap \bar{O}, \quad (3.15) \]

and, for \((t, x, y) \in [t_0, t_0 + r] \times (B_r(x_0) \cap \bar{O}) \times B_r(\varphi(t_0, x_0))\),

\[ \hat{\pi}(x, y, D\varphi(t, x)) \in N(x, y, D\varphi(t, x)), \varphi(t, x) - g(t, x) \geq \eta, \quad (3.16) \]

where

\[ \varphi(t, x) := \phi(t, x) + \lambda d(x) - \frac{d^2(x)}{\epsilon} + \frac{1}{2} |x - x_0|^2 \]

with \( \epsilon, \lambda > 0 \) small enough.

In view of (i) and (iv) of Assumption 2.2, we also have, after possibly changing \( r > 0 \) and sending \( \epsilon \) to 0,

\[ y\hat{\pi}(x, y, D\varphi(t, x))^\top \mu(x) - \mathcal{L}\varphi(t, x) \geq \eta \quad (3.17) \]

for \((t, x, y) \in [t_0, t_0 + r] \times (B_r(x_0) \cap \bar{O}) \times B_r(\varphi(t_0, x_0)) =: B\).

The fact that \( v^* - \varphi \) achieves a strict local maximum at \((t_0, x_0)\), recall (3.15), and that \( g(t, x) = v(t, x) \) on \( \partial_x D \) imply that

\[ \sup_{\partial_p B} (v - \varphi) < 0, \quad (3.18) \]

where \( \partial_p B := \{ t_0 + r \} \times (B_r(x_0) \cap \bar{O}) \cup [t_0, t_0 + r] \times \partial(B_r(x) \cap \bar{O}) \).

Let \((t_n, x_n)\) be a sequence in \([0, T) \times \bar{O}\) converging to \((t_0, x_0)\) such that

\[ v(t_n, x_n) \to v^*(t_0, x_0) . \]
and let $(\hat{t}_n, \hat{x}_n)$ be a strict maximum point of $v^* - \varphi$ on $[t_n - n^{-1}, t_n] \times \tilde{O}$. One easily checks that $\hat{t}_n < T$ and that $(\hat{t}_n, \hat{x}_n) \to (t_0, x_0)$.

Obviously, we can assume that $(\hat{t}_n, \hat{x}_n) \in B$. Let $(X^n, Y^n)$ denote the solution of (2.1) and (2.3) associated to the Markovian control $\hat{\pi}(X^n, Y^n, D\varphi(\cdot, X^n))$ and the initial conditions $(X^n(\hat{t}_n), Y^n(\hat{t}_n)) = (\hat{x}_n, \hat{y}_n)$. Note that these processes are well defined on $[\hat{t}_n, \tau_n]$ where

$$\tau_n := \inf \left\{ t \geq \hat{t}_n : (t, X^n(t)) \notin B \text{ or } |Y^n(t) - \varphi(t, X^n(t))| \geq r \right\}.$$ 

Applying Itô’s Lemma to $Y^n - \varphi(\cdot, X^n)$ and arguing as in the proof of Lemma 3.6, we then deduce that (3.17) and (3.18) lead to a contradiction to (DP2) of Corollary 2.1.

2. The case where $(t_0, x_0) \in \{T\} \times O$ can be treated similarly by similar argument as in previous step, after replacing $\varphi$ by $(t, x) \mapsto \phi(t, x) + \sqrt{T - t}$. 

\[ \square \]

### 3.3 A comparison result

In this section, we prove Theorem 2.3. This is a consequence of the following comparison result and the growth property for $v$ which was derived in Remark 2.7.

**Proposition 3.1** Let Assumption 2.4 hold. Let $V$ (resp. $U$) be a non-negative lower-semicontinuous (resp. upper-semicontinuous) locally bounded map on $[0, T] \times \tilde{O}$ satisfying (2.16). Assume that $V$ (resp. $U$) is a viscosity supersolution (resp. subsolution) of (2.15) on $[0, T] \times \tilde{O}$. Then, $V \geq U$ on $[0, T] \times \tilde{O}$.

In order to prove Proposition 5.1, we need a technical Lemma:

**Lemma 3.8** Let Assumption 2.3 and (iv) of Assumption 2.4 hold. Fix $x_0 \in \partial O$. Then there exists $(\rho, \tau) \in S(x_0, \hat{O})$ such that

$$Dd(x_0)^\top \text{diag} [x_0] \rho_0 > 0 \text{ and } \rho(0) = \rho_0.$$ 

(3.19)

Moreover, we could choose $\tau > 0$ small enough such that $(\rho, \tau) \in S(x, \tilde{O})$ for all $x \in B_r(x_0)$ with some $r > 0$.

**Proof.** Note that the condition (iv) of Assumption 2.3 allows us to define $\rho_0 := \sigma(x_0)\tilde{\rho}_0/|\sigma(x_0)|\tilde{\rho}_0|$, where $\tilde{\rho}_0$ satisfies $\sigma(x_0)^\top \text{diag} [x_0] Dd(x_0)\tilde{\rho}_0 > 0$. It then follows from (iv) of Assumption 2.4 that we can find $r > 0$ and a Lipschitz continuous map $\hat{\rho}$ such that

$$Dd(x)^\top \text{diag} [x_0] \rho_0 > 0 \text{ and } \delta_x(\hat{\rho}(x)) < \infty \forall x \in B_r(x_0) \cap \tilde{O},$$

define $\hat{\chi}$ as $\chi^\rho_{x_0}$ for the Markovian control $\rho = \hat{\rho}(\hat{\chi})$ and take $\rho(s) = \hat{\rho}(\hat{\chi}(s))$. 

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It remains to prove that there exists $\tau > 0$ such that

$$\chi^\rho_x(s) \in \bar{O}$$

for $s \leq \tau$ and $x \in B_r(x_0) \cap \bar{O}$.

The fact that $\chi^\rho_x(s) = x + s\text{diag}[x] \rho + o(s)$ implies that there exists $\bar{x} \in B_r(x_0)$ such that

$$d(\chi^\rho_x(s)) = d(x) + sDd(\bar{x})^\top \text{diag}[x] \rho + o(s).$$

This leads to required result by taking $\tau$ small enough.

Proof of Proposition 5.1

1. As usual, we first fix $\kappa > 0$ and introduce the functions $\tilde{U}(t,x) := e^{\kappa t}U(t,x)$, $\tilde{V}(t,x) := e^{\kappa t}V(t,x)$ and $\tilde{g}(t,x) := e^{\kappa t}g(t,x)$.

Let $\lambda > 1$ and $\bar{\gamma} \in \bar{O}$ be as in Assumption 2.4. Let $\beta$ be defined by

$$\beta(t,x) = e^{\tau(T-t)} (1 + x^\lambda \bar{\gamma}),$$

for some $\tau > 0$ to be chosen below, and observe that the fact that $\delta_x \geq 0$ and $\lambda \bar{\gamma} \in K_x$ for all $x \in \bar{O}$ implies that, for all $(t,x) \in [0,T] \times \bar{O}$,

$$M\beta(t,x) = \inf_{\rho_0 \in K_x} \left( \delta_x(\rho_0) + x^\lambda \delta_x(\rho_0) - \rho_0^\top (\lambda \bar{\gamma}) \right) e^{\tau(T-t)} \geq 0. \quad (3.20)$$

Moreover, one easily checks, by using the fact that $\mu$ and $\sigma$ are bounded, and $K$ is compact, that we can choose $\tau$ large enough so that, on $[0,T] \times \bar{O}$,

$$\left\{ -2L|D\beta(t,x)| \right\} \text{diag}[x] \mu(x) + \kappa \beta(t,x) - L\beta(t,x) \geq 0,$$

$$\kappa \beta(t,x) - \partial_t \beta(t,x) - \frac{1}{2} \text{Tr} \left[ a(x) D^2 \beta(t,x) \right] \geq 0,$$

where $a(z) := \text{diag}[z] \sigma(z) \sigma(z)^\top \text{diag}[z]$, and $L$ is as in (v.a) of Assumption 2.4 if it holds and $L = 0$ otherwise.

2. In order to show that $U \leq V$, we argue by contradiction. We therefore assume that

$$\sup_{[0,T] \times \bar{O}} (U - V) > 0 \quad (3.22)$$

and work towards a contradiction.

2.1. Using the growth condition on $\tilde{U}$ and $\tilde{V}$, and (5.4), we deduce that

$$0 < 2m := \sup_{[0,T] \times \bar{O}} (\tilde{U} - \tilde{V} - 2\alpha \beta) < \infty \quad (3.23)$$

for $\alpha > 0$ small enough. Fix $\varepsilon > 0$ and let $f$ be defined on $\bar{O}$ by

$$f(x) = \sum_{i=1}^d (x_i)^{-2}. \quad (3.24)$$

Arguing as in the proof of Proposition 6.9 in [9], see also below for similar arguments, we obtain that

$$\Phi^\varepsilon := \tilde{U} - \tilde{V} - 2(\alpha \beta + \varepsilon f)$$
admits a maximum \((t_\varepsilon, x_\varepsilon)\) on \([0, T] \times \mathcal{O}\), which, for \(\varepsilon > 0\) small enough, satisfies
\[
\Phi( t_\varepsilon, x_\varepsilon ) \geq m > 0 , \tag{3.25}
\]
as well as
\[
\limsup_{\varepsilon \to 0} \varepsilon \left( |f(x_\varepsilon)| + |\text{diag } [x_\varepsilon] Df(x_\varepsilon)| + |\mathcal{L}f(x_\varepsilon)| \right) = 0 . \tag{3.26}
\]

2.2. It follows Lemma 3.8 that there exists \((\rho_\varepsilon, \tau_\varepsilon) \in \mathcal{S}(x_\varepsilon, \mathcal{O})\) and \(\tau_\varepsilon > 0\) such that
\[
\begin{cases}
\rho_\varepsilon = 0 & \text{if } x_\varepsilon \in \mathcal{O}, \\
Dd(x) \text{diag } [x_\varepsilon] \rho_\varepsilon > 0 & \text{if } x_\varepsilon \in \partial \mathcal{O}, \\
\chi^{\rho_\varepsilon}(s) \in \mathcal{O} & \text{for } s \leq \tau_\varepsilon ,
\end{cases}
\tag{3.27}
\]
for all \(x \in B_{\tau_\varepsilon}(x_\varepsilon)\).

For \(n \geq 1\) and \(\zeta \in (0, 1)\) such that \(\zeta / n \leq \tau_\varepsilon\), we then define the function \(\Psi_n^{\varepsilon, \zeta}\) on \([0, T] \times (0, \infty)^{2d}\) by
\[
\Psi_n^{\varepsilon, \zeta}(t, x, y) := \Theta(t, x, y) - \varepsilon(f(x) + f(y)) - \zeta(|x - x_\varepsilon|^2 + |t - t_\varepsilon|^2) - n^2|\chi_n^{\rho_\varepsilon}(\zeta/n) - y|^2 ,
\]
where
\[
\Theta(t, x, y) := \tilde{U}(t, x) - \tilde{V}(t, y) - \alpha(\beta(t, x) + \beta(t, y)) .
\]
It follows from the growth condition on \(\tilde{U}\) and \(\tilde{V}\) that \(\Psi_n^{\varepsilon, \zeta}\) attains its maximum at some \((t_n^{\varepsilon, x_n^{\varepsilon}, y_n^{\varepsilon}}) \in [0, T] \times \bar{O}^2\). From now, we define \(\chi_n^{\varepsilon} := \chi_n^{\rho_\varepsilon}\) and \(\chi^1 := \chi_1^{\rho_\varepsilon}\). Note that
\[
\chi_n^{\varepsilon} = \text{diag } [x_n^{\varepsilon}] \chi^1 . \tag{3.28}
\]
Moreover, the inequality \(\Psi_n^{\varepsilon, \zeta}(t_n^{\varepsilon, x_n^{\varepsilon}, y_n^{\varepsilon}}) \geq \Psi_n^{\varepsilon, \zeta}(t_\varepsilon, x_\varepsilon, \chi_n^{\rho_\varepsilon}(\zeta/n))\) implies that
\[
\Theta(t_n^{\varepsilon, x_n^{\varepsilon}, y_n^{\varepsilon}}) \geq \Theta(t_\varepsilon, x_\varepsilon, \chi_n^{\rho_\varepsilon}(\zeta/n)) - 2\varepsilon f(x_\varepsilon) + n^2|\chi_n^{\rho_\varepsilon}(\zeta/n) - y_n|^2
+ \zeta \left(|x^{\varepsilon}_n - x_\varepsilon|^2 + |t^{\varepsilon}_n - t_\varepsilon|^2\right) + \varepsilon\left(f(x_n^{\varepsilon}) + f(y_n^{\varepsilon})\right). \tag{3.27}
\]
Using the growth property of \(\tilde{U}\) and \(\tilde{V}\) again, we deduce that the term on the second line is bounded in \(n\) so that, up to a subsequence,
\[
\chi_n^{\varepsilon}(\zeta/n), y_n^{\varepsilon} \longrightarrow \bar{x}^{\varepsilon} \in \bar{O} \quad \text{and} \quad t_n^{\varepsilon} \longrightarrow \bar{t}^{\varepsilon} \in [0, T] .
\]
Recalling similarly as Lemma 3.4 and using the fact that \((\rho_\varepsilon, \tau_\varepsilon) \in \mathcal{S}(x_\varepsilon, \mathcal{O})\) show that
\[
\tilde{V}(t_\varepsilon, x_\varepsilon) \geq \tilde{V}(t_\varepsilon, \chi_n^{\rho_\varepsilon}(\zeta/n)) e^{-\int_0^{\bar{t}^{\varepsilon}} \delta_{\chi_n^{\rho_\varepsilon}(\zeta/n)} ds} .
\]
This, together with the maximum property of \((t_\varepsilon, x_\varepsilon)\), implies that
\[
0 \geq \Phi^\varepsilon(\bar{t}_n, \bar{x}_n) - \Phi^\varepsilon(t_\varepsilon, x_\varepsilon) \\
\geq \limsup_{n \to \infty} \left( |\chi_n^\varepsilon(\zeta/n) - y_n^\varepsilon| + \zeta \left( |x_n^\varepsilon - x_\varepsilon|^2 + |t_n^\varepsilon - t_\varepsilon|^2 \right) \right).
\]
We get
\[
\begin{align*}
(a) \quad & n^2 |\chi_n^\varepsilon(\zeta/n) - y_n^\varepsilon|^2 + \zeta \left( |x_n^\varepsilon - x_\varepsilon|^2 + |t_n^\varepsilon - t_\varepsilon|^2 \right) \to 0, \\
(b) \quad & \bar{U}(t_n^\varepsilon, x_n^\varepsilon) - \bar{V}(t_n^\varepsilon, y_n^\varepsilon) \to \left( \bar{U} - \bar{V} \right)(t_\varepsilon, x_\varepsilon) \geq m + 2\alpha\beta(t_\varepsilon, x_\varepsilon) + 2\varepsilon f(x_\varepsilon),
\end{align*}
\]
where we used (5.5) for the last assertion.

3.1. We now show that, after possibly passing to a subsequence,
\[y_n^\varepsilon \in \mathcal{O}. \tag{3.29}\]
Note that, above result is true when \(x_\varepsilon \in \mathcal{O}\). It remains to show it in the case \(x_\varepsilon \in \partial \mathcal{O}\). It follows from (a) that
\[y_n^\varepsilon = x_n^\varepsilon + \sum_n \text{diag} [x_n^\varepsilon] \rho_\varepsilon + o(n^{-1}),\]
which implies that
\[d(y_n^\varepsilon) = d(x_n^\varepsilon) + \sum_n (Dd(x_n^\varepsilon))^{\top} \text{diag} [x_n^\varepsilon] \rho_\varepsilon + \epsilon_n,\]
for some \(\epsilon_n \to 0\) as \(n \to \infty\). Then, (3.29) follows from (3.27).

3.2. Assume that, after possibly passing to a subsequence,
\[
\begin{align*}
\min \left\{ M \left( x_n^\varepsilon, \bar{U}(t_n^\varepsilon, x_n^\varepsilon), p_n^\varepsilon \right), \bar{U}(t_n^\varepsilon, x_n^\varepsilon) - \bar{g}(t_n^\varepsilon, x_n^\varepsilon) \right\} & \leq 0, \\
\min \left\{ M \left( y_n^\varepsilon, \bar{V}(t_n^\varepsilon, y_n^\varepsilon), q_n^\varepsilon \right), \bar{V}(t_n^\varepsilon, y_n^\varepsilon) - \bar{g}(t_n^\varepsilon, y_n^\varepsilon) \right\} & \geq 0,
\end{align*}
\]
where
\[
\begin{align*}
p_n^\varepsilon & := 2n^2 \text{diag} \left[ \chi^1(\zeta/n) \right] (\chi_n^\varepsilon(\zeta/n) - y_n^\varepsilon) + 2\zeta (x_n^\varepsilon - x_\varepsilon) + \alpha D\beta(t_n^\varepsilon, x_n^\varepsilon) + \varepsilon Df(x_n^\varepsilon) \\
q_n^\varepsilon & := 2n^2 (\chi_n^\varepsilon(\zeta/n) - y_n^\varepsilon) - \alpha D\beta(t_n^\varepsilon, y_n^\varepsilon) - \varepsilon Df(y_n^\varepsilon).
\end{align*}
\]
Assuming that, after possibly passing to a subsequence, \(\bar{U}(t_n^\varepsilon, x_n^\varepsilon) \leq \bar{g}(t_n^\varepsilon, x_n^\varepsilon)\) for all \(n \geq 1\), we get a contradiction to (b) since \(\bar{V}(t_n^\varepsilon, y_n^\varepsilon) \geq \bar{g}(t_n^\varepsilon, y_n^\varepsilon)\) so that passing to the limit, recall (a) and the fact that \(g\) is continuous, implies \(\bar{U}(t_n^\varepsilon, x_\varepsilon) \leq \bar{g}(t_n^\varepsilon, x_\varepsilon) \leq \bar{V}(t_n^\varepsilon, x_\varepsilon)\).

We can therefore assume that \(\bar{U}(t_n^\varepsilon, x_n^\varepsilon) > \bar{g}(t_n^\varepsilon, x_n^\varepsilon)\) for all \(n\). Using the two above inequalities, we then deduce that
\[
0 \geq M \left( x_n^\varepsilon, \bar{U}(t_n^\varepsilon, x_n^\varepsilon), p_n^\varepsilon \right) - M \left( y_n^\varepsilon, \bar{V}(t_n^\varepsilon, y_n^\varepsilon), q_n^\varepsilon \right),
\]
which implies that we can find \( \hat{\rho}_{x_n}^\varepsilon \in \tilde{K}_{\lambda_\varepsilon(n/\varepsilon)} \) such that for all \( \rho_n^\varepsilon \in \tilde{K}_{y_n} \), we have

\[
0 \geq \delta_{x_n}(\hat{\rho}_{x_n}^\varepsilon) \left[ \Theta(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon) - \varepsilon(f(x_n^\varepsilon) + f(y_n^\varepsilon)) \right] \\
+ (\delta_{x_n}(\hat{\rho}_{x_n}^\varepsilon) - \delta_{y_n}(\rho_n^\varepsilon)) \left( \tilde{V}(T, y_n^\varepsilon) + \alpha \beta(y_n^\varepsilon) + \varepsilon f(y_n^\varepsilon) \right) \\
+ \alpha M(x_n^\varepsilon, \beta(x_n^\varepsilon), D\beta(t_n^\varepsilon, x_n^\varepsilon)) + \varepsilon M(x_n^\varepsilon, f(x_n^\varepsilon), f(y_n^\varepsilon)) \\
+ \alpha M(y_n^\varepsilon, \beta(y_n^\varepsilon), D\beta(t_n^\varepsilon, y_n^\varepsilon)) + \varepsilon M(y_n^\varepsilon, f(y_n^\varepsilon), f(y_n^\varepsilon)) \\
- 2\zeta(\hat{\rho}_{x_n}^\varepsilon)^\top \text{diag}[x_n^\varepsilon] (x_n^\varepsilon - x_\varepsilon) \\
- 2n^2 (\hat{\rho}_{x_n}^\varepsilon)^\top \text{diag} \left[ \chi_n^\varepsilon(\varepsilon/\eta) \right] - (\rho_n^\varepsilon)^\top \text{diag} [y_n^\varepsilon]) (x_n^\varepsilon e^{\tilde{\eta} W_n^\varepsilon} - y_n^\varepsilon) \\
\]

In view of (iv) of Assumption 2.4 and (a) above, we can choose \( \rho_n^\varepsilon \) such that, for some \( C > 0 \),

\[
|\hat{\rho}_{x_n}^\varepsilon - \rho_n^\varepsilon| \leq C|x_n^\varepsilon - y_n^\varepsilon| \quad \text{and} \quad \delta_{x_n}(\hat{\rho}_{x_n}^\varepsilon) - \delta_{y_n}(\rho_n^\varepsilon) \geq -\varepsilon(x_n^\varepsilon, y_n^\varepsilon) \to 0 .
\]

Using (a), (b), (iii) of Assumption 2.4, the fact that \( \tilde{V}, \beta, f \geq 0 \), (3.20) and (3.26), the previous inequality applied to \( \varepsilon > 0 \) small enough and \( n \) large enough leads to

\[
0 \geq c_K(m/2)
\]

which contradicts (5.5).

**3.2.** In view of the above point, we can now assume, after possibly passing to a subsequence, that \( t_n^\varepsilon < T \) for all \( n \geq 1 \). From Ishii’ Lemma, see Theorem 8.3 in [18], we deduce that, for each \( \eta > 0 \), there are real coefficients \( b_{1,n}^\varepsilon, b_{2,n}^\varepsilon \) and symmetric matrices \( \chi_{n}^{\varepsilon,\eta} \) and \( \chi_{n}^{\varepsilon,\eta} \) such that

\[
(b_{1,n}^\varepsilon, b_{2,n}^\varepsilon, \chi_{n}^{\varepsilon,\eta}) \in \mathcal{P}_\varepsilon^+ \tilde{U}(t_n^\varepsilon, x_n^\varepsilon) \quad \text{and} \quad (-b_{2,n}^\varepsilon, q_n^\varepsilon, \chi_{n}^{\varepsilon,\eta}) \in \mathcal{P}_\varepsilon^{-} \tilde{V}(t_n^\varepsilon, y_n^\varepsilon) ,
\]

see [18] for the standard notations \( \mathcal{P}_\varepsilon^+ \) and \( \mathcal{P}_\varepsilon^- \), \( b_{1,n}^\varepsilon, b_{2,n}^\varepsilon, \chi_{n}^{\varepsilon,\eta} \) and \( \chi_{n}^{\varepsilon,\eta} \) satisfy

\[
\left\{ \begin{array}{l}
\begin{array}{l}
b_{1,n}^\varepsilon + b_{2,n}^\varepsilon = 2\zeta(t_n^\varepsilon - t_\varepsilon) - \alpha \tau(\beta(t_n^\varepsilon, x_n^\varepsilon) + \beta(t_n^\varepsilon, y_n^\varepsilon)) \\
\chi_{n}^{\varepsilon,\eta} 0 \\
0 -\chi_{n}^{\varepsilon,\eta}
\end{array} \\
\leq \left( A_n^\varepsilon + B_n^\varepsilon \right) + \eta(A_n^\varepsilon + B_n^\varepsilon)^2
\end{array} \right. \tag{3.30}
\]

with

\[
A_n^\varepsilon := \begin{pmatrix}
2n^2 \text{diag} \left[ \chi^1(\varepsilon/\eta) \right]^2 + 2\zeta I_d & -2n^2 \text{diag} \left[ \chi^1(\varepsilon/\eta) \right] \\
-2n^2 \text{diag} \left[ \chi^1(\varepsilon/\eta) \right] & 2n^2 I_d
\end{pmatrix}
\]

\[
B_n^\varepsilon := \begin{pmatrix}
\alpha D^2 \beta(t_n^\varepsilon, x_n^\varepsilon) + \varepsilon D^2 f(x_n^\varepsilon) & 0 \\
0 & \alpha D^2 \beta(t_n^\varepsilon, y_n^\varepsilon) + \varepsilon D^2 f(y_n^\varepsilon)
\end{pmatrix},
\]

and \( I_d \) stands for the \( d \times d \) identity matrix.
3.2.a. Assume that, after possibly passing to a subsequence, either

\[ N(x_n^ε, \hat{U}(t_n^ε, x_n^ε), p_n^ε) = \emptyset \]

or

\[ M \left( x_n^ε, \hat{U}(t_n^ε, x_n^ε), p_n^ε \right) \leq 0. \]

It then follows from Lemma 3.2 that, in both cases, \( M \left( x_n^ε, \hat{U}(t_n^ε, x_n^ε), p_n^ε \right) \leq 0 \). Since the supersolution property of \( \hat{V} \) ensures that \( M \left( y_n^ε, \hat{V}(t_n^ε, y_n^ε), q_n^ε \right) \geq 0 \), arguing as in Step 2.2. above leads to a contradiction. Similarly, we cannot have \( U(t_n^ε, x_n^ε) \leq g(t_n^ε, y_n^ε) \) along a subsequence since \( V \geq g \) and \( g \) is continuous, recall (a) and (b).

We can then assume that \( N(x_n^ε, \hat{U}(t_n^ε, x_n^ε), p_n^ε) \neq \emptyset \), \( M \left( x_n^ε, \hat{U}(t_n^ε, x_n^ε), p_n^ε \right) > 0 \) and \( U(t_n^ε, x_n^ε) > g(t_n^ε, x_n^ε) \) for all \( n \geq 1 \), after possibly passing to a subsequence. It then follows from the super- and subsolution properties of \( \hat{V} \) and \( \hat{U} \), see Step 1., the fact that \( K \) is compact and Lemma 3.2 that there exists \( \pi(x_n^ε), \pi(y_n^ε) \in K \) such that

\[ \pi(x_n^ε) \in N(x_n^ε, \hat{U}(t_n^ε, x_n^ε), p_n^ε) \quad \text{and} \quad \pi(y_n^ε) \in N(y_n^ε, \hat{V}(t_n^ε, y_n^ε), q_n^ε) \quad (3.31) \]

and, recall (5.6),

\[
\kappa \left( \hat{U}(t_n^ε, x_n^ε) - \hat{V}(t_n^ε, y_n^ε) \right) \leq -\hat{U}(t_n^ε, x_n^ε)\pi(x_n^ε)^T \mu(x_n^ε) + \hat{V}(t_n^ε, y_n^ε)\pi(y_n^ε)^T \mu(y_n^ε) \\
+ b^1_n + b^2_n \\
+ \tilde{\mu}(x_n^ε)^T p_n^ε - \tilde{\mu}(y_n^ε)^T q_n^ε + \frac{1}{2} \text{Tr} \left[ a(x_n^ε)X_n^ε - a(y_n^ε)Y_n^ε \right] \\
\leq -\hat{U}(t_n^ε, x_n^ε)\pi(x_n^ε)^T \mu(x_n^ε) + \hat{V}(t_n^ε, y_n^ε)\pi(y_n^ε)^T \mu(y_n^ε) \\
+ 2\tilde{\zeta}(t_n^ε - x_n^ε) - \alpha \tau (\beta(t_n^ε, x_n^ε) + \beta(t_n^ε, y_n^ε)) + 2\tilde{\mu}(x_n^ε)^T p_n^ε - \tilde{\mu}(y_n^ε)^T q_n^ε \\
+ \frac{1}{2} \text{Tr} \left[ \Xi(t_n^ε, x_n^ε, y_n^ε) (A_n^ε + B_n^ε + \eta(A_n^ε + B_n^ε)^2) \right] \\
\]

where \( \tilde{\sigma}(z) := \text{diag}[z] \sigma(z), \tilde{\mu}(z) = \text{diag}[z] \mu(z) \) and the positive semi-definite matrix \( \Xi(t_n^ε, x_n^ε, y_n^ε) \) is defined by

\[
\Xi(x_n^ε, y_n^ε) := \begin{pmatrix}
\tilde{\sigma}(x_n^ε)^T & \tilde{\sigma}(y_n^ε)^T \\
\tilde{\sigma}(x_n^ε)^T & \tilde{\sigma}(y_n^ε)^T
\end{pmatrix}.
\]

3.2.b. We now assume that (v.a) of Assumption 2.4 holds. Then, for \( n \) large enough, we can choose \( \pi(x_n^ε) \) such that

\[
\left| \hat{V}(t_n^ε, y_n^ε)\pi(y_n^ε)^T \mu(y_n^ε) - \hat{U}(t_n^ε, x_n^ε)\pi(x_n^ε)^T \mu(x_n^ε) \right| \\
\leq L \left[ (q_n^ε)^T \text{diag}[y_n^ε] \mu(y_n^ε) - (p_n^ε)^T \text{diag}[x_n^ε] \mu(x_n^ε) \right].
\]
It then follows from (3.31) and (3.32) that
\[
\kappa \left( \tilde{U}(t^\varepsilon_n, x^\varepsilon_n) - \tilde{V}(t^\varepsilon_n, y^\varepsilon_n) \right) \leq L |(q^\varepsilon_n)^\top \text{diag} \left[ y^\varepsilon_n \right] \mu(y^\varepsilon_n) - (p^\varepsilon_n)^\top \text{diag} \left[ x^\varepsilon_n \right] \mu(x^\varepsilon_n) | \\
+ 2\zeta (t^\varepsilon_n - x^\varepsilon_n) - \alpha \tau \left( \beta(t^\varepsilon_n, x^\varepsilon_n) + \beta(t^\varepsilon_n, y^\varepsilon_n) \right) \\
+ \bar{\mu}(x^\varepsilon_n)^\top p^\varepsilon_n - \bar{\mu}(y^\varepsilon_n)^\top q^\varepsilon_n \\
+ \frac{1}{2} \text{Tr} \left[ \Xi(x^\varepsilon_n, y^\varepsilon_n) \left( A_n^\varepsilon + B_n^\varepsilon + \eta(A_n^\varepsilon + B_n^\varepsilon)^2 \right) \right].
\]

Using (a)-(b), (3.21) and (3.26), we deduce that we can find \( C > 0 \) independent of \( (\eta, \zeta) \) such that for \( \varepsilon \) small and \( n \) large enough
\[
\kappa \frac{m}{2} \leq \kappa \left( \tilde{U}(t^\varepsilon_n, x^\varepsilon_n) - \tilde{V}(t^\varepsilon_n, y^\varepsilon_n) - (\alpha\beta + \varepsilon f)(t^\varepsilon_n, x^\varepsilon_n) - (\alpha\beta + \varepsilon f)(t^\varepsilon_n, y^\varepsilon_n) \right) \\
\leq C(1 + \zeta)\theta(\varepsilon, n) + \frac{1}{2} \text{Tr} \left[ \Xi(x^\varepsilon_n, y^\varepsilon_n) \left( A_n^\varepsilon + \eta(A_n^\varepsilon + B_n^\varepsilon)^2 \right) \right]
\]
where \( \theta(\varepsilon, n) \) is independent of \( (\eta, \zeta) \) and satisfies
\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} |\theta(\varepsilon, n)| = 0.
\]
Sending \( \eta \to 0 \) in the previous inequality provides
\[
\kappa \frac{m}{2} \leq C(1 + \zeta)\theta(\varepsilon, n) + \frac{1}{2} \text{Tr} \left[ \Xi(x^\varepsilon_n, y^\varepsilon_n)A_n^\varepsilon \right],
\]
so that
\[
\kappa \frac{m}{2} \leq C(1 + \zeta)\theta(\varepsilon, n) + \zeta \text{Trace} \left[ \bar{\sigma}(x^\varepsilon_n)\bar{\sigma}'(x^\varepsilon_n) \right] \\
+ n^2 |\text{diag} \left[ x^\varepsilon_n (\zeta/n) \right] \sigma(x^\varepsilon_n) - \text{diag} \left[ y^\varepsilon_n \right] \sigma(y^\varepsilon_n) |^2.
\]
Finally, using (a) and the Lipschitz continuity of the coefficients, we obtain by sending \( n \) to \( \infty \) and then \( \zeta \) to 0 in the last inequality that \( km \leq 0 \), which is the required contradiction and concludes the proof.

3.2.c. In the case where (v.b) of Assumption 2.4 holds. Then, (3.31) and (3.32) imply that
\[
\kappa \left( \tilde{U}(t^\varepsilon_n, x^\varepsilon_n) - \tilde{V}(t^\varepsilon_n, y^\varepsilon_n) \right) \leq 2\zeta (t^\varepsilon_n - x^\varepsilon_n) - \alpha \tau \left( \beta(t^\varepsilon_n, x^\varepsilon_n) + \beta(t^\varepsilon_n, y^\varepsilon_n) \right) \\
+ \frac{1}{2} \text{Tr} \left[ \Xi(t^\varepsilon_n, x^\varepsilon_n, y^\varepsilon_n) \left( A_n^\varepsilon + B_n^\varepsilon + \eta(A_n^\varepsilon + B_n^\varepsilon)^2 \right) \right],
\]
and the proof is concluded as in 3.2.b. above by using the fact that the right hand-side in the min in of (3.21) is non-negative (instead of the left hand-side as above).

3.4 Proof of the “face-lifted” representation

In this Section, we prove Corollary 2.2 and Corollary 4.2. We start with some preliminary results.
Lemma 3.9 Let (i)-(iv) of Assumption 2.4 hold. Fix $t_0 \in [0,T]$ and let $V$ (resp. $U$) be a lower-semicontinuous (resp. upper-semicontinuous) viscosity supersolution (resp. subsolution) on $\bar{O}$ of

$$\min \{ M_d \phi, \phi - g(t_0, \cdot) \} \geq 0 \quad (3.34)$$

(resp. $\min \{ M \phi, \phi - g(t_0, \cdot) \} \leq 0$).

Assume that $U$ and $V$ are non-negative and satisfy the growth condition (2.16). Then, $U \leq V$ on $\bar{O}$.

Proof. This follows from the same line of arguments as in Steps 1., 2. and 3.1. of the proof of Proposition 5.1.

Lemma 3.10 Let (i)-(iv) of Assumption 2.4 hold. Then,

(i) There exists $C > 0$ such that

$$|\hat{g}(t,x)| \leq C(1 + x^\gamma) \text{ for all } (t,x) \in [0,T] \times \bar{O}. \quad (3.36)$$

(ii) Assume further that the assumptions of Corollary 2.2 hold. Then, for each $t_0 \in [0,T]$, $\hat{g}(t_0, \cdot)$ is continuous on $\bar{O}$ and is the unique viscosity solution of (3.34)-(3.35) satisfying the growth condition (3.36).

Proof. 1. Fix $\rho \in L_1^0(\text{Leb})$ and $\tau \geq 0$. It follows from Assumption 2.4 that we can find $C > 0$ such that

$$e^{-\int_0^\tau \delta \chi^\tau_{(\cdot)}(\rho(s))ds} g(t_0, \chi^\tau_{(\cdot)}(\tau)) \leq C \left( 1 + \prod_{i=1}^d (x^i)^{\gamma_i} e^{\int_0^\tau \gamma_i \rho_i ds} \right) e^{-\int_0^\tau \delta \chi^\tau_{(\cdot)}(\rho(s))ds} \leq C (1 + x^\gamma)$$

where we used the fact $\delta \chi^\tau_{(\cdot)} \geq 0$ since $0 \in K_z$ for all $z \in \bar{O}$, and the fact that $\gamma^T \rho - \delta \chi^\tau_{(\cdot)}(\rho) \leq 0$ since $\gamma \in K \subset K_{\chi^\tau_{(\cdot)}}$.

2. We now prove that $\hat{g}(t_0, \cdot)$ is a (discontinuous) viscosity solution of (3.34)-(3.35). Let $\hat{g}_*(t_0, \cdot)$ (resp. $\hat{g}^*(t_0, \cdot)$) be the lower-semicontinuous (resp. upper-semicontinuous) envelope of $x \mapsto \hat{g}(t_0, x)$.

Note that $\hat{g}(t_0, \cdot)$ satisfies the dynamic programming principle

$$\hat{g}(t_0, x) = \sup_{(\rho, \tau) \in S(x, \bar{O})} e^{-\int_0^\tau h \delta \chi^\tau_{(\cdot)}(\rho(s))ds} (g(t_0, \chi^\tau_{(\cdot)}(\tau)) \mathbf{1}_{\tau \leq h} + \hat{g}(t_0, \chi^\tau_{(\cdot)}(h)) \mathbf{1}_{\tau > h}), \ h > 0. \quad (3.37)$$

2.1. We start with the supersolution property. Note that $\hat{g} \geq g$ by construction (take $\rho = 0$ and $\tau = 0$ in the definition of $\hat{g}$ or in (3.37)). Let $\phi$ be a non-negative smooth function and let $x_0 \in \bar{O}$ be a strict minimum point of $\hat{g}_*(t_0, \cdot) - \phi$ such that $\hat{g}_*(t_0, x_0) - \phi(x_0) = 0$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\bar{O}$ such that $\hat{g}(t_0, x_n) \to \hat{g}_*(t_0, x_0)$.

Assume that

$$\min \{ \min_{\rho_0 \in S_0(x_0, \bar{O})} (\delta \chi_{(\cdot)}(\rho_0)) \phi(x_0) - \rho_0 \mathbf{1} \text{diag} [x_0] D \phi(x_0), \phi(x_0) - g(t_0, x_0) \} < 0. \quad (3.38)$$
3. PROOF OF THE PDE CHARACTERIZATION

We shall prove the required result in following cases:

**Case 1.** $x_0 \in \mathcal{O}$:

Note that in this case $S_0(x_0, \tilde{\mathcal{O}}) = \tilde{K}_{x_0}$. Then there exists $\rho_0 \in \tilde{K}_{x_0}$ so that

$$\delta_{x_0}(\rho_0)\phi(x_0) - \rho_0^\top \text{diag} [x_0] D\phi(x_0), \phi(x_0) - g(t_0, x_0) < 0.$$ 

It then follows from (iv) of Assumption 2.4 that we can find $r, m > 0$ and a Lipschitz continuous map $\hat{\rho}$ such that, for some $\eta > 0$,

$$\delta_x(\hat{\rho}(x)) \leq \eta$$

and

$$\min \{\delta_x(\hat{\rho}(x))\phi(x) - \hat{\rho}(x)^\top \text{diag} [x] D\phi(x), \phi(x) - g(t_0, x)\} < -m, \quad (3.39)$$

for all $x \in B_r(x_0) \subset \mathcal{O}$.

This implies that $\chi_n$ defined as $\chi_n = \hat{\rho}(\chi_n)$ for the Markovian control $\rho = \hat{\rho}(\chi_n)$ satisfies

$$\phi(x_n) < e^{-\int_0^{h_n} \delta_{\chi_n(s)}(\hat{\rho}(\chi_n(s)))ds} (\phi(\chi_n(h_n))) - mh_n$$

where

$$h_n := \inf \{s \geq 0 : \hat{\chi}_n(s) \notin B_r(x_0)\} \wedge h$$

for some $h > 0$. Since $x_0$ is a strict minimum point of $\tilde{g}(t_0, \cdot) - \phi$, we can then find $\zeta > 0$ such that

$$\phi(x_n) < e^{-\int_0^{h_n} \delta_{\chi_n(s)}(\hat{\rho}(\chi_n(s)))ds} (-\zeta 1_{h_n < h} + \tilde{g}(t_0, \hat{\chi}_n(h_n)) - mh_n) \ .$$

Using the left-hand side of (3.39), we then obtain

$$\phi(x_n) < e^{-\int_0^{h_n} \delta_{\chi_n(s)}(\hat{\rho}(\chi_n(s)))ds} \tilde{g}(t_0, \hat{\chi}_n(h_n)) - e^{-h\eta} (\zeta 1_{h_n < h} + mh 1_{h_n = h}) \ .$$

Since $\tilde{g}(t_0, x_n) - \phi(x_n) \to 0$, this leads to a contradiction to (3.37) for $n$ large enough.

**Case 2.** $x_0 \in \partial \mathcal{O}$:

In this case, we fix $(\rho, \tau) \in \mathcal{S}(x_0, \tilde{\mathcal{O}})$ such that $\rho(0) = \rho_0$ and $\tau > 0$.

It follows from (ii) of Assumption 2.2 that there exists a sequence $(\rho_n, \tau_n) \to (\rho, \tau)$ such that, up to a subsequence,

$$(\rho_n, \tau_n) \in \mathcal{S}(x_n, \tilde{\mathcal{O}}) \text{ for } n \geq 1.$$ 

This implies that

$$\tau_n > 0 \text{ for } n \text{ large enough.} \quad (3.40)$$

Recalling (3.38) that we can find $r, m, \eta > 0$

$$\delta_{\chi_n(s)}(\rho_n(s)) < \eta$$
and
\[-m > \min \left\{ \delta_{\chi_n(s)}(\rho_n(s))\phi(\chi_n(s)) - \rho_n(s)^\top \text{diag} [\chi_n(s)] D\phi(\chi_n(s)), \right. \]
\[\left. \phi(\chi_n(s)) - g(t_0, \chi_n(s)) \right\}, \]
for \(s \leq h_n\) when \(n\) is large enough, where
\[\chi_n := \chi^p_{x_n} \text{ and } h_n := \inf\{s \geq 0 : \chi_n(s) \notin B_r(x_0)\} \land \tau_n.\]

This implies that, for some \(\zeta > 0\),
\[\phi(x_n)e^{t_n^0 \delta_{\chi_n(s)}(\rho_n(s))ds} < \phi(\chi_n(h_n)) - mh_n \]
\[< [g(t_0, \chi_n(h_n)) - \zeta]1_{h_n < \tau_n} + [g(t_0, \chi_n(h_n)) - m]1_{h_n \geq \tau_n} - mh_n\]
which leads to a contradiction of (3.37) since \(\tau_n > 0\).

2.2. We now turn to the subsolution property. Let \(\phi\) be a non-negative smooth function and let \(x_0 \in \bar{\mathcal{O}}\) be a strict maximum point of \(\tilde{g}^*(t_0, \cdot) - \phi\) such that \(\tilde{g}^*(t_0, x_0) - \phi(x_0) = 0\). Let \((x_n)_n\) be a sequence in \(\mathcal{O}\) such that \(\tilde{g}(t_0, x_n) \rightarrow \tilde{g}^*(t_0, x_0)\). Assume that
\[\min \left\{ \inf_{\rho \in K_{x_0}} (\delta_{x_0}(\rho)\phi(x_0) - \rho_0^\top \text{diag} [x_0] D\phi(x_0)), \phi(x_0) - g(t_0, x_0) \right\} > 0. \quad (3.41)\]

Since \(g \geq 0\), this implies that \(\phi > 0\) on \(B_r(x_0)\), for some \(r > 0\). Moreover, the fact that \(K_{x_0}\) is compact implies that, after possibly changing \(r > 0\), we can find \(\varepsilon > 0\) such that \(\phi(x) - \varepsilon > 0\) on \(B_r(x_0)\) and \(M(x_0, \phi(x_0) - \varepsilon, D\phi(x_0)) > 0\). In view of Lemma 3.2, this implies that \((\phi(x_0) - \varepsilon)^{-1} \text{diag} [x_0] D\phi(x_0) \in \text{int}(K_{x_0})\). It follows from Assumption 2.1 that, after possibly changing \(r > 0\), \(N(x, \phi(x) - \varepsilon, D\phi(x)) \neq \emptyset\) on \(B_r(x_0)\), which, by Lemma 3.2 again, implies that \(M(x, \phi(x) - \varepsilon, D\phi(x)) \geq 0\) on \(B_r(x_0)\). Since \(\delta_x \geq c_K > 0\), recall (iii) of Assumption 2.4, we deduce that \(M(x, \phi(x), D\phi(x)) \geq \varepsilon c_K\) on \(B_r(x_0)\). Using Lemma 3.2, the continuity of \(g\) and (3.41), we finally obtain
\[\min \left\{ \inf_{\rho \in K_x} (\delta_x(\rho_0)\phi(x) - \rho_0^\top \text{diag} [x] D\phi(x)), \phi(x) - g(t_0, x) \right\} \geq m \text{ on } B_r(x_0) \quad (3.42)\]
for some \(m > 0\). Moreover, it follows from Assumption (ii) of Corollary 2.2 that
\[\sup \{\delta_x(\rho_0), \rho \in \bar{K}_x\} \leq \eta \text{ on } B_r(x_0) \quad (3.43)\]
for some \(\eta > 0\). We now consider a sequence \((x_n)_n\) in \(\bar{\mathcal{O}}\) such that \(\tilde{g}(t_0, x_n) \rightarrow \tilde{g}^*(t_0, x_0)\). Given \((\rho, \tau) \in \mathcal{S}(x_0, \bar{\mathcal{O}})\), we set \(h_n := \inf\{s \geq 0 : \chi^p_{x_n}(s) \notin B_r(x_0)\}\)
4. THE EXTENSION TO THE AMERICAN OPTION WITHOUT A BARRIER

for some \( h > 0 \). Then, (3.42) and the fact that \( x_0 \) is a strict maximum point of \( \tilde{g}^*(t_0, \cdot) - \phi \) implies that we can find \( \zeta > 0 \) such that, for all \( \tau \geq 0 \),

\[
\begin{align*}
\phi(x_n) & \geq e^{-\int_{0}^{h_n \land \tau} \tilde{g}(\tau) ds} \left( \phi(x_n^\rho(h_n \land \tau)) + m(h_n \land \tau) \right) \\
& \geq e^{-\int_{0}^{h_n \land \tau} \tilde{g}(\tau) ds} \left( \tilde{g}(t_0, x_n^\rho(h_n)) \right) 1_{h_n > \tau} + g(t_0, x_n^\rho(\tau)) 1_{\tau \leq h_n} + \zeta \land m \land (mh) .
\end{align*}
\]

Since \( \phi(x_n) - \tilde{g}(t_0, x_n) \to 0 \), the above inequality combined with (3.43) leads to a contradiction to (3.37) for \( n \) large enough, by arbitrariness of \( \tau \) and \( \rho \). \( \Box \)

We can now conclude the proof of Corollary 2.2.

**Proof of Corollary 2.2** The subsolution property follows from Theorem 2.2 since \( \tilde{g} \geq g \). As for the supersolution property, we note that Theorem 2.2 implies that, for fixed \( t_0 \in [0, T] \), \( v_*(t_0, \cdot) \) is a supersolution of \( \min \{ \varphi - g(t_0, \cdot), M_d \varphi \} \geq 0 \). It thus follows from Lemma 3.9 that \( v_* \geq \tilde{g} \). The supersolution property then follows.

To conclude, we note that the comparison result of Proposition 5.1 obviously still holds if we replace \( g \) by \( \tilde{g} \) since \( \tilde{g} \) is continuous with respect to its first variable by assumption, and with respect to its second one by Lemma 3.10. \( \Box \)

**Remark 3.2** Note that the supersolution property \( v_* \geq \tilde{g}(T, \cdot) \) could be proved as a direct consequence of Lemma 3.4. However, above proof is provide to focus on studying \( \tilde{g}(t_0, \cdot) \) is continuous on \( \tilde{O} \) and is the unique viscosity solution of (3.34)-(3.35) and without required the viscosity solution of \( v \).

4 The extension to the American option without a barrier

In this section, we explain how the obstacle version of geometric dynamic programming principle in Theorem 2.1 of Chapter 1 can be used to relate the super-hedging price of an American option under constraints without a barrier, i.e. \( \mathcal{O} = \mathbb{R}^d \), which has been further studied by [15]. The super-hedging price is thus defined as

\[
\bar{v}(t, x) := \inf \left\{ y \in \mathbb{R}_+ : \exists \pi \in \mathcal{A} \text{ s.t. } Y_{t,x,y}^\pi(\tau) \geq g(\tau, X_{t,x}(\tau)) \quad \forall \tau \in \mathcal{T}_{t,T} \right\} . \quad (4.1)
\]

Observe that \( \bar{v}(t, x) \) coincides with the lower bound of the set \( \{ y \in \mathbb{R}_+ : (x, y) \in \tilde{V}(t) \} \) where

\[
\tilde{V}(t) := \{ (x, y) \in (0, \infty)^d \times \mathbb{R}_+ : \exists \pi \in \mathcal{A} \text{ s.t. } Z_{t,x,y}^\pi(\tau) \in G(\tau) \quad \forall \tau \in \mathcal{T}_{\tau,\tau} \} ,
\]

It follows from the obstacle geometric dynamic programming principle of Theorem 2.1 that
Corollary 4.1 Fix \((t,x,y) \in [0,T] \times (0,\infty)^d \times \mathbb{R}_+\).

\((DP1)\) If \(y > \bar{v}(t,x)\), then there exists \(\pi \in \mathcal{A}\) such that

\[ Y_{t,x,y}^\pi(\theta) \geq \bar{v}(\theta, X_{t,x}(\theta)) \text{ for all } \theta \in \mathcal{T}_{[t,T]} \] .

\((DP2)\) If \(y < \bar{v}(t,x)\), then \(\exists \tau \in \mathcal{T}_{[t,T]}\) such that

\[ P\left[ Y_{t,x,y}^\pi(\theta \land \tau) > \bar{v}(\theta, X_{t,x,y}(\theta))1_{\theta<\tau} + g(\tau, X_{t,x,y}(\tau))1_{\theta \geq \tau} \right] < 1 \]

for all \(\theta \in \mathcal{T}_{[t,T]}\) and \(\pi \in \mathcal{A}\).

We observe that \(S_0(x,\mathbb{R}^d) = \tilde{K}_x\) and therefore \(\mathcal{M}_d v(t,x)\) coincides to \(\mathcal{M} v(t,x)\) for all \((t,x) \in [0,T] \times \mathbb{R}^d\), recall Remark 2.6. This allows us to provide a PDE characterization of \(\bar{v}\) without worrying about the operator \(\mathcal{M}_d\).

Theorem 4.1 Let the conditions of Theorem 2.3 hold. Then, \(\bar{v}\) is a discontinuous viscosity solution of

\[
\begin{align*}
\min\{ & H\varphi, \mathcal{M}\varphi \} = 0 \quad \text{on } [0,T) \times (0,\infty)^d; \\
\min\{ & \varphi(T,x) - g(T,x), \mathcal{M}\varphi(T,x) \} = 0 \quad \text{on } (0,\infty)^d \\
& \text{and satisfies the boundary condition} \\
& \bar{v}_t(T,x) = \bar{v}^*(T,x) = \hat{g}(T,x) \text{ on } (0,\infty)^d. 
\end{align*}
\] (4.2)

Moreover, it is the unique viscosity solution of (4.2)-(4.3) in the class of non-negative functions satisfying

\[ |\bar{v}(t,x)| \leq C (1 + x^\gamma) \quad \forall (t,x) \in [0,T] \times (0,\infty)^d. \] (4.3)

In the case where \(\sigma\) is invertible, the above discussion already shows that \(\hat{g} = \tilde{g}\). If moreover \(\mu\) and \(\sigma\) are constant, we can actually interpret the super-hedging price as the price of an American option with payoff \(\hat{g}\), without taking the portfolio constraints into account. This phenomenon was already observed for plain vanilla or barrier european options, see e.g. [9] and [13]. It comes from the fact that, when the parameters are constant, the gradient constraint imposed at \(T\) by the terminal condition \(v(T,\cdot) = \hat{g}\) propagates into the domain. It is therefore automatically satisfied and we retrieve the result of Corollary 1 in [13].

Corollary 4.2 Let the conditions of Theorem 2.3 hold. Assume further that \(\mu\) and \(\sigma\) are constant and that \(\sigma\) is invertible. Then, \(\hat{g} = \tilde{g}\) and

\[
v(t,x) = \sup_{\tau \in \mathcal{T}_{t,x}} \mathbb{E}_Q[\hat{g}(\tau, X_{t,x}(\tau))] \] (4.4)

where \(Q \sim P\) is defined by

\[
\frac{dQ}{dP} := e^{-\frac{1}{2}\sigma^{-1}\mu^2T + (\sigma^{-1}\mu)^TW_T} .
\]
Remark 4.1 Since \( \sigma \) is constant and invertible, the condition that \( \mu \) is constant could be relaxed, under mild assumptions, by performing a suitable initial change of measure.

Proof of Corollary 4.2. The fact that \( \hat{g} = \hat{g} \) follows from the discussion at the beginning of Section 2.4. Note that the continuity of \( g \) implies that \( \hat{g} \) is continuous too. Also observe that the fact that \( \sigma \) is invertible implies that, for a smooth function \( \varphi \),

\[
\sup_{\pi \in \mathcal{N}(\tau, t)} (\pi^T \mu(t, x) - \mathcal{L}\varphi(t, x)) = -\partial_t \varphi - \frac{1}{2} \text{Trace} \left[ \text{diag} \left( \sigma \right) \text{diag} \left( \sigma \right)^T \text{diag} \left( \sigma \right)^2 \varphi \right] \tag{4.5}
\]

Let us now observe that the map \( w \) defined on \([0, T] \times (0, \infty)^d \) by

\[
w(t, x) := \sup_{\tau \in [t, T]} \mathbb{E}^Q \left[ \hat{g}(\tau, X_{t,x}(\tau)) \right]
\]

is a viscosity solution on \([0, T] \times (0, \infty)^d \) of

\[
\min \left\{ -\partial_t \varphi - \frac{1}{2} \text{Trace} \left[ \text{diag} \left( \sigma \right) \text{diag} \left( \sigma \right)^T \text{diag} \left( \sigma \right)^2 \varphi \right] , \varphi - \hat{g} \right\} = 0. \tag{4.6}
\]

In particular, it is a subsolution of (4.2), recall (4.5). We next deduce from the definition of \( \hat{g} \) that, for all \( \rho \in \mathbb{R}^d \),

\[
w(t, xe^\rho) = \sup_{\tau \in [t, T]} \mathbb{E}^Q \left[ \hat{g}(\tau, X_{t,x}(\tau)) e^\rho \right] \leq \sup_{\tau \in [t, T]} \mathbb{E}^Q \left[ e^{\delta(\rho)} \hat{g}(\tau, X_{t,x}(\tau)) \right] = e^{\delta(\rho)} w(t, x).
\]

It follows that \( w(t, x) \geq e^{-\delta(\rho)} w(t, xe^\rho) \) for all \( \rho \in \mathbb{R}^d \) which implies that \( w \) is a viscosity supersolution of \( \mathcal{M} \varphi = 0 \). Hence, \( w \) is a supersolution of (4.2), recall (4.5). Finally, (3.36) and standard estimates show that \( w \) satisfy the growth condition (2.16). It thus follows from Corollary 2.2 that \( \tilde{v}_s = \tilde{v}^* = \bar{w} \). \( \square \)
CHAPITRE 2. APPLICATION TO AMERICAN OPTION PRICING UNDER CONSTRAINTS
Deuxième partie

The stochastic target problems with multiple constraints in probability
1 Introduction

Option pricing (in incomplete financial markets or markets with frictions) and optimal management decisions have to be based on some risk criterion or, more generally, on some choice of preferences. In the academic literature, one usually models the attitude of the financial agents toward risk in terms of an utility or loss function. However, practitioners have in general no idea of “their utility function”. Even the choice of a loss function is somehow problematic. On the other hand, they have a rough idea on the type of P&L they can afford, and indeed have as a target. This is the case for traders, for hedge-fund managers,...

The aim of this chapter is to provide a direct PDE characterization of the minimal initial endowment required so that the terminal wealth of a financial agent (possibly diminished by the payoff of a random claim) can match a set of constraints in probability. In practice, this set of constraints has to be viewed as a rough description of a targeted P&L distribution.

To be more precise, let us consider the problem of a trader who would like to hedge a European claim of the form $g(X_{t,x}(T))$, where $X_{t,x}$ models the evolution of some risky assets, assuming that their value is $x$ at time $t$. The aim of the trader is to find an initial endowment $y$ and a hedging strategy $\nu$ such that the terminal value of his hedging portfolio $Y_{t,x,y}(T)$ diminished by the liquidation value of the claim $g(X_{t,x}(T))$ matches an a-priori distribution of the form

$$\mathbb{P} \left[ Y_{t,x,y}(T) - g(X_{t,x}(T)) \geq -\gamma^i \right] \geq p^i , \ i \leq \kappa ,$$

where $\gamma^\kappa \geq \cdots \geq \gamma^2 \geq \gamma^1 \geq 0$, for some $\kappa \geq 1$. The minimal initial endowment
required to achieve the above constraints is given by:

\[ v(t, x, p) = \inf \{ y : \exists \nu \text{ s.t. } Y_{t,x,y}^\nu(T) \geq \ell, \mathbb{P} [Y_{t,x,y}^\nu(T) - g(X_{t,x}(T)) \geq -\gamma_i] \geq p^i \ \forall \ i \leq \kappa \}, \]

where we used the notation \( p := (p^1, \ldots, p^\kappa) \) and \( \ell \in \mathbb{R} \) is a given lower bound that is imposed in order to avoid that the wealth goes too negative, even if it is with small probability.

In the case \( \kappa = 1 \), such a problem is referred to as the “quantile hedging problem”. It has been widely studied by Föllmer and Leukert [23] who provided an explicit description of the optimal terminal wealth \( Y_{t,x,y}^\nu(T) \) in the case where the underlying financial market is complete. This result is derived from a clever use of the Neyman-Pearson Lemma in mathematical statistics and applies to non-Markovian frameworks. A direct approach, based on the notion of stochastic target problems, has then been proposed by Bouchard, Elie and Touzi [12]. It allows to provide a PDE characterization of the pricing function \( v \), even in incomplete markets or in cases where the stock price process \( X_{t,x} \) can be influenced by the trading strategy \( \nu \), see e.g. [10]. The problem (0.8) is a generalization of this work to the case of multiple constraints in probability.

As in Bouchard, Elie and Touzi [12], the first step consists in rewriting the stochastic target problem with multiple constraints in probability (0.8) as a stochastic target problem in the \( \mathbb{P} \)-a.s. sense. This is achieved by introducing a suitable family of \( d \)-dimensional bounded martingales \( \{P_{t,p}^\alpha, \alpha\} \) and by re-writing \( v \) as

\[ v(t, x, p) = \inf \{ y : \exists (\nu, \alpha) \text{ s.t. } Y_{t,x,y}^\nu(T) \geq \ell, \min_{i \leq \kappa} (\Delta^i(X_{t,x}(T), Y_{t,x,y}^\nu(T)) - P_{t,p}^{\alpha,i}(T)) \geq 0 \}, \]

where \( \Delta^i(x, y) := 1_{\{y - g(x) \geq -\gamma_i\}} \) and \( P_{t,p}^{\alpha,i} \) denotes the \( i \)-th component of \( P_{t,p}^\alpha \). As in [12], “at the optimum” each process \( P_{t,p}^{\alpha,i} \) has to be interpreted as the martingale coming from the martingale representation of \( 1_{\{Y_{t,x,y}^\nu(T) - g(X_{t,x}(T)) \geq -\gamma_i\}} \). The above reduction allows to appeal to the Geometric dynamic programming principle (GDPP) of Soner and Touzi [46], which leads to the PDE characterization stated in Theorem 2.1 below, with suitable boundary conditions.

In this chapter, we focus on the case where the market is complete but the amount of money that can be invested in the risky assets is bounded. The incomplete market case could be discussed by following the lines of the paper, but will add extra complexity. Since the proofs below are already complex, we decided to restrict to the complete market case. The fact that the amount of money that can be invested in the risky assets is bounded could also be relaxed. It does not really simplifies the arguments. On the other hand, it is well-known that quantile hedging type strategies can lead to the explosion of the number of risky asset to hold in the portfolio near
the maturity. This is due to the fact that it typically leads to hedging discontinuous payoffs, see the example of a call option in the Black-and-Scholes model in [23]. In our multiple constraint case, we expect to obtain a similar behavior. The constraint on the portfolio is therefore imposed to avoid this explosion, which leads to strategies that can not be implemented in practice.

2 PDE characterization of the P&L matching problem

2.1 Problem formulation

Let \( W \) be a standard \( d \)-dimensional Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with \( d \geq 1 \). We denote by \( \mathcal{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T} \) the \( \mathbb{P} \)-complete filtration generated by \( W \) on some time interval \([0, T]\) with \( T > 0 \). Given \((t, x) \in [0, T] \times (0, \infty)^d\), the stock price process \( X_{t,x} \), starting from \( x \) at time \( t \), is assumed to be the unique strong solution of

\[
X(s) = x + \int_t^s \text{diag}[X(r)] \mu(X(r)) dr + \int_t^s \text{diag}[X(r)] \sigma(X(r)) dW_r, \quad (2.1)
\]

where

\[
x \in (0, \infty)^d \mapsto \text{diag}[x] (\mu(x), \sigma(x)) =: (\mu_X(x), \sigma_X(x)) \in \mathbb{R}^d \times \mathbb{M}^d
\]

is Lipschitz continuous and \( \sigma \) is invertible. All over this paper, we shall assume that there exists some \( L > 0 \) such that

\[
|\mu| + |\sigma| + |\sigma^{-1}| \leq L \quad \text{on} \quad (0, \infty)^d. \quad (2.2)
\]

A financial strategy is described by an element \( \nu \) of the set \( \mathcal{U} \) of progressively measurable processes taking values in some fixed subset \( U \subset \mathbb{R}^d \) which are independent on \( \mathcal{F}_t \), each component \( \nu_i \) at time \( r \) representing the amount of money invested in the \( i \)-th risky asset \( r \). Importantly, we shall assume all over this paper that

\[
U \quad \text{is convex closed, its interior contains 0 and} \quad \sup\{|u|, \, u \in U\} \leq L. \quad (2.3)
\]

This (important) assumption will be commented in Remarks 2.1 below. In the above, we label by \( L \) the different bounds because this constant will be used hereafter.

For sake of simplicity, we assume that the risk free interest rate is equal to zero. The associated wealth process, starting with the value \( y \) at time \( t \), is thus given by

\[
Y(s) = y + \int_t^s \nu^T_r \text{diag}[X_{t,x}(r)]^{-1} dX_{t,x}(r)
\]

\[
= y + \int_t^s \mu_Y(X_{t,x}(r), \nu_r) dr + \int_t^s \sigma_Y(X_{t,x}(r), \nu_r) dW_r, \quad (2.4)
\]
where
\[ \mu_Y(x, u) := u^\top \mu(x) \text{ and } \sigma_Y(x, u) := u^\top \sigma(x), \quad (x, u) \in \mathbb{R}_+^d \times U. \]

The aim of the trader is to hedge an European option of payoff \( g(X_{t,x}(T)) \) at time \( T \), where
\[ g : (0, \infty)^d \mapsto \mathbb{R} \text{ is Lipschitz continuous}. \] (2.5)

Here, the price is chosen so that the net wealth \( Y_{t,x,y}(T) - g(X_{t,x}(T)) \) satisfies a P&L constraint. Namely, given a collection of thresholds \( \gamma := (\gamma^i)_{i \leq \kappa} \in \mathbb{R}^\kappa \) and of probabilities \( (p^i)_{i \leq \kappa} \in [0, 1]^\kappa \), for some \( \kappa \geq 1 \), the price of the option is defined as the minimal initial wealth \( y \) such that there exists a strategy \( \nu \in \mathcal{U} \) satisfying
\[ \mathbb{P}[Y_{t,x,y}(T) \geq g^i(X_{t,x}(T))] \geq p^i \text{ for all } i \in \{1, \ldots, \kappa\} =: \mathcal{K}, \] (2.6)

where
\[ g^i := g - \gamma^i, \quad i \in \mathcal{K}. \] (2.7)

Obviously, we can assume without loss of generality that
\[ 0 \leq \gamma^1 \leq \gamma^2 \leq \cdots \leq \gamma^\kappa. \] (2.8)

This means that the net hedging loss should not exceed \(-\gamma^i\) with probability more than \(p^i\). This coincides with a constraint on the distribution of the P&L of the trader, in the sense that it should match the constraints imposed by the discrete histogram associated to \((\gamma, p)\). In order to avoid that the wealth process goes too negative, even with small probability, we further impose that \( Y_{t,x,y}(T) \geq \ell \) for some \( \ell \in \mathbb{R}_- \). The price is then defined, for \((t, x, p) \in [0, T] \times (0, \infty)^d \times [0, 1]^{\kappa}\), as :
\[ v(t, x, p) := \inf\{y \geq \ell : \exists \nu \in \mathcal{U} \text{ s.t } Y_{t,x,y}^\nu(T) \geq \ell, \ \mathbb{P}[Y_{t,x,y}^\nu(T) \geq g^i(X_{t,x}(T))] \geq p^i \ \forall i \in \mathcal{K}\}. \] (2.9)

Note that, after possibly changing \( g \) and \( \gamma \), one can always reduce to the case where
\[ g^1 \geq g^2 \geq \cdots \geq g^\kappa \geq \ell. \] (2.10)

We further assume that \( g \) is bounded from above and that \( g^\kappa > \ell \) uniformly, which, after possibly changing the constant \( L \) can be written as
\[ \ell + L^{-1} \leq g^\kappa \leq g \leq L. \] (2.11)
Remark 2.1 The above criteria extends the notion of quantile hedging discussed in [23] to multiple constraints in probability. In [23], it is shown that the optimal strategy associated to a quantile hedging problem may lead to the hedging of a discontinuous payoff. This is in particular the case in the Black and Scholes model when one wants to hedge a call option, only with a given probability of success. This typical feature is problematic in practice as it leads to a possible explosion of the \textit{delta} near the maturity. This explains why we have deliberately imposed that $U$ is compact, i.e. that the amount of money invested in the stocks is bounded.

Remark 2.2 Since $U$ is bounded, see (2.3), $Y_{t,x,y}^{\nu}$ is a $Q_{t,x}$-martingale for $Q_{t,x} \sim P$ defined by
\[
\frac{dQ_{t,x}}{dP} = E \left( \int_{t}^{T} -\left( \mu \sigma^{-1} \right) (X_{t,x}(s)) dW_s \right),
\]
recall (2.2). The constraint $Y_{t,x,y}^{\nu}(T) \geq \ell$ thus implies that $Y_{t,x,y}^{\nu} \geq \ell$ on $[t,T]$. In particular, the restriction to $y \geq \ell$ is redundant. We write it only for sake of clarity.

2.2 Problem reduction and domain decomposition

As in [12], the first step consists in converting our stochastic target problem under probability constraints into a stochastic target problem in standard form as studied in [47]. This will allow us to appeal to the Geometric Dynamic Programming Principle to provide a PDE characterization of $v$. In our context, such a reduction is obtained by adding a family of $\kappa$-dimensional martingales defined by
\[
P_{t,p}^{\alpha}(s) = p + \int_{s}^{t} \alpha_r dW_r, \quad (t,p,\alpha) \in [0,T] \times [0,1]^\kappa \times \mathcal{A},
\]
where $\mathcal{A}$ is the set of predictable processes $\alpha$ in $L^2([0,T], \mathbb{M}^{\kappa,d})$. Given $(t,p) \in [0,T] \times [0,1]^\kappa$, we further denote by $\mathcal{A}_{t,p}$ the set of elements $\alpha \in \mathcal{A}$ which are independent on $\mathcal{F}_t$ and $P_{t,p}^{\alpha} \in [0,1]^\kappa$ $dt \times d\mathbb{P}$-a.e. on $[t,T]$, and define
\[
G(x,p) := \inf \{ y \geq \ell : \min_{i \in K} \{1_{\{y \geq p_i(x)\}} - p_i\} \geq 0 \}, \quad (x,p) \in (0,\infty)^d \times \mathbb{R}^\kappa.
\]
Note that
\[
G(\cdot, p) = \infty \text{ for } p \notin (-\infty, 1]^\kappa, \text{ and } G(\cdot, p_1) \geq G(\cdot, p_2) \text{ if } p_1^i \lor 0 \geq p_2^i \forall i \in K. \quad (2.12)
\]

Proposition 2.1 For all $(t,x,p) \in [0,T] \times (0,\infty)^d \times [0,1]^\kappa$,\[
v(t,x,p) = \inf \{ y \geq \ell : \exists (\nu, \alpha) \in \mathcal{U} \times \mathcal{A}_{t,p} \text{ s.t. } Y_{t,x,y}^{\nu}(T) \geq G(X_{t,x}(T), P_{t,p}^{\alpha}(T)) \}, \quad (2.13)
\]
\[
= \inf \{ y \geq \ell : \exists (\nu, \alpha) \in \mathcal{U} \times \mathcal{A} \text{ s.t. } Y_{t,x,y}^{\nu}(T) \geq G(X_{t,x}(T), P_{t,p}^{\alpha}(T)) \}. \quad (2.14)
\]
Proof. The proof follows from the same arguments as in [12]. We provide it for completeness. We fix \((t, x, p) \in [0, T] \times (0, \infty)^d \times [0, 1]^\kappa\), set \(v := v(t, x, p)\) for ease of notations, and denote by \(w_1\) and \(w_2\) the right-hand side of (2.13) and (2.14) respectively. The fact that \(w_1 \geq w_2\) is obvious. Conversely, if \(Y_{t,x,y}^\nu(T) \geq G(X_{t,x}(T), P_{t,p}^\alpha(T))\) for some \((\nu, \alpha) \in \mathcal{U} \times \mathcal{A}\), then (2.12) implies that \(P_{t,p}^{\alpha,i}(T) \leq 1\) for all \(i \in \mathcal{K}\). Since \(P_{t,p}^\alpha\) is a martingale, it takes values in \([-\infty, 1]^\kappa\) on \([t, T]\). Moreover, we can find \(\hat{\alpha} \in \mathcal{A}\) such that \(P_{t,p}^{\hat{\alpha},i}(T) = 0\) on \(A_i := \{\min_{[t,T]} P_{t,p}^{\alpha,i} \leq 0\}\) and \(P_{t,p}^{\hat{\alpha},i}(T) = P_{t,p}^{\alpha,i}(T)\) on \(A_i^c\) for \(i \in \mathcal{K}\). It follows from the above discussion and the martingale property of \(P_{t,p}^\alpha\) that it takes values in \([0, 1]^\kappa\) on \([t, T]\), so that \(\hat{\alpha} \in \mathcal{A}_{t,p}\). Since \(P_{t,p}^{\hat{\alpha},i}(T) \leq P_{t,p}^{\alpha,i}(T) \vee 0\), the inequality \(Y_{t,x,y}^\nu(T) \geq G(X_{t,x}(T), P_{t,p}^\alpha(T))\) together with (2.12) imply that \(Y_{t,x,y}^\nu(T) \geq G(X_{t,x}(T), P_{t,p}^{\hat{\alpha},i}(T))\). This shows that \(w_2 \geq w_1\), so that \(w_2 = w_1\). It remains to show that \(v = w_2\). The inequality \(w_2 \geq v\) is an immediate consequence of the martingale property of \(P_{t,p}^\alpha\). On the other hand, for \(y > v\), we can find \(v \in \mathcal{U}\) such that \(\hat{p}^i := \mathbb{P}[Y_{t,x,y}^\nu(T) \geq g^i(X_{t,x}(T))] \geq p^i\) for all \(i \in \mathcal{K}\). Set \(\hat{p} := (\hat{p}^i)_{i \in \mathcal{K}}\). Then, the martingale representation theorem implies that we can find \(\alpha \in \mathcal{A}\) such that \(P_{t,p}^{\alpha,i}(T) = 1_{\{Y_{t,x,y}^\nu(T) \geq g^i(X_{t,x}(T))\}}\) for each \(i \in \mathcal{K}\). We conclude by observing that \(P_{t,p}^{\alpha,i}(T) \geq P_{t,p}^{\hat{\alpha},i}(T)\) for each \(i \in \mathcal{K}\). \(\square\)

Remark 2.3 As in [12], the new controlled process \(P_{t,p}^\alpha\) should be interpreted as the martingale with components given by \((\mathbb{P}[Y_{t,x,y}^\nu(T) \geq g^i(X_{t,x}(T))] | \mathcal{F}_s)\) \(s \in [t, T]\), at least when the controls \(\nu\) and \(\alpha\) are optimal. This is rather transparent in the above proof. The fact that we can restrict to the set of controls \(\mathcal{A}_{t,p}\) is therefore clear since a conditional probability should take values in \([0, 1]\).

Remark 2.4 Note that \(\alpha \in \mathcal{A}_{t,p}\) implies that \(\alpha^i \equiv 0\) for all \(i \in \mathcal{K}\) such that \(p^i \in \{0, 1\}\), since \(P_{t,p}^\alpha\) is a martingale.

The representation (2.14) coincides with a stochastic target problem in standard form but with unbounded controls as studied in [12], unbounded referring to the fact that \(\alpha\) can not be bounded a-priori since it comes from the martingale representation theorem. In particular, a PDE characterization of the value function \(v\) in the parabolic interior of the domain

\[ D := [0, T) \times (0, \infty)^d \times (0, 1)^\kappa \]

follows from the general results of [12]. The main difference comes from the fact that the constraints \(P_{t,p}^{\alpha,i} \in [0, 1]\) introduce boundary conditions that have to be discussed separately. In order to deal with these boundary conditions, we first divide the closure of the domain \(\bar{D}\) into different regions corresponding to its parabolic interior \(D\) and the different boundaries associated to the level of conditional probabilities. Namely, given

\[ \mathcal{P}_\kappa := \{(I, J) \in \mathcal{K}^2 : I \cap J = \emptyset \text{ and } I \cup J \subset \mathcal{K}\}, \]
we set, for \((I, J) \in \mathcal{P}_\kappa\),
\[
D_{IJ} := [0, T) \times (0, \infty)^d \times B_{IJ},
\]
where
\[
B_{IJ} := \{ p \in [0,1]^\kappa : p^i = 0 \text{ for } i \in I, \ p^j = 1 \text{ for } j \in J, \text{ and } 0 < p^l < 1 \text{ for } l \notin I \cup J \}.
\]
Then,
\[
[0, T) \times (0, \infty)^d \times [0, 1]^\kappa = \cup_{(I, J) \in \mathcal{P}_\kappa} D_{IJ}.
\]

The interpretation of the partition is the following. For \((t, x, p) \in D\), any \(p^i\) takes values in \((0, 1)\) so that \(P_{t,p}^{\alpha,i}\) is not constrained locally at time \(t\) by the state constraints which appears in (2.13), namely \(P_{t,p}^{\alpha,i} \in [0, 1]\). This means that the control \(\alpha^i\) can be chosen arbitrarily, at least locally around the initial time \(t\). When \((t, x, p) \in D_{IJ}\) with \(I \cup J \neq \emptyset\), then there is at least one \(i \leq \kappa\) such that \(p^i = 0\) or \(p^i = 1\). In this case the state constraints \(P_{t,p}^{\alpha,i} \in [0, 1]\) on \([t, T]\) imposes to choose \(\alpha^i = 0\) on \([t, T]\), see Remark 2.4. Hence, letting \(\pi_{IJ}\) be the operator defined by
\[
p \in [0, 1]^\kappa \mapsto \pi_{IJ}(p) = (p^i 1_{i \notin I,J} + 1_{i \in I,J})_{i \in \mathcal{K}}
\]
for \((I, J) \in \mathcal{P}_\kappa\), we have
\[
v = v_{IJ} := v(\cdot\,; \pi_{IJ}(\cdot)) \text{ on } D_{IJ},
\]
where, for \((t, x, p) \in \bar{D}\),
\[
v_{IJ}(t, x, p) = \inf \{ y \geq \ell : Y_{t,x,y}^\nu(T) \geq G(X_{t,x}(T), P_{t,\pi_{IJ}(p)}^\nu(T)) \text{ for some } (\nu, \alpha) \in \mathcal{U}^t \times \mathcal{A}_{t,\pi_{IJ}(p)}^{IJ}\}
\]
with
\[
\mathcal{A}_{t,\pi_{IJ}(p)}^{IJ} := \{ \alpha \in \mathcal{A}_{t,p} : \alpha^i = 0 \text{ for all } i \in I \cup J \},
\]
recall Remark 2.4.

In the rest of the paper, we shall write \((I, J) \in \mathcal{P}_\kappa^k\) when \((I, J) \in \mathcal{P}_\kappa\) and \(|I| + |J| = k, k \leq \kappa\). We shall also use the notations \((I', J') \supset (I, J)\) when \(I' \supset I\) and \(J' \supset J\). If in addition, \((I', J') \neq (I, J)\), then we will write \((I', J') \supset (I, J)\).

**Remark 2.5** It is clear that \(v\) and each \(v_{IJ}\), \((I, J) \in \mathcal{P}_\kappa\), are non-decreasing with respect to their \(p\)-parameter. In particular, \(v_{IJ} \geq v_{IJ'} \geq v_{IJ''}\) for \((I', J') \supset (I, J)\).

**Remark 2.6** Since \(g^i \geq g^j\) for \(i \leq j\), it would be natural to restrict to the case where \(p^i \leq p^j\) for \(i \leq j\). From the PDE point of view, this would lead to the introduction of boundary conditions on the planes for which \(p^i = p^j\) for some \(i \neq j\). Since this restriction does not appear to be necessary in our approach, we deliberately do not use this formulation. From the pure numerical point of view, one could however use the fact that \(v(\cdot, p) = v(\cdot, \hat{p})\) where \(\hat{p}\) is defined by \(\hat{p}^i = \max_{i \leq j} p^i\) for \(i \leq \kappa\).
Remark 2.7 Note that, as defined above on $\tilde{D}_{IJ}$, the function $v_{IJ}$ depends on its $p$-parameters only through the components $(p^l)_{l \in I \cup J}$. However, for ease of notations, we shall always use the notation $v_{IJ}(\cdot, p)$ instead of a more transparent notation such as $v_{IJ}(t, x, (p^l)_{l \in I \cup J})$. Similarly, a test function on $\tilde{D}_{IJ}$ depends on the $p$-parameter only through $(p^l)_{l \in I \cup J}$.

Remark 2.8 Note that, for any $J \subset K$, 

$$v_{J \cap \mathcal{K}} = \inf \{ y \geq \ell : Y_{t,x,y}^\nu(T) \geq g_J(X_{t,x}(T)) \text{ for some } \nu \in \mathcal{U} \},$$

where 

$$g_J := \max_{j \in J} g_j \vee \ell \quad (2.19)$$

coincides with the super-hedging price of the payoff $g_J(X_{t,x}(T))$, while 

$$v_{K \cap \emptyset} = \inf \{ y \geq \ell : \exists \nu \in \mathcal{U} \text{ s.t. } 1_{\{Y_{t,x,y}^\nu(T) \geq \max_{i \leq \kappa} g_i(X_{t,x}(T)) \geq 0 \text{ and } Y_{t,x,y}^\nu(T) \geq \ell \} = \ell} \right.$$

2.3 PDE characterization

As already mentioned, stochastic target problems of the form (2.18) have been studied in [12] which provides a PDE characterization of each value function $v_{IJ}$ on $D_{IJ}$. In order to state it, we first need to introduce some additional notations. For ease of notations, we set 

$$\mu_{X,P} := \begin{pmatrix} \mu_X \\ 0_\kappa \end{pmatrix} \quad \text{and} \quad \sigma_{X,P}(\cdot, a) := \begin{pmatrix} \sigma_X \\ a \end{pmatrix} \quad \text{for } a \in M_{\kappa,d},$$

where $0_\kappa := (0, \ldots, 0) \in \mathbb{R}^\kappa$.

Given $(I, J) \in \mathcal{P}_\kappa$ and $\varepsilon > 0$, we then define 

$$F^\varepsilon_{IJ} := \sup_{(u,a) \in N^\varepsilon_{IJ}} L^{u,a},$$

where, for $(u, a) \in U \times M_{\kappa,d}$ and $\theta := (x, q, Q) \in \Theta := (0, \infty)^d \times \mathbb{R}^{d+\kappa} \times M_{d+\kappa,d+\kappa}$,

$$L^{u,a}(\theta) := \mu_Y(x, u) - \mu_{X,P}(x)^T q - \frac{1}{2} \text{Trace } [(\sigma_{X,P}(x,a)Q),$$

and 

$$N^\varepsilon_{IJ} := \{(u, a) \in U \times A_{IJ} : |N^{u,a}| \leq \varepsilon\}$$

with 

$$N^{u,a}(x, q) := \sigma_Y(x, u) - q^T \sigma_{X,P}(x,a) \quad \text{and} \quad A_{IJ} := \{a \in M_{\kappa,d} : a^k = 0 \text{ for } k \in I \cup J\}.$$

The main result of [12] states that $v_{IJ}$ is a discontinuous viscosity solution of the PDE 

$$\min\{\varphi - \ell, -\partial_t \varphi + F^0_{IJ}(\cdot, \varphi, \partial^2 \varphi)\} = 0 \text{ on } D_{IJ},$$
where, for a smooth function \( \varphi : (t, x, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \), \( D\varphi \) and \( D^2\varphi \) stand for the gradient and the Hessian matrix with respect to \((x, p)\), and \( \partial_i\varphi \) stands for the time derivative. The label “discontinuous viscosity solution” means that it has been stated in terms of the upper- and lower-semicontinuous envelopes of \( v \), see Definition 2.2 below, and that we need to relax the operator \( F_{I,J}^0 \), which may not be continuous, by considering the upper- and lower-semicontinuous envelopes \( F_{I,J}^+ \) and \( F_{I,J}^- \),

\[
F_{I,J}^+(\theta) := \limsup_{(\theta', \varepsilon') \to (\theta, 0)} F_{I,J}^0(\theta') \quad \text{and} \quad F_{I,J}^-(\theta) := \liminf_{(\theta', \varepsilon') \to (\theta, 0)} F_{I,J}^0(\theta') .
\]

This leads to a system, hereafter called \((S)\), of PDEs, each stated on a sub-domain \( D_{I,J} \), with appropriate boundary conditions, see Theorem 2.2 and Corollary 2.1 below.

Before defining precisely what we mean by a solution of \((S)\), we need to introduce an extra technical object to which we will appeal when we define the notion of subsolution.

**Definition 2.1** Given \((I, J) \in \mathcal{P}_\kappa \) and \((t, x, p) \in D_{I,J} \), we denote by \( C_{I,J}(t, x, p) \) the set of \( C^{1,2} \) functions \( \varphi \) with the following property: for all \( \varepsilon > 0 \), all open set \( B \) such that \((x, D\varphi(t, x, p)) \in B \) and \( N_{I,J}^0 \neq \emptyset \) on \( B \), and all \((\tilde{u}, \tilde{a}) \in N_{I,J}^0(x, D\varphi(t, x, p))\), there exists an open neighborhood \( B' \) of \((x, D\varphi(t, x, p))\) and a locally Lipschitz map \((\tilde{u}, \tilde{a})\) such that \(|(\tilde{u}, \tilde{a})(x, D\varphi(t, x, p)) - (\tilde{u}, \tilde{a})| \leq \varepsilon \) and \((\tilde{u}, \tilde{a}) \in N_{I,J}^0(x, D\varphi(t, x, p))\) on \( B' \).

**Remark 2.9** Fix \((I, J) \in \mathcal{P}_\kappa \) such that there exists \( i \in \mathcal{K} \setminus (I \cup J) \). Let \( \varphi \) be a smooth function such that \( D_{p^0}\varphi \neq 0 \) on a neighborhood of \((t, x, p) \in \bar{D} \). Then, \((u, a) \in N_{I,J}^0(x, D\varphi(t, x, p))\) is equivalent to

\[
a^i = \left( \sigma_Y(x, u) - D_x\varphi(t, x, p)^\top \sigma_X(x) - \sum_{j \not\in I \cup J \cup \{i\}} a^j D_{p^0}\varphi(t, x, p) \right) / D_{p^0}\varphi(t, x, p).
\]

Since \( D_{p^0}\varphi \neq 0 \) on a neighborhood of \((t, x, p)\), this readily implies that \( \varphi \in C_{I,J}(t, x, p) \).

A viscosity solution of \((S)\) is then defined as follows.

**Definition 2.2** (i) Given a locally bounded map \( V \) defined on \( \bar{D} \) and \((I, J) \in \mathcal{P}_\kappa \), we define \( V_{I,J} := V(\cdot, \pi_{I,J}(\cdot)) \) and

\[
V_{I,J}^+(t, x, p) := \limsup_{(t', x', p') \to (t, x, p)} V_{I,J}(t', x', p') \quad \text{and} \quad V_{I,J}^-(t, x, p) := \liminf_{(t', x', p') \to (t, x, p)} V_{I,J}(t', x', p'),
\]

and

\[
V_{I,J}^+(t, x, p) := \limsup_{(t', x', p') \to (t, x, p)} V_{I,J}(t', x', p') \quad \text{and} \quad V_{I,J}^-(t, x, p) := \liminf_{(t', x', p') \to (t, x, p)} V_{I,J}(t', x', p').
\]
for \((t,x,p) \in D_{IJ}\).

(ii) We say that \(V\) is a discontinuous viscosity supersolution of (S) if \(V_{I,J}\) is a viscosity supersolution of

\[
\min \left\{ \varphi - \ell, -\partial_t \varphi + F_{I,J}^* (\cdot, D\varphi, D^2 \varphi) \right\} = 0 \quad \text{on} \quad D_{IJ}, \tag{2.20}
\]

for each \((I,J) \in \mathcal{P}_\kappa\).

(iii) We say that \(V\) is a discontinuous viscosity subsolution of (S) if \(V_{I,J}^*\) is a viscosity subsolution of

\[
\min \left\{ \varphi - \ell, -\partial_t \varphi + F_{I,J} (\cdot, D\varphi, D^2 \varphi) \right\} = 0 \quad \text{if} \quad \varphi \in C_{IJ}, \quad \text{on} \quad D_{IJ}, \tag{2.21}
\]

for each \((I,J) \in \mathcal{P}_\kappa\).

(iv) We say that \(V\) is a discontinuous viscosity solution of (S) if it is both a discontinuous super- and subsolution of (S).

We can now state our first result which is a direct Corollary of Theorem 2.1 in [12].

**Theorem 2.1** The function \(v\) is a discontinuous viscosity solution of (S).

**Proof.** The above result is an immediate consequence of Theorem 2.1 in [12]. Note that we replaced their condition Assumption 2.1 by the condition \(\varphi \in C_{IJ}\), which is equivalent, in the statement of the subsolution property, see Remark 2.9.

**Remark 2.10** Fix \((I,J) \in \mathcal{P}_\kappa\) such that \(I \cup J = K\). Then, \((u,a) \in N_{f}^e(x, D\varphi(t,x,p))\) implies that

\[
|u^\top \sigma(x) - D_x \varphi(t,x,p)^\top \text{diag} [x] \sigma(x)| \leq \varepsilon.
\]

Since \(u \in U\) and \(\sigma(x)\) is invertible by assumption, one easily checks that (2.20) implies

\[
\text{diag} [x] D_x \varphi(t,x,p) \in U, \tag{2.22}
\]

recall the usual convention \(\sup \emptyset = -\infty\). This is the classical gradient constraint that appears in super-hedging problems with constraints on the strategy, see e.g. [20], where it is written in terms of proportions of the wealth invested in the risky assets.
2. PDE CHARACTERIZATION OF THE P&L MATCHING PROBLEM

Remark 2.11 Let $\varphi$ be a smooth function. If $D_{p_i} \varphi(t, x, p) = 0$ for $i \notin I \cup J$, then $N_{IJ}^e(x, D\varphi(t, x, p))$ takes the form $U_e \times \mathbb{M}^{\kappa, d}$ for some $U_e \subset U$, $\varepsilon > 0$. Thus the optimization over $a \in \mathbb{M}^{\kappa, d}$ in the definition of $F_{IJ}^e$ is performed over an unbounded set. On the other hand, if $D_{p_i} \varphi(t, x, p) > 0$ for $i \notin I \cup J$, then the same arguments as in Remark 2.9 imply that at least one line of $a$ is given by the other ones. In particular, for $|I| + |J| = \kappa - 1$, the sequence of sets $(N_{IJ}^e(x, D\varphi(t, x, p)))_{0 \leq \varepsilon \leq 1}$ is contained in a compact subset of $U \times \mathbb{M}^{\kappa, d}$. This implies that $F_{IJ}^* \neq F_{IJ*}$ in general.

As already mentioned the main difficulty comes from the boundary conditions. We first state the space boundary condition in the $p$-variable, whose proof is provided later in Section 3.

Theorem 2.2 Fix $(I, J), (I', J') \in \mathcal{P}_\kappa$ such that $(I', J') \supset (I, J)$, we have

(i) $v_{IJ*}$ is viscosity supersolution of

$$
\min \{ \varphi - \ell, -\partial_t \varphi + F_{IJ'}^*(\cdot, D\varphi, D^2\varphi) \} = 0 \text{ on } D_{IJ'}^e,
$$

(ii) $v_{IJ*}^*(t, x, p) \leq v_{IJ'}^*(t, x, p)$, for $(t, x, p) \in D_{IJ} \cap [0, T] \times (0, \infty)^d \times B_{IJ'}.$

We now discuss the boundary condition as $t$ approaches $T$.

In the case where $I \cup J = K$ with $|J| > 0$, the map $v_{IJ}$ coincides with the superhedging problem associated to the payoff $g_J$ as defined in (2.19), recall Remark 2.8. One could therefore expect that $v_{IJ}(T-\cdot) = g_J$. However, as usual, see e.g. [20], the terminal condition for $v_{IJ}$ is not the natural one since the gradient constraint that appears implicitly in (2.21), see Remark 2.10, should propagate up to the time boundary. The natural boundary condition should be given by the smallest function $\phi$ above $g_J$ that satisfies the associated gradient constraint $\text{diag}[x] D_x \phi \in U$. This leads to the introduction of the “face-lifted” version of $g_J$ defined by:

$$
\hat{g}_J(x) := \sup_{\zeta \in \mathbb{R}^d} \left[ g_J(xe^\zeta) - \delta_U(\zeta) \right],
$$

where

$$
\delta_U(\zeta) := \sup_{u \in U} u^\top \zeta, \ z \in \mathbb{R}^d
$$

is the support function of the convex closed set $U$ and $x e^\zeta = (x^i e^\zeta)_i \leq d$.

When $I \cup J \neq K$, the above mentioned gradient constraint does not appear anymore in (2.21), see e.g. Remark 2.9, and the terminal boundary condition can
be naturally stated in terms of
\[ \hat{G}(x, p) := \inf \{ y \geq \ell : y \geq g^i(x)1_{0<p^i<1} + \hat{g}^i(x)1_{p^i=1}, \text{ for all } i \in K \} \]
\[ = \max_{i \in K} (\ell 1_{p^i=0} + g^i(x)1_{0<p^i<1} + \hat{g}^i(x)1_{p^i=1}) . \tag{2.25} \]

Corollary 2.1 \( v_*(T, \cdot) \geq \hat{G}_* \) and \( v^*(T, \cdot) \leq \hat{G}^* \) on \((0, \infty)^d \times [0, 1]^{\kappa} \).

Proof. It is a consequence of Proposition 1.2 and Theorem 1.1 in the next chapter.

Remark 2.12 In the case of \( \kappa = 1 \) and \( g^1 \geq \ell = 0 \), it is shown in [23] and [12] that the terminal condition should be face-lifted with respect to the \( p \)-variable when the set \( U \) in which controls take values is \( \mathbb{R}^d \). This follows from the convexity of the value function in its \( p \)-variable. Namely, the terminal condition as \( t \to T \) is then given by \( p^1 g^1 \). Corollary 2.1 shows that it is no more the case when we restrict to a compact set \( U \).

Remark 2.13 Combining Theorem 2.1, Theorem 2.2 and Corollary 2.1 provides a PDE characterization of the value function \( v \). However, the following should be noted:

1. It is clear that \( \hat{G}_* < \hat{G}^* \) for some \( p \in \partial[0, 1]^{\kappa} \).
2. The boundary conditions induced by Theorem 2.2 may not lead to \( v_{IJ,*} \geq v^*_{IJ} \) on the boundary in the \( p \)-variable.
3. The operator \( F_{IJ} \) in (2.20) and (2.21) is in general discontinuous when \( I \cup J \neq K \), see Remark 2.11 above.

This prevents us from proving a general comparison result for super- and subsolutions of (S). We are therefore neither able to prove that \( v \) is the unique solution of (S) in a suitable class, nor to prove the convergence of standard finite difference numerical schemes. In order to surround this difficulty, we shall introduce in the next chapter a sequence of convergent approximating problems which are more regular and for which convergent schemes can be constructed.

3 Proof of the boundary condition in the \( p \)-variable

We first recall the geometric dynamic programming principle of [46], see also [47] and [49], to which we will appeal to prove the boundary conditions. We next report the proof of the supersolution properties in Proposition 3.1, and that of the subsolution properties in Proposition 3.2.
Corollary 3.1 Fix \((t, x, p) \in \tilde{D}_{IJ}\).

\begin{enumerate}[(GDP1)]
    
    \item If \(y \geq \ell\) and \((\nu, \alpha) \in \mathcal{U}_t \times \mathcal{A}_{t, \pi, IJ}(p)\) are such that
    \[
    Y_{t, x, y}^\nu(T) \geq G(X_{t, x}(T), P_{t, p}^\alpha(T))
    \]
    then
    \[
    Y_{t, x, y}^\nu(\theta) \geq v_{IJ}^\nu(\theta, X_{t, x}(\theta), P_{t, p}^\alpha(\theta)), \quad \text{for all } \theta \in \mathcal{T}_{[t, T]}^i.
    \]

    \item For \(y < v_{IJ}^\nu(t, x, p), \theta \in \mathcal{T}_{[t, T]}^i\) and \((\nu, \alpha) \in \mathcal{A}_{t, \pi, IJ}(p),\)
    \[
    \mathbb{P}\left[Y_{t, x, y}^\nu(\theta) > v_{IJ}^\nu(\theta, X_{t, x}(\theta), P_{t, p}^\alpha(\theta))\right] < 1.
    \]
\end{enumerate}

We now turn to the boundary condition in the \(p\)-variable, i.e. as \(p \to \partial B_{IJ}\).

Proposition 3.1 For all \((I, J), (I', J') \in \mathcal{P}_\kappa\) such that \((I', J') \supset (I, J)\), we have

\[
v_{IJ}^* \leq v_{I', J'}^* \text{ on } D_{I', J'}.
\]

**Proof.** Since \(v\) is non-decreasing with respect to each variable \(p^i, i \leq k\), we have \(v_{IJ} \leq v_{I', J'}\) for \(J' \supset J\). Hence it suffices to show the result for \(J = J'\). We also assume that \(I' \neq I\), since otherwise there is nothing to prove. Moreover, we claim that it is enough to show that

\[
v_{IJ}^* \leq \bar{v}_{IJ}^* := \max\left\{ v_{(I \cup K)J}^* : K \subset I' \setminus I, K \neq \emptyset \right\} \text{ on } D_{I', J'}.
\]

Indeed, if the above holds, then there exists \(K_1 \subset I' \setminus I\) such that \(K_1 \neq \emptyset\) and \(v_{IJ}^* \leq v_{(I \cup K_1)J}^*\). If \(K_1 \cup I = I'\), the result is proved. If not, then applying the same result to \(I \cup K_1\) instead of \(I\) implies that there exists \(K_2 \subset I' \setminus (I \cup K_1)\) such that \(K_2 = K_1 \cup K_1\) strictly contains \(K_1\) and for which \(v_{IJ}^* \leq v_{(I \cup K_2)J}^*\). The result then follows by iterating this procedures so as to construct an increasing sequence of sets \(K_n \subset I' \setminus I\) such that \(I \cup K_n = I'\) for a finite \(n\).

We proceed in two steps.

**Step 1.** We first show that for any smooth function \(\tilde{\varphi}\) on \(\tilde{D}\) and \((\tilde{t}, \tilde{x}, \tilde{p}) \in D_{I', J}\) such that \(D_{p^i}\tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{p}) \neq 0\) for some \(i \in (I \cup J)^c\) and

\[
\max\left(\text{strict}\right)(v_{IJ}^* - \tilde{\varphi}) = (v_{IJ}^* - \tilde{\varphi})(\tilde{t}, \tilde{x}, \tilde{p}) = 0,
\]

we have

\[
\min\{\tilde{\varphi} - \bar{v}_{IJ}^*, -\partial_t \tilde{\varphi} + F_{IJ, \pi}\tilde{\varphi}\}(\tilde{t}, \tilde{x}, \tilde{p}) \leq 0.
\]

Assume to the contrary that there exists \(\eta > 0\) s.t.

\[
\min\{\tilde{\varphi} - \bar{v}_{IJ}^*, -\partial_t \tilde{\varphi} + F_{IJ, \pi}\tilde{\varphi}\}(\tilde{t}, \tilde{x}, \tilde{p}) \geq 2\eta.
\]

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In view of Remark 2.9, this implies that there exists \( \varepsilon > 0 \) and a locally Lipschitz map \((\hat{u}, \hat{a})\) such that

\[
\min\{\tilde{\varphi} - v_{t,J}^\prime, -\partial_t \tilde{\varphi} + L^{(\hat{u}, \hat{a})}(\cdot, D\tilde{\varphi})(\cdot, D^2 \tilde{\varphi})\}(t, x, p) \geq \eta, \tag{3.4}
\]

\[
(\hat{u}, \hat{a})(x, D\tilde{\varphi}(t, x, p)) \in N^0(t, x, D\tilde{\varphi}(t, x, p)), \tag{3.5}
\]

for all \((t, x, p) \in B := B_\varepsilon(\hat{i}, \hat{x}, \hat{p}) \cap \tilde{D}_{IJ}\).

Let \((t_n, x_n, p_n)\) be a sequence in \(B\) that converges to \((\hat{i}, \hat{x}, \hat{p})\) such that

\[v_{IJ}(t_n, x_n, p_n) \to v_{IJ}^*(\hat{i}, \hat{x}, \hat{p})\]

and set \(y_n := v_{IJ}(t_n, x_n, p_n) - n^{-1}\) so that

\[\gamma_n := y_n - \tilde{\varphi}(t_n, x_n, p_n) \to_n \infty 0.\]

We denote by \((X^n, P^n, Y^n)\) the solution of the (2.1), (2.4) and (2.12) associated to the initial condition \((t_n, x_n, p_n)\) and the Markovian control

\[(v^n, a^n) = (\hat{u}, \hat{a})(X^n, D\tilde{\varphi}(\cdot, X^n, P^n))\]

and define the stopping time

\[\theta_n := \theta_{n1} \wedge \theta_{n2},\]

where

\[\theta_{n1} := \inf\{s \geq t_n : \min_{i \in I' \setminus I} P^{n,i}(s) = 0\},\]

\[\theta_{n2} := \inf\{s \geq t_n : (s, X^n(s), P^n(s)) \notin B \cap D_{IJ}\}.
\]

Note that, since \((\hat{i}, \hat{x}, \hat{p})\) achieves a strict local maximum of \(v_{IJ}^* - \tilde{\varphi}\), we have

\[v_{IJ}^* - \tilde{\varphi} \leq -\zeta \text{ on } \partial B = \partial(B \cap D_{IJ}), \text{ for some } \zeta > 0.\]

Using (3.4), we then deduce that

\[Y^n(\theta_n) - \gamma_n \geq \tilde{\varphi}(\theta_n, X^n(\theta_n), P^n(\theta_n))\]

\[\geq \left(\tilde{v}_{IJ}^*(\theta_n, X^n(\theta_n), P^n(\theta_n)) + \eta\right) 1_{\theta_n = \theta_{n1}} + (v_{IJ}^*(\theta_n, X^n(\theta_n), P^n(\theta_n)) + \zeta) 1_{\theta_n < \theta_{n1}}.\]

We now observe that, by definition of \(\theta_{n1}\) and \(\theta_{n2}\), \((\theta_{n2}, X^n(\theta_{n2}), P^n(\theta_{n2})) \in D_{IJ}\) and therefore \(v_{IJ}(\theta_n, X^n(\theta_n), P^n(\theta_n)) = v(\theta_n, X^n(\theta_n), P^n(\theta_n))\) on \(\{\theta_n < \theta_{n1}\}\). On the other hand, letting \(K\) be the random subset of \(I' \setminus I\) such that \(P^{n,i}(\theta_{n1}) = 0\) for \(i \in K\), we have \(\tilde{v}_{IJ}^*(\theta_n, X^n(\theta_n), P^n(\theta_n)) \geq v_{IJ\cup K}(\theta_n, X^n(\theta_n), P^n(\theta_n)) = v(\theta_n, X^n(\theta_n), P^n(\theta_n))\) on \(\{\theta_n = \theta_{n1}\}\). It then follows from the previous inequality that

\[Y^n(\theta_n) - \gamma_n \geq v(\theta_n, X^n(\theta_n), P^n(\theta_n)) + \zeta \wedge \eta.\]
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Since $\gamma_n \to 0$, this leads to a contradiction to GDP2 for $n$ large.

**Step 2.** The rest of the proof is similar to the proof of Section 6.2 in [12]. We provide the main arguments for completeness. It remains to show that, for any 
smooth function $\bar{\phi}$ and $(\tilde{t}, \tilde{x}, \tilde{p}) \in D_{I,J}$ so that

$$\max_{D_{I,J}}(v^*_IJ - \bar{\phi}) = v^*_IJ(\tilde{t}, \tilde{x}, \tilde{p}) - \bar{\phi}(\tilde{t}, \tilde{x}, \tilde{p}) = 0,$$

we have

$$\bar{\phi}(\tilde{t}, \tilde{x}, \tilde{p}) \leq v^{**}_{IJ}(\tilde{t}, \tilde{x}, \tilde{p}).$$

We argue by contradiction and assume that

$$\bar{\phi}(\tilde{t}, \tilde{x}, \tilde{p}) > v^{**}_{IJ}(\tilde{t}, \tilde{x}, \tilde{p}).$$

(3.7)

Given $\rho > 0$ and $k \geq 1$, we define the modified test function

$$\phi_k(t, x, p) := \bar{\phi}(t, x, p) + |x - \tilde{x}|^4 + |t - \tilde{t}|^2 + \sum_{i \in I \cup J} |p_i - \tilde{p}_i|^4 + \sum_{i \in I' \setminus I} \psi_k(1 - p_i),$$

where

$$\psi_k(z) := -k\rho \int_z^1 e^{2k} e^{-e^{2k+s} - e^{2k+1}} ds, \quad \text{for all } z \in \mathbb{R},$$

(3.8)

Let $(t_k, x_k, p_k) \in \bar{D}_{I,J}$ be such that it maximizes of $(v^*_IJ - \phi_k)$ on $\bar{D}_{I,J}$ and observe that

$$-2\rho k \leq \psi'_k \leq -\frac{\rho k}{2(e-1)},$$

(3.9)

$$\psi'' < 0,$$

(3.10)

$$\lim_{k \to \infty} \frac{(\psi'_k(z_k))^2}{|\psi'_k(z_k)|} = \rho, \quad \text{if } (z_k)_{k \geq 1} \subset (0, 1) \text{ satisfies } \lim_{k \to \infty} k(1 - z_k) = 0.$$ (3.11)

Standard arguments then show that

$$(t_k, x_k, p_k) \to (\tilde{t}, \tilde{x}, \tilde{p}) \text{ and } kp_k \to 0,$$

see e.g. Step 2 in Section 6.1 of [12]. Note that (3.9) implies that $D_{p^i} \phi_k(t_k, x_k, p_k) < 0$ for $i \in I' \setminus I$, for $k$ large enough. It then follows from Step 1, Theorem 2.1 and (3.7) that

$$-\partial_t \phi_k(t_k, x_k, p_k) + F^\phi_k(t_k, x_k, p_k) \leq 0.$$

Then, there exist $\varepsilon_k, q_k \in \mathbb{R}^{d+\kappa}$ and $A_k \in \mathbb{M}^{d+\kappa}$ such that

$$\varepsilon_k \to 0$$

$$|\langle q_k, A_k \rangle - (D \phi_k, D^2 \phi_k)(t_k, x_k, p_k)| \leq \frac{1}{k},$$

(3.12)

$$-\partial_t \phi_k(t_k, x_k, p_k) + F^\phi_k(t_k, p_k, q_k, A_k) \leq \frac{1}{k}.$$
Given an arbitrary \( u \in U \), fix \( i_0 \in I' \setminus I \) and \( \alpha_k \in \mathbb{M}^{\alpha, d} \) such that \( \alpha_k^j = 0 \) for \( j \neq i_0 \) and

\[
\alpha_k^{i_0} := (\sigma_U(x_k, u) - q_k^0(t_k, x_k, p_k)\nabla \sigma_X(x_k)) / q_k^{p_0},
\]

where \( q_k^e \) stands for the first \( d \) components of \( q_k \) and \( q_k^{p_0} \) stands for (by abuse of notations) the \( d + i_0 \) component of \( q_k \). Note that \( (u, \alpha_k) \in N_{IJ}^0(x_k, q_k) \). Combined with the third inequality in (3.12), this implies that

\[
k^{-1} \geq -\partial_{t'} \varphi_k(t_k, x_k, p_k) + \mu_X(x_k, u) - \mu_X(x_k)^\top q_k^e - \frac{1}{2} \text{Trace} [\sigma_X(x_k)\sigma_X(x_k)^\top A_{kx}^{xx}]
\]

\[
- \frac{1}{2} (\alpha_k^{i_0})^2 A_{k \sigma}^{p_0 p_0} - \sigma_X(x_k)^\top A_{k \sigma}^{p_0} \alpha_k^{i_0},
\]

where \( A_{kx}^{xx} = (A_k^{ij})_{i,j \leq d}, A_k^{p_0 p_0} = A_k^{d+i_0, d+i_0} \) and \( A_k^{p_0} = (A_k^{i,j})_{i \leq d} \). Sending \( k \to \infty \), using (3.9), (3.11), the definition of \( \alpha_k^{i_0} \), (3.12) and recalling that \( D_{p_0} \tilde{\varphi} = 0 \) then leads to

\[
0 \geq -\partial_{t'} \tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{p}) + \mu_Y(\tilde{x}, u) - \mu_X(\tilde{x})^\top D_{x'} \tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{p})
\]

\[
- \frac{1}{2} \text{Trace} [\sigma_X(\tilde{x})\sigma_X(\tilde{x})^\top D_{xx} \varphi(\tilde{t}, \tilde{x}, \tilde{p})]
\]

\[
+ \frac{1}{2} \rho^{-1} |\sigma_Y(\tilde{x}, u) - D_{x'} \varphi(\tilde{t}, \tilde{x}, \tilde{p})|^2 \sigma_X(\tilde{x})|^2.
\]

Since \( \rho > 0 \) and \( u \in U \) are arbitrary, this implies that

\[
|u^\top \sigma(\tilde{x}) - D_{x'} \varphi(\tilde{t}, \tilde{x}, \tilde{p})^\top \sigma_X(\tilde{x})|^2 = 0 \text{ for all } u \in U.
\]

This leads to a contradiction since \( \sigma \) is assumed to be invertible.

\[\square\]

**Proposition 3.2** For all \( (I, J), (I', J') \in \mathcal{P}_\kappa \) such that \( J' \supset J \), \( v_{IJ^*} \) is a supersolution on \( D_{IJ} \) of

\[
\min \{ \varphi - \ell, -\partial \varphi + F_{IJ}^* \varphi \} \geq 0 \text{ on } D_{IJ'}.
\]

**Proof.** By definition, we have \( v_{IJ^*} \geq \ell \). The rest of proof is divided in several steps.

**Step 1.** We first show that, for a smooth function \( \tilde{\varphi} \) on \( D_{IJ} \) and \( (\tilde{t}, \tilde{x}, \tilde{p}) \in \bar{D}_{IJ} \cap D_{IJ'} \) so that

\[
\min \{(\text{strict})_{D_{IJ}} (v_* - \tilde{\varphi}) = (v_* - \tilde{\varphi})(\tilde{t}, \tilde{x}, \tilde{p}) = 0,
\]

we have

\[
\max \{\tilde{\varphi} - v_{IJ}^{J^*}, -\partial_t \tilde{\varphi} + F_{IJ}^* \tilde{\varphi} \}(\tilde{t}, \tilde{x}, \tilde{p}) \geq 0,
\]

where

\[
v_{IJ^*} \geq v_{IJ}^{J^*} := \min \{v_{I(J \cup K)}: K \subset J' \setminus J, K \neq \emptyset\}. \quad (3.15)
\]
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We argue by contradiction and assume that there exists $\varepsilon, \eta > 0$ such that
\[
\max\{\tilde{\phi} - L_j I_j, -\partial_t \tilde{\phi} + F_{I,J} \tilde{\phi}\}(t, x, p) \leq -\eta, \tag{3.16}
\]
\[
\forall (t, x, p) \in B := B_\epsilon(\tilde{t}, \tilde{x}, \tilde{p}) \cap \bar{D}_{I,J}.
\]

Note that, since $(\tilde{t}, \tilde{x}, \tilde{p})$ achieves a strict local minimum of $v_{I,J*} - \tilde{\phi}$ on $\bar{D}_{I,J}$, we have
\[
v_{I,J*} - \tilde{\phi} \geq \zeta \text{ on } \partial B = \partial(B \cap D_{I,J}), \tag{3.17}
\]
for some $\zeta > 0$. Let $(t_n, x_n, p_n)$ be a sequence in $B \cap D_{I,J}$ that converges to $(\tilde{t}, \tilde{x}, \tilde{p})$ such that
\[
v_{I,J}(t_n, x_n, p_n) \to v_{I,J*}(\tilde{t}, \tilde{x}, \tilde{p})
\]
and set $y_n := v_{I,J}(t_n, x_n, p_n) + n^{-1}$ so that
\[
\gamma_n := y_n - \tilde{\phi}(t_n, x_n, p_n) \to 0.
\]
Since $y^n > v_{I,J}(t_n, x_n, p_n)$, there exists $(\nu^n, \alpha^n) \in \mathcal{U} \times \mathcal{A}_{t_n,p_n}$ such that
\[
Y^n(T) \geq G(X^n(T), P^n(T)),
\]
where $(Y^n, X^n, P^n) := (Y^n_{t_n,x_n,y_n}, X^n_{t_n,x_n}, P^n_{t_n,x_n}.)$

Let us now define
\[
\theta_n := \theta_{n1} \wedge \theta_{n2}
\]
where
\[
\theta_{n1} := \inf\{s \geq t_n : \max_{i \in J 
\supseteq J} P^n_i(s) = 1\},
\]
\[
\theta_{n2} := \inf\{s \geq t_n : (s, X^n(s), P^n(s)) \notin B \cap D_{I,J}\}.
\]

It then follows from GDP1 that
\[
Y^n(\theta_n) \geq v(\theta_n, X^n(\theta_n), P^n(\theta_n)).
\]

We now observe that, by definition of $\theta_{n1}$ and $\theta_{n2}$, $(\theta_n, X^n(\theta_{n1}), P^n(\theta_{n2})) \in D_{I,J}$ and therefore
\[
v_{I,J}(\theta_n, X^n(\theta_n), P^n(\theta_n)) = v(\theta_n, X^n(\theta_n), P^n(\theta_n)) \text{ on } \{\theta_n < \theta_{n1}\}.
\]

On the other hand, letting $K$ be the random subset of $J \setminus J$ such that $P^n_{i}(\theta_{n1}) = 1$ for $i \in K$, we have $v_{I,J}'(\theta_n, X^n(\theta_n), P^n(\theta_n)) \leq v_{I,J,K}(\theta_n, X^n(\theta_n), P^n(\theta_n)) = v(\theta_n, X^n(\theta_n), P^n(\theta_n))$ on $\{\theta_n = \theta_{n1}\}$. It then follows from the previous inequality that
\[
Y^n(\theta_n) \geq v_{I,J}(\theta_n, X^n(\theta_n), P^n(\theta_n))1_{\theta_n < \theta_{n1}} + v_{I,J}'(\theta_n, X^n(\theta_n), P^n(\theta_n))1_{\theta_n = \theta_{n1}}.
\]

We now appeal to (3.16) and (3.17) to deduce that
\[
Y^n(\theta_n) \geq \tilde{\phi}(\theta_n, X^n(\theta_n), P^n(\theta_n)) + \zeta \wedge \eta.
\]
The required contradiction then follows from the same arguments as in Section 5.1 of [12].

**Step 2.** We now show that for any smooth function $\tilde{\varphi}$ on $\bar{D}_{I,J}$ and $(\tilde{t}, \tilde{x}, \tilde{p}) \in \bar{D}_{I,J} \cap D_{I,J'}$ such that

$$\min(\text{strict})_{D_{I,J}} (v_* - \tilde{\varphi}) = (v_* - \tilde{\varphi})(\tilde{t}, \tilde{x}, \tilde{p}) = 0,$$

(3.18)

we have

$$\max\{\tilde{\varphi} - v_{I,J*}, -\partial_t \tilde{\varphi} + F_{I,J*}^* \tilde{\varphi}\}(\tilde{t}, \tilde{x}, \tilde{p}) \geq 0.$$

To see this, assume that

$$(-\partial_t \tilde{\varphi} + F_{I,J*}^* \tilde{\varphi})(\tilde{t}, \tilde{x}, \tilde{p}) < 0.$$  

(3.19)

Then, it follows from Step 1 that $v_{I,J*}(\tilde{t}, \tilde{x}, \tilde{p}) = \tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{p}) \geq v_{I(J \cup K_1)*}(\tilde{t}, \tilde{x}, \tilde{p})$ for some $K_1 \subseteq J' \setminus J$ such that $K_1 \neq \emptyset$. If $J \cup K_1 = J'$, then this proves the required result. If not, then we use the fact that $\nu$ is non-decreasing in its $p^j$ components to deduce that $v_{I,J*} \leq v_{I(J \cup K_1)*}$. It follows that $v_{I,J*}(\tilde{t}, \tilde{x}, \tilde{p}) = v_{I(J \cup K_1)*}(\tilde{t}, \tilde{x}, \tilde{p})$ and that $(\tilde{t}, \tilde{x}, \tilde{p})$ is also a minimum point of $v_{I(J \cup K_1)*} - \tilde{\varphi}$ on $\bar{D}_{I(J \cup K_1)} \subset D_{I,J'}$. In view of Step 1, this implies that

$$\max\{\tilde{\varphi} - \nu_{I(J \cup K_1)}^*, -\partial_t \tilde{\varphi} + F_{I,J*}^* \tilde{\varphi}\}(\tilde{t}, \tilde{x}, \tilde{p}) \geq 0,$$

which, by (3.19) and the inequality $F_{I,J*}^* \leq F_{I,J*}^*$, implies that

$$\tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{p}) \geq \nu_{I(J \cup K_1)}^*(\tilde{t}, \tilde{x}, \tilde{p}).$$

After at most $\kappa$ iterations of this argument, we finally obtain $\tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{p}) \geq v_{I,J*}(\tilde{t}, \tilde{x}, \tilde{p})$.

**Step 3.** Repeating the arguments of Section 6.1 of [12], we then deduce from Step 2 that, for any smooth function $\tilde{\varphi}$ on $\bar{D}_{I,J'}$ and $(\tilde{t}, \tilde{x}, \tilde{p}) \in D_{I,J'} \cap \bar{D}_{I,J}$ such that

$$\min(\text{strict})_{D_{I,J}} (v_{I,J*} - \tilde{\varphi}) = (v_{I,J*} - \tilde{\varphi})(\tilde{t}, \tilde{x}, \tilde{p}) = 0,$$

(3.20)

we have

$$\max\{\tilde{\varphi} - v_{I,J*}, -\partial_t \tilde{\varphi} + F_{I,J*}^* \tilde{\varphi}\}(\tilde{t}, \tilde{x}, \tilde{p}) \geq 0.$$  

(3.21)

If $v_{I,J*}(\tilde{t}, \tilde{x}, \tilde{p}) = \tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{p}) < v_{I,J*}(\tilde{t}, \tilde{x}, \tilde{p})$ then

$$(-\partial_t \tilde{\varphi} + F_{I,J*}^* \tilde{\varphi})(\tilde{t}, \tilde{x}, \tilde{p}) \geq 0.$$  

(3.22)

Otherwise, $v_{I,J*}(\tilde{t}, \tilde{x}, \tilde{p}) = \tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{p}) = v_{I,J*}(\tilde{t}, \tilde{x}, \tilde{p})$ so that $(\tilde{t}, \tilde{x}, \tilde{p})$ is a local minimizer of $v_{I,J*} - \tilde{\varphi}$ on $\bar{D}_{I,J} \supset \bar{D}_{I,J'}$. In this case, we then deduce from Theorem 2.1 that (3.22) holds too. □
4. On the unbounded control

In this section, we consider our problem in the case where the amount of money that can be invested in the risky assets is unbounded, i.e. \( U = \mathbb{R}^d \). In this case, we find an initial endowment \( y \) and a hedging strategy \( \nu \) such that the terminal value of his hedging portfolio \( Y_{t,x,y}^{\nu}(T) \) diminished by the liquidation value of the claim \( g(X_{t,x}(T)) \) matches an a-priori distribution of the form
\[
\mathbb{P}\left[Y_{t,x,y}^{\nu}(T) - g(X_{t,x}(T)) \geq -\gamma_i \right] \geq p_i, \ i \leq \kappa
\]
and
\[
Y_{t,x,y}^{\nu}(T) \in [\ell, L]
\]
The minimal initial endowment required to achieve the above constraints is given by:
\[
\hat{\nu}(t, x, p) = \inf\{y \geq \ell : \exists (\nu, \alpha) \in \mathcal{U}_t \times \mathcal{A}_{t,p} \text{ s.t. } Y_{t,x,y}^{\nu}(T) \geq G(X_{t,x}(T), P_{t,p}(T))\}.
\]
Note that the boundedness that \( \ell \leq Y_{t,x,y}^{\nu}(T) \leq L \) and that \( 0 \leq P_{t,p}(T) \leq 1 \) implies that
\[
\int_t^T |\nu(s)|^2 + |\alpha(s)|^2 \, ds < \infty,
\]
therefore, the conditions \( Z1 - Z5 \) and \( G1, G2 \) in Theorem 2.1 of Chapter 1 are satisfied. This allows us to apply the geometric dynamic programming principle and collect a PDE characterization of \( \hat{v} \) as a similar form in the case of bounded controls, recall Theorem 2.1 and Theorem 2.2. However, contrary to Remark 2.13, the fact that the control could take any value in \( \mathbb{R}^d \) leads to a comparison result for the associated PDE, the convexity with respect to \( p \) of \( \hat{v} \) and also a face-lift form for the terminal boundary condition as \( t \to T \):
\[
v^*(T, \cdot) = v_*(T, \cdot) = l(1 - \max_{i \leq \kappa} p^i) + \sum_{i=0}^{\kappa-1} g^{i+1}(p^{i+1} - \max_{k \leq i} p^k)_+ =: \tilde{G}.
\]
This leads directly to a PDE characterization of \( v \) under a terminal boundary condition as \( t \to T \):

**Theorem 4.1** The function \( \hat{v} \) is continuous and is the unique bounded discontinuous viscosity solution of the system (S) in the class of bounded discontinuous solutions \( V \) which are non-decreasing in their last variable and satisfy \( V_*(T, \cdot) = V^*(T, \cdot) = \tilde{G} \) for all \( (I, J) \), \( (I', J') \) such that \( (I', J') \supseteq (I, J) \).

In order to prove above result, we report the proof of the terminal boundary condition and next the corresponding comparison. It start with the terminal boundary condition as \( t \to T \).
**Proposition 4.1** For all \((x, p) \in \mathbb{R}^d \times [0, 1]^\kappa\), we have
\[
v_{IJ*}(T, x, p) = v_{IJ*}(T, x, p) = \tilde{G}(x, p).
\]

**Proof.** We consider the required result on each domain \(D_{IJ}\) for \((I, J) \in \mathcal{P}_\kappa\). The proof as following steps:

1. We first prove the required result in the case where \(\kappa = 1\). It follows from (ii) of Theorem 2.2 that \(v^*(T, x, 1) \leq g_1(x)\) and \(v^*(T, x, 0) \leq \ell\). And the subsolution property of \(v^*\) implies that \(v^*(t, x, \cdot)\) is convex for \((t, x) \in [0, T) \times \mathbb{R}^d\). Then
\[
v^*(T, x, p^1) \leq p^1 g^1(x) + \ell(1 - p^1).
\]

It remains to prove that
\[
v_*(T, x, p^1) \geq p^1 g^1(x) + \ell(1 - p^1).
\]

Let \((t_n, x_n, p_n)_n \subset D\) be a sequence that converges to \((T, x, p)\) and such that \(v(t_n, x_n, p_n) \to v_*(T, x, p)\). We define \(y_n := v(t_n, x_n, p_n) + n^{-1}\) so that, for each \(n\), there exists \((\nu_n, \alpha_n) \in \mathcal{U}^n \times \mathcal{A}_{t_n, p_n}\) satisfying \(y_n + Y_n(T) \geq \ell\) and
\[
Y_n(T) \geq g^1(X_n(T))1_{P^1(T) > 0}.
\]

where \((X_n, Y_n, P^1_n) := (X_{t_n, x_n}, Y_{t_n, x_n, y_n}, P_{t_n, p_n}^{\nu_n, \alpha_n})\). This implies that
\[
Y_n(T) \geq P^1_n(T)g^1(X_n(T)) + \ell(1 - P^1_n(T)).
\]

Let \(Q_n\) be the \(\mathbb{P}\)-equivalent martingale measure defined be
\[
dQ_n/d\mathbb{P} = \mathcal{E}\left(\int \mu^{-1}(X_n(s))ds\right) = H_n(T).
\]

We have
\[
y_n \geq \mathbb{E}^{Q_n}[P^1_n(T)g^1(X_n(T)) + \ell(1 - P^1_n(T))]
\]
\[
\geq p_n g^1(x_n) + \ell(1 - p_n) - \mathbb{E}[H_n(T)g^1(X_n(T)) - g^1(x_n)] + \ell[H_n(T) - 1],
\]

Passing to the limit, \(v_*(T, x, p^1) \geq \tilde{G}(x, p^1)\).

2. To conclude the proof, we now assume that the required result holds if \(\kappa \leq k\) for some \(k \geq 1\), and show that it holds if \(\kappa = k + 1\). We fix \((x, p) \in \mathbb{R}^d \times [0, 1]^{k+1}\) and consider to different situations separately.

a. In the case that \(\{p^i\}_{i \geq 1}\) is not a non-decreasing sequence, this means that there exists \(i' < j'\) such that \(p^{i'} > p^{j'}\). Note that, since \(g^{i'} > g^{j'}\), \(v\) does not depend on \(p^{j'}\) when \(p^{i'}\) belongs to the interval \([0, p^{j'}]\). Then,
\[
v^*(t, x, p) \leq v^*(t, x, \tilde{p}) \leq v_{(j')}^*(t, x, \tilde{p})\text{ and }v_*(t, x, p) \geq v_*(t, x, \tilde{p}) \geq v_{(j')}^*(t, x, \tilde{p}),
\]

where \(\tilde{p}\) is the latest possible value for \(p^{i'}\) before \(j'\).
where \( \tilde{p} := \pi_{(p)}(p) \). The required result follows from the recurrent assumption for \( \tilde{p} \).

b. In the case that \( \{p^i\}_{i \geq 1} \) is a non-decreasing sequence, we now prove that

\[
v_\ast(T, x, p) \geq \tilde{G}(x, p).
\]

Let \((t_n, x_n, p_n)_n \subset D\) be a sequence that converges to \((T, x, p)\) and such that

\[
v(t_n, x_n, p_n) \to v_\ast(T, x, p).
\]

We define \( y_n := v(t_n, x_n, p_n) + n^{-1} \) so that, for each \( n \), there exists \((\nu^\ast, \alpha^\ast) \in \mathcal{U}^{\nu_n} \times \mathcal{A}_{\nu_n, p_n}\) satisfying \( y_n + Y_n(T) \geq \ell \) and

\[
Y_n(T) \geq \max_{i \leq k} g^i(X_n(T))1_{P_n(T) > 0},
\]

where \((X_n, Y_n, P_n) := (X_{t_n, x_n}, Y_{t_n, x_n, y_n}, P_{t_n, p_n})\). This implies that

\[
Y_n(T) \geq \sum_{i \leq k} g^i(X_n(T))(P_n(T) - P_{n-1}^i(T)) + \ell(1 - P_n^\ast(T)).
\]

Let \( Q_n \) be the \( \mathbb{P} \)-equivalent martingale measure defined be

\[
dQ_n/d\mathbb{P} = \mathcal{E}(\int \gamma(X_n(s))ds) = H_n(T).
\]

We have

\[
y_n \geq \mathbb{E}^{Q_n}[\sum_{i \leq k} g^i(X_n(T))(P_n(T) - P_{n-1}^i(T)) + \ell(1 - P_n^\ast(T))]
\]

\[
\geq \mathbb{E}^{Q_n}[\sum_{i \leq k} (g^i(X_n(T)) - g^{i+1}(X_n(T)))P_n(T) + \ell]
\]

\[
\geq \tilde{G}(x_n, p_n) - \sum_{i \leq k} \mathbb{E}[|H_n(T)(g^i(X_n(T)) - g^{i+1}(X_n(T))) - (g^i(x_n) - g^{i+1}(x_n))|],
\]

where \( g^{k+1} = \ell \). Passing to the limit, \( v_\ast(T, x, p) \geq \tilde{G}(x, p) \). It remains to prove that

\[
v_\ast(T, x, p) \leq \tilde{G}(x, p).
\]

Recall that \( v_\ast \) is convex in \( p \). This, together with the fact that \( p^1 \leq .. \leq p^{k+1} \), implies that

\[
v_\ast(t, x, p) \leq p^{k+1}v_\ast(t, x, \frac{p}{p^{k+1}}) + (1 - p^{k+1})v_\ast(t, x, 0_{k+1})
\]

\[
\leq p^{k+1}\left[p^{k}\frac{p}{p^{k+1}}v_\ast(t, x, \tilde{p}, 1, 1) + (1 - \frac{p^k}{p^{k+1}})v_\ast(t, x, 0_{k+1})\right]
\]

\[
+(1 - p^{k+1})v_\ast(t, x, 0_{k+1})
\]

\[
\leq p^{k+1}\left[p^{k}\frac{p}{p^{k+1}}v^{\ast}_{(k, k+1)}(t, x, \tilde{p}) + (1 - \frac{p^k}{p^{k+1}})v^{\ast}_{(1,...,k)}(t, x, 1)\right]
\]

\[
+(1 - p^{k+1})v^{\ast}_{(1,...,k)}(t, x)
\]

\[
\leq p^{k}v^{\ast}_{(k, k+1)}(t, x, \tilde{p}) + (p^{k+1} - p^{k})v^{\ast}_{(1,...,k)}(t, x, 1)
\]

\[
+(1 - p^{k+1})v^{\ast}_{(1,...,k)}(t, x)
\]
where \( \tilde{p} := (p^1, ..., p^{k-1})/p_k \) and \( 0_k := (0)_{i \leq k} \).

Considering a consequence \((t_n, x_n, p_n) \in [0, T) \times \mathbb{R}^d \times (0, 1)^{k+1} \) such that \((t_n, x_n, p_n) \to (T, x, p)\) and \(v^*(t_n, x_n, p_n) \to v^*(T, x, p)\) and using the recurrent assumption, we have

\[
v^*(T, x, p) \leq p_k \hat{G}(x, \tilde{p}, 1, 0) + (p_k^{k+1} - p_k^k) g^{k+1} + (1 - p_k^{k+1}) \ell = \hat{G}(x, p).
\]

Before introducing the comparison result, we convert the associated PDE \((S)\) to a new form

\[
L^{u,a}(x, q, Q) = q^\top p a \sigma(x)^{-1} \mu(x) - \frac{1}{2} \Xi^a(x, Q) =: \hat{L}^a(x, q, \Xi^a(x, Q))
\]

where

\[
\Xi^a(x, Q) := \text{Trace} [\sigma_X p \sigma_X^\top (x, a) Q],
\]

see also the subsection 2.1 of next chapter. Note that such reformulation does not depend on the control variable \(u\). This leads to a comparison result for \((S)\) as follows:

**Proposition 4.2** Fix \((I, J) \in \mathcal{P}_n\). Let \(V_1\) (resp. \(V_2\)) be a bounded lower-semicontinuous (resp. upper-semicontinuous) viscosity supersolution of on \(D_{IJ}\),

\[
\sup_{a \in A_{IJ}} \max \{ \varphi - L, \min \{ \varphi - \ell, -\partial_t \varphi + \hat{L}^a(x, D\varphi, \text{Tr}[\sigma_X p (\sigma_X) \hat{G}(x, p)] \} \} = 0.
\]

such that

\[
v_{IJ,*}(T, x, p) = v_{IJ,*}(T, x, p) = \hat{G}(x, p).
\]

Assume that \(V_1 \geq V_2\) on \(\partial D_{IJ}\). Then, \(V_1 \geq V_2\) on \(\bar{D}_{IJ}\).

**Proof.** As usual, we first fix \(\kappa > 0\) and introduce the functions \(\hat{V}(t, x, p) := e^{\kappa t} V_1(t, x, p)\), \(\hat{V}_2(t, x, p) := e^{\kappa t} V_2(t, x, p)\) and \(\hat{G}^\kappa(t, x, p) := e^{\kappa t} \hat{G}(x, p)\), so that the function \(\hat{V}_1\) (resp. \(\hat{V}_2\)) is a viscosity supersolution (resp. subsolution) of

\[
\sup_{a \in A_{IJ}} \left\{ -\partial_t \varphi + \kappa \varphi + \hat{L}^a(x, D\varphi, \text{Tr}[\sigma_X p (\sigma_X) \hat{G}(x, p)] \} = 0
\]

on \([0, T) \times \mathbb{R}^d \times [0, 1]^k\) and

\[
\hat{V}_1 \geq \hat{G}^\kappa \geq \hat{V}_2 \text{ on } \mathbb{R}^d \times [0, 1]^k.
\]

We need to show that

\[
\sup_{[0, T] \times \mathbb{R}^d \times [0, 1]^k} (\hat{V}_2 - \hat{V}_1) \leq 0.
\]

Arguing by contradiction, we therefore assume that

\[
\sup_{[0, T] \times \mathbb{R}^d \times [0, 1]^k} (\hat{V}_2 - \hat{V}_1) =: 2m > 0.
\]
and work towards a contradiction. We choose a family \(\{\beta_r\}_{r \geq 0}\) of \(C^2\) functions on \(\mathbb{R}^d\) parameterized by \(r \geq 0\) with following properties:

\[
\begin{align*}
(i) & \quad \beta_r \geq 0, \\
(ii) & \quad \liminf_{|x| \to \infty} \beta_r(x)/|x| \leq 2L, \\
(iii) & \quad \|D_{xx}\beta_r\| \leq C, \\
(iv) & \quad \lim_{r \to \infty} \beta_r(x) = 0, \text{ for } x \in \mathbb{R}^d.
\end{align*}
\]

It follows from (4.1)(ii)-(iii) that \(\beta := e^{-rt}\beta_r(x)\) satisfies, for some \(\kappa, \alpha, r > 0\),

\[
-\partial_t \beta + \kappa \beta - \frac{1}{2} \text{Trace } [\sigma\sigma^\top D_{xx}\beta] \geq 0,
\]

and function \(\Phi := U - V - 2\beta\) achieves its maximum at \((\hat{t}, \hat{x}, \hat{p})\) on \([0, T] \times \mathbb{R}^d \times [0, 1]^k\) with

\[
\Phi(\hat{t}, \hat{x}, \hat{p}) \geq m.
\]

2. For \(n \geq 1\), we then define the function \(\Psi_n\) on \([0, T] \times \mathbb{R}^{2n} \times [0, 1]^{2k}\) by

\[
\Psi_n(t, x, y, p, q) := \Theta(t, x, y, p, q) - (|x - \bar{x}|^2 + |t - \bar{t}|^2 + |p - \bar{p}|^2)
\]

\[
- n^2(|x - y|^2 - |p - q|^2),
\]

where

\[
\Theta(t, x, y, p, q) := \tilde{V}_2(t, x, p) - \tilde{V}_1(t, y, q) - (\beta(t, x) + \beta(t, y)).
\]

It follows from the boundedness property of \(\tilde{V}_2\) and \(\tilde{V}_1\) that \(\Psi_n\) achieves its maximum at some \((t^n, x^n, y^n, p^n, q^n) \in [0, T] \times \mathbb{R}^{2n} \times [0, 1]^{2k}\). Moreover, the inequality \(\Psi_n(t^n, x^n, y^n, p^n, q^n) \geq \Psi_n(\hat{t}, \hat{x}, \hat{p}, \hat{p})\) implies that

\[
\Theta(t^n, x^n, y^n, p^n, q^n) \geq \Theta(\hat{t}, \hat{x}, \hat{x}, \hat{p}, \hat{p}) + n^2(|x^n - y^n|^2 + |p^n - q^n|^2)
\]

\[
+ (|x^n - \bar{x}|^2 + |t^n - \bar{t}|^2 + |p^n - \bar{p}|^2).
\]

Using the boundedness property of \(\tilde{V}_2\) and \(\tilde{V}_1\) again together with the fact that \([0, 1]^k\) is compact, we deduce that the term on the second line is bounded in \(n\) so that, up to a subsequence,

\[
x^n, y^n \overset{n \to \infty}{\longrightarrow} \bar{x} \in \mathbb{R}^d, \quad p^n, q^n \overset{n \to \infty}{\longrightarrow} \bar{p} \in [0, 1]^k, \quad \text{and } t^n \overset{n \to \infty}{\longrightarrow} \bar{t} \in [0, T].
\]

Sending \(n \to \infty\) in the previous inequality and using the maximum property of \((\hat{t}, \hat{x}, \hat{p})\), we also get

\[
0 \geq \Phi(\hat{t}, \hat{x}, \hat{p}) - \Phi(\hat{t}, \hat{x}, \hat{p})
\]

\[
\geq \limsup_{n \to \infty} \left( n^2(|x^n - y^n|^2 + |p^n - q^n|^2) + |p^n - \bar{p}|^2 + |x^n - \bar{x}|^2 + |t^n - \bar{t}|^2 \right),
\]

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which shows that $(\bar{t}, \bar{x}, \bar{p}) = (\bar{t}, \bar{x}, \bar{p})$, and

\[
(a) \quad n^2|x^n - y^n|^2 + n^2|p^n - q^n|^2 + |x^n - \bar{x}|^2 + |t^n - \bar{t}|^2 + |p^n - \bar{p}|^2 \xrightarrow{n \to \infty} 0,
\]

\[
(b) \quad \tilde{V}_2(t^n, x^n, p^n) - \tilde{V}_1(t^n, y^n, q^n) \xrightarrow{n \to \infty} \left( \tilde{V}_2 - \tilde{V}_1 \right) (\tilde{t}, \tilde{x}, \tilde{p}) \geq m + 2\beta(\tilde{t}, \tilde{x}) > 0.
\]

Assume that, after possibly passing to a subsequence, $t^n = T$, for all $n \geq 1$. Then, Ishii’s Lemma, see e.g. [1],

\[
\tilde{V}_2(T, x^n, p^n) \leq \tilde{g}(T, x^n, p^n) \quad \text{and} \quad \tilde{V}_1(T, y^n, q^n) \geq \tilde{g}(T, y^n, q^n),
\]

which leads to a contradiction to (b). In view of the above point, we can now assume, after possibly passing to a subsequence, that $t^n < T$ for all $n \geq 1$. From Ishii’s Lemma, see Theorem 8.3 in [18], we deduce that, for each $\epsilon > 0$, there are real coefficients $b_{1,n}, b_{2,n}$ and symmetric matrices $X^{n,\epsilon}$ and $Y^{n,\epsilon}$ such that

\[
(b_{1,n}, a_{1,n}, X^{n,\epsilon}) \in \mathcal{D}^+ \tilde{V}_2(t^n, x^n, p^n) \quad \text{and} \quad (b_{2,n}, a_{2,n}, Y^{n,\epsilon}) \in \mathcal{D}^- \tilde{V}_1(t^n, y^n, q^n),
\]

see [18] for the standard notations $\mathcal{D}^+$ and $\mathcal{D}^-$, where

\[
a_{1,n} := 2n^2(p^n - q^n) + 2(p^n - \bar{p}),
\]

\[
a_{2,n} := -2n^2(p^n - q^n),
\]

and $b_{1,n}, b_{2,n}, X^{n,\epsilon}$ and $Y^{n,\epsilon}$ satisfy

\[
\begin{align*}
&\left( \begin{array}{c}
\mathcal{X}^{n,\epsilon}_{x,x} \\
\mathcal{X}^{n,\epsilon}_{x,p} \\
\mathcal{X}^{n,\epsilon}_{p,p}
\end{array} \right)
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\leq (A_n + B_n) + \epsilon(A_n + B_n)^2. \quad (4.4)
\end{align*}
\]

with

\[
A_n := 2n^2
\begin{pmatrix}
Id_x & 0 & -Id_x & 0 \\
0 & Id_p & 0 & Id_p \\
-Id_x & 0 & -Id_x & 0 \\
0 & Id_p & 0 & Id_p
\end{pmatrix}
\]

\[
B_n :=
\begin{pmatrix}
D_{x,x} \beta(t^n, x^n) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & D_{x,x} \beta(t^n, y^n) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

where $Id_p$ stands for the identical matrix with dimension $k \times k$.

This implies that

\[
0 \geq -b_{1,n} + \kappa \tilde{V}_2(t^n, x^n, p^n) - \frac{1}{2} \text{Trace} \left[ \sigma(x^n) \sigma(x^n)^T \mathcal{X}^{n,\epsilon}_{x,x} \right]
\]

\[- \inf_{\alpha \in M^{x \times d}} \left\{ -\alpha a_{1,n} \sigma(x^n)^{-1} \mu(x^n) + \text{Trace} \left[ \sigma(x^n) \alpha \mathcal{X}^{n,\epsilon}_{x,p} \right] + \frac{1}{2} \text{Trace} \left[ \alpha \alpha^T \mathcal{X}^{n,\epsilon}_{p,p} \right] \right\}, \quad (A)
\]

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We also obtain that

\[ 0 \leq b_{2,n} + \kappa \tilde{V}_1(t^n, y^n, q^n) - \frac{1}{2} \text{Trace} \left[ \sigma(y^n)\sigma(y^n)^T \mathcal{Y}_{x,x}^n \right] \]

\[ -n^2 \inf_{\alpha \in \mathbb{R}^{k \times d}} \{ 2\alpha(p^n - q^n)\top \sigma(x^n)^{-1} \mu(x^n) + (1 + \epsilon) \text{Trace} [\alpha \alpha^\top] \}. \quad (B) \]

We choose \( \alpha^n_{i,j} := -(p^n_j - q^n_j)(\sigma^{-1}\mu)_{i,j}(y^n)/ (1 + \epsilon) \)
so that the inferior in (B) achieves the minimum at \( \alpha^n \) and

\[ -\alpha^n(p^n - q^n)^\top (\sigma^{-1}\mu)(y^n) = (1 + \epsilon) \text{Trace} [\alpha^n(\alpha^n)^\top]. \]

This, together with (A), implies that

\[ 0 \geq -(b_{2,n} + b_{1,n}) + \kappa (\tilde{V}_2(t^n, x^n, p^n) - \tilde{V}_1(t^n, y^n, q^n)) \]

\[ + \frac{1}{2} \text{Trace} \left[ \sigma(y^n)\sigma(y^n)^T \mathcal{Y}_{x,x}^n \right] - \frac{1}{2} \text{Trace} \left[ \sigma(x^n)\sigma(x^n)^\top \mathcal{X}_{x,x}^n \right] \]

\[ -2n^2 \alpha^n(p^n - q^n)^\top [ (\sigma^{-1}\mu)(x^n) + (\sigma^{-1}\mu)(y^n) ] \]

\[ -\alpha^n(p^n - \tilde{p})^\top (\sigma^{-1}\mu)(x^n). \]

Since \( \sigma^{-1}\mu \) is bounded in \( \mathbb{R}^d \), we take \( \epsilon \to 0 \) and deduce that

\[ 0 \geq -2(t^n - \tilde{t}) + \kappa [ \tilde{V}_2(t^n, x^n, p^n) - \tilde{V}_1(t^n, y^n, q^n) - \beta(t^n, x^n) - \beta(t^n, y^n) ] \]

\[ -\partial_t \beta(t^n, x^n) + \kappa \beta(t^n, x^n) - \frac{1}{2} \text{Trace} \left[ \sigma(x^n)\sigma(x^n)^\top D_{x,x} \beta(t^n, x^n) \right] \]

\[ -\partial_t \beta(t^n, y^n) + \kappa \beta(t^n, y^n) - \frac{1}{2} \text{Trace} \left[ \sigma(y^n)\sigma(y^n)^\top D_{x,x} \beta(t^n, y^n) \right] \]

\[ + n^2 |\sigma(y^n) - \sigma(x^n)|^2 - C(n^2 |p^n - q^n|^2 + |p^n - \tilde{p}|^2) \]

\[ \geq \kappa m + \lambda(n), \]

where \( \lim_{n \to \infty} \lambda(n) = 0 \). This leads a contradiction. \[ \square \]
Chapitre 4

The approximating problems and numerical schemas

We shall see that both the associated Hamilton-Jacobi-Bellman operator and the boundary conditions in Theorem 2.1 and Corollary 2.1 of previous chapter are discontinuous, which leaves little hope to be able to establish a comparison result, and therefore build a convergent numerical scheme directly based on this PDE characterization. We therefore introduce a sequence of approximating problems that are more regular and for which we can prove comparison. We show that they converge to the value function at any continuity point in the $p$-variable, or, more precisely, to its right and left limits in the $p$-variable, depending on the chosen approximating sequence. In particular, we will show that it allows to approximate point-wise the relaxed problems:

\[ v(t, x, p) \]  
\[ := \inf \left\{ y : \forall \varepsilon > 0 \ \exists \nu^\varepsilon \ \text{s.t.} \ Y_{t,x,y}^{\nu^\varepsilon}(T) \geq \ell, \ P \left[ Y_{t,x,y}^{\nu^\varepsilon}(T) - g(X_{t,x}(T)) \geq -\gamma^i \right] \geq p^i - \varepsilon \ \forall \ i \right\} \]  

and

\[ \bar{v}(t, x, p) \]  
\[ := \inf \{ y : \exists \nu \ \text{s.t.} \ Y_{t,x,y}^{\nu}(T) \geq \ell, \ P \left[ Y_{t,x,y}^{\nu}(T) - g(X_{t,x}(T)) \geq -\gamma^i \right] > p^i \ \forall \ i \leq \kappa \}. \]

The first value function $v$ is indeed shown to be the left-limit in $p$ of $v$, while $\bar{v}$ is the right-limit in $p$ of $v$. In cases where $v$ is continuous, then $\bar{v} = v = \underline{v}$ and our schemes converge to the original value function. However the continuity of $v$ in its $p$-variable seems are a-priori difficult to prove by lack of convexity and strict monotonicity of the indicator function, and may fail in general. Still, one of the two approximations can be chosen to solve practical problems.
1 The approximating problems

1.1 Definition and convergence properties

Set $\Lambda := (0, (L^{-1} \land 1)/2)^e$. Our approximating sequence $(v^\lambda)_{\lambda \in \Lambda}$ is a sequence of value functions associated to regularized stochastic target problems with controlled loss. Namely, for $\lambda \in \Lambda$, we set

$$v^\lambda(t,x,p) := \inf\{y \geq \ell \colon \exists \nu \in \mathcal{U}^t \text{ s.t. } Y_{t,x,y}^\nu(T) \geq \ell, \ E[\Delta^i_{\lambda}(X_{t,x}(T),Y_{t,x,y}^\nu(T))] \geq p^i \text{ for } i \in \mathcal{K}\},$$

where

$$\Delta^i_{\lambda}(x,y) = \begin{cases} 
0 & \text{if } y < \ell \\
\frac{\lambda^i(y-\ell)}{g^i(x)-2\lambda^i} & \text{if } \ell \leq y < g^i(x) - 2\lambda^i \\
\lambda^i + \frac{\lambda^i(y-g^i(x)+2\lambda^i)}{g^i(x)-g^i(x)+\lambda^i} & \text{if } g^i(x) - 2\lambda^i \leq y < g^i(x) - \lambda^i \\
(1 - \lambda^i) + \frac{\lambda^i(y-g^i(x)+\lambda^i)}{g^i(x)-g^i(x)+\lambda^i} & \text{if } g^i(x) - \lambda^i \leq y < \hat{g}^i(x) \\
1 & \text{if } y \geq \hat{g}^i(x)
\end{cases}$$

(1.1)

is well-defined for $\lambda \in \Lambda = (0, (L^{-1} \land 1)/2)^e$ as in Figure 4.1, recall (2.7), (2.8) and (2.11) of previous chapter.
The convergence \((v^\lambda)_{\lambda \in \Lambda}\) as \(\lambda \downarrow 0\) is an immediate consequence of the linearity of \(Y^\nu\) with respect to its initial condition.

**Proposition 1.1** *For all* \(\lambda \in \Lambda\) *and* \((t, x, p) \in [0, T] \times (0, \infty)^d \times [0, 1]^\kappa\),

\[
v^\lambda(t, x, p \oplus \lambda) + 2 \max_{i \leq \kappa} \lambda^i \geq v(t, x, p) \geq v^\lambda(t, x, p \ominus \lambda) - \max_{i \leq \kappa} \lambda^i, \tag{1.2}
\]

*where*

\[p \oplus \lambda := (((p^i + \lambda^i) \wedge 1) \vee 0)_{i \leq \kappa}\] and \(p \ominus \lambda := (((p^i - \lambda^i) \wedge 1) \vee 0)_{i \leq \kappa}\).

**Proof.** This follows easily from the linearity of \(Y^\nu\) with respect to its initial condition and the fact that

\[1_{\{y - g(x) + 2\lambda^i \geq 0\} + \lambda^i \geq \Delta^i_\lambda(x, y) \geq 1_{\{y - g(x) + \lambda^i \geq 0\}} - \lambda^i, \text{ for } (y, x) \in \mathbb{R} \times (0, \infty)^d.\]

\[\square\]

As an immediate consequence, we deduce that the sequences \((v^\lambda(\cdot, \cdot \oplus \lambda))_{\lambda \in \Lambda}\) and \((v^\lambda(\cdot, \cdot \ominus \lambda))_{\lambda \in \Lambda}\) allows to approximate \(v\) at any continuity points in its \(p\)-variable. More precisely, the following holds.

**Corollary 1.1** *For all* \((t, x, p) \in [0, T] \times (0, \infty)^d \times [0, 1]^\kappa\),

\[v(t, x, p-) = \lim_{\lambda \downarrow 0} v^\lambda(t, x, p \ominus \lambda) \text{ and } v(t, x, p+) = \lim_{\lambda \downarrow 0} v^\lambda(t, x, p \oplus \lambda), \tag{1.3}\]

*where*

\[v(\cdot, p-) := \lim_{\varepsilon \downarrow 0} v(\cdot, p \ominus \varepsilon 1_\kappa) \text{ and } v(\cdot, p+) := \lim_{\varepsilon \downarrow 0} v(\cdot, p \oplus \varepsilon 1_\kappa)\]

*with* \(1_\kappa = (1, \ldots, 1) \in \mathbb{R}^\kappa\).

Proving the continuity in its \(p\)-variable of the initial value function \(v\) by probabilistic arguments, and therefore the point-wise convergence of our approximation seems very difficult, and is beyond the scope of this paper. A standard approach could be to derive the continuity of \(v\) by using its PDE characterization and by applying a suitable comparison theorem which would imply that \(v_* = v^\ast\). As explained at the end of Subsection 2.3 of Chapter 3, this also does not seem to be feasible.

Note however that the right- and left-limits of \(v\) in its \(p\)-variable have interpretations (0.1) and (0.2) in terms of natural relaxed version of the original problem (2.9) in Chapter 3.

**Proposition 1.2** *For all* \((t, x, p) \in [0, T] \times (0, \infty)^d \times (0, 1)^\kappa\),

\[v(t, x, p+) = \bar{v}(t, x, p) \geq \underline{v}(t, x, p) = v(t, x, p-).\]

**Proof.** It is obvious that \(\bar{v} \geq \underline{v} \geq \underline{w}\). Moreover, any \(y > v(t, x, p + \varepsilon 1_\kappa)\) for some \(\varepsilon > 0\), satisfies \(y \geq \bar{v}(t, x, p)\). Hence, for \(\varepsilon > 0\) small enough, \(v(t, x, p + \varepsilon 1_\kappa) \geq \bar{v}(t, x, p)\), so that \(v(t, x, p+) \geq \bar{v}(t, x, p)\). Similarly, \(y > \underline{w}(t, x, p)\) implies \(y \geq v(t, x, p - \varepsilon 1_\kappa)\), for any \(\varepsilon > 0\) small enough, and therefore \(v(t, x, p) \geq v(t, x, p-)\). \(\square\)
1.2 PDE characterization of the approximating problems

The reason for introducing the sequence approximating problems \((v^\lambda)_{\lambda \in \Lambda}\) is that they are more regular:

1. \(\Delta^\lambda (x,y+h) - \Delta^\lambda (x,y) \leq C_\lambda |h|\), for \((x,y,h) \in (0,\infty)^d \times \mathbb{R} \times \mathbb{R}\). \((1.4)\)

where

\[
C_\lambda := \max_{i \in K} \max \{\lambda^i / (L^{-1} - 2\lambda^i) , 1 / \lambda^i , 1\}, \tag{1.5}
\]

recall (2.8) and (2.11) of previous chapter.

2. Its inverse with respect to its y-variable is Lipschitz continuous too. Hence, the natural boundary condition at \(T\) is given by a continuous function

\[
G^\lambda (x,p) := \inf \{ y \geq \ell : \min_{i \in K} (\Delta^\lambda (x,y) - p^i) \geq 0 \}. \tag{1.6}
\]

Item 2. above will allow us to prove that the boundary condition as \(t \to T\) is indeed given by the continuous function \(G^\lambda\), compare with 1. of Remark 2.13 in Chapter 3.

**Proposition 1.3** The function \(v^\lambda\) satisfies

\[
v^\lambda (T,\cdot) = v^\lambda (\cdot,\cdot) = G^\lambda \text{ on } (0,\infty)^d \times [0,1]^\kappa. \tag{1.7}
\]

**Proof.** See Section 3 below. \(\square\)

Item 1. above induces a gradient constraint on \(v^\lambda\) with respect its \(p\)-variable, showing that it is strictly increasing with respect to this variable, in a suitable sense, which will allow us to prove a comparison result for the related PDE, compare with Remark 2.11 and 3. of Remark 2.13 in previous chapter. We could not obtain this for the original problem by lack of continuity and local strict monotonicity of the indicator function. More precisely, we shall prove in Subsection 3.3 below the following.

**Proposition 1.4** Set

\[
\varrho := 4L^2(T \vee 1). \tag{1.8}
\]

Fix \((I,J) \in \mathcal{P}_\kappa \setminus \mathcal{P}_\kappa^c\) and assume that \(t \geq 0\) is such that \(v^\lambda_{ij}(t,x,p) > v^\lambda_{ij}(t,x,p) + \varphi\). Let \(\varphi\) be a smooth function such that \((t,x,p)\) achieves a maximum of \(v^\lambda_{ij} - \varphi\). Then,

\[
\sum_{i \in I \cup J} D_p^i \varphi \geq \frac{t}{C_\lambda (t + \varrho)} =: \omega_\lambda(t), \tag{1.9}
\]

where \(C_\lambda\) is defined as in (1.5).
1. THE APPROXIMATING PROBLEMS

Note that the above can be translated in terms of the operator $M^\lambda_{IJ}$ defined as:

$$(y,z,q_p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \mapsto M^\lambda_{IJ}(y,z,q_p) := \max_{\iota \geq 0} \min\{ y - z - \iota, \kappa(\iota) - \sum_{i \in I \cup J} q^i_p \}.$$

**Corollary 1.2** Fix $(I,J) \in \mathcal{P}_\kappa \setminus \mathcal{P}^n_\kappa$. Then $v^\lambda_{IJ}$ is a viscosity subsolution on $D_{IJ}$ of

$$M^\lambda_{IJ}(\varphi, v^\lambda_{IJ}, D_p\varphi) = 0.$$

In view of Theorem 2.1 in [12], this implies that $v^\lambda$ is a discontinuous viscosity solution of the system $(S^\lambda)$ defined as follows, where we use the convention

$$M^\lambda_{IJ} = -\infty \text{ for } I \cup J = \mathcal{K}.$$  \hspace{1cm} (1.10)

**Definition 1.1** Let $V$ be a locally bounded map defined on $\overline{D}$.

(i) We say that $V$ is a discontinuous viscosity supersolution of $(S^\lambda)$ if, for each $(I,J) \in \mathcal{P}_\kappa$, $V_{IJ}^s$ is a viscosity supersolution on $D_{IJ}$ of

$$H_{IJ}^\lambda[\varphi, V_{IJ}^s] := \max \left\{ \min \{ \varphi - \ell, -\partial_t \varphi + F^s_{IJ}(\cdot, D\varphi, D^2\varphi) \} , M^\lambda_{IJ}(\varphi, V_{IJ}^s, D_p\varphi) \right\} = 0.$$  \hspace{1cm} (1.11)

(ii) We say that $V$ is a discontinuous viscosity subsolution of $(S^\lambda)$ if, for each $(I,J) \in \mathcal{P}_\kappa$, $V_{IJ}^s$ is a viscosity subsolution on $D_{IJ}$ of

$$H_{IJ}^\lambda[\varphi, V_{IJ}^s] := \max \left\{ \min \{ \varphi - \ell, -\partial_t \varphi + F_{IJ}(\cdot, D\varphi, D^2\varphi) \} , M^\lambda_{IJ}(\varphi, V_{IJ}^s, D_p\varphi) \right\} = 0.$$  \hspace{1cm} (1.12)

(iii) We say that $V$ is a discontinuous viscosity solution of $(S^\lambda)$ if it is both a discontinuous super- and subsolution of $(S^\lambda)$.

**Remark 1.1** The convention (1.10) means that a supersolution of (1.11) (resp. a subsolution of (1.12)) for $I \cup J = \mathcal{K}$ is indeed a supersolution of (2.20) (resp. a subsolution of (2.21)) of Chapter 3.

**Remark 1.2** Note that a viscosity supersolution of (2.20) on $D_{IJ}$ is also a viscosity supersolution of (1.11) on $D_{IJ}$. As already argued,

$$v^\lambda$$

is a discontinuous solution of $(S)$ \hspace{1cm} (1.13)

by Theorem 2.1 in [12], so that Corollary 1.2 implies that it is a discontinuous solution of $(S^\lambda)$. From the supersolution point of view, the latter characterization is weaker. Still we shall use it because, first, it is sufficient and, second, we shall appeal to it when discussing the convergence of a finite difference approximation scheme below.
Combining the above results, we obtain:

**Theorem 1.1** The function $v^\lambda$ is a discontinuous viscosity solution of $(S^\lambda)$. Moreover, it satisfies

$$v^\lambda(T, \cdot) = v_*(T, \cdot) = G^\lambda \text{ on } (0, \infty)^d \times [0, 1]^\kappa. \quad (1.14)$$

The fact that the above Theorem allows to characterize uniquely $v^\lambda$ is a consequence of the following comparison result, in the viscosity sense.

**Theorem 1.2** (i) Let $V$ be a bounded function on $[0, T) \times (0, \infty)^d \times [0, 1]^\kappa$ which is non-decreasing with respect to its last parameter. Assume that $V$ is a discontinuous viscosity supersolution of $(S^\lambda)$ such that $V_*(T, \cdot) \geq G^\lambda$ and $V_{IJ} \geq V_{I'J'}$ on $\partial D_{IJ} \cap D_{I'J'}$ for all $(I, J), (I', J') \in \mathcal{P}_\kappa$ such that $(I', J') \supseteq (I, J)$. Then, $V \geq v^\lambda$ on $\bar{D}$.

(ii) Let $V$ be a bounded function on $[0, T) \times (0, \infty)^d \times [0, 1]^\kappa$ which is non-decreasing with respect to its last parameter. Assume that $V$ is a discontinuous viscosity subsolution of $(S^\lambda)$ such that $V^*(T, \cdot) \leq G^\lambda$ and $V_{IJ}^* \leq V_{I'J'}^*$ on $\partial D_{IJ} \cap D_{I'J'}$ for all $(I, J), (I', J') \in \mathcal{P}_\kappa$ such that $(I', J') \supseteq (I, J)$. Then, $V \leq v^\lambda$ on $\bar{D}$.

**Proof.** See Subsection 3.5 below. \qed

Combining the above results leads to the following characterization.

**Theorem 1.3** The function $v^\lambda$ is continuous and is the unique bounded discontinuous viscosity solution of the system $(S^\lambda)$ in the class of bounded discontinuous solutions $V$ which are non-decreasing in their last variable and satisfy $V_*(T, \cdot) = V^*(T, \cdot) = G^\lambda$, $V_{IJ} \leq V_{I'J'}$, and $V_{IJ} \geq V_{I'J'}$ on $\partial D_{IJ} \cap D_{I'J'}$ for all $(I, J), (I', J') \in \mathcal{P}_\kappa$ such that $(I', J') \supseteq (I, J)$.

## 2 Finite differences approximation

In this section, we construct an explicit finite difference scheme and prove its convergence.

### 2.1 PDE reformulation

We first reformulate the PDEs associated to $v^\lambda$ in a more tractable way, which will allow us to define naturally a monotone scheme. To this purpose, we introduce
the support function $\delta_U$ associated to the closed convex (and bounded) set $U$ as in (2.24) of Chapter 3. Since $0 \in \text{int} U$, $\delta_U$ characterizes $U$ in the following sense

$$u \in \text{int} U \iff \min_{|\zeta|=1} (\delta_U(\zeta) - \zeta^\top u) > 0 \quad \text{and} \quad u \in U \iff \min_{|\zeta|=1} (\delta_U(\zeta) - \zeta^\top u) \geq 0,$$

see e.g. Rockafellar [39].

Moreover $(u, a) \in N_{IJ}(x, q)$ with $q^\top = (q_x^\top, q_p^\top)$, if and only if there exists $\xi \in \mathbb{R}^d$ such that $|\xi| \leq \epsilon$ for which

$$u^\top = \bar{u}(x, a, q)^\top + \xi^\top \sigma(x)^{-1} \in U \quad \text{and} \quad a \in A_{IJ},$$

where

$$\bar{u}(x, q, a)^\top := q_x^\top \text{diag}[x] + q_p^\top a \sigma(x)^{-1}.$$

It follows that

$$-r + F^*_{IJ}(x, q, Q) \geq 0 \iff \bar{K}^*_{IJ}(x, r, q, Q) \geq 0$$

and

$$-r + F_{IJ*}(x, q, Q) \leq 0 \iff \bar{K}_{IJ*}(x, r, q, Q) \leq 0$$

where $\bar{K}_{IJ}$ and $\bar{K}_{IJ*}$ are the upper- and lower-semicontinuous envelopes of

$$\bar{K}_{IJ}(x, r, q, Q) := \sup_{a \in A_{IJ}} \min \left\{ -r + L^{\bar{u}(x, q, a)}(x, q, Q), R^a(x, q) \right\}$$

with

$$R^a(x, q) := \inf_{|\zeta|=1} \left( R^{a, \zeta}(x, q) \right) \quad \text{and} \quad R^{a, \zeta}(x, q) := \delta_U(\zeta) - \zeta^\top \bar{u}(x, q, a).$$

**Remark 2.1** For later use, note that, for $q^\top = (q_x^\top, q_p^\top)$,

$$L^{\bar{u}(x, q, a)}(x, q, Q) = q_p^\top a \sigma(x)^{-1} \mu(x) - \frac{1}{2} \Xi^a(x, Q) := L^a(x, q_p, \Xi^a(x, Q)).$$

where

$$\Xi^a(x, Q) := \text{Trace} \left[ \sigma_{X,P} \sigma_{X,P}^\top(x, a)Q \right],$$

does not depend on $q_x$.

It follows that $V$ is a viscosity supersolution of (1.11) if and only if it is a viscosity supersolution of

$$\bar{H}_{IJ}^\lambda[\varphi, V_{J*}] := \max \left\{ \min \left\{ \varphi - \ell, \bar{K}_{IJ*}(\cdot, \partial_t \varphi, D \varphi, D^2 \varphi) \right\}, M_{IJ}^\lambda(\varphi, V_{J*}, D_p \varphi) \right\} = 0,$$

and that $V$ is a viscosity sub-solution of (1.12) if and only if it is a viscosity sub-solution of

$$\bar{H}_{IJ*}^\lambda[\varphi, V^*_J] := \max \left\{ \min \left\{ \varphi - \ell, \bar{K}_{IJ*}(\cdot, \partial_t \varphi, D \varphi, D^2 \varphi) \right\}, M_{IJ}^\lambda(\varphi, V^*_J, D_p \varphi) \right\} = 0.$$
2.2 Scheme construction

We now define a monotone finite difference scheme for the formulation obtained in the previous section. In the following, we write \( h \) to denote an element of the form \( h = (h_0, h_1, h_2) \in (0, 1)^3 \).

a. The discretization in the time variable.

Given \( n_0 \in \mathbb{N} \), we first introduce a discretization time-step \( h_0 := T/n_0 \) together with a grid

\[
T^h := \{ (T - (n_0 - i)h_0), \ i = 0, \ldots, n_0 \}.
\]

The time derivative is approximated as usual by

\[
\partial_t^h \phi(t, x, p) := h_0^{-1}(\varphi(t + h_0, x, p) - \varphi(t, x, p)).
\]

b. Discretization in the space variable.

The grids in the space variables are defined as

\[
X^h_{cX} := \{ e^{-cX^+(n-i)h_1}, \ i = 0, \ldots, n_X \}^d \quad \text{and} \quad P^h := \{ 1 - (n-i)h_1, \ i = 0, \ldots, n_P \}^e,
\]

for some \( c_X, n \in \mathbb{N} \) and where \( h_1 := 1/n, \ (n_X, n_P) := n(2c_X, 1) \).

Note that the space discretization in the \( x \) variable amounts to performing a logarithmic change of variable. Taking this into account, the first order derivatives with respect to \( x \) and \( p \) are then approximated as follows, with \( \{ e_i \}_{i \leq d} \) (resp. \( \{ \ell_j \}_{j \leq e} \)) denoting the canonical basis of \( \mathbb{R}^d \) (resp. \( \mathbb{R}^e \)):

\[
\partial_{p,j}^{L,h} \phi(t, x, p) := h_1^{-1} \left\{ \begin{array}{ll}
\varphi(t + h_0, x, p \oplus h_1 \ell_j) - \varphi(t, x, p) & \text{if } \mu^T (a_{\sigma^{-1}}(x)) \ell_j \leq 0 \\
\varphi(t, x, p) - \varphi(t + h_0, x, p \ominus h_1 \ell_j) & \text{if } \mu^T (a_{\sigma^{-1}}(x)) \ell_j > 0
\end{array} \right.
\]

\[
\partial_{x,i}^{R,h} \zeta(t, x, p) := h_1^{-1} \text{diag} [x]^{-1} \left\{ \begin{array}{ll}
\varphi(t + h_0, x \hat{\ominus} h_1 e_i, p) - \varphi(t, x, p) & \text{if } e_i^\top \zeta \geq 0 \\
\varphi(t, x, p) - \varphi(t + h_0, x \hat{\ominus} h_1 e_i, p) & \text{if } e_i^\top \zeta < 0
\end{array} \right.
\]

\[
\partial_{p,j}^{R,h} \zeta(t, x, p) := h_1^{-1} \left\{ \begin{array}{ll}
\varphi(t + h_0, x, p \oplus h_1 \ell_j) - \varphi(t, x, p) & \text{if } \ell_j^T a_{\sigma^{-1}}(x) \zeta \geq 0 \\
\varphi(t, x, p) - \varphi(t + h_0, x, p \ominus h_1 \ell_j) & \text{if } \ell_j^T a_{\sigma^{-1}}(x) \zeta < 0
\end{array} \right.
\]

\[
\partial_{p,j}^{M,h} \phi(t, x, p) := h_1^{-1}(\varphi(t + h_0, x, p \oplus h_1 \ell_j) - \varphi(t, x, p)),
\]

where the operators \( \oplus \) and \( \hat{\ominus} \) are given in Proposition 1.1 and

\[
x \hat{\ominus} y := ((x^i e^y^i) \lor e^{-cX})_{i \leq d}
\]

and

\[
x \hat{\ominus} y := ((x^i e^{-y^i}) \lor e^{cX})_{i \leq d},
\]

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for \((x, y) \in (0, \infty)^d \times \mathbb{R}^d\).

We denote by \(\partial_p^{L,h}, \partial_x^{K,h}, \partial_p^{K,h}\) and \(\partial_p^{M,h}\) the corresponding vectors.

As for the second order term, we use the Camilli and Falcone approximation [16], in order to ensure that the scheme is monotone. Namely, we first introduce an approximation parameterized by \(h_2 > 0\) of Trace \(\sigma(X,P) \sigma^\top(X,P)(x,a) D^2 \varphi(t + h_0, x, p)\) as follows

\[
\tilde{\Delta}^h[\varphi, a](t, x, p) := h_2^{-1} \sum_{i=1}^d \varphi(t + h_0, x \oplus \sqrt{h_2} a^i(x), p \oplus \sqrt{h_2} a^i(x)) - 2h_1^{-1} \sum_{i=1}^d |a^i(x)|^2 (\varphi(t_0, x, p) - \varphi(t + h_0, x \ominus h_1 e_i, p))
\]

(2.3)

where \(\sigma^i\) and \(a^i\) denote the \(i\)-th column of \(\sigma_X\) and \(a\).

Note that the above approximation of the second order term requires the computation of the approximated value function at points outside of the grid. It therefore requires an interpolation procedure. In this paper, we use a local linear interpolation based on the Coxeter-Freudenthal-Kuhn triangulation, see e.g. [36]. It consists in first constructing the set of simplices \(\{S_j\}_j\) associated to the regular triangulation of \(\ln[e^{-c_X}, e^{c_X}]^d \times [0, 1]^\kappa\) with the set of vertices \(\ln X_{c_X} h \times P^h\). Here the \(\ln\) operator means that we take the \(\ln\) component-wise, recall that we use a logarithmic scale. In such a way, we can then provide an approximating function belonging to the set \(S^h\) of the functions which are continuous in \(\ln[e^{-c_X}, e^{c_X}]^d \times [0, 1]^\kappa\) and piecewise affine inside each simplex \(S_j\) (in \(\ln\) scale for the \(x\)-variable). More precisely, each point \((y, p) \in [-c_X, c_X]^d \times [0, 1]^\kappa\) can be expressed as a weighted combination of the corners of the simplex \(S_j\) it lies in. We can thus write

\[
(y, p) = \sum_{\zeta \in \ln X_{c_X} h \times P^h} \omega(y, p \mid \zeta) \zeta,
\]

where \(\omega\) is a non negative weighting function such that

\[
\sum_{\zeta \in \ln X_{c_X} h \times P^h} \omega(y, p \mid \zeta) = 1.
\]

Given a map \(\varphi\) defined on \(T^h \times X_{c_X} h \times P^h\), we then approximate it at \((t, x, p) \in T^h \times [e^{-c_X}, e^{c_X}]^d \times [0, 1]^\kappa\) by

\[
\varphi(t, x, p) := \sum_{(\zeta_X, \zeta_P) \in \ln X_{c_X} h \times P^h} \omega(\ln x, p \mid \zeta) \varphi(t, e^{\zeta_X}, \zeta_P)
\]
in which the exponential is taken component by component.

This leads to the approximation of $\hat{\Delta}^h[\varphi, a](t, x, p)$ by

$$
\Delta^h[\varphi, a](t, x, p) := h_2^{-1} \sum_{i=1}^{d} \sum_{(\zeta_x, \zeta_p) \in \mathbb{X}_h \times \mathbb{P}^h} \omega(x^i_{h+}, p^i_{h+}[a] \mid (\zeta_x, \zeta_p)) \varphi(t + h_0, e^{\zeta_x}, \zeta_p)
$$

$$
= h_2^{-1} \sum_{i=1}^{d} \sum_{(\zeta_x, \zeta_p) \in \mathbb{X}_h \times \mathbb{P}^h} \omega(x^i_{h-}, p^i_{h-}[a] \mid (\zeta_x, \zeta_p)) \varphi(t + h_0, e^{\zeta_x}, \zeta_p)
$$

$$
- 2dh_2^{-1} \varphi(t + h_0, x, p)
$$

$$
- h_1^{-1} \sum_{i=1}^{d} |\sigma^i(x)|^2 (\varphi(t + h_0, x, p) - \varphi(t + h_0, x \hat{\otimes} h_1 e_i, p))
$$

where

$$
x^i_{h+} := x \hat{\otimes} \sqrt{h_2} \sigma^i(x), \quad p^i_{h+}[a] := p \oplus h_2 a^i,
$$

and

$$
x^i_{h-} := x \hat{\otimes} \sqrt{h_2} \sigma^i(x), \quad p^i_{h-}[a] := p \oplus h_2 a^i.
$$

c. The approximated operator.

Given $\hat{a} > 0$, we then approximate $\hat{H}^J_I$ by $\hat{H}^{h,cx,\hat{a}}_I$ defined as

$$
\hat{H}^{h,cx,\hat{a}}_I[\varphi, \psi] := \max \left\{ \min \left\{ \varphi - \ell, \sup_{a \in A^\hat{a}_I} \hat{K}^a_h \varphi \right\}, \hat{M}^I_h[\varphi, \psi] \right\}
$$

with

$$
A^\hat{a}_I := \{ a \in A_I : |a| \leq \hat{a} \},
$$

and

$$
\hat{K}^a_h \varphi := \min \left\{ -\partial_t^{L,h} \varphi + \bar{L}^a(\cdot, \partial_p^{L,h} \varphi, \Delta^{L,h}[\varphi, a]), \min_{|i|=1} R^a(\cdot, \partial_x^{R,h} [\varphi, \zeta], \partial_p^{R,h} [\varphi, a, \zeta]) \right\}.
$$

The resolution is done as follows:

(i). For $(I, J) \in \mathcal{P}_h$, we define $w^{\hat{a},cx,h}_{I,J} \in S^h$ as the solution of

$$
\begin{cases}
\begin{align*}
w^{\hat{a},cx,h}_{I,J}(T, \cdot) &= G^h(\cdot, \pi_{IJ}) \quad &\text{on} & \mathbb{X}^h_{cx} \times \mathbb{P}^h \\
\max\{w^{\hat{a},cx,h}_{I,J} - L, \hat{H}^{h,cx,\hat{a}}_{I,J}[w^{\hat{a},cx,h}_{I,J}, 0]\} &= 0 &\text{on} & \mathbb{T}_- \times \mathbb{X}^h_{cx} \times \mathbb{P}^h \\
w^{\hat{a},cx,h}_{I,J} &= \hat{g}_J &\text{on} & \mathbb{T}_- \times (\mathbb{X}^h_{cx} \setminus \mathbb{X}^h_{cx,\hat{a}}) \times \mathbb{P}^h
\end{align*}
\end{cases}
$$

where we use the notations

$$
\mathbb{T}_-:= \{(T - (n_0 - i)h_0), \quad i = 0, \ldots, n_0 - 1\}
$$
We then proceed by backward induction on \( |I| + |J| \). Once \( w^{\bar{a},c_X,h}_{IJ} \in S^h \) constructed for \( (I', J') \in \mathcal{P}_n^k \) for all \( l \geq k \), for some \( 1 \leq k \leq \kappa \), we define \( w^{\bar{a},c_X,h}_{IJ} \) for \( (I, J) \in \mathcal{P}_\kappa^{k-1} \) as the solution of

\[
\begin{aligned}
\left\{ \begin{array}{l}
\bar{w}^{\bar{a},c_X,h}_{IJ}(T, \cdot) = G^h(\cdot, \pi_{IJ}) \text{ on } X^h \times P^h \\
(w^{\bar{a},c_X,h}_{IJ} - w^{\bar{a},c_X,h}_{J'J}) \wedge \max\{w^{\bar{a},c_X,h}_{IJ} - L, \bar{H}^{\bar{a},c_X,h} [w^{\bar{a},c_X,h}_{IJ}, w^{\bar{a},c_X,h}_{J'J}]\} = 0 \text{ on } D_\bar{h} \cap D_{IJ}
\end{array} \right.
\end{aligned}
\]

\[
\bar{w}^{\bar{a},c_X,h}_{IJ} = \hat{g}_J \text{ on } (T^h \times (X^h \setminus X^h_{JI\bar{h}}) \times P^h) \cap D_{IJ}
\]

\[
\bar{w}^{\bar{a},c_X,h}_{IJ} = w^{\bar{a},c_X,h}_{J'J} \text{ on } D^h \cap \partial D_{IJ} \cap D_{J'I} \text{ for } (I', J') \supseteq (I, J)
\]

One easily checks that

\[
|\bar{\Delta}^h[\bar{a}, c](t, x, p) - \Delta^h[\bar{a}, c](t, x, p)| \leq O(h_1/h_2), \tag{2.4}
\]

which implies that the numerical scheme is monotone and consistent whenever

\[
h_0 = o(h_1) \text{ and } h_1 = o(h_2). \tag{2.5}
\]

2.3 Convergence of the approximating scheme

The convergence of the scheme is obtained as \( h = (h_0, h_1, h_2) \to 0 \) and \( c_X \to \infty \), with the convention (2.5), and then \( \bar{a} \to \infty \). We therefore define the relaxed semi-limits, for \((t, x, p) \in \bar{D}_{IJ}, (I, J) \in \mathcal{P}_\kappa, \)

\[
\begin{aligned}
\bar{w}^{\bar{a}}_{IJ}(t, x, p) &:= \limsup_{(t', x', p') \to (t, x, p)} \bar{w}^{\bar{a},c_X,h}_{IJ}(t', x', p'), \\
\bar{w}^{\bar{a}}_{IJ}(t, x, p) &:= \liminf_{(t', x', p') \to (t, x, p)} \bar{w}^{\bar{a},c_X,h}_{IJ}(t', x', p')
\end{aligned}
\]

and

\[
\begin{aligned}
\bar{w}^{\bar{a}}_{IJ}(t, x, p) &:= \limsup_{(t', x', p') \to (t, x, p)} \bar{w}^{\bar{a}}_{IJ}(t', x', p'), \\
\bar{w}^{\bar{a}}_{IJ}(t, x, p) &:= \liminf_{(t', x', p') \to (t, x, p)} \bar{w}^{\bar{a}}_{IJ}(t', x', p')
\end{aligned}
\]

in which the limits are taken along sequences of points \((t', x', p') \in \bar{D}_{IJ}\) and \( h \) satisfying (2.5). Note that \( w^{\bar{a},c_X,h}_{IJ} \) takes values in \([\ell, L]\), so that the above are well-defined and bounded. Moreover, it is convergent:

**Theorem 2.1** *For all \((I, J) \in \mathcal{P}_\kappa, \bar{w}^{\bar{a}}_{IJ} = \bar{w}^{\bar{a}}_{IJ} = v^\lambda_{IJ} \text{ on } \bar{D}_{IJ}.*
Proof. See Subsection 4.2 below.

We conclude this section with some numerical illustration in the Black and Scholes model, where the stock price $X$ is defined as

$$X_{t,x}(s) = x + \int_t^s X_{t,x}(r) dW_r$$

for $s \in [t, 1]$, the payoff $g(X) = (K - X)_+$ with the strike price $K = 3$, the thresholds $\gamma = \{\gamma^1, \gamma^2\} = \{0, 0.5\}$.

Example 2.1 We study the case $U = [-1, 1]$.

Then, the “face-lifted” version of $g$ is defined by

$$\hat{g}(x) = \begin{cases} 
3 - x & \text{if } x \in (0, 1] \\
2 - \ln(x) & \text{if } x \in [1, e^2] \\
0 & \text{if } x \geq e^2
\end{cases}$$

Taking $\lambda = 1/32$ and $\ell = -1$, the Figure 4.2 plots an estimated value of $v^\lambda(0, x, p^1, p^2)$ when we fix $x = e$. 

Figure 4.2 – $v^\lambda$ with $U = [-1, 1]$
Example 2.2 When $U = [-5, 5]$, the “face-lifted” version of $g$ is equal to $g$ on $\mathbb{R}_+$. The Figure 4.3 plots an estimated value of $v^\lambda(0, x, p^1, p^2)$ when we take $\lambda = 1/32$, $\ell = -1$ and $x = 1$. In the Figure 4.4, we describe its graph when $p^2 = 0$ in the same setting.

![Figure 4.3 – $v^\lambda$ with $U = [-5, 5]$](image)

3 Proof of the PDE characterizations

In this section, we collect the proofs of Proposition 1.3, Proposition 1.4, Theorem 1.2, Theorem 1.3 and Theorem 2.1. The boundary conditions in the space variable $p$ follows the same argument as in Section 3 in the previous chapter. It remains to prove the boundary conditions in time and the gradient estimates in the viscosity sense. We first recall the geometric dynamic programming principle of [46], see also [47] and [49], to which we will appeal to prove the boundary conditions. We next report the proof of the supersolution properties in subsection 3.2, and that of the subsolution properties in subsection 3.1. The gradient estimates in the viscosity sense and the corresponding comparison result are proved in next subsection.
Corollary 3.1 Fix \((t, x, p) \in \bar{D}_{I,J}\).

(GDP1') If \(y \geq \ell\) and \((\nu, \alpha) \in \mathcal{U}_t \times A_{t,\pi I,J(p)}\) are such that
\[
\Delta_{\lambda}(X_{t,x}, (T), Y_{t,x,y}^\nu(T)) \geq P_{t,p}^\alpha(T)
\]
then
\[
Y_{t,x,y}^\nu(\theta) \geq v_{I,J}^\lambda(\theta, X_{t,x}(\theta), P_{t,p}^\alpha(\theta)) , \text{ for all } \theta \in \mathcal{T}_{[t,T]}^I .
\]

(GDP2') For \(y < v_{I,J}^\lambda(t, x, p), \theta \in \mathcal{T}_{[t,T]}^I \) and \((\nu, \alpha) \in \mathcal{U}_t \times A_{t,\pi I,J(p)}\),
\[
\mathbb{P} \left[ Y_{t,x,y}^\nu(\theta) > v_{I,J}^\lambda(\theta, X_{t,x}(\theta), P_{t,p}^\alpha(\theta)) \right] < 1.
\]

3.1 Boundary condition for the upper-semicontinuous envelope

We start with the boundary condition as \(t \to T\).

Proposition 3.1 For all \((I, J), (I', J') \in \mathcal{P}_n\) such that \((I', J') \supset (I, J),\) we have
\[
v_{I,J}^\lambda(T, \cdot) \leq G^\lambda \text{ on } (0, \infty)^d \times \bar{B}_{I',J'}.
\]
Hence, it suffices to show that
\[ \psi^I_T = w := \inf \{ y \geq \ell : \exists \nu \in \mathcal{U} \text{ s.t. } Y^\nu(T) \geq \hat{g}_I(X(T)) \} . \]

Hence, it suffices to show that
\[ w^*(T, \cdot) \leq \hat{g}_I, \quad (3.1) \]
where \( w^*(T, x) := \lim_{\varepsilon \to 0} \sup \{ w(t', x') : (t', x') \in (T - \varepsilon, T] \times B(x') \} \). We only sketch the proof of (3.1) as it follows from the same arguments as in [9], up to obvious modifications. In the following, we let \( (t_n, x_n) \) be a sequence in \([0, T] \times (0, \infty)^d\) such that \( (t_n, x_n) \to (T, x) \) and \( w(t_n, x_n) \to w^*(T, x) \).

It follows from the dual formulation of [22] that, for each \( n \geq 1 \), we can find a predictable process \( \vartheta^n \) with values in \( \mathbb{R}^d \) such that
\[ H_n^{\vartheta_n} := \mathcal{E} \left( -\int_{t_n}^T \sigma^{-1}_X (\mu_X - \vartheta^n_s) (X_n(s)) dW_s \right) T \in L^1(\mathbb{P}) , \]
where \( X_n := X_{t_n, x_n} \), and
\[ w(t_n, x_n) \leq \mathbb{E} \left[ H_n^{\vartheta_n}(T) \left( g_J(X_n(T)) - \int_{t_n}^T \delta_U(\vartheta^n_s) ds \right) \right] + n^{-1} . \]

Since \( \delta_U \) is homogeneous of degree 1 and convex, this implies that
\[ w(t_n, x_n) \leq \mathbb{E} \left[ H_n^{\vartheta_n}(T) \left( g_J(X_n(T)) - \delta_U(\int_{t_n}^T \vartheta^n_s) ds \right) \right] + n^{-1} \]
so that, by definition of \( \hat{g}_J \) in (2.23),
\[ w(t_n, x_n) \leq \mathbb{E} \left[ H_n^{\vartheta_n}(T) \hat{g}_J(Z_n^{\vartheta_n}(T)) \right] + n^{-1} , \]
where \( Z_n := X_n e^{-\int_{t_n}^T \vartheta^n_s ds} \). It remains to prove that
\[ \lim_{n \to \infty} \mathbb{E} \left[ H_n^{\vartheta_n}(T) \hat{g}_J(Z_n^{\vartheta_n}(T)) \right] \leq \hat{g}_J(x) . \]
To show this, it suffices to follow line by line the arguments contained after the equation (6.7) in the proof of Proposition 6.7 in [9].

**Step 2.** We now consider the case \( I \cup J \neq K \). We assume that
\[ y_0 := \psi^{\lambda^*_I}(T, x, p) > G^\lambda(x, p) \quad (3.2) \]
and work towards a contradiction. It follows from Step 1 that \( \hat{g}_J(x) \geq \psi^{\lambda^*_I}(T, x) \). In view of (3.2) and (1.6), this leads to \( \psi^{\lambda^*_I}(T, x, p) > \psi^{\lambda^*_I}(T, x) \). Hence, there exists a
sequence \((t_n, x_n, p_n)_n \subset D_{IJ}\) which converges to \((T, x, p)\) such that \(v^{\lambda}_{IJ}(t_n, x_n, p_n) \rightarrow v^{\lambda}_{IJ}(T, x, p)\) and
\[
v^\lambda_{I'J'}(t_n, x_n, p_n) < v^\lambda_{I'J'}(T, x, p) + \epsilon < y_n \quad \text{for all } n \geq 1, \text{ for some } \epsilon > 0,
\]
where \(y_n := v^\lambda_{IJ}(t_n, x_n, p_n) - n^{-1}\). We can then find \(\nu_n \in \mathcal{U}\) such that
\[
Y_n(T) \geq \hat{g}_{IJ}(X_n(T)) \geq \ell,
\]
where \((X_n, Y_n) := (X_{t_n,x_n}, Y_{t_n,x_n,y_n}).\) Moreover, since \(\Delta^\lambda\) is strictly increasing on \(\{(x', y') : \Delta^\lambda(x', y') \in (0, 1)\}\) and \(y_0 > G^\lambda(x, p)\), we have \(\Delta^\lambda(x, y_0) > p_l^I\) for \(l \notin I' \cup J'\). Since \((X_n(T), Y_n(T)) \rightarrow (x, y_0)\) in law, up to a subsequence, because \(U\) is bounded and by the Lipschitz continuity of \((\mu_X, \sigma_X)\), we deduce that
\[
\mathbb{E}[\Delta^\lambda(X_n(T), Y_n(T))] \geq p_l^I \quad \text{for all } l \notin I' \cup J', \text{ and } n \text{ large enough. Finally, } Y_n(T) \geq \ell
\]
so that \(\mathbb{E}[\Delta^\lambda(X_n(T), Y_n(T))] \geq 0\) for \(l \in I'\). This contradicts the fact that \(y_n < v^\lambda_{IJ}(t_n, x_n, p_n)\). \(\square\)

We now turn to the boundary condition in the \(p\)-variable, i.e. as \(p \rightarrow \partial B_{IJ}\).

**Proposition 3.2** For all \((I, J), (I', J') \in \mathcal{P}_n\) such that \((I', J') \supset (I, J)\), we have
\[
v^\lambda_{IJ} \leq v^\lambda_{I'J'} \quad \text{on } D_{I'J'}.
\]

**Proof.** The proof follows the same argument as in Proposition 3.1 of the previous chapter.

### 3.2 Boundary condition for the lower-semicontinuous envelope

We start with the boundary condition as \(t \rightarrow T\).

**Proposition 3.3** For all \((I, J), (I', J') \in \mathcal{P}_n\) such that \((I', J') \supset (I, J)\), we have
\[
v^\lambda_{I'J'}(T, \cdot) \geq G^\lambda \quad \text{on } (0, \infty)^d \times \bar{B}_{I'J'}.
\]

**Proof.** Fix \((x, p) \in (0, \infty)^d \times \bar{B}_{I'J'}\). Since \(v^\lambda_{I'J'} \geq \ell\), the required result is trivial when \(p = 0\). We thus consider the case where \(p \neq 0\), and fix \(l \in \mathcal{K}\) such that that \(p_l^I > 0\). Let \((t_n, x_n, p_n)_n \subset D_{IJ}\) be a sequence that converges to \((T, x, p)\) and such that \(v^\lambda_{IJ}(t_n, x_n, p_n) \rightarrow v^\lambda_{I'J'}(T, x, p)\). We define \(y_n := v^\lambda_{IJ}(t_n, x_n, p_n) + n^{-1}\) so that, for each \(n\), there exists \((\nu^\alpha, \alpha^\gamma) \in \mathcal{U} \times \mathcal{A}_{t_n,p_n}\) satisfying \(y_n + Y_n(T) \geq \ell\) and
\[
\mathbb{E}[\Delta^\lambda(X_n(T), Y_n(T))] \geq p_l^I,
\]
where \((X_n, Y_n) := (X_{t_n,x_n,Y_{t_n,x_n,y_n}}).\) Using the fact that \(U\) is bounded and that \((\mu_X, \sigma_X)\) is Lipschitz continuous, one easily checks that, after possibly passing to a
subsequence, \((X_n(T), Y_n(T))\) converges to \((x, v^\lambda_{t,x,p}(T, x, p))\) \(\mathbb{P}\)-a.s. and in law. Since \(\Delta^\lambda\) is continuous, this implies that

\[
\Delta^\lambda_1(x, v^\lambda_{t,x,p}(T, x, p)) \geq p^1 > 0 .
\]

By arbitrariness of \(l\) such that \(p^l \neq 0\), this leads to the required result. \(\square\)

In order to discuss the boundary condition in the \(p\)-variable, we follow \cite{12} and first provide a supersolution property for \(v^\lambda_{t,x,p}\) on the boundary \(\bar{D}_{IJ} \cap D_{I',J'}\) for \((I', J') \supset (I, J)\). A more precise statement will be deduced from the following one and the comparison result of Proposition 3.6 below, see Subsection 3.5.

**Proposition 3.4** For all \((I, J), (I', J') \in \mathcal{P}_\kappa\) such that \(J' \supset J\), \(v^\lambda_{t,x,p}\) is a supersolution on \(\bar{D}_{IJ}\) of

\[
\min\{\varphi - \ell, -\partial_t \varphi + F^*_{t,x,p} \varphi\} \geq 0 \quad \text{on} \quad D_{I',J'} .
\]

**Proof.** The proof follows the same argument as in Proposition 3.2 of the previous chapter.

### 3.3 Gradient estimates

In this section, we prove Proposition 1.4. It is based on the following growth estimate.

**Proposition 3.5** Fix \((I, J), (I', J') \in \mathcal{P}_\kappa\) such that \(I \cup J \neq \mathcal{K}\) and \(J \subset J'\). Let \((t, x, p) \in D_{IJ}\) be such that \((v^\lambda_{t,x,p} - v^\lambda_{t',x,p})(t, x, p) > \ell \geq 0\). Let \(\varrho > 0\) be defined as in (1.8). Then,

\[
v^\lambda_{t,x,p}(t, x, p) - v^\lambda_{t',x,p}(t, x, p - C_\delta (\ell + \varrho) \mathbf{1}_{IJ}) \geq \delta \ell \quad \text{for all} \ 0 \leq \delta \leq 1 ,
\]

where \(\mathbf{1}_{IJ}\) stands for \((\mathbf{1}_{\{i \notin I \cup J\}})_{i \leq \kappa}\).

**Proof.** Fix \((t, x, p) \in D_{IJ}, y > v^\lambda_{t,x,p}(t, x, p)\) and \(\ell \geq 0\) such that \(v^\lambda_{t,x,p}(t, x, p) - \ell > v^\lambda_{t',x,p}(t, x, p)\). Then, we can find \(\nu \in \mathcal{U}\) such that

\[
Y_{t,x,y}(T) \geq \ell \quad \text{and} \quad \mathbb{E}[\Delta^\lambda_1(X_{t,x}(T), Y_{t,x,y}(T))] \geq p^1 \quad \text{for all} \ i \leq \kappa ,
\]

and \(\nu' \in \mathcal{U}\) such that

\[
Y_{t,x,y-\ell}(T) \geq \ell \quad \text{and} \quad Y_{t,x,y-\ell}(T) \geq \hat{g}^j(X_{t,x}(T)) \quad \text{for all} \ j \in J' ,
\]

recalling (1.1). Set \(\nu_\delta := (1 - \delta)\nu + \delta \nu' \in \mathcal{U}\), recall that \(U\) is convex, and \(y_\delta := (1 - \delta)y + \delta(y - \ell) = y - \delta \ell\). Note that

\[
Y_{t,x,y_\delta}(T) = (1 - \delta)Y_{t,x,y}(T) + \delta Y_{t,x,y-\ell}(T) ,
\]
by (2.4). Combined with the above inequalities and the fact that \( p^i = 1 \) for \( i \in J \subset J' \), this readily implies that

\[
Y_{t,x,y}(T) \geq \ell \quad \text{and} \quad Y_{t,x,y}(T) \geq \hat{g}(X_{t,x}(T)) \quad \text{for} \quad i \in J .
\]

(3.5)

Since \( \Delta_\chi \) is \( C_\chi \)-Lipschitz with respect to \( y \), see (1.4), we also have

\[
\begin{align*}
\mathbb{E}[\Delta_\chi(X_{t,x}(T), Y_{t,x,y}(T)) ] &= \mathbb{E}[\Delta_\chi \left( X_{t,x}(T), Y_{t,x,y}(T) + \delta(Y_{t,x,y}(T) - Y_{t,x,y}(T)) \right)] \\
&\geq \mathbb{E}[\Delta_\chi(X_{t,x}(T), Y_{t,x,y}(T))] - C_\chi \delta \mathbb{E} \left[ Y_{t,x,y}(T) - Y_{t,x,y}(T) \right] \\
&\geq p^i - C_\chi \delta \mathbb{E} \left[ Y_{t,x,y}(T) - Y_{t,x,y}(T) \right], \quad \text{for} \quad i \notin J ,
\end{align*}
\]

where

\[
\begin{align*}
\mathbb{E} \left[ Y_{t,x,y}(T) - Y_{t,x,y}(T) \right] &\leq \ell + \mathbb{E} \left[ \int_t^T (\nu_s - \nu_s')^T \mu(X_{t,s}(s)) ds + \int_t^T (\nu_s - \nu_s')^T \sigma(X_{t,s}(s)) dW_s \right] .
\end{align*}
\]

Recalling (2.2) and (2.3), standard estimates imply that the right-hand side term is bounded by \( \rho \) as defined in (1.8). Hence

\[
\mathbb{E}[\Delta_\chi(X_{t,x}(T), Y_{t,x,y}(T))] \geq p^i - C_\chi \delta (\ell + \rho) , \quad \text{for} \quad i \notin J .
\]

(3.6)

We now combine (3.5) and (3.6) to deduce that

\[
y_s \geq v_{IJ}^\chi(t, x, p \ominus C_\chi \delta (\ell + \rho) 1_{JJ}) .
\]

By arbitrariness of \( y > v_{IJ}^\chi(t, x, p) \), this implies the required result. \( \square \)

**Proof of Proposition 1.4.**

Fix \((I, J) \in \mathcal{P}_\kappa \) and \((t, x, p) \in D_{IJ}\) such that \( v_{IJ}^\lambda(t, x, p) > v_{IJ}^\nu(t, x, p) + \ell \) for some \( \ell \geq 0 \). Let \( \varphi \) be a smooth function and assume that \((t, x, p)\) achieves the maximum of \( v_{IJ}^\nu - \varphi \).

Since \( v_{IJ}^\lambda \) is non-decreasing with respect to its \( p \)-variable, and by definition of \( v_{IJ}^\nu \), there exists \((t_n, x_n, p_n) \to (t, x, p)\) such that \( v_{IJ}^\lambda(t_n, x_n, p_n) \to v_{IJ}^\lambda(t, x, p) \) and \( v_{IJ}^\nu(t_n, x_n, p_n + C_\chi \delta (\ell + \rho) 1_{JJ}) > v_{IJ}^\nu(t_n, x_n, p_n + C_\chi \delta (\ell + \rho) 1_{JJ}) + \ell \) for \( \delta > 0 \) small enough. By applying Proposition 3.5 at the point \((t_n, x_n, p_n + C_\chi \delta (\ell + \rho) 1_{JJ})\) for \((I', J') = (J', J)\), we deduce that

\[
v_{IJ}^\lambda(t_n, x_n, p_n + C_\chi \delta (\ell + \rho) 1_{JJ}) - v_{IJ}^\lambda(t_n, x_n, p_n) \geq \delta \ell ,
\]

and therefore

\[
\varphi(t, x, p + C_\chi \delta (\ell + \rho) 1_{JJ}) - \varphi(t, x, p) \geq \delta \ell .
\]

Dividing by \( \delta \) and sending \( \delta \) to 0 leads to the required result for \( \varphi \) defined as in (1.9) above. \( \square \)
3. PROOF OF THE PDE CHARACTERIZATIONS

3.4 Comparison results for the system of PDEs \((S^\lambda)\)

We provide a comparison result for \((S^\lambda)\). Additional technical improvements will be considered in the next section to discuss the convergence of the numerical scheme defined in Section 2.

Proposition 3.6 Let \(\psi_1 \geq \psi_2\) be two functions such that \(\psi_1\) and \(-\psi_2\) are lower-semicontinuous. Fix \((I, J) \in \mathcal{P}_\kappa\). Let \(V_1\) be a bounded lower-semicontinuous viscosity supersolution of

\[ H_{IJ}^{\lambda} [\varphi, \psi_1] = 0 \quad \text{on } D_{IJ}, \quad (3.7) \]

and let \(V_2\) be a bounded upper-semicontinuous viscosity subsolution of

\[ H_{IJ}^{\lambda} [\varphi, \psi_2] = 0 \quad \text{on } D_{IJ}. \quad (3.8) \]

Assume that \(V_1 \geq V_2\) on \(\partial D_{IJ}\). Assume further that either \(V_1 \geq \psi_2\) on \(D_{IJ}\) or that \((I, J) \in \mathcal{P}_\kappa^\kappa\). Then, \(V_1 \geq V_2\) on \(\overline{D}_{IJ}\).

Proof.

Part 1: \((I, J) \notin \mathcal{P}_\kappa^\kappa\). As usual, we first fix \(\rho > 0\) and introduce the functions \(\tilde{V}_1(t, x, p) := e^{\rho t} V_1(t, x, p)\) and \(\tilde{V}_2(t, x, p) := e^{\rho t} V_2(t, x, p)\). Arguing by contradiction, we assume that

\[ \sup_{\overline{D}_{IJ}} (\tilde{V}_2 - \tilde{V}_1) =: m > 0. \]

and work towards a contradiction.

1. For \(n, k \geq 1\) and \(\varepsilon > 0\), we then define the function \(\Psi_{n, \varepsilon}^k\) on \([0, T] \times \mathbb{R}^{2n} \times [0, 1]^{2\kappa}\) by

\[ \Psi_{n, \varepsilon}^k(t, x, y, p, q) := \tilde{V}_2(t, x, p) - \tilde{V}_1(t, y, q) - \Theta_{n, \varepsilon}^k(t, x, y, p, q), \]

where

\[ \Theta_{n, \varepsilon}^k(t, x, y, p, q) := \frac{n^2}{2} |x - y|^2 + \frac{k^2}{2} |p - q|^2 + f_\varepsilon(x), \]

with

\[ f_\varepsilon(x) := \varepsilon \left( |x|^2 + \sum_{i \leq d} (x^i)^{-1} \right). \]

It follows from the boundedness of \(\tilde{V}_2\) and \(\tilde{V}_1\) that \(\Psi_{n, \varepsilon}^k\) achieves its maximum at some \((t_{n, \varepsilon}^k, x_{n, \varepsilon}^k, y_{n, \varepsilon}^k, p_{n, \varepsilon}^k, q_{n, \varepsilon}^k) \in [0, T] \times (0, \infty)^{2d} \times \overline{B}_{IJ}^2\). Similarly, the map

\[ (t, x, p, q) \in [0, T] \times (0, \infty)^d \times \overline{B}_{IJ}^2 \mapsto \tilde{V}_2(t, x, p) - \tilde{V}_1(t, x, q) - \frac{k^2}{2} |p - q|^2 - f_\varepsilon(x) \]
achieves a maximum at some \((\tilde{t}^k, \tilde{x}^k, \tilde{p}^k, \tilde{q}^k) \in [0, T] \times (0, \infty)^d \times \bar{B}^2_{IJ}\). Moreover, the inequality 

\[
\Psi_{n, \varepsilon}^k(t_{n, \varepsilon}, x_{n, \varepsilon}, y_{n, \varepsilon}, p_{n, \varepsilon}, q_{n, \varepsilon}) \geq \Psi_{n, \varepsilon}^k(t^k, x^k, y^k, p^k, q^k)
\]

implies that 

\[
\tilde{V}_2(t_{n, \varepsilon}, x_{n, \varepsilon}, y_{n, \varepsilon}, p_{n, \varepsilon}, q_{n, \varepsilon}) - \tilde{V}_1(t_{n, \varepsilon}, y_{n, \varepsilon}, q_{n, \varepsilon}) \geq \tilde{V}_2(t_{n, \varepsilon}, x_{n, \varepsilon}, y_{n, \varepsilon}, q_{n, \varepsilon}) - \tilde{V}_1(t^k, x^k, y^k, q^k) - \frac{k^2}{2} |p^k - q^k|^2 - f_\varepsilon(x_{n, \varepsilon}) + f_\varepsilon(x^k) + n^2 |y_{n, \varepsilon} - y^k|^2 + \frac{k^2}{2} |p_{n, \varepsilon} - q_{n, \varepsilon}|^2.
\]

Using the boundedness of \(\tilde{V}_2\) and \(\tilde{V}_1\) again together with the fact that \(\bar{B}_{I,J}\) is compact, we deduce that the term on the second line is bounded in \(n\) so that, up to a subsequence,

\[
(t_{n, \varepsilon}, x_{n, \varepsilon}, y_{n, \varepsilon}, p_{n, \varepsilon}, q_{n, \varepsilon}) \rightarrow (\tilde{t}^k, \tilde{x}^k, \tilde{p}^k, \tilde{q}^k) \text{ as } n \rightarrow \infty,
\]

for some \((\tilde{t}^k, \tilde{x}^k, \tilde{p}^k, \tilde{q}^k) \in [0, T] \times (0, \infty)^d \times \bar{B}^2_{I,J}\). By sending \(n \rightarrow \infty\) in the previous inequality, we also obtain 

\[
\tilde{V}_2(t_{n, \varepsilon}, x_{n, \varepsilon}, y_{n, \varepsilon}, p_{n, \varepsilon}, q_{n, \varepsilon}) - \tilde{V}_1(t_{n, \varepsilon}, y_{n, \varepsilon}, q_{n, \varepsilon}) \leq \tilde{V}_2(t_{n, \varepsilon}, x_{n, \varepsilon}, y_{n, \varepsilon}, q_{n, \varepsilon}) - \tilde{V}_1(t^k, x^k, y^k, q^k) - \frac{k^2}{2} |p^k - q^k|^2 - f_\varepsilon(x_{n, \varepsilon}) - \liminf_{n \rightarrow \infty} \frac{n^2}{2} |y_{n, \varepsilon} - y^k|^2.
\]

It then follows from the maximum property at \((t^k, x^k, p^k, q^k)\) that the last term on the right-hand side converges to 0 and that we can assume, without loss of generality, that \((\tilde{t}^k, \tilde{x}^k, \tilde{p}^k, \tilde{q}^k) = (t^k, x^k, p^k, q^k)\), i.e.

\[
(t_{n, \varepsilon}, x_{n, \varepsilon}, y_{n, \varepsilon}, p_{n, \varepsilon}, q_{n, \varepsilon}) \xrightarrow{n \rightarrow \infty} (t^k, x^k, p^k, q^k) \text{ and } n^2 |y_{n, \varepsilon} - y^k|^2 \xrightarrow{n \rightarrow \infty} 0. \tag{3.9}
\]

It follows from similar arguments that we could choose \((x^k)_{\varepsilon \geq 0}\) such that, up to a subsequence,

\[
f_\varepsilon(x_{\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ and } (t^k, p^k, q^k) \xrightarrow{\varepsilon \rightarrow 0} (t_k, p_k, q_k), \tag{3.10}
\]

and 

\[
\lim_{\varepsilon \rightarrow 0} \Psi_{n, \varepsilon}^k(t_{n, \varepsilon}, x_{n, \varepsilon}, y_{n, \varepsilon}, p_{n, \varepsilon}, q_{n, \varepsilon}) = \sup_{[0, T] \times (0, \infty)^d \times [0, 1]^{2n}} \Psi_{0,0}^k(t, x, p, q) \geq m. \tag{3.11}
\]

For later use, note that the left-hand side in (3.10) together with the definition of \((\mu_X, \sigma_X)\) and the fact that \((\mu, \sigma)\) is bounded implies

\[
|D_x f_\varepsilon(x_{\varepsilon})^\top \mu_X(x_{\varepsilon})| + |D_x f_\varepsilon(x_{\varepsilon})^\top \sigma_X(x_{\varepsilon})| + |\Trace [\sigma_X \tilde{\sigma}_X(x_{\varepsilon}) D_x^2 f_\varepsilon(x_{\varepsilon})]| \xrightarrow{\varepsilon \rightarrow 0} 0. \tag{3.12}
\]
Similarly, we must have
\[
\lim_{k \to \infty} k^2 |p_k - q_k|^2 = 0. \tag{3.13}
\]

Since \( V_1(T, \cdot) \geq V_2(T, \cdot) \), the above implies that we cannot have \( t_{n, \varepsilon}^k = T \) along a subsequence. Since \( V_2 \geq V_1 \) on \( \partial D_{1J} \), we obtain a similar contradiction if, up to a subsequence, \((t_{n, \varepsilon}^k, x_{n, \varepsilon}^k, p_{n, \varepsilon}^k, q_{n, \varepsilon}^k)_{\varepsilon, k, n} \in \partial D_{IJ} \) or \((t_{n, \varepsilon}^k, y_{n, \varepsilon}^k, q_{n, \varepsilon}^k)_{\varepsilon, k, n} \in \partial D_{IJ} \) for all \( \varepsilon, n, k \). We can therefore assume from now on that \( t_{n, \varepsilon}^k < T \), \((t_{n, \varepsilon}^k, x_{n, \varepsilon}^k, p_{n, \varepsilon}^k, q_{n, \varepsilon}^k)_{\varepsilon, k, n} \notin \partial D_{IJ} \) and \((t_{n, \varepsilon}^k, y_{n, \varepsilon}^k, q_{n, \varepsilon}^k)_{\varepsilon, k, n} \notin \partial D_{IJ} \) for all \( k, n, \varepsilon \).

2. For ease of notations, we now set \( z_{n, \varepsilon}^k := (t_{n, \varepsilon}^k, x_{n, \varepsilon}^k, y_{n, \varepsilon}^k, p_{n, \varepsilon}^k, q_{n, \varepsilon}^k) \). From Ishii’s Lemma, see Theorem 8.3 in [18], we deduce that, for each \( \eta > 0 \), there are real coefficients \( a_{n, \varepsilon}^k, b_{n, \varepsilon}^k \) and symmetric matrices \( \mathcal{X}_{n, \varepsilon}^k \) and \( \mathcal{Y}_{n, \varepsilon}^k \) such that
\[
\left( a_{n, \varepsilon}^k, D_{(x,p)} \Theta_{n, \varepsilon}^k(z_{n, \varepsilon}^k), \mathcal{X}_{n, \varepsilon}^k \right) \in \bar{\mathcal{P}}^+ \bar{V}_2(t_{n, \varepsilon}^k, x_{n, \varepsilon}^k, p_{n, \varepsilon}^k)
\]
and
\[
\left( -b_{n, \varepsilon}^k, -D_{(y,q)} \Theta_{n, \varepsilon}^k(z_{n, \varepsilon}^k), \mathcal{Y}_{n, \varepsilon}^k \right) \in \bar{\mathcal{P}}^- \bar{V}_1(t_{n, \varepsilon}^k, y_{n, \varepsilon}^k, q_{n, \varepsilon}^k),
\]
see [18] for the standard notations \( \bar{\mathcal{P}}^+ \) and \( \bar{\mathcal{P}}^- \), where
\[
D_x \Theta_{n, \varepsilon}^k(z_{n, \varepsilon}^k) = n^2(x_{n, \varepsilon}^k - y_{n, \varepsilon}^k) + D_x f_{\varepsilon}(x_{n, \varepsilon}^k), \tag{3.14}
\]
\[
D_y \Theta_{n, \varepsilon}^k(z_{n, \varepsilon}^k) = k^2(p_{n, \varepsilon}^k - q_{n, \varepsilon}^k), \tag{3.15}
\]
\[
-D_y \Theta_{n, \varepsilon}^k(z_{n, \varepsilon}^k) = n^2(x_{n, \varepsilon}^k - y_{n, \varepsilon}^k), \tag{3.16}
\]
\[
-D_q \Theta_{n, \varepsilon}^k(z_{n, \varepsilon}^k) = k^2(p_{n, \varepsilon}^k - q_{n, \varepsilon}^k), \tag{3.17}
\]
and \( a_{n, \varepsilon}^k, b_{n, \varepsilon}^k, \mathcal{X}_{n, \varepsilon}^k \) and \( \mathcal{Y}_{n, \varepsilon}^k \) satisfy
\[
\begin{cases}
    a_{n, \varepsilon}^k + b_{n, \varepsilon}^k = 0 \\
    \mathcal{X}_{n, \varepsilon}^k \\
    \begin{pmatrix}
    A_{n, \varepsilon}^k \leq \mathcal{Y}_{n, \varepsilon}^k
    \end{pmatrix}
\end{cases}
\leq A_{n, \varepsilon}^k + \eta(A_{n, \varepsilon}^k)^2. \tag{3.18}
\]
with
\[
A_{n, \varepsilon}^k := \begin{pmatrix}
    n^2 I_d + D_x^2 f_{\varepsilon}(x_{n, \varepsilon}^k) & 0 & -n^2 I_d & 0 \\
    0 & k^2 I_\kappa & 0 & -k^2 I_\kappa \\
    -n^2 I_d & 0 & n^2 I_d & 0 \\
    0 & -k^2 I_\kappa & 0 & k^2 I_\kappa
\end{pmatrix},
\]
where \( I_d \) and \( I_\kappa \) stand for the \( d \times d \) and \( \kappa \times \kappa \) identity matrices.

We now study different cases:

**Case 1.** If, up to a subsequence,
\[
M_{ij}^\Lambda(V_1(t_{n, \varepsilon}^k, y_{n, \varepsilon}^k, q_{n, \varepsilon}^k), \psi_1(t_{n, \varepsilon}^k, y_{n, \varepsilon}^k, q_{n, \varepsilon}^k), -e^{-\rho_{n, \varepsilon}^k} D_q \Theta_{n, \varepsilon}^k(z_{n, \varepsilon}^k)) \geq 0,
\]

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then there exists \( \iota_{n,\varepsilon}^k \geq 0 \) such that

\[
\min \left\{ V_1(t_{n,\varepsilon}^k, y_{n,\varepsilon}^k, q_{n,\varepsilon}^k) - \psi_1(t_{n,\varepsilon}^k, y_{n,\varepsilon}^k, q_{n,\varepsilon}^k) - \iota_{n,\varepsilon}^k, \quad e^{-p_{n,\varepsilon}^k} \varpi(\iota_{n,\varepsilon}^k) + \sum_{i \in I \cup J} D_q \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) \right\} \geq 0. 
\] (3.19)

If, up to a subsequence, \( V_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) - \psi_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) \leq \iota_{n,\varepsilon}^k \), we obtain a contradiction by using (3.19), the fact that \( \psi_1 \geq \psi_2 \), \( \psi_1 \) and \( -\psi_2 \) are lower-semicontinuous, and by (3.9), (3.10), (3.11) and (3.13).

We then assume that

\[
V_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) - \psi_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) > \iota_{n,\varepsilon}^k.
\]

Then, there exists \( \tilde{\iota}_{n,\varepsilon}^k > \iota_{n,\varepsilon}^k \) satisfying

\[
V_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) - \psi_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) > \tilde{\iota}_{n,\varepsilon}^k,
\]

so that, by the subsolution property of \( V_2 \),

\[
e^{-p_{n,\varepsilon}^k} \varpi(\tilde{\iota}_{n,\varepsilon}^k) - \sum_{i \in I \cup J} D_q \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) \leq 0.
\]

Since \( D_p \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) = -D_q \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) \) by (3.15) and (3.17), (3.19) implies that \( \varpi(\tilde{\iota}_{n,\varepsilon}^k) \leq \varpi(\iota_{n,\varepsilon}^k) \). Since \( \tilde{\iota}_{n,\varepsilon}^k > \iota_{n,\varepsilon}^k \) and \( \varpi \) is strictly increasing, recall (1.9), this leads to a contradiction.

From now on, we assume that

\[
M_{I \cup J}^\lambda(V_1(t_{n,\varepsilon}^k, y_{n,\varepsilon}^k, q_{n,\varepsilon}^k), \psi_1(t_{n,\varepsilon}^k, y_{n,\varepsilon}^k, q_{n,\varepsilon}^k), -e^{-p_{n,\varepsilon}^k} D_q \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k)) < 0. 
\] (3.20)

**Case 2.** If, up to a subsequence,

\[
V_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) \leq \ell \lor \psi_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k).
\]

It follows from the supersolution property of \( V_1 \) that \( V_1(t_{n,\varepsilon}^k, y_{n,\varepsilon}^k, q_{n,\varepsilon}^k) \geq \ell \). Since we also have \( V_1 \geq \psi_2 \) by assumption, passing to the limit leads to a contradiction as above.

**Case 3.** From now on, we can therefore assume that

\[
V_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) > \ell \lor \psi_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k),
\]

and (3.20) holds. In particular, the subsolution property of \( V_2 \) and (3.15)-(3.17) imply that

\[
\sum_{i \in I \cup J} D_p \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) = - \sum_{i \in I \cup J} D_q \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) \geq \varpi(\iota_{n,\varepsilon}^k) > 0
\] (3.21)
where

$$i_{n,\varepsilon}^k := (V_2 - \psi_2)(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k)/2 > 0 .$$

For later use, note that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} i_{n,\varepsilon}^k > 0 \quad (3.22)$$

since otherwise, we would get a contradiction to (3.11) as above since $V_1 \geq \psi_2$ by assumption.

The inequality (3.21) implies that there must exist some $i_{n,\varepsilon}^k \notin I \cup J$ such that

$$D_p \Theta_{n,\varepsilon}^k (z_{n,\varepsilon}) = -D_q \Theta_{n,\varepsilon}^k (z_{n,\varepsilon}) \geq \pi(i_{n,\varepsilon}^k)/\kappa > 0 \quad (3.23)$$

recall (3.15)-(3.17). Let us now fix $(u_{n,\varepsilon}^k, \alpha_{n,\varepsilon}^k) \in U \times A_{IJ}$ such that

$$(u_{n,\varepsilon}^k, \alpha_{n,\varepsilon}^k) \in N_{IJ}^{1/n}(y_{n,\varepsilon}^k, -e^{-\rho t_n^k} D_y \Theta_{n,\varepsilon}^k (z_{n,\varepsilon}), -e^{-\rho t_n^k} D_q \Theta_{n,\varepsilon}^k (z_{n,\varepsilon})) \quad (3.24)$$

i.e.

$$(u_{n,\varepsilon}^k)^\top \sigma(y_{n,\varepsilon}^k) = -e^{-\rho t_n^k} D_y \Theta_{n,\varepsilon}^k (z_{n,\varepsilon})^\top \sigma_X(y_{n,\varepsilon}^k) - e^{-\rho t_n^k} D_q \Theta_{n,\varepsilon}^k (z_{n,\varepsilon})^\top \alpha_{n,\varepsilon} + \xi_{n,\varepsilon}^k$$

for some

$$\xi_{n,\varepsilon}^k \in \mathbb{R}^d \text{ such that } |\xi_{n,\varepsilon}^k| \in [-n^{-1}, n^{-1}] . \quad (3.25)$$

Using (3.15)-(3.17) and (3.23), we see that $\tilde{\alpha}_{n,\varepsilon}^k$ defined as

$$(\tilde{\alpha}_{n,\varepsilon}^k)^i := (\alpha_{n,\varepsilon}^k)^i \text{ for } i \neq i_{n,\varepsilon}^k$$

and

$$D_p \Theta_{n,\varepsilon}^k (z_{n,\varepsilon})(\tilde{\alpha}_{n,\varepsilon}^k - \alpha_{n,\varepsilon}^k)^i_{n,\varepsilon} := e^{\rho t_n^k}(u_{n,\varepsilon}^k)^\top (\sigma(x_{n,\varepsilon}) - \sigma(y_{n,\varepsilon}^k))$$

$$- D_x \Theta_{n,\varepsilon}^k (z_{n,\varepsilon})^\top (\sigma_X(x_{n,\varepsilon}) - \sigma_X(y_{n,\varepsilon}^k))$$

$$- D_x f_{n,\varepsilon}(x_{n,\varepsilon})^\top \sigma_X(y_{n,\varepsilon}^k) - e^{\rho t_n^k} \xi_{n,\varepsilon}^k$$

satisfies $(u_{n,\varepsilon}^k, \tilde{\alpha}_{n,\varepsilon}^k) \in N_{IJ}^{0}(x_{n,\varepsilon}, e^{-\rho t_n^k} D_x \Theta_{n,\varepsilon}^k (z_{n,\varepsilon}), e^{-\rho t_n^k} D_q \Theta_{n,\varepsilon}^k (z_{n,\varepsilon}))$.

Using the super- and subsolution properties of $V_1$ and $V_2$, we can then choose $(u_{n,\varepsilon}^k, \alpha_{n,\varepsilon}^k)$ such that

$$-\frac{1}{n} \leq b_{n,\varepsilon}^k + \rho \tilde{V}_1(t_{n,\varepsilon}^k, y_{n,\varepsilon}^k, q_{n,\varepsilon}^k)$$

$$+ e^{\rho t_n^k} \mu_Y(y_{n,\varepsilon}^k, u_{n,\varepsilon}^k) + \mu_X(y_{n,\varepsilon}^k)^\top D_y \Theta_{n,\varepsilon}^k (z_{n,\varepsilon}^k)$$

$$- \frac{1}{2} \text{Trace}[\sigma_X \sigma_X^\top (y_{n,\varepsilon}^k, \alpha_{n,\varepsilon}^k) J_{n,\varepsilon}] ,$$
and
\[
\frac{1}{n} \geq -a_{n,\varepsilon}^k + \rho \tilde{V}_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) \\
+ e^{\alpha_{n,\varepsilon}} \mu_Y(x_{n,\varepsilon}^k, u_{n,\varepsilon}^k) - \mu_X(x_{n,\varepsilon}^k)^\top D_x \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) \\
- \frac{1}{2} \text{Trace} \left[ \sigma_{X,P}^\top \sigma_{X,P}(x_{n,\varepsilon}^k, \bar{\alpha}_{n,\varepsilon}^k \lambda_{n,\varepsilon}^k) \right].
\]

Hence,
\[
-\frac{\eta}{n} \leq \eta V_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) - \tilde{V}_1(t_{n,\varepsilon}^k, y_{n,\varepsilon}^k, q_{n,\varepsilon}^k) \\
- e^{\alpha_{n,\varepsilon}} (\mu_Y(x_{n,\varepsilon}^k, u_{n,\varepsilon}^k) - \mu_Y(y_{n,\varepsilon}^k, u_{n,\varepsilon}^k)) \\
+ \mu_X(x_{n,\varepsilon}^k)^\top D_x \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) + \mu_X(y_{n,\varepsilon}^k)^\top D_y \Theta_{n,\varepsilon}^k(z_{n,\varepsilon}^k) \\
+ \frac{1}{2} \text{Trace} \left[ \sigma_{X,P}^\top \sigma_{X,P}(x_{n,\varepsilon}^k, \bar{\alpha}_{n,\varepsilon}^k \lambda_{n,\varepsilon}^k) \right] \\
- \frac{1}{2} \text{Trace} \left[ \sigma_{X,P}^\top \sigma_{X,P}(y_{n,\varepsilon}^k, \bar{\alpha}_{n,\varepsilon}^k \lambda_{n,\varepsilon}^k) \right].
\]

Using (5.6), and then letting \( \eta \to 0 \),
\[
-\frac{\eta}{n} \leq -\rho \tilde{V}_2(t_{n,\varepsilon}^k, x_{n,\varepsilon}^k, p_{n,\varepsilon}^k) - \tilde{V}_1(t_{n,\varepsilon}^k, y_{n,\varepsilon}^k, q_{n,\varepsilon}^k) \\
- e^{\alpha_{n,\varepsilon}} (\mu_Y(x_{n,\varepsilon}^k) - \mu_Y(y_{n,\varepsilon}^k)) + n^2 (\mu_X(x_{n,\varepsilon}^k) - \mu_X(y_{n,\varepsilon}^k))(x_{n,\varepsilon}^k - y_{n,\varepsilon}^k) \\
+ D_x f_\varepsilon(x_{n,\varepsilon}^k)^\top \mu_X(x_{n,\varepsilon}^k) + \frac{1}{2} \text{Trace} \left[ \sigma_{X}(x_{n,\varepsilon}^k) \sigma_{X}(x_{n,\varepsilon}^k)^\top D_x^2 f_\varepsilon(x_{n,\varepsilon}^k) \right] \\
+ \frac{n^2}{2} \text{Trace} \left[ (\sigma_Y(x_{n,\varepsilon}^k) - \sigma_Y(y_{n,\varepsilon}^k))(\sigma_Y(x_{n,\varepsilon}^k) - \sigma_Y(y_{n,\varepsilon}^k))^\top \right] \\
+ \frac{k^2}{2} \| \bar{\alpha}_{n,\varepsilon}^k - \bar{\alpha}_{n,\varepsilon}^k \|^2.
\]

We now send \( n \to \infty \) and then \( \varepsilon \to 0 \) in the above inequality, and deduce from (3.9), (3.10), (3.12), (3.26), (3.11), (3.22), (3.23), (3.25) and the Lipschitz continuity of \( (\mu_X, \sigma_X) \) that
\[
0 \leq -\rho m,
\]
which contradicts the fact that \( \rho, m > 0 \).

**Part 2.** We now consider the case \( I \cup J = \mathcal{K} \). Part of the arguments being similar as in Part 1, we only sketch them.

**Step 1.** In the case \( I \cup J = \mathcal{K} \), we can work as if \( V_1 \) and \( V_2 \) do not depend on \( p \). Indeed, \( a \in A_{IJ} \) implies \( a = 0 \), so that the derivatives in \( p \) do not appear in the operator. Moreover, recalling the convention (1.10) and the discussion of subsection 2.1, we see that a function \( w \) is a viscosity supersolution (resp. subsolution) of (2.20) (resp. (2.21)) if and only if it is a viscosity supersolution (resp. subsolution) of
\[
\min \left\{ \varphi - \ell, -\partial_t \varphi + \tilde{F}_{IJ}(\cdot, D\varphi, D^2 \varphi); R(x, q) \right\} = 0 \text{ on } D_{IJ}, \quad (3.27)
\]
where
\[
\bar{F}_{I,j}(x,q,Q) := L^q \cdot \text{diag}[x],0(x,q,Q)
\]
and
\[
R(x,q) := \inf_{|\zeta| \leq 1} \delta_U(\zeta) - \zeta^\top \cdot \text{diag}[x]q.
\]
Note that here we do not need to consider the semicontinuous envelopes of the operator since the unbounded control \(a\) does not play any role and \(U\) is bounded.

Given \(\rho > 0\), we set \(\tilde{V}_1(t,x) := e^{\rho t} V_1(t,x)\) and \(\tilde{V}_2(t,x) := e^{\rho t} V_2(t,x)\). We now choose \(u \in \text{int} U \cap (-\infty, 0)^d\), which is possible since \(0 \in \text{int} U\) by assumption, \(\delta \in (0,1)\) and define
\[
\tilde{V}_\delta := (1 - \delta) \tilde{V}_1 + \delta \psi
\]
where, for some \(\epsilon > 0\),
\[
\psi(x) := \epsilon e^\sum_{i \leq d} u^i x^i.
\]
Note that, since \(\text{diag}[x] \psi(x)\) is bounded on \((0,\infty)^d\), \(\text{int} U\) is convex and contains \(0\), we can choose \(\epsilon > 0\) small enough so that
\[
0 < \nu \leq \delta_U(\zeta) - \zeta^\top e^{-\rho t} \cdot \text{diag}[x] \psi(x) u = \delta_U(\zeta) - \zeta^\top e^{-\rho t} \cdot \text{diag}[x] D_x \psi(x) (3.28)
\]
where \(\nu > 0\) does not depend on \(\zeta\) and \((t,x) \in [0,T] \times (0,\infty)^d\).

In order to show that \(V_2 \leq V_1\), we argue by contradiction. We therefore assume that
\[
\sup_{[0,T] \times (0,\infty)^d} \left( \tilde{V}_2 - \tilde{V}_\delta \right) =: 2m > 0, \quad (3.29)
\]
for \(\delta\) small enough, and work towards a contradiction.

Using the boundedness of \(\tilde{V}_2\) and \(\tilde{V}_\delta\), and (5.4), we deduce that
\[
\Phi^\epsilon := \tilde{V}_2 - \tilde{V}_\delta - f^\epsilon,
\]
where \(f^\epsilon\) is defined as in Part 1, admits a maximum \((t^\epsilon, x^\epsilon)\) on \([0,T] \times (0,\infty)^d\), which, for \(\epsilon > 0\) small enough, satisfies
\[
\Phi^\epsilon(t^\epsilon, x^\epsilon) \geq m > 0. \quad (3.30)
\]
Without loss of generality, we can choose \((x^\epsilon)_{\epsilon > 0}\) such that
\[
f^\epsilon(x^\epsilon) \xrightarrow[\epsilon \rightarrow 0]{} 0, \quad (3.31)
\]
which implies (see Part 1)
\[
|D_x f^\epsilon(x^\epsilon)^\top \cdot \text{diag}[x^\epsilon]| + |D_x f^\epsilon(x^\epsilon)^\top \mu_X(x^\epsilon)| + |D_x f^\epsilon(x^\epsilon)^\top \sigma_X(x^\epsilon)| \xrightarrow[\epsilon \rightarrow 0]{} 0. \quad (3.32)
\]
We can then assume from now on that $t \rightarrow 0$

By similar arguments as in Part 1, we can not have $t \rightarrow 0$

Lemma to deduce that, for each $\eta > 0$

Step 2. If, up to a subsequence, $\tilde{t}_n, x_n^\varepsilon, y_n^\varepsilon \in [0, T] \times (0, \infty)^{2d}$. Moreover, the same arguments as in Part 1 imply that, up to a subsequence and after possibly changing $(t^\varepsilon, x^\varepsilon)_{\varepsilon > 0}$,

$$x_n^\varepsilon, y_n^\varepsilon \rightarrow x^\varepsilon (0, \infty)^d, \quad t_n^\varepsilon \rightarrow t^\varepsilon,$$  \hspace{1cm} (3.33)

and

$$n^2|x_n^\varepsilon-y_n^\varepsilon|^2 \rightarrow 0, \quad \tilde{V}_2(t_n^\varepsilon, x_n^\varepsilon) - \tilde{V}_\delta(t_n^\varepsilon, y_n^\varepsilon) \rightarrow \tilde{V} - \tilde{V}_\delta \geq m + f(x^\varepsilon) > 0.$$ \hspace{1cm} (3.35)

By similar arguments as in Part 1, we can not have $t_n^\varepsilon = T$, up to a subsequence. We can then assume from now on that $t_n^\varepsilon < T$ for all $\varepsilon, n$.

Step 2. Since $t_n^\varepsilon < T$ and $(x_n^\varepsilon, y_n^\varepsilon) \in (0, \infty)^{2d}$, see (3.33), we can appeal to Ishii’s Lemma to deduce that, for each $\eta > 0$, there are real coefficients $b_{1,n}, b_{2,n}$ and symmetric matrices $X_{n,\varepsilon}^\varepsilon, \eta$ and $Y_{n,\varepsilon}^\varepsilon, \eta$ such that

$$(b_{1,n}^\varepsilon, p_n^\varepsilon, X_{n,\varepsilon}^\varepsilon) \in \mathcal{P}+\tilde{V}_2(t_n^\varepsilon, x_n^\varepsilon) \quad \text{and} \quad (-b_{2,n}^\varepsilon, q_n^\varepsilon, Y_{n,\varepsilon}^\varepsilon) \in \mathcal{P}-\tilde{V}_\delta(t_n^\varepsilon, y_n^\varepsilon),$$

where

$$(3.36)$$

and $b_{1,n}^\varepsilon, b_{2,n}^\varepsilon, X_{n,\varepsilon}^\varepsilon, \eta$ and $Y_{n,\varepsilon}^\varepsilon, \eta$ satisfy

$$\begin{cases} 
 b_{1,n}^\varepsilon + b_{2,n}^\varepsilon = 0 \\
 X_{n,\varepsilon}^\varepsilon \quad 0 \\
 0 \quad -Y_{n,\varepsilon}^\varepsilon 
\end{cases} \leq A_n^\varepsilon + \eta(A_n^\varepsilon)^2 \quad \hspace{1cm} (3.37)$$

with

$$A_n^\varepsilon := \left( \begin{array} {cc} 2n^2I_d + D^2f(x_n^\varepsilon) & -2n^2I_d \\
-2n^2I_d & 2n^2I_d \end{array} \right).$$

We now study in different cases:

Case 1. If, up to a subsequence, $\tilde{V}_2(t_n^\varepsilon, x_n^\varepsilon) \leq \ell e^{\delta t_n^\varepsilon}$, then we get a contradiction for $n$ large and $\delta$ small, since $\tilde{V}_1(t_n^\varepsilon, y_n^\varepsilon) \geq \ell e^{\delta t_n^\varepsilon} \text{ and } \psi \geq 0 \geq \ell e^{\delta t_n^\varepsilon}$. 

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Case 2. If, up to a subsequence, 
\[ R(x_n^\varepsilon, e^{-\rho t_n} p_n^\varepsilon) > 0, \]
then we must have
\[ \rho \tilde{V}_2(t_n^\varepsilon, x_n^\varepsilon) - b_{1,n}^\varepsilon + \tilde{F}_{IJ}(x_n^\varepsilon, p_n^\varepsilon, X_n^\varepsilon, q_n^\varepsilon) \leq 0. \]

Since \( \psi, x \in (0, \infty)^d \mapsto (\text{diag}[x]D_x \psi(x), \text{diag}[x]^2 D_x^2 \psi(x)) \) and \((\mu, \sigma)\) are bounded, one easily checks that the supersolution property of \( V_1 \) implies that
\[ \rho \tilde{V}_2(t_n^\varepsilon, y_n^\varepsilon) + b_{2,n}^\varepsilon + \tilde{F}_{IJ}(y_n^\varepsilon, q_n^\varepsilon, Y_n^\varepsilon) \geq O(\delta). \]

Standard arguments based on (3.32), (3.33), (3.34), (3.35) and (3.37) then leads to a contradiction for \( \delta > 0 \) small enough, after sending \( \eta \rightarrow 0, n \rightarrow \infty \) and then \( \varepsilon \rightarrow 0. \)

Case 3. We can now assume that \( R(x_n^\varepsilon, e^{-\rho t_n} p_n^\varepsilon) \leq 0 \). By the supersolution property of \( V_1 \), we have
\[ R \left( y_n^\varepsilon, e^{-\rho t_n} q_n^\varepsilon - \delta D_y \psi(y_n^\varepsilon) \right) \geq 0. \]

We can then find \( \zeta_n^\varepsilon \) such that \( |\zeta_n^\varepsilon| = 1 \) and
\[ 0 \geq (e^{\rho t_n^\varepsilon} \delta_U(\zeta_n^\varepsilon) - (\zeta_n^\varepsilon)\top \text{diag}[x_n^\varepsilon]^2 p_n^\varepsilon - e^{\rho t_n^\varepsilon} \delta_U(\zeta_n^\varepsilon) + (\zeta_n^\varepsilon)\top \text{diag}[y_n^\varepsilon]^2 q_n^\varepsilon) \]
\[ + \delta e^{\rho t_n^\varepsilon} \left( \delta_U(\zeta_n^\varepsilon) - (\zeta_n^\varepsilon)\top e^{-\rho t_n^\varepsilon} \text{diag}[y_n^\varepsilon]^2 D_y \psi(y_n^\varepsilon) \right). \]

In view of (3.28) and (3.36), this implies that
\[ (\zeta_n^\varepsilon)\top \left( \text{diag}[x_n^\varepsilon - y_n^\varepsilon]^2 (x_n^\varepsilon - y_n^\varepsilon) + \text{diag}[x_n^\varepsilon]^2 D_x f_n(x_n^\varepsilon) \right) \]
\[ = (\zeta_n^\varepsilon)\top \left( \text{diag}[x_n^\varepsilon]^2 p_n^\varepsilon - \text{diag}[y_n^\varepsilon]^2 q_n^\varepsilon \right) \]
\[ \geq v \delta e^{\rho t_n^\varepsilon}. \]

Using (3.32), (3.33), (3.34) and (3.36), this leads to a contradiction as \( n \rightarrow \infty \) and then \( \varepsilon \rightarrow 0. \)

3.5 Proof of Theorem 1.2

In order to complete the proof of Theorem 1.2, we first show that \( v^\lambda \) is continuous on \( \bar{D} \).

Proposition 3.7 The function \( v^\lambda \) is continuous on \( \bar{D} \).

Proof. We argue by induction.

Step 1. We first notice that \( v^\lambda_{IJ} \) is continuous on \( \bar{D} \) for \((I, J) \in \mathcal{P}_n^\kappa \). This is a direct consequence of Theorem 1.1 and Proposition 3.6.
Step 2. We now assume that \( v^\lambda_{IJ} \) is continuous on \( \bar{D}_{IJ} \) if \((I, J) \in \mathcal{P}^\kappa_{\forall} \) for some \( 1 \leq k < \kappa \), and show that this implies that it holds for \((I, J) \in \mathcal{P}^\kappa_{\forall} \).

By Step 1, we know that \( v^\lambda_{IJ} \) is continuous on \( \bar{D} \). Moreover, \( v^\lambda_{IJ} \geq v^\lambda_{IJ} \) since \( v^\lambda \) is non-decreasing with respect to its \( p \)-variable. In view of Theorem 1.1 and Proposition 3.6, it thus suffices to show that \( v^\lambda_{IJ} \geq v^\lambda_{IJ} \) on \( \partial D_{IJ} \cap [0, T) \times (0, \infty)^d \times [0, 1]^\kappa \).

By Proposition 3.2 and our induction assumption, we have

\[
\forall (I, J), (I', J') \in \mathcal{P}_\kappa \text{ such that } (I', J') \supseteq (I, J), \text{ hence, it suffices to show that } \forall (I, J), (I', J') \in \mathcal{P}_\kappa \text{ such that } (I', J') \supseteq (I, J), \text{ since } v^\lambda_{IJ} \geq v^\lambda_{IJ}.
\]

We now fix \((I', J') \in \mathcal{P}_\kappa \text{ such that } (I', J') \supseteq (I, J). Since } v^\lambda_{IJ} \geq v^\lambda_{IJ}, \text{ it suffices to restrict to the case } I = I'. By Proposition 3.4, } v^\lambda_{IJ} \text{ is a viscosity supersolution of}

\[
\min \{ \varphi - \ell, -\partial_t \varphi + F_{IJ'}^*(\varphi) \} \geq 0 \text{ on } D_{IJ'}.
\]

On the other hand, Step 1 and Theorem 1.1 imply that \( v^\lambda_{IJ} \) is continuous on \( D_{IJ'} \) and is a viscosity subsolution of

\[
\min \{ \varphi - \ell, -\partial_t \varphi + F_{IJ'}^*(\varphi) \} \leq 0 \text{ on } D_{IJ'}.
\]

(i) First assume that \( I \cup J' = K \). Then, Theorem 1.1 and Proposition 3.6 imply that \( v^\lambda_{IJ} \geq v^\lambda_{IJ} \) on \( D_{IJ'} \).

(ii) We now assume that \( v^\lambda_{IJ} \geq v^\lambda_{IJ} \) on \( D_{IJ'} \) if \( |I| + |J'| = n \in (\kappa - k, \kappa] \) and show that this implies that the result also holds for \( |I| + |J'| = n - 1 \). Our recursion assumption implies that \( v^\lambda_{IJ} \geq v^\lambda_{IJ} \) on \( D_{IJ''} \) for all \( J'' \supseteq J', J'' \neq J' \). Since \( v^\lambda_{IJ} \leq v^\lambda_{IJ} \), we have \( v^\lambda_{IJ} \geq v^\lambda_{IJ} \) on \( \partial D_{IJ'} \). Moreover, (i) above together with the fact that \( I \subset J'^c \), since \( |I| + |J'| \leq \kappa \), imply that \( v^\lambda_{IJ} \geq v^\lambda_{IJ} \) on \( D_{IJ'} \).

On the other hand, by Theorem 1.1 and our induction assumption, \( v^\lambda_{IJ} \) is continuous and is a subsolution on \( D_{IJ'} \) of

\[
\max \left\{ \min \left\{ \varphi - \ell, -\partial_t \varphi + F_{IJ'}^*(\varphi, D\varphi, D^2\varphi) \right\}, M_{\lambda_{IJ}}(\varphi, v^\lambda_{IJ'c}, D\varphi) \right\} = 0
\]

while, by Proposition 3.4, \( v^\lambda_{IJ} \) is a supersolution on \( D_{IJ'} \) of

\[
\max \left\{ \min \left\{ \varphi - \ell, -\partial_t \varphi + F_{IJ'}^*(\varphi, D\varphi, D^2\varphi) \right\}, M_{\lambda_{IJ}}(\varphi, v^\lambda_{IJ'c}, D\varphi) \right\} = 0.
\]

The fact that \( v^\lambda_{IJ} \geq v^\lambda_{IJ} \) is then a consequence of Proposition 3.6. \( \square \)

Proof of Theorem 1.2.
4. THE PROOF OF THE CONVERGENCE RESULT

We only prove item (i) of the theorem, the second one being proved similarly. We argue by induction as in the above proof.

**Step 1.** The fact that $V_{IJ} \geq v^\lambda_{IJ}$ on $\bar{D}_{IJ}$ when $(I, J) \in P_\kappa$ is an immediate consequence of Theorem 1.1 and Proposition 3.6.

**Step 2.** We now assume that $V_{IJ} \geq v^\lambda_{IJ}$ on $\bar{D}_{IJ}$ if $(I, J) \in P_\kappa - k$ for some $1 \leq k < \kappa$, and show that this implies that it holds for $(I, J) \in P_\kappa - k - 1$.

By Step 1 and the fact that $V$ is non-decreasing with respect to its $p$-parameter, we know that $V_{IJ} \geq V_{J^c I^c} \geq v^\lambda_{J^c I^c}$ which is upper-semicontinuous by Proposition 3.7. Moreover, we have by assumption that $V_{IJ} \geq V_{I' J'}$ on $\partial D_{IJ} \cap D_{I' J'}$ for $(I', J') \in P_\kappa$ such that $I' \supset I$ and $J' \supset J$ with $(I', J') \neq (I, J)$, and that $V_{IJ}(T, \cdot) \geq G^\lambda$. Since $v^\lambda$ is continuous by Proposition 3.7, our induction assumption then leads to $V_{IJ} \geq v^\lambda_{IJ}$ on $\partial D_{IJ}$. The fact that $V_{IJ} \geq v^\lambda_{IJ}$ on $\bar{D}_{IJ}$ is then a consequence of Theorem 1.1 and Proposition 3.6. 

4 The proof of the convergence result

We first provide some additional technical results.

4.1 Additional technical results for the convergence of the finite differences scheme

We start with a simple remark concerning the PDEs obtained in the interior of the domains in Section 2.3.

**Remark 4.1** (i) The result of Proposition 3.6 still holds if $V_1$ is a supersolution on $\bar{D}_{IJ}$ of

$$(\varphi - w) \wedge \max\{\varphi - \omega, H^\lambda_{IJ}[\varphi, \psi_1]\} = 0 \text{ on } D_{IJ},$$

and $V_2$ is a subsolution on $\bar{D}_{IJ}$ of

$$(\varphi - w) \wedge \max\{\varphi - \omega, H^\lambda_{IJ}[\varphi, \psi_2]\} = 0 \text{ on } D_{IJ},$$

for some continuous map $w, \omega$. In this case, the proof of Proposition 3.6 can be easily modified by studying simple additional cases. In the case where $w \geq \psi_2 = \psi_1$, the assumption $V_1 \geq \psi_2$ on $D_{IJ}$ in not necessary anymore in Proposition 3.6, since it is induced by the super-solution property of $V_1$.

(ii) Obviously, the result of Proposition 3.6 still holds if $V_1$ is a supersolution on $\bar{D}_{IJ}$ of

$$\max\{\varphi - \omega, H^\lambda_{IJ}[\varphi, \psi_1]\} = 0 \text{ on } D_{IJ},$$
and \( V_2 \) is a subsolution on \( \tilde{D}_{IJ} \) of
\[
\max\{\varphi - \omega, H_{IJ}^0[\varphi, \psi_2]\} = 0 \quad \text{on } D_{IJ},
\]
for some continuous map \( \omega \).

(iii) Note that in the two above cases, we can replace \( H_{IJ}^0 \) by \( H_{IJ}^{\tilde{a}} \) defined as \( H_{IJ}^\lambda \) but with \( A_{IJ}^0 \) instead of \( A_{IJ} \), \( \tilde{a} > 0 \). The proof follows line by line the one of Proposition 3.6, up to the study of simple additional cases as mentioned in (i) and (ii) above.

We now discuss the boundary condition as \( t \to T \).

**Lemma 4.1** Fix \( (I, J) \in \mathcal{P}_n \).

(i) Let \( V_1 \) be a supersolution on \( \tilde{D}_{IJ} \) of
\[
\max\left\{ \max\{\varphi - L, \min\{\varphi - \ell, -\partial_t \varphi + F_{IJ}^{\tilde{a}}(\cdot, D\varphi, D^2\varphi)\}, M_{IJ}^\lambda(\varphi, v_{ij,J}, D_p\varphi)\}, \varphi - G^\lambda \right\} = 0 \quad \text{on } \{T\} \times (0, \infty)^d \times \tilde{B}_{IJ}.
\]

(ii) Let \( V_2 \) be a subsolution on \( \tilde{D}_{IJ} \) of
\[
\min\left\{ \max\{\varphi - L, \min\{\varphi - \ell, -\partial_t \varphi + F_{IJ}^{\tilde{a}}(\cdot, D\varphi, D^2\varphi)\}, M_{IJ}^\lambda(\varphi, v_{ij,J}, D_p\varphi)\}, \varphi - G^\lambda \right\} = 0 \quad \text{on } \{T\} \times (0, \infty)^d \times \tilde{B}_{IJ}.
\]

**Proof.** The proof is standard. We start with item (i). Given a test function \( \varphi \) such that \((T, x_0, p_0) \in \{T\} \times (0, \infty)^d \times \tilde{B}_{IJ} \) achieves a minimum of \( V_1 - \varphi \) on \( D_{IJ} \), we define \( \varphi_n(t, x, p) := \varphi(t, x, p) - n(T - t), n \geq 1 \). We assume that
\[
\max\{\varphi - L, M_{IJ}^\lambda(\varphi, v_{ij,J}, D_p\varphi), \varphi - G^\lambda\}(T, x_0, p_0) < 0.
\]
Since \((T, x_0, p_0)\) also achieves a minimum of \( V_1 - \varphi_n \), the supersolution property of \( V_1 \) implies that
\[
\min\left\{ \varphi - \ell, -n - \partial_t \varphi + F_{IJ}^{\tilde{a}}(\cdot, D\varphi, D^2\varphi) \right\}(T, x_0, p_0) \geq 0.
\]
Sending \( n \to \infty \), we obtain a contradiction since \( F_{IJ}^{\tilde{a}}(\cdot, D\varphi, D^2\varphi)(T, x_0, p_0) < \infty \), recall that \( U \) and \( A_{IJ}^0 \) are bounded. We finally use the fact that \( G^\lambda \leq L \), since \( g \leq L \) by assumption.

We now discuss item (ii). By similar arguments as above, we first obtain that \( V_2 \) is a subsolution on \( \tilde{D}_{IJ} \) of
\[
\min\left\{ \max\{\varphi - L, \varphi - \ell, M_{IJ}^\lambda(\varphi, v_{ij,J}, D_p\varphi)\}, \varphi - G^\lambda \right\} = 0
\]
on \( \{T\} \times (0, \infty)^d \times \tilde{B}_{IJ} \). We conclude by using the fact that \( G^\lambda \geq \ell \). \( \square \)
Lemma 4.2 Let $V_1$ and $V_2$ be as in Lemma 4.1 for some $(I, J) \in \mathcal{P}_K$. Assume that they take values in $[\ell, L]$. Then, $G^\lambda \geq V_2$ on $\{T\} \times (0, \infty)^d \times \bar{B}_{IJ}$. If in addition $(I, J) \in \mathcal{P}_K^\kappa$, then $G^\lambda \leq V_1$ on $\{T\} \times (0, \infty)^d \times \bar{B}_{IJ}$.

Proof. The fact that $V_2 \leq G^\lambda$ on $\{T\} \times (0, \infty)^d \times \bar{B}_{IJ}$ follows from Lemma 4.1. We now show that $V_1 \geq v_{IJ}^\lambda$ on $\{T\} \times (0, \infty)^d \times \bar{B}_{IJ}$. To see this, note that $V_1(T, x, p) \geq G^\lambda(x, p)$ implies $V_1(T, x, p) \geq v_{IJ}^\lambda(T, x, p)$ by Theorem 1.3. Recalling the convention (1.10), this concludes the proof for $(I, J) \in \mathcal{P}_K^\kappa$. \hfill \Box

We finally discuss the boundary condition as $p \to \partial B_{IJ}$.

Lemma 4.3 Fix $(I, J) \in \mathcal{P}_K \setminus \mathcal{P}_K^\kappa$ and $(I', J') \supseteq (I, J)$.

(i) Let $V_1$ be a bounded supersolution on $\bar{D}_{IJ}$ of

$$\max \{ (\varphi - v_{IJ}^\lambda) \wedge \max \{ \varphi - L, \bar{H}^{\lambda}_{IJ}[\varphi, v_{IJ}^\lambda] \} , \varphi - v_{IJ}^\lambda \} = 0 \quad \text{on} \quad \partial D_{IJ} \cap D_{I'J'}$$

$$\max \{ (\varphi - v_{IJ}^\lambda) \wedge \max \{ \varphi - L, \bar{H}^{\lambda}_{IJ}[\varphi, v_{IJ}^\lambda] \} , \varphi - G^\lambda \} = 0 \quad \text{on} \quad \partial_T B_{I'J'}.$$

Then, $V_1$ is a bounded supersolution on $\bar{D}_{I'J'}$ of

$$\max \{ (\varphi - v_{I'J'}^\lambda) \wedge \max \{ \varphi - L, \bar{H}^{\lambda}_{I'J'}[\varphi, v_{I'J'}^\lambda] \} , \varphi - v_{I'J'}^\lambda \} = 0 \quad \text{on} \quad D_{I'J'}$$

$$\varphi - G^\lambda = 0 \quad \text{on} \quad \partial_T B_{I'J'}.$$  

(ii) Let $V_2$ be a bounded subsolution on $\bar{D}_{IJ}$ of

$$\min \{ \varphi - v_{IJ}^\lambda , \max \{ \varphi - L , \bar{H}^{\lambda}_{IJ}[\varphi, v_{IJ}^\lambda] \} , \varphi - v_{IJ}^\lambda \} = 0 \quad \text{on} \quad \partial D_{IJ} \cap D_{I'J'}$$

$$\min \{ \varphi - v_{IJ}^\lambda , \max \{ \varphi - L , \bar{H}^{\lambda}_{IJ}[\varphi, v_{IJ}^\lambda] \} , \varphi - G^\lambda \} = 0 \quad \text{on} \quad \partial_T B_{I'J'}.$$

Then, $V_2 \leq v_{I'J'}^\lambda$ on $D_{I'J'}$ and $V_2 \leq G^\lambda$ on $\{T\} \times (0, \infty)^d \times B_{I'J'}$.

Proof. (i) It follows from the same argument as in the proof of Lemma 4.1 that $V_1$ is a bounded supersolution on $\bar{D}_{IJ}$ of

$$\max \{ (\varphi - v_{IJ}^\lambda) \wedge \max \{ \varphi - L , \bar{H}^{\lambda}_{IJ}[\varphi, v_{IJ}^\lambda] \} , \varphi - v_{IJ}^\lambda \} = 0 \quad \text{on} \quad \bar{D}_{IJ} \cap D_{I'J'},$$

$$\max \{ (\varphi - v_{IJ}^\lambda) \wedge \max \{ \varphi - L , \bar{H}^{\lambda}_{IJ}[\varphi, v_{IJ}^\lambda] \} , \varphi - G^\lambda \} = 0 \quad \text{on} \quad \partial_T B_{I'J'}.$$

By following, up to minor modifications (related to the fact that their test function has a derivative in $p$ that converges to $\infty$, see also Step 2. of the proof of Proposition 3.2 for an adaptation to our context), the arguments used in Section 6.1 of [12], we deduce that $V_1$ is a bounded supersolution on $\bar{D}_{IJ}$ of

$$\max \left\{ (\varphi - v_{IJ}^\lambda) \wedge \max \{ \varphi - L , -\partial_p \varphi + \tilde{F}^{\lambda}_{IJ'}(\cdot, D\varphi, D^2\varphi) \} , \varphi - v_{IJ}^\lambda \right\} = 0 \quad \text{on} \quad \bar{D}_{IJ} \cap D_{I'J'},$$

$$\varphi - G^\lambda = 0 \quad \text{on} \quad \{T\} \times (0, \infty)^d \times (\bar{B}_{IJ} \cap B_{I'J'}).$$
We conclude by using the fact that \(-\partial_t \varphi + \bar{F}_{ij}^\alpha(\cdot, D \varphi, D^2 \varphi) \leq \bar{H}_{ij}^\alpha[\varphi, v_{ij}^\lambda, r_i, r_j]\).

(ii) Since \(v_{ij}^\lambda \leq v_{ij}^\lambda\) on \(\partial D_{ij} \cap D_{ij}^\prime\) and \(v_{ij}^\lambda \leq G^\lambda\) on \(\{T\} \times (0, \infty)^d \times (\bar{B}_{ij} \cap B_{ij}^\prime)\), see Theorem 1.3 and recall that \(v^\lambda\) is non-decreasing in \(\rho, V^2\) is indeed a bounded subsolution on \(\bar{D}_{ij}\) of

\[
\min \{\max \{\varphi - L, H^\lambda_{ij}[\varphi, v_{ij}^\lambda]\}, \varphi - v_{ij}^\lambda\} = 0 \quad \text{on} \quad \partial D_{ij} \cap D_{ij}^\prime,
\]

\[
\min \{\max \{\varphi - L, H^\lambda_{ij}[\varphi, v_{ij}^\lambda]\}, \varphi - G^\lambda\} = 0 \quad \text{on} \quad \partial T \times (0, \infty)^d \times B_{ij}^\prime.
\]

By the same argument as in the proof of Lemma 4.1, we then deduce that \(V_2\) is a bounded subsolution on \(\bar{D}_{ij}\) of

\[
\min \{\max \{\varphi - L, H^\lambda_{ij}[\varphi, v_{ij}^\lambda]\}, \varphi - v_{ij}^\lambda\} = 0 \quad \text{on} \quad D_{ij}^\prime,
\]

\[
\varphi - G^\lambda = 0 \quad \text{on} \quad \{T\} \times (0, \infty)^d \times B_{ij}^\prime.
\]

We then argue as in Step 2. of the proof of Proposition 3.2 above to deduce that \(V_2 \leq v_{ij}^\lambda\) on \(D_{ij}^\prime\).

\[\square\]

4.2 Proof of Theorem 2.1

We first prove the convergence for \((I, J) \in \mathcal{P}_\kappa^\kappa\).

Proposition 4.1 Fix \((I, J) \in \mathcal{P}_\kappa^\kappa\). Then, \(\bar{w}_{ij}^* \leq v_{ij}^\lambda \leq \bar{w}_{ij}^\lambda, v_{ij}^\lambda\) on \(\bar{D}_{ij}\). In particular,

\[
\bar{w}_{ij}^* = v_{ij}^\lambda = \bar{w}_{ij}^\lambda, v_{ij}^\lambda\) on \(\bar{D}_{ij}\).
\]

Proof. 1. Recall that \(\bar{w}_{ij}^*\) is well-defined and takes values in \([\ell, L]\). Moreover, the numerical scheme defined above is monotone and consistent under (2.5), recall in particular (2.4). Arguing as in [3], it follows that \(\bar{w}_{ij}^*\) is a viscosity subsolution on \(\bar{D}_{ij}\) of

\[
\max \{\varphi - L, H^\lambda_{ij}[\varphi, 0]\} = 0 \quad \text{on} \quad D_{ij},
\]

\[
\min \{\max \{\varphi - L, H^\lambda_{ij}[\varphi, 0]\}, \varphi - G^\lambda\} = 0 \quad \text{on} \quad \{T\} \times (0, \infty)^d \times \bar{B}_{ij}.
\]

Appealing to Lemma 4.2, the above operator can be reduced to

\[
\max \{\varphi - L, H^\lambda_{ij}[\varphi, 0]\} = 0 \quad \text{on} \quad D_{ij},
\]

\[
\varphi - G^\lambda = 0 \quad \text{on} \quad \{T\} \times (0, \infty)^d \times \bar{B}_{ij}.
\]

The fact that \(\bar{w}_{ij}^* \leq v_{ij}^\lambda\) then follows from Theorem 1.3 and Remark 4.1.
2. By the same arguments and Lemma 4.2 again, we deduce that \( \bar{w}_{IJ}^{a} \) is a bounded viscosity supersolution on \( \bar{D}_{IJ} \) of

\[
\max \{ \varphi - L , \ H_{IJ}^{a} [\varphi , 0] \} = 0 \quad \text{on} \quad D_{IJ} \\
\varphi - G^{a} = 0 \quad \text{on} \quad \{ T \} \times (0, \infty)^{d} \times \bar{B}_{IJ}.
\]

Since \((I, J) \in \mathcal{P}_{\kappa} \setminus \mathcal{P}_{\kappa}^{c} , A_{IJ} = \{ 0 \} \) so that \( \bar{w}_{IJ}^{a} \) and \( H_{IJ}^{a} \) do not depend on \( a \). The fact that \( v_{IJ}^{a} \leq \bar{w}_{IJ}^{a} \) on \( \bar{D}_{IJ} \) then follows from Remark 4.1 and the fact that \( v_{IJ}^{a} \) is a subsolution of the latter, by Theorem 1.3.

We now complete the proof by an induction argument.

**Proposition 4.2** Fix \((I, J) \in \mathcal{P}_{\kappa} \setminus \mathcal{P}_{\kappa}^{c} \). Then, \( \bar{w}_{IJ}^{a} \leq v_{IJ}^{a} \leq \bar{w}_{IJ}^{a} \) on \( \bar{D}_{IJ} \). In particular, \( \bar{w}_{IJ}^{a} = v_{IJ}^{a} = \bar{w}_{IJ}^{a} \) on \( \bar{D}_{IJ} \).

**Proof.** In view of Proposition 4.1, we can argue by induction. We therefore assume that

\[
\bar{w}_{IJ}^{a} \geq v_{IJ}^{a} \geq \bar{w}_{IJ}^{a} \quad \text{for all} \quad (I, J) \in \mathcal{P}_{\kappa}^{k}
\]  

(4.2) for some \( 1 \leq k \leq \kappa \), and show that this implies that it holds for \((I, J) \in \mathcal{P}_{\kappa}^{k-1} \) as well.

1. By the same argument as in Proposition 4.1, Lemma 4.3, and (4.1), we first obtain that \( \bar{w}_{IJ}^{a} \) is a bounded viscosity subsolution on \( \bar{D}_{IJ} \) of

\[
\begin{align*}
(\varphi - v_{IJ}^{a}) \wedge \max \{ \varphi - L , \ H_{IJS}^{a} [\varphi , v_{IJ}^{a}] \} &= 0 \quad \text{on} \quad D_{IJ} \\
\varphi - v_{IJ}^{a} &= 0 \quad \text{on} \quad D_{IJ} \cup \{ T \} \times (0, \infty)^{d} \times \bar{B}_{IJ} \\
\varphi - G^{a} &= 0 \quad \text{on} \quad \{ T \} \times (0, \infty)^{d} \times \bar{B}_{IJ}.
\end{align*}
\]

(4.3)

On the other hand, Theorem 1.3 implies that \( v_{IJ}^{a} \) is a supersolution of (4.3) on \( \bar{D}_{IJ} \) with \( H_{IJS}^{a} \) in place of \( H_{IJ}^{a} \), that \( v_{IJ}^{a} = v_{IJ}^{a} \) on \( \partial D_{IJ} \cap \bar{D}_{IJ} \) and \( v_{IJ}^{a} \) is a supersolution of \( \bar{D}_{IJ} \) on \((0, \infty)^{d} \times \bar{B}_{IJ} \). Finally, \( v_{IJ}^{a} \geq v_{IJ}^{a} \) since it is non-decreasing in its \( p \)-parameter. The fact that \( \bar{w}_{IJ}^{a} \leq v_{IJ}^{a} \) on \( \bar{D}_{IJ} \) then follows from Remark 4.1.

2. By the same reasoning, recall in particular Lemma 4.3 and the first assertion of Proposition 4.1, \( \bar{w}_{IJ}^{a} \) is a bounded viscosity supersolution on \( \bar{D}_{IJ} \) of

\[
\begin{align*}
(\varphi - v_{IJ}^{a}) \wedge \max \{ \varphi - L , \ H_{IJS}^{a} [\varphi , v_{IJ}^{a}] \} &= 0 \quad \text{on} \quad D_{IJ} \\
(\varphi - v_{IJ}^{a}) \wedge \max \{ \varphi - L , H_{IJS}^{a} [\varphi , v_{IJ}^{a}] \} &= 0 \quad \text{on} \quad D_{IJ} \cup \{ T \} \times (0, \infty)^{d} \times \bar{B}_{IJ} \\
\varphi - G^{a} &= 0 \quad \text{on} \quad \{ T \} \times (0, \infty)^{d} \times \bar{B}_{IJ}
\end{align*}
\]

where \( v_{IJ}^{a} = \bar{w}_{IJ}^{a} = \bar{w}_{IJ}^{a} \) and \( v_{IJ}^{a} = \bar{w}_{IJ}^{a} = \bar{w}_{IJ}^{a} \) are continuous.

On the other hand, \( v_{IJ}^{a} \) is a subsolution of \( \max \{ \varphi - L , H_{IJS}^{a} [\varphi , v_{IJ}^{a}] \} = 0 \) on \( \bar{D}_{IJ} \) and satisfies the boundary condition \( v_{IJ}^{a} = v_{IJ}^{a} \) on \( \partial D_{IJ} \cap D_{IJ} \) and \( v_{IJ}^{a} = G^{a} \).
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on \( \{T\} \times (0, \infty)^d \times \bar{B}_{IJ}, \) recall Theorem 1.3. In view of Remark 4.1, and the fact that
\( \bar{w}_{IJ}^{\alpha} \geq v_{I,J}^{\lambda} \) on \( D_{IJ} \) and \( \bar{w}_{IJ}^{\beta} \geq G^\lambda \) on \( \{T\} \times (0, \infty)^d \times \bar{B}_{IJ}, \) by its supersolution
property, it only remains to prove that \( \bar{w}_{IJ}^{\alpha} \geq v_{I,J}^{\lambda} \) on \( \partial D_{IJ} \cap \bar{D}_{IJ}. \)

Since \( v_{I,J}^{\lambda} \geq v_{I',J'}^{\lambda}, \) it suffices to consider the case \( I = I'. \) If \( (I, J') \in \mathcal{P}_\kappa \), then the
result follows from Remark 4.1, Theorem 1.3 and the second and third equations in
the system above. Assuming that \( \bar{w}_{IJ}^{\alpha} \geq v_{I,J'}^{\lambda} \) on \( \partial D_{IJ} \cap D_{IJ'} \) for \( (I, J') \in \mathcal{P}_k \) with
\( |I| + |J| + 2 \leq k \leq \kappa, \) then we deduce similarly that it holds for \( (I, J') \in \mathcal{P}_{k-1} \), since
our induction assumption guarantees the required boundary conditions. \( \Box \)
Troisième partie

Optimal control of direction of reflection for jump diffusions
Chapitre 5

The SDEs with controlled oblique reflection

1 Introduction

The aim of this chapter is to study a class of optimal control problem for reflected processes on the boundary of a bounded domain $\mathcal{O}$, whose direction of reflection can be controlled. When the direction of reflection $\gamma$ is not controlled, the existence of a solution to reflected SDEs was studied in the case where the domain $\mathcal{O}$ is a half space by El Karoui and Marchan [17], and, Ikeda and Watanabe [26]. Tanaka [50] considered the case of convex sets. More general domains have been discussed by Dupuis and Ishii [21], where they proved the strong existence and uniqueness of solutions in two cases. In the first case, the direction of reflection $\gamma$ at each point of the boundary is single valued and varies smoothly, even if the domain $\mathcal{O}$ may be non smooth. In the second case, the domain $\mathcal{O}$ is the intersection of a finite number of domains with relatively smooth boundaries. Motivated by applications in financial mathematics, Bouchard [8] then proved the existence of a solution to a class of reflected SDEs, in which the oblique direction of reflection is controlled. This result is restricted to Brownian SDEs and to the case where the control is a deterministic combination of an Itô process and a continuous process with bounded variation. In this paper, we extend Bouchard’s result to the case of jump diffusion and allow the control to have discontinuous paths.

As a first step, we start with an associated deterministic Skhorokhod problem:

$$
\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s), \varepsilon(s))1_{\phi(s) \in \partial \mathcal{O}} d\eta(s), \quad \phi(t) \in \mathcal{O},
$$

(1.1)

where $\eta$ is a non decreasing function and $\gamma$ is controlled by a control process $\varepsilon$ taking values in a given compact set $E$ of $\mathbb{R}^l$. Bouchard [8] proved the strong existence of a solution for such problems in the family of continuous functions when $\varepsilon$ is a continuous function with bounded variation. Extending this result, we consider the
Skhorokhod problem in the family of càdlàg functions with finite number of points of discontinuity. The difficulty comes from the way the solution map is defined at the jump times. In this paper, we will investigate on a particular class of solutions, which is parameterized through the choice of a projection operator $\pi$. If the value $\phi(s-) + \Delta \psi(s)$ is out of the closure of the domain at a jump time $s$, we simply project this value on the boundary $\partial \mathcal{O}$ of the domain along the direction $\gamma$. The value after the jump of $\phi$ is chosen as $\pi(\phi(s-) + \Delta \psi(s), \varepsilon(s))$, where the projection $\pi$ along the oblique direction $\gamma$ satisfies $y = \pi(y,e) - l(y,e)\gamma(\pi(y,e),e)$, for all $y \notin \bar{\mathcal{O}}$ and $e \in E$, for some suitable positive function $l$. This leads to

$$\phi(s) = (\phi(s-) + \Delta \psi(s)) + \gamma(\phi(s),\varepsilon(s))\Delta \eta(s),$$

with $\Delta \eta(s) = l(\phi(s-) + \Delta \psi(s), \varepsilon(s))$. When the direction of reflection is not oblique and the domain $\mathcal{O}$ is convex, the function $\pi$ is just the usual projection operator and $l(y)$ coincides with the distance to the closure of the domain $\bar{\mathcal{O}}$.

We next consider the stochastic case. Namely, we prove the existence of an unique pair formed by a reflected process $X^\varepsilon$ and a non decreasing process $L^\varepsilon$ satisfying

$$\left\{ \begin{array}{l}
X(r) = x + \int_t^r F(X(s-))dZ_s + \int_t^r \gamma(X(s),\varepsilon(s))1_{X(s)\in \partial \mathcal{O}}dL(s), \\
X(r) \in \bar{\mathcal{O}}, \text{ for all } r \in [t,T]
\end{array} \right. \quad (1.2)$$

where $Z$ is the sum of a drift term, a Brownian stochastic integral and an adapted compound Poisson process, and the control process $\varepsilon$ belongs to the class $\mathcal{E}$ of $E$-valued càdlàg predictable processes with bounded variation and finite activity. As in the deterministic case, we only study a particular class of solutions, which is parameterized by $\pi$. This means that whenever $X$ is not in the domain $\mathcal{O}$ because of a jump, it is projected on the boundary $\partial \mathcal{O}$ along the direction $\gamma$ and the value after the jump is also chosen as $\pi(\phi(s) + F(X(s-))\Delta Z_s, \varepsilon(s))$.

...
In the case where the direction of reflection $\gamma$ is not controlled, i.e. $\gamma$ is a smooth function from $\mathbb{R}^d$ to $\mathbb{R}^d$ satisfying $|\gamma| = 1$ which does not depend on $\varepsilon$, Dupuis and Ishii [21] proved the strong existence of a solution to the SP when $\mathcal{O}$ is a bounded open set and there exists $r \in (0, 1)$ such that
\[
\bigcup_{0 \leq \lambda \leq r} B(x - \lambda \gamma(x), \lambda r) \subset \mathcal{O}^c, \; \forall x \in \partial \mathcal{O}.
\] (2.1)

In the case of controlled directions of reflection, Bouchard [8] showed that the existence holds whenever the condition (2.1) is imposed uniformly in the control variable:

**G1.** $\mathcal{O}$ is a bounded open set, $\gamma$ is a smooth function from $\mathbb{R}^d \times E$ to $\mathbb{R}^d$ satisfying $|\gamma| = 1$, and there exists some $r \in (0, 1)$ such that
\[
\bigcup_{0 \leq \lambda \leq r} B(x - \lambda \gamma(x, e), \lambda r) \subset \mathcal{O}^c, \; \forall (x, e) \in \partial \mathcal{O} \times E.
\] (2.2)

In all this paper, $E$ denotes a given compact subset of $\mathbb{R}^l$ for some $l \geq 1$.

In order to extend this result to the case where $\psi$ is a deterministic càdlàg function with finite number of points of discontinuity, we focus on the definition of the solution value at the jump times. At each jump time $s$, the value after the jump of $\phi$ is chosen as $\pi(\phi(s-) + \Delta \psi(s), \varepsilon(s))$, where $\pi$ is the image of a projection operator on the boundary $\partial \mathcal{O}$ along the direction $\gamma$ satisfying following conditions:

**G2.** For any $y \in \mathbb{R}^d$ and $e \in E$, there exists $(\pi(y, e), l(y, e)) \in \bar{\mathcal{O}} \times \mathbb{R}_+$ satisfying
\[
\begin{cases} 
\text{if } y \in \bar{\mathcal{O}}, & \pi(y, e) = y, \; l(y, e) = 0, \\
\text{if } y \notin \bar{\mathcal{O}}, & \pi(y, e) \in \partial \mathcal{O} \text{ and } y = \pi(y, e) - l(y, e)\gamma(\pi(y, e), e) \end{cases}
\]
Moreover, $\pi$ and $l$ are Lipchitz continuous functions with respect to their first variable and uniformly in the second one.

This means that the value of $\phi$ just after the jump at time $s$ is defined as
\[
\phi(s) = (\phi(s-) + \Delta \psi(s)) + \gamma(\phi(s), \varepsilon(s))\Delta \eta(s),
\]
where $\Delta \eta(s) = l(\phi(s-) + \Delta \psi(s), \varepsilon(s))$, or equivalently
\[
\Delta \phi(s) = \Delta \psi(s) + \gamma(\phi(s), \varepsilon(s))\Delta \eta(s).
\]

In view of the existence result of Bouchard [8], we already know that the existence of a solution to (SP) is guaranteed between the jump times and that the uniqueness between the jump times holds if $(\psi, \varepsilon) \in BV^f([0, T], \mathbb{R}^d) \times BV^f([0, T], \mathbb{R}^l)$. By pasting together the solutions at the jumps times according to the above rule, we clearly obtain an existence on the whole time interval $[0, T]$ when $\psi$ and $\varepsilon$ have only a finite number of discontinuous points.
Lemma 2.1 Assume that $G1$ and $G2$ hold and fix $\psi, \varepsilon \in D^f([0,T], \mathbb{R}^d)$. Then, there exists a solution $(\phi, \eta)$ to (SP) associated to $(\pi, l)$, i.e. there exists $(\phi, \eta) \in D^f([0,T], \mathbb{R}^d) \times D^f([0,T], \mathbb{R})$ such that, for $s \in [0,T]$,

(i) $\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s), \varepsilon(s))1_{\phi(s) \in \partial O}d\eta(s),$

(ii) $\phi(t) \in \bar{O}$ for $t \in [0,T],$

(iii) $\eta$ is a non decreasing function,

(iv) $\phi(s) = \pi(\phi(s-), \Delta \psi(s), \varepsilon(s))$ and $\Delta \eta(s) = l(\phi(s-), \Delta \psi(s), \varepsilon(s))$

Moreover, the uniqueness holds if $(\psi, \varepsilon) \in BV^f([0,T], \mathbb{R}^d) \times BV^f([0,T], \mathbb{R}^l)$.

Remark 2.1 Note that

1. When the domain $O$ is convex and the reflection is not oblique, i.e. $\gamma(x, e) \equiv n(x)$ for all $x \in \partial O$ with $n(x)$ standing for the inward normal vector at the point $x \in \partial O$, then the usual projection operator together with the distance function to the boundary $\partial O$ satisfies assumption $G2$.

2. Clearly, the Lipschitz continuity conditions of $\pi$ and $l$ are not necessary for proving the existence of a solution to (SP). They will be used later to provide some technical estimates on the controlled processes, see Proposition 2.1 below.

2.2 SDEs with oblique reflection.

We now consider the stochastic version of (SP). Let $W$ be a standard $n$-dimensional Brownian motion and $\mu$ be a Poisson random measure on $\mathbb{R}^d$, which are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $W$ and $\mu$ are independent. We denote by $\mathcal{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}$ the $\mathbb{P}$-complete filtration generated by $(W, \mu)$. We suppose that $\mu$ admits a deterministic $(\mathbb{P}, \mathcal{F})$-intensity kernel $\tilde{\mu}(dz)ds$ which satisfies

\[ \int_0^T \int_{\mathbb{R}^d} \tilde{\mu}(dz)ds < \infty. \quad (2.3) \]

The aim of this section is to study the existence and uniqueness of a solution $(X, L)$ to the class of reflected SDEs with controlled oblique reflection $\gamma$:

\[ X(t) = x + \int_0^t F_s(X(s-))dZ_s + \int_0^t \gamma(X(s), \varepsilon(s))1_{X(s) \in \partial O}dL(s), \quad (2.4) \]

where $(F_s)_{s \leq T}$ is a predictable process with values in the set of Lipchitz functions from $\mathbb{R}^d$ to $\mathbb{M}^d$ such that $F(0)$ is essentially bounded, and $Z$ is a $\mathbb{R}^d$-valued càdlàg Lévy process defined as

\[ Z_t = \int_0^t b_sds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}^d} \beta_s(z)\mu(dz, ds), \quad (2.5) \]
where \((t, z) \in [0, T] \times \mathbb{R}^d \mapsto (b_t, \sigma_t, \beta_t(z))\) is a deterministic bounded map with values in \(\mathbb{R}^d \times \mathbb{M}^d \times \mathbb{R}^d\).

As already mentioned, an existence of solutions was proved for Itô processes, i.e. \(\beta = 0\), and the controls \(\varepsilon\) with continuous path and essentially bounded variation in [8]. In this paper, we extend this result to the case of jump diffusion and to the case where the controls \(\varepsilon\) can have discontinuous paths with a.s. finite activity. As in the deterministic case, we only consider a particular class of solutions which is parameterized by the projection operator \(\pi\). Namely, \(X\) is projected on the boundary \(\partial \mathcal{O}\) through the projection operator \(\pi\) whenever it is out of the domain because of a jump. The value after the jump of \(X\) is chosen as \(\pi(X(s-) + F_s(X(s-))\Delta Z_s, \varepsilon(s))\).

In order to state rigorously the main result of this section, we first need to introduce some additional notations and definitions. For any Borel set \(K\), we denote by \(D_F([0, T], K)\) the set of \(K\)-valued adapted càdlàg semimartingales with finite activity and \(BV_F([0, T], K)\) the set of processes in \(D_F([0, T], K)\) with a.s. bounded variation on \([0, T]\). We set \(\mathcal{E} := BF_F([0, T], E)\) for case of our notations.

**Definition 2.1** Given \(x \in \mathcal{O}\) and \(\varepsilon \in \mathcal{E}\), we say that \((X, L) \in D_F([0, T], \mathbb{R}^d) \times D_F([0, T], \mathbb{R})\) is a solution of the reflected SDEs with direction of reflection \(\gamma\), projection operator \((\pi, l)\) and initial condition \(x\), if

\[
\begin{align*}
X(t) &= x + \int_0^t F_s(X(s-))dZ_s + \int_0^t \gamma(X(s), \varepsilon(s))1_{X(s)\in\partial\mathcal{O}}dL(s), \\
X(s) &\in \mathcal{O} \quad \forall s \leq T, \quad \text{\(L\) is a non decreasing process}, \\
X(s) &= \pi(X(s-) + F_s(X(s-))\Delta Z_s, \varepsilon(s)), \\
\Delta L(s) &= l(X(s-) + F_s(X(s-))\Delta Z_s, \varepsilon(s)) \forall s \leq T.
\end{align*}
\]

We now state the main result of this section.

**Theorem 2.1** Fix \(x \in \mathcal{O}\) and \(\varepsilon \in \mathcal{E}\), and assume that \(G1\) and \(G2\) hold. Then, there exists an unique solution \((X, L) \in D_F([0, T], \mathbb{R}^d) \times D_F([0, T], \mathbb{R})\) of the reflected SDEs (2.6) with oblique direction of reflection \(\gamma\), projection operator \((\pi, l)\) and initial condition \(x\).

**Proof.** Let \(\{T_k\}_{k \geq 1}\) be the jump times of \((Z, \varepsilon)\). Assume that (2.6) admits a solution on \([T_0, T_{k-1}]\) for \(k \geq 1\), with \(T_0 = 0\). It then follows from Theorem 2.2 in [8] that there exists an unique solution \((X, L)\) of (2.6) on \([T_{k-1}, T_k]\). We set

\[
X(T_k) = \pi(X(T_k-) + F_{T_k}(X(T_k-))\Delta Z_{T_k}, \varepsilon(T_k))
\]

so that

\[
X(T_k) = X(T_k-) + F_{T_k}(X(T_k-))\Delta Z_{T_k} + \gamma(X(T_k), \varepsilon(T_k))1_{X(T_k)\in\partial\mathcal{O}}\Delta L(T_k),
\]

with \(\Delta L(T_k) = l(X(T_k-) + F_{T_k}(X(T_k-))\Delta Z_{T_k}, \varepsilon(T_k))\). Since \(N_{[0, T]}^{(Z, \varepsilon)} < \infty\) \(\mathbb{P}\)- a.s., an induction leads to an existence result on \([0, T]\). Uniqueness follows from the uniqueness of the solution on each interval \([T_{k-1}, T_k]\), see Theorem 2.2 in [8].
Optimal control of direction of reflection for jump diffusions

In this chapter, we then introduce an optimal control problem, which extends the framework of [8] to the jump diffusion case,

\[ v(t, x) = \sup_{\varepsilon \in \mathcal{E}} J(t, x; \varepsilon) \]

where the cost function \( J(t, x; \varepsilon) \) is defined as

\[ \mathbb{E} \left[ \beta^\varepsilon_{t,x}(T) g(X^\varepsilon_{t,x}(T)) + \int_t^T \beta^\varepsilon_{t,x}(s) f(X^\varepsilon_{t,x}(s)) ds \right] \]

with \( \beta^\varepsilon_{t,x}(s) = e^{-\int_t^s \rho(X^\varepsilon_{t,x}(r))dL^\varepsilon_{t,x}(r)} \), \( f, g, \rho \) are some given functions, and the pair \((X^\varepsilon_{t,x}, L^\varepsilon_{t,x})\) satisfies

\[ X(t) = x + \int_0^t F_s(X(s)) dZ_s + \int_0^t \gamma(X(s), \varepsilon(s)) 1_{X(s) \in \partial O} dL_s(s), \]

\[ X(s) \in \bar{O} \forall s \leq T, \quad L \text{ is a non decreasing process}, \]

\[ X(s) = \pi(X(s)) + F_s(X(s)) \Delta Z_s, \varepsilon(s)), \]

\[ \Delta L(s) = l(X(s)) + F_s(X(s)) \Delta Z_s, \varepsilon(s)) \forall s \leq T. \]

As usual, the technical key for deriving the associated PDEs is the dynamic programming principle (DPP). The formal statement of the DPP may be written as follows, for \( \tau \) in the set \( \mathcal{T}(t, T) \) of stopping times taking values in \([t, T]\),

\[ v(t, x) = \tilde{v}(t, x), \]

where

\[ \tilde{v}(t, x) := \sup_{\varepsilon \in \mathcal{E}} \mathbb{E} \left[ \beta^\varepsilon_{t,x}(\tau) v(\tau, X^\varepsilon_{t,x}(\tau)) + \int_t^\tau \beta^\varepsilon_{t,x}(s) f(X^\varepsilon_{t,x}(s)) ds \right], \]

see [8], Fleming and Soner [24] and Lions [34]. Bouchard and Touzi [14] recently discussed a weaker version of the classical DPP (0.2), which is sufficient to provide a viscosity characterization of the associated value function, without requiring
the usual heavy measurable selection argument nor the a priori continuity on the associated value function. In this paper, we apply their result to our context:

\[ v(t,x) \leq \sup_{\epsilon \in \mathcal{E}} \mathbb{E}\left[ \beta_{t,x}(\tau)[v^*,g](\tau,X_{t,x}^\epsilon(\tau)) + \int_{t}^{\tau} \beta_{t,x}(s)f(X_{t,x}^\epsilon(s))ds \right], \quad (0.3) \]

and, for every upper semi-continuous function \( \varphi \) such that \( \varphi \leq v^* \),

\[ v(t,x) \geq \sup_{\epsilon \in \mathcal{E}} \mathbb{E}\left[ \beta_{t,x}(\tau)[\varphi,g](\tau,X_{t,x}^\epsilon(\tau)) + \int_{t}^{\tau} \beta_{t,x}(s)f(X_{t,x}^\epsilon(s))ds \right], \quad (0.4) \]

where \( v^* \) (resp. \( v_* \)) is the upper (resp. lower) semi-continuous envelope of \( v \), and \([w,g](s,x) := w(s,x)1_{s<T} + g(x)1_{s=T}\) for any map \( w \) define on \([0,T] \times \bar{O}\). This allows us to provide a PDE characterization of the value function \( v \) in the viscosity sense. We finally extend the comparison principle of Bouchard [8] to our context.

### 1 The optimal control problem

We now introduce the optimal control problem which extends the one considered in [8]. The set of control processes \( \zeta := (\alpha, \epsilon) \) is defined as \( \mathcal{A} \times \mathcal{E} \), where \( \mathcal{A} \) is the set of predictable processes taking values in a given compact subset \( A \) of \( \mathbb{R}^m \), for some \( m \geq 1 \). We recall some necessary assumptions in the previous chapter:

**G1.** \( \mathcal{O} \) is a bounded open set, \( \gamma \) is a smooth function from \( \mathbb{R}^d \times \mathcal{E} \) to \( \mathbb{R}^d \) satisfying \( |\gamma| = 1 \), and there exists some \( r \in (0,1) \) such that

\[ \bigcup_{0 \leq \lambda \leq r} B(x - \lambda\gamma(x,e), \lambda r) \subset \mathcal{O}^c, \quad \forall (x,e) \in \partial \mathcal{O} \times \mathcal{E}. \]

**G2.** For any \( y \in \mathbb{R}^d \) and \( e \in \mathcal{E} \), there exists \( (\pi(y,e), l(y,e)) \in \bar{\mathcal{O}} \times \mathbb{R}_+ \) satisfying

\[
\begin{cases}
\text{if } y \in \bar{\mathcal{O}}, & \pi(y,e) = y, \ l(y,e) = 0, \\
\text{if } y \notin \bar{\mathcal{O}}, & \pi(y,e) \in \partial \mathcal{O} \text{ and } y = \pi(y,e) - l(y,e)\gamma(\pi(y,e),e)
\end{cases}
\]

Moreover, \( \pi \) and \( l \) are Lipchitz continuous functions with respect to their first variable and uniformly in the second one.

The family of controlled processes \( (X_{t,x}^{\alpha,\epsilon}, L_{t,x}^{\alpha,\epsilon}) \) is defined as follows. Let \( b, \sigma \) and \( \chi \) be continuous maps on \( \bar{\mathcal{O}} \times \mathcal{A} \) and \( \mathcal{O} \times \mathcal{A} \times \mathbb{R}^d \) with values in \( \mathbb{R}^d \), \( \mathbb{M}^d \) and \( \mathbb{R}^d \) respectively. We assume that they are Lipchitz continuous with respect to their first variable, uniformly in the others, and that \( \chi \) is bounded with respect to its last component. It then follows from Theorem 2.1 of Chapter 5 that, for \((t,x) \in [0,T] \times \mathcal{O} \) and \((\alpha, \epsilon) \in \mathcal{A} \times \mathcal{E} \), there exists an unique pair \((X_{t,x}^{\alpha,\epsilon}, L_{t,x}^{\alpha,\epsilon}) \in 142
\[ D_{\mathcal{F}}([0,T],\mathbb{R}^d) \times D_{\mathcal{F}}([0,T],\mathbb{R}) \text{ which satisfies, for } r \in [t,T], \]
\[ X(r) = x + \int_t^r b(X(s),\alpha(s))ds + \int_t^r \sigma(X(s),\alpha(s))dW_s \]
\[ + \int_t^r \int_{\mathbb{R}^d} \chi(X(s^-),\alpha(s),z)\mu(dz,ds) + \int_t^r \gamma(X(s),\varepsilon(s))1_{X(s)\in\partial\mathcal{O}}dL(s) \]
\[ X(r) \in \hat{\mathcal{O}}, \ L \text{ is non decreasing}, \]
\[ (X(r),DL(s)) = \int_{\mathbb{R}^d} (\pi,l)(X(s^-) + \chi(X(s^-),\alpha(s),z),\varepsilon(s))\mu(dz,\{s\}). \]

Let \(\rho, f, g\) be bounded Borel measurable real valued maps on \(\hat{\mathcal{O}} \times E, \hat{\mathcal{O}} \times A\) and \(\hat{\mathcal{O}}\), respectively. We assume that \(g\) is Lipchitz continuous and \(\rho \geq 0\). The functions \(\rho\) and \(f\) are also assumed to be Lipchitz continuous in their first variable, uniformly in their second one. We then define the cost function
\[ J(t,x;\zeta) := \mathbb{E} \left[ \beta_{t,x}^\zeta(T)g(X_{t,x}^\zeta(T)) + \int_t^T \beta_{t,x}^\zeta(s)f(X_{t,x}^\zeta(s),\alpha(s))ds \right], \quad (1.4) \]
where \(\beta_{t,x}^\zeta(s) := e^{-\int_t^s \rho(X_{t,y}^\zeta(r^-),\varepsilon(r))dL_{t,y}^\zeta(r)}\), for \(\zeta = (\alpha,\varepsilon) \in A \times E\).

The aim of the controller is to maximize \(J(t,x;\zeta)\) over the set \(A_t \times E_t\) of controls in \(A \times E\) which are independent on \(\mathcal{F}_t\), compare with [14]. The associated value function is then defined as
\[ v(t,x) := \sup_{\zeta \in A_t \times E_t} J(t,x;\zeta). \]

## 2 Dynamic programming.

In order to provide a PDE characterization of the value function \(v\), we shall appeal as usual to the dynamic programming principle. The classical DPP (0.2) relates the time-\(t\) value function \(v(t,\cdot)\) to the later time-\(\tau\) value \(v(\tau,\cdot)\), for any stopping time \(\tau \in \mathcal{T}(t,T)\). Recently Bouchard and Touzi [14] provided a weaker version of the DPP, which is sufficient to provide a viscosity characterization of \(v\). This version allows us to avoid the technical difficulties related to the use of non-trivial measurable selection arguments or the a-priori continuity of \(v\).

From now, for \(t \leq T\), we denote by \(\mathcal{T}_t(\tau_1,\tau_2)\) the set of elements in \(\mathcal{T}(\tau_1,\tau_2)\) which are independent on \(\mathcal{F}_t\). The weak version of the DPP reads as follows:

**Theorem 2.1** Fix \((t,x) \in [0,T] \times \hat{\mathcal{O}}\) and \(\tau \in \mathcal{T}_t(t,T)\), then
\[ v(t,x) \leq \sup_{\zeta \in A_t \times E_t} \mathbb{E} \left[ \beta_{t,x}^\zeta(\tau)[v^*,g](\tau,X_{t,x}^\zeta(\tau)) + \int_t^\tau \beta_{t,x}^\zeta(s)f(X_{t,x}^\zeta(s))ds \right], \quad (2.1) \]
and
\[ v(t,x) \geq \sup_{\zeta \in A_t \times E_t} \mathbb{E} \left[ \beta_{t,x}^\zeta(\tau)[\varphi,g](\tau,X_{t,x}^\zeta(\tau)) + \int_t^\tau \beta_{t,x}^\zeta(s)f(X_{t,x}^\zeta(s))ds \right], \quad (2.2) \]
for any upper semi continuous function \(\varphi\) such that \(v \geq \varphi\).
Arguing as in [14], the result follows once \( J(\cdot; \zeta) \) is proved to be lower semicontinuous for all \( \zeta \in A \times \mathcal{E} \). In our setting, one can actually prove the continuity of the above map.

**Proposition 2.1** Fix \((t_0, x_0) \in [0, T] \times \tilde{O} \) and \( \zeta \in A \times \mathcal{E} \). Then, we have

\[
\lim_{(t,x) \to (t_0,x_0)} J(t, x; \zeta) = J(t_0, x_0; \zeta). \tag{2.3}
\]

Proposition 2.1 will be proved later in the Section 3. Before providing the proof of the DPP, we verify the consistency with deterministic initial data assumption, see Assumption A4 in [14].

**Lemma 2.1** (i) Fix \((t, x) \in [0, T] \times \tilde{O}, (\zeta, \theta) \in A_t \times \mathcal{E}_t \times T_t(t, T)\). For \( \mathbb{P} \)-a.e \( \omega \in \Omega \), there exists \( \tilde{\zeta}_\omega \in A_{\theta(\omega)} \times \mathcal{E}_{\theta(\omega)} \) such that

\[
\mathbb{E} \left[ \beta_{t,x}^{\zeta}(T) g(X_{t,x}^{\zeta}(T)) + \int_t^T \beta_{t,x}^{\zeta}(s) f(X_{t,x}^{\zeta}(s), \alpha(s)) ds \mid \mathcal{F}_\theta \right] (\omega)
\]

\[
= \beta_{t,x}^{\zeta}(\theta(\omega)) J(\theta(\omega), X_{t,x}^{\zeta}(\theta(\omega)); \tilde{\zeta}_\omega) + \int_t^{\theta(\omega)} \beta_{t,x}^{\zeta}(s) f(X_{t,x}^{\zeta}(s), \alpha(s)(\omega)) ds.
\]

(ii) For \( t \leq s \leq T, \theta \in T_t(t, s), \tilde{\zeta} \in A_s \times \mathcal{E}_s \) and \( \tilde{\zeta} := \zeta_{1,[t,s]} + \zeta_{1,[s,T]} \), we have, for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \),

\[
\mathbb{E} \left[ \beta_{t,x}^{\zeta}(T) g(X_{t,x}^{\zeta}(T)) + \int_t^T \beta_{t,x}^{\zeta}(s) f(X_{t,x}^{\zeta}(s), \alpha(s)) ds \mid \mathcal{F}_\theta \right] (\omega)
\]

\[
= \beta_{t,x}^{\zeta}(\theta(\omega)) J(\theta(\omega), X_{t,x}^{\zeta}(\theta(\omega)); \tilde{\zeta}) + \int_t^{\theta(\omega)} \beta_{t,x}^{\zeta}(s) f(X_{t,x}^{\zeta}(s), \alpha(s)(\omega)) ds.
\]

**Proof.** In this proof, we consider the space \((\Omega, \mathcal{F}, \mathbb{P})\) as being the product space \(C([0, T], \mathbb{R}^d) \times S([0, T], \mathbb{R}^d)\), where \( S([0, T], \mathbb{R}^d) := \{(t_i, z_i)_{i \geq 1} : t_i \uparrow T, z_i \in \mathbb{R}^d\}\), equipped with the product measure \( \mathbb{P} \) induced by the Wiener measure and the Poisson random measure \( \mu \). We denote by \( \omega \) or \( \tilde{\omega} \) a generic point. We also define the stopping operator \( \omega^r := \omega_{r \land t} \) and the translation operator \( T_r(\omega) := \omega_{r + t} \omega_r \). We then obtain from direct computations that, for \( \zeta = (\alpha, \varepsilon) \in A_t \times \mathcal{E}_t \):

\[
\mathbb{E} \left[ \beta_{t,x}^{\zeta}(T) g(X_{t,x}^{\zeta}(T)) + \int_t^T \beta_{t,x}^{\zeta}(s) f(X_{t,x}^{\zeta}(s), \alpha(s)) ds \mid \mathcal{F}_\theta \right] (\omega)
\]

\[
= \beta_{t,x}^{\zeta}(\theta(\omega)) \int \left[ \beta_{t,x}^{\zeta(\theta(\omega) + T_\theta(\omega)(\omega))}(T) g \left( X_{t,x}^{\zeta(\theta(\omega) + T_\theta(\omega)(\omega))}(T) \right) \right.
\]

\[
\left. + \int_t^T \beta_{t,x}^{\zeta(\theta(\omega) + T_\theta(\omega)(\omega))}(s) f \left( X_{t,x}^{\zeta(\theta(\omega) + T_\theta(\omega)(\omega))}(s), \alpha(\theta(\omega) + T_\theta(\omega)(\omega)) s \right) \right] d\mathbb{P}(T_\theta(\omega)(\omega)) + \int_t^{\theta(\omega)} \beta_{t,x}^{\zeta}(s) f(X_{t,x}^{\zeta}(s), \alpha(s)) ds
\]

\[
= \beta_{t,x}^{\zeta}(\theta(\omega)) J(\theta(\omega), X_{t,x}^{\zeta}(\theta(\omega)); \tilde{\zeta}_\omega) + \int_t^{\theta(\omega)} \beta_{t,x}^{\zeta}(s) f(X_{t,x}^{\zeta}(s), \alpha(s)) ds,
\]
where \( \tilde{\omega} \in \Omega \mapsto \tilde{\zeta}_\omega(\tilde{\omega}) := \zeta(\omega^{\theta(\omega)} + T_{\theta(\omega)}(\tilde{\omega})) \in \mathcal{A}_{\theta(\omega)} \times \mathcal{E}_{\theta(\omega)} \).

This leads proves (i). The assertion (ii) is proved similarly by using the fact that \( \theta(\omega) \in [t, s] \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \).

**Proof of Theorem 2.1**

The inequality (2.1) is clearly a consequence of (i) in Lemma 2.1. So it remains to prove the inequality (2.2). Fix \( \epsilon > 0 \). In view of definition of \( J \) and \( v \), there exists a family \( \{\zeta(s, y)\}_{(s, y) \in [0, T] \times \mathcal{O}} \) such that

\[
J(s, y; \zeta(s, y)) \geq v(s, y) - \epsilon/3, \text{ for all } (s, y) \in [0, T] \times \mathcal{O}.
\]

Using Proposition 2.1 and the upper semi continuity of \( \varphi \), we can choose a family \( \{r(s, y)\}_{(s, y) \in [0, T] \times \mathcal{O}} \subset (0, \infty) \) such that

\[
J(s, y; \zeta(s, y)) - J(\cdot; \zeta(s, y)) \leq \epsilon/3 \text{ and } \varphi - \varphi(s, y) \leq \epsilon/3 \text{ on } U(s, y; r(s, y)),
\]

where \( U(s, y; r) := [(s - r) \vee 0, s] \times B(y, r) \). Hence,

\[
J(\cdot; \zeta(s, y)) \geq \varphi - \epsilon \text{ on } U(s, y; r(s, y)).
\]

Note that \( \{U(s, y; r) : (s, y) \in [0, T] \times \mathcal{O}, 0 < r < r(s, y)\} \) is a Vitali covering of \( [0, T] \times \mathcal{O} \). It then follows from the Vitali’s covering Theorem that there exists a countable sequence \( \{t_i, x_i\}_{i \in \mathbb{N}} \) so that \( [0, T] \times \mathcal{O} \subset \bigcup_{i \in \mathbb{N}} U(t_i, x_i; r_i) \) with \( r_i := r(t_i, x_i) \).

We can then extract a partition \( \{B_i\}_{i \in \mathbb{N}} \) of \( [0, T] \times \mathcal{O} \) and a sequence \( \{t_i, x_i\}_{i \geq 1} \) satisfying \( (t_i, x_i) \in B_i \) for each \( i \in \mathbb{N} \), such that \( (t, x) \in B_i \) implies \( t \leq t_i \), and

\[
J(\cdot; \zeta_i) - \varphi \geq \epsilon \text{ on } B_i, \text{ for some } \zeta_i \in \mathcal{A}_{t_i} \times \mathcal{E}_{t_i}.
\]

We now fix \( \zeta \in \mathcal{E}_t \times \mathcal{A}_t \) and \( \tau \in \mathcal{T}_t(t, T) \) and set

\[
\tilde{\zeta} := 1_{\mathcal{I}_t(t, \tau)} + 1_{[\tau, T]} \sum_{n \geq 1} \zeta_n 1_{\{s, X_{t,x}^\zeta(s) \in B_n\}}.
\]

Note that \( \tilde{\zeta} \in \mathcal{A}_t \times \mathcal{E}_t \) since \( \tau \), \( (\zeta_n)_{n \geq 1} \) and \( X_{t,x}^\zeta \) are independent of \( \mathcal{F}_t \). It then follows from (ii) of Lemma 2.1 and (2.5) that

\[
J(t, x, \tilde{\zeta}) - \mathbb{E}[g(X_{t,x}^\zeta(T))1_{\{\tau = T\}}] - \mathbb{E}\left[ \int_t^\tau \beta^\zeta_{t,x}(s) f(X_{t,x}^\zeta(s))ds \right]
\]

\[
\geq \mathbb{E}\left[ \beta^\zeta_{t,x}(\tau) J(\tau, X_{t,x}^\zeta(\tau); \zeta) 1_{\{\tau < T\}} \right]
\]

\[
\geq \mathbb{E}\left[ \beta^\zeta_{t,x}(\tau) \varphi(\tau, X_{t,x}^\zeta(\tau)) 1_{\{\tau < T\}} \right] - \epsilon.
\]

This implies that

\[
v(t, x) \geq \mathbb{E}\left[ \beta^\zeta_{t,x}(\tau) [\varphi, g](\tau, X_{t,x}^\zeta(\tau)) + \int_t^\tau \beta^\zeta_{t,x}(s) f(X_{t,x}^\zeta(s))ds \right].
\]

\( \square \)
3 Proof of the continuity of the cost function

In this section, we prove the continuity of the cost function in the \((t,x)\)-variable as follows.

1. We first show that the map \(J(\cdot; \alpha, \varepsilon)\) is continuous if \(\mu\) and \(\varepsilon\) are such that

\[
N_{[t,T]}^\mu \leq m \mathbb{P} - \text{a.s and } \varepsilon \in \mathcal{E}_k^b, \text{for some } m, k \geq 1,
\]

where \(\mathcal{E}_k^b\) is defined as the set of \(\varepsilon \in \mathcal{E}\) such that \(|\varepsilon| \leq k\) and the number of jump times of \(\varepsilon\) is a.s smaller than \(k\), and \(N_{[t,T]}^\mu := \mu(\mathbb{R}^d, [t,T])\). This result is proved as a consequence of the estimates of \(X\) and \(\beta\) in Lemma 3.1 presented below together with the Lipchitz continuity conditions on \(f\), \(g\) and \(\rho\).

**Lemma 3.1** Fix \(k, m \in \mathbb{N}\). Assume that \(G1\) and \(G2\) hold, \(N_{[t,T]}^\mu \leq m \mathbb{P}-\text{a.e and } \varepsilon \in \mathcal{E}_k^b\). Then, there exist a constant \(M > 0\) and a function \(\lambda\) so that, for all \(t \leq t' \leq T\) and \(x, x' \in \mathcal{O}\), we have

\[
\mathbb{E} \left[ \sup_{t' \leq s \leq T} |X_{t,x}^{\alpha,\varepsilon}(s) - X_{t',x'}^{\alpha,\varepsilon}(s)|^4 \right] \leq M|x - x'|^4 + \lambda(|t - t'|), \tag{3.1}
\]

\[
\mathbb{E} \left[ \sup_{t' \leq s \leq T} \left| \ln \beta_{t,x}^{\alpha,\varepsilon}(s) - \ln \beta_{t',x'}^{\alpha,\varepsilon}(s) \right| \right] \leq M|x - x'| + \lambda(|t - t'|), \tag{3.2}
\]

where \(\lim_{t \to 0} \lambda(t) = 0\).

**Proof.** In order to prove the result, we use a similar argument as in Proposition 3.1 [8] on the time intervals where \((Z, \varepsilon)\) is continuous. We focus on the differences which come from the points of discontinuity of \((Z, \varepsilon)\). From now, we denote \((X, L) := (X_{t,x}^{\alpha,\varepsilon}, L_{t,x}^{\alpha,\varepsilon})\) and \((X', L') := (X_{t',x'}^{\alpha,\varepsilon}, L_{t',x'}^{\alpha,\varepsilon})\). We only prove the first assertion, the second one follows from the same line of arguments.

Let \(\{T_i\}_{i \geq 1}\) be the sequence of jump times on \([t', T]\) of \((Z, \varepsilon)\) and \(T_0 := t'\). By the same argument as in Proposition 3.1 of [8] we obtain that

\[
\mathbb{E} \left[ \sup_{r \in [T_i, T_{i+1}]} |X(r) - X'(r)|^4 \right] \leq C_1 \mathbb{E} \left[ |X(T_i) - X'(T_i)|^4 \right], \text{ for some } C_1 > 0. \tag{3.3}
\]

It follows from (1.3) that

\[
\mathbb{E} \left| X(T_{i+1}) - X'(T_{i+1}) \right|^4 \leq \mathbb{E} \left[ \left| \int_{\mathbb{R}^d} \pi(\chi(X(T_{i+1}^-), \alpha(T_{i+1}), z), \varepsilon(T_{i+1})) \right. \right.
\]

\[
- \pi(\chi(X'(T_{i+1}^-), \alpha(T_{i+1}), z), \varepsilon(T_{i+1})) \left. \right| \mu(dz, \{T_{i+1}\}) \right|^4.
\]

This, together with (2.3) and the Lipchitz continuity assumption on \(\chi\) and \(\pi\), implies that

\[
\mathbb{E} \left| X(T_{i+1}) - X'(T_{i+1}) \right|^4 \leq C_2 \mathbb{E} \left| X(T_{i+1}^-) - X'(T_{i+1}^-) \right|^4, \text{ for some } C_2 > 0. \tag{3.4}
\]
Using the previous inequality and (3.3), we deduce that there exists $C > 0$ s.t
\[ \mathbb{E}\left[ \sup_{r \in [T_i, T_{i+1}]} |X(r) - X'(r)|^4 \right] \leq C \mathbb{E}[|X(T_i) - X'(T_i)|^4]. \]

Applying an induction argument, it implies to
\[ \mathbb{E}\left[ \sup_{r \in [T_i, T_{i+1}]} |X(r) - X'(r)|^4 \right] \leq C^i \mathbb{E}[|X(t') - X'(t')|^4]. \]

Since $N^\mu_{[t,T]} \leq m$ and $N^\varepsilon_{[t,T]} \leq k$, we have
\[ \mathbb{E}\left[ \sup_{t \leq s \leq T} |X(s) - X'(s)|^4 \right] \leq C^{m+k} \mathbb{E}[|X(t') - x'|^4], \text{ for some } C > 1. \tag{3.5} \]

It remains to estimate $\mathbb{E}[|X(t') - x'|^4]$. Let $\{\bar{T}_i\}_{i \geq 1}$ be the sequence of jump times on $[t, t']$ and set $\bar{T}_0 := t$. Using a similar argument as in Proposition 3.1 [8], we deduce that there exists $\bar{C}_1 > 0$ such that
\[ \mathbb{E}\left[ \sup_{r \in [T_i, T_{i+1}]} |X(r) - x'|^4 \right] \leq \bar{C}_1 \mathbb{E}[|X(\bar{T}_i) - x'|^4 + |ar{T}_{i+1} - \bar{T}_i|]. \tag{3.6} \]

Recalling (1.3) and the Lipschitz continuity assumption on $\pi$, we deduce that there exists $\bar{C}_2 > 0$ so that
\[
\mathbb{E}[|X(\bar{T}_{i+1}) - x'|^4]
\leq \bar{C}_2 \mathbb{E}\left[ |X(\bar{T}_{i+1}) - x'|^4 + \int_{\mathbb{R}^d} |\chi(\bar{T}_{i+1} - z)|^4 |\mu|_{(\bar{T}_{i+1})}|dz\right].
\]

The previous inequality together with (3.6) leads to
\[
\mathbb{E}\left[ \sup_{r \in [T_i, T_{i+1}]} |X(r) - x'|^4 \right] \leq C \mathbb{E}\left[ |X(\bar{T}_i) - x'|^4 + |\bar{T}_{i+1} - \bar{T}_i| + \int_{\mathbb{R}^d} |\chi(\bar{T}_{i+1} - z)|^4 |\mu|_{(\bar{T}_{i+1})}|dz\right],
\]
for some $C > 0$.

Then,
\[
\mathbb{E}\left[ \sup_{t \leq r \leq t'} |X(r) - x'|^4 \right] \leq C^{m+k} |x - x'|^4 + C |t' - t| + C \mathbb{E}\left[ \int_t^{t'} \int_{\mathbb{R}^d} |\chi(X(s) - z)|^4 |\mu|_{(s)}|dz|ds\right],
\]
for some $C > 1$.

Since $\chi$ is bounded with respect to $z$, we then conclude that
\[
\mathbb{E}\left[ \sup_{t \leq r \leq t'} |X(r) - x'|^4 \right] \leq M' |x - x'|^4 + \lambda(|t' - t|),
\]
2. We now provide the proof of Proposition 2.1 in the general case. Fix \((\alpha, \varepsilon) \in A \times \mathcal{E}\). We denote by \(\{T_m\}_{m \geq 1}\) the sequence of jump times of \(\mu\) on the interval \([t, T]\) of \((Z, \varepsilon)\), \(T_0 := t\) and

\[
\mu^{(m)}(A, B) := \mu(A, B \cap [t, T_m]) \quad \text{for} \quad A \in \mathcal{B}_{\mathbb{R}^d}, B \in \mathcal{B}_{[0, T]}
\]

where \(\mathcal{B}_{\mathbb{R}^d}\) and \(\mathcal{B}_{[0, T]}\) denote the Borel tribes of \(\mathbb{R}^d\) and \([0, T]\) respectively. Let \((X^{(m)}_{t,x}, L^{(m)}_{t,x}, J_m(t, x; \alpha, \varepsilon^{(m)}))\) be defined as \((X^\alpha_{t,x}, L^\alpha_{t,x}, \beta^\alpha_{t,x}, J(t, x; \alpha, \varepsilon))\) in (1.1), (1.1), (1.3) and (1.4) with \(\mu^{(m)}\) in place of \(\mu\) and \(\varepsilon^{(m)} := \varepsilon 1_{[t, T_m]} + \varepsilon (T_m) 1_{[T_m, T]}\) in place of \(\varepsilon\), where

\[
T'_m := \sup\{s \geq t : N^{(m)}_{[t, s]} \leq m, |\varepsilon(s)| \leq m\} \wedge T_m.
\]

Since \(f, g\) and \(\beta\) are bounded on the domain \(\mathcal{O}\), there exists a constant \(M > 0\) such that, for all \((t, x) \in [0, T] \times \mathbb{R}^d\), we have

\[
|J_m(t, x; \alpha, \varepsilon^{(m)}) - J(t, x, y; \alpha, \varepsilon)| \leq M \mathbb{E}
\begin{align*}
&\left| \beta^{(m)}(T) g(X^{(m)}(T)) - \beta^\alpha_{t,x}(T) g(X^\alpha_{t,x,y}(T)) \right| 1_{\mathcal{A}^{(m)}_t} \\
+ & \mathbb{E} \left| \int_t^T \beta^{(m)}(u) f(X^{(m)}(u)) - \beta^\alpha_{t,x}(u) f(X^\alpha_{t,x}(u)) du \right| 1_{\mathcal{A}^{(m)}_t} \\
& \leq M \mathbb{P}\left\{A^{(m)}_t \right\} \leq M \mathbb{P}\left\{A^{(m)}_0 \right\},
\end{align*}
\]

where \(A^{(m)}_t := \{\omega \in \Omega : N^m_{[t, T]} \leq m, N^m_{[t, s]} \leq m, |\varepsilon|(T) \leq m\}\). Since \(N^m_{[0, T]} \leq \infty\) \(\mathbb{P}\)-a.s and \(\varepsilon\) is a process with bounded variation and finite activity, then \(\mathbb{P}\{A^{(m)}_0 \}\) converges to 0 when \(m\) goes to \(\infty\).

Hence,

\[
\sup_{(t, x) \in [0, T] \times \mathcal{O}} |J_m(\cdot; \alpha, \varepsilon^{(m)}) - J(\cdot; \alpha, \varepsilon)| \to 0, \quad \text{when} \quad m \to \infty. \tag{3.7}
\]

It then follows the first part of proof that \(J_m(\cdot; \alpha, \varepsilon^{(m)})\) is continuous map. This leads to the required result (2.3). \(\square\)

4 The PDE characterization.

We are now ready to provide a PDE characterization for \(v\). Note that the fact that the process \(X^\alpha_{t,x}\) is projected through the projection operator \(\pi\) whenever it exists the domain \(\mathcal{O}\) because of a jump implies that the associated Dynkin operator is given, for values \((a, e)\) of the control process, by

\[
\mathcal{L}^{a,e} \varphi(s, x) := \partial_t \varphi(s, x) + \langle b(x, a), D\varphi(s, x) \rangle + \frac{1}{2} \text{Trace} \left[ \sigma(x, a) \sigma^*(x, a) D^2 \varphi(s, x) \right] + \int_{\mathbb{R}^d} e^{-\rho(x, e) \varphi^*(z)} \varphi(s, \pi^a_{x,e}(z)) - \varphi(s, x) \right| \overline{\mu}(dz),
\]

for some \(M' > 0\) and a function \(\lambda\) satisfying \(\lim_{a \to 0} \lambda(a) = 0\). \(\square\)
for smooth functions \( \varphi \), where

\[
(\pi_{x}^{a,e}(z), I_{x}^{a,e}(z)) := (\pi, I)(x + \chi(x, a, z), e).
\]

Also note that the probability of having a jump at the time where the boundary is reached is 0. It follows that the reflection terms does not play the same role, from the PDE point of view, depending whenever the reflection operate at a point of continuity or at a jump time. In the first case, it corresponds to a Neumman type boundary condition, while, in the second case, it only appears in the Dynkin operator which drives the evolution of the value function in the domain as described above. This formally implies that \( v \) should be a solution of

\[
\mathcal{B} \varphi = 0,
\]

where

\[
\mathcal{B} \varphi := \begin{cases}
\min_{(a,e)\in A \times E} \{-\mathcal{L}^{a,e} \varphi - f(\cdot, a)\} & \text{on } [0, T) \times \mathcal{O} \\
\min_{e\in E} \mathcal{H}^{e} \varphi & \text{on } [0, T) \times \partial \mathcal{O} \\
\varphi - g & \text{on } \{T\} \times \bar{\mathcal{O}}
\end{cases}
\]

and, for a smooth function \( \varphi \) on \([0, T] \times \bar{\mathcal{O}}\) and \((a, e)\in A \times E\),

\[
\mathcal{H}^{e}(x, y, p) := \rho(x, e) y - \langle \gamma(x, e), p \rangle \text{ and } \mathcal{H}^{e} \varphi(t, x) := \mathcal{H}^{e}(x, \varphi(t, x), D \varphi(t, x)).
\]

Since \( v \) is not known to be smooth a-priori, we shall appeal as usual to the notion of viscosity solutions. Also note that the above operator \( \mathbb{B} \) is not continuous, so that we have to consider its upper- and lower-semicontinuous envelopes to properly define the PDE. From now, given a function \( w \) on \([0, T] \times \bar{\mathcal{O}}\), we set

\[
\begin{cases}
w_{*}(t, x) = \liminf_{(t', x') \to (t, x), \ (t', x') \in [0, T] \times \mathcal{O}} w(t', x') \\
w^{*}(t, x) = \limsup_{(t', x') \to (t, x), \ (t', x') \in [0, T] \times \mathcal{O}} w(t', x'), \text{ for } (t, x) \in [0, T] \times \bar{\mathcal{O}}.
\end{cases}
\]

**Definition 4.1** A lower semicontinuous (resp. upper semicontinuous ) function \( w \) on \([0, T] \times \bar{\mathcal{O}}\) is a viscosity super-solution (resp. sub-solution )of (4.1) if, for any test function \( \varphi \in C^{1,2}([0, T] \times \bar{\mathcal{O}}) \) and \((t_{0}, x_{0})\in [0, T] \times \bar{\mathcal{O}}\) that achieves a local minimum (resp. maximum) of \( w - \varphi \) so that \((w - \varphi)(t_{0}, x_{0}) = 0\), we have \( \mathcal{B}_{\varphi} \geq 0 \) (resp. \( \mathcal{B}_{\varphi} \leq 0 \)), where

\[
\mathcal{B}_{\varphi} := \begin{cases}
\mathcal{B} \varphi & \text{on } [0, T] \times \mathcal{O} \\
\min_{(a,e)\in A \times E} \max \{-\mathcal{L}^{a,e} \varphi - f(\cdot, a), \mathcal{H}^{e} \varphi\} & \text{on } [0, T) \times \partial \mathcal{O} \\
\varphi - g & \text{on } \{T\} \times \partial \mathcal{O}
\end{cases}
\]

and

\[
\mathcal{B}_{\varphi} := \begin{cases}
\mathcal{B} \varphi & \text{on } [0, T] \times \mathcal{O} \\
\min \{-\mathcal{L}^{a,e} \varphi - f(\cdot, a), \min_{e\in E} \mathcal{H}^{e} \varphi\} & \text{on } [0, T) \times \partial \mathcal{O} \\
\min\{\varphi - g, \min_{e\in E} \mathcal{H}^{e} \varphi\} & \text{on } \{T\} \times \partial \mathcal{O}
\end{cases}
\]
A local bounded function \( w \) is a discontinuous viscosity solution of (4.1) if \( w_* \) (resp. \( w^* \)) is a super-solution (resp. sub-solution) of (4.1).

We can now state our main result.

**Theorem 4.1** Assume that G1 and G2 hold. Then, \( v \) is a discontinuous viscosity solution of (4.1).

The proof of this result is reported in the subsequent sections.

### 4.1 The super-solution property.

Let \( \varphi \in C^{1,2}([0,T] \times \bar{O}) \) and \((t_0, x_0) \in [0, T] \times \bar{O}\) be such that

\[
\min(\text{strict})_{[0,T] \times \bar{O}} (v_* - \varphi) = (v_* - \varphi)(t_0, x_0) = 0.
\]

1. We first prove the required result in the case where \((t_0, x_0) \in [0, T] \times \partial O\).

Then, arguing by contradiction, we suppose that

\[
\min_{(a, e) \in A \times E} \max\{ -\mathcal{L}^{a,e}_{t_0} \varphi(t_0, x_0) - f(x_0, a), \mathcal{H}^{e}_{t_0} \varphi(t_0, x_0) \} \leq -2\epsilon < 0.
\]

Define \( \phi(t, x) := \varphi(t, x) - |t - t_0|^2 - \eta |x - x_0|^4 \), so that, for \( \eta > 0 \) small enough, we have

\[
\min_{(a, e) \in A \times E} \max\{ -\mathcal{L}^{a,e}_{t_0} \phi(t_0, x_0) - f(x_0, a), \mathcal{H}^{e}_{t_0} \phi(t_0, x_0) \} \leq -2\epsilon,
\]

recall (2.3). This implies that there exists \((a_0, e_0) \in A \times E\) and \( \delta > 0 \) such that \( t_0 + \delta < T \) and

\[
\max\{ -\mathcal{L}^{a_0,e_0}_{t_0} \phi(t, x) - f(x, a_0), \mathcal{H}^{e_0}_{t_0} \phi(t, x) \} \leq -\epsilon \quad \text{on} \quad \bar{B} \cap [0, T] \times \bar{O},
\]

where \( B := [t_0 - \delta, t_0 + \delta) \times B(x_0, \delta) \).

Let \((t_n, x_n)\) be a sequence in \([0, T] \times O\) converging to \((t_0, x_0)\) such that

\[
v(t_n, x_n) \rightarrow v_*(t_0, x_0),
\]

and set \((X_n, L_n, \beta_n) := (X^{a_0,e_0}_{t_n,x_n}, L^{a_0,e_0}_{t_n,x_n}, \beta^{a_0,e_0}_{t_n,x_n})\). Obviously, we can assume that \((t_n, x_n) \in B\). Let \( \tau_n \) and \( \theta_n \) be the first exit times of \((s, X_n(s))_{s \geq t_n}\) and \((X_n(s))_{s \geq t_n}\) from \( B \) and \( O \), respectively.

Using Itô’s Lemma, we have

\[
\mathbb{E}[\beta_n(\tau_n)\phi(\tau_n, X_n(\tau_n))] = \phi(t_n, x_n) + \mathbb{E} \left[ \int_{t_n}^{\tau_n} \beta_n(s) \mathcal{L}^{a_0,e_0}_{s} \phi(s, X_n(s)) ds \right] - \mathbb{E} \left[ \int_{t_n}^{\tau_n} \beta_n(s) \mathcal{H}^{e_0}_{s} \phi(s, X_n(s)) dL^e_n(s) \right],
\]

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where $L^c_n$ denotes the continuous part of $L_n$. It follows from (4.2) that
\[
\mathbb{E}[\beta_n(\tau_n)\phi(\tau_n, X_n(\tau_n))]
\geq \phi(t_n, x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), a_0)ds\right] + \mathbb{E}\left[\int_{t_n}^{\tau_n} \epsilon \beta_n(s)dL_n^c(s)\right].
\]
Since $\rho \geq 0$, the function $\beta_n(\cdot)$ is non increasing. Hence
\[
\mathbb{E}\left[\beta_n(\tau_n)\phi(\tau_n, X_n(\tau_n)) + \int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), a_0)ds\right] \geq \phi(t_n, x_n) + \mathbb{E}[\epsilon \beta_n(\tau_n)L_n^c(\tau_n)].
\]
Since $\mathcal{O}$ is bounded,
\[
\varphi - \phi \geq \zeta \text{ on } \partial^c \mathcal{B} \text{ for some } \zeta > 0,
\]
where $\partial^c \mathcal{B} : = [t_0 - \delta, t_0 + \delta] \times (B(x_0, \delta) \cap \mathcal{O}) \cup \{t_0 + \delta\} \times \mathcal{O}$.
This, together with the fact that $\beta_n(\tau_n) = 1$, $L_n(\tau_n) = 0$ on $\{\tau_n < \theta_n\}$, leads to
\[
\mathbb{E}\left[\beta_n(\tau_n)\varphi(\tau_n, X_n(\tau_n)) + \int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), a_0)ds\right] \\
\geq \phi(t_n, x_n) + \mathbb{E}[\epsilon_1_{\tau_n < \theta_n} + \beta_n(\tau_n)(\zeta + \epsilon L_n^c(\tau_n))1_{\tau_n \geq \theta_n}].
\]
Let $c > 0$ be a positive constant satisfying $|\rho| \leq c$, we have
\[
\mathbb{E}\left[\beta_n(\tau_n)\varphi(\tau_n, X_n(\tau_n)) + \int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), a_0)ds\right] \\
\geq \phi(t_n, x_n) + \mathbb{E}[\epsilon_1_{\tau_n < \theta_n} + \epsilon L_n^c(\tau_n)(\zeta + \epsilon L_n^c(\tau_n))1_{\tau_n \geq \theta_n}] \\
\geq \phi(t_n, x_n) + \min\{\zeta; \mathbb{E}e^{-\epsilon \sum_{s \leq \tau_n} \Delta L_n(s)}\inf_{k \geq 0} (e^{-ck}(\zeta + \epsilon k))\}.
\]
It follows from Jensen’s inequality, the Lipschitz property of $l$, (2.3) and the fact that $l(\cdot, e_0) = 0$ on $\mathcal{O}$ that
\[
\ln \mathbb{E}\left[e^{-\epsilon \sum_{s \leq T} \Delta L_n(s)}\right] \\
\geq -c\mathbb{E}\left[\sum_{s \leq T} \Delta L_n(s)\right] \\
\geq -c\mathbb{E}\left[\sum_{s \leq T} l(X_n(s-)) + \chi(X_n(s-), a_0, \Delta Z(s)), e_0\right] \\
\geq -c\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} l(X_n(s-)) + \chi(X_n(s-), a_0, z), e_0)\bar{\mu}(dz)ds\right] \\
\geq -c\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} [l(X_n(s-)) + \chi(X_n(s-), a_0, z), e_0) - l(X_n(s-), e_0)]\bar{\mu}(dz)ds\right] \\
\geq -c'\int_0^T \sup_{\mathcal{O} \times \mathbb{R}^d} |\chi| \int_{\mathbb{R}^d} \bar{\mu}(dz)ds > -\infty.
\]
Then, there exists $\epsilon_0 > 0$ so that
\[
\mathbb{E} \left[ \beta_n(\tau_n) \phi(\tau_n, X_n(\tau_n)) + \int_{\tau_n}^{\tau_n} \beta_n(s) f(X_n(s), a_0) ds \right] \geq \phi(t_n, x_n) + \epsilon_0.
\]
For $n$ large enough, this leads to a contradiction to the statement (2.2) of Theorem 2.1.

2. The proof is similar for $(t_0, x_0) \in [0, T) \times \mathcal{O}$. Indeed, by a similar localization as above, we can restrict to the case where $X_n$ does not escape the domain $\mathcal{O}$, expect possible by a jump, i.e. $L^n(\tau_n) = 0$. It follows that a contradiction can be obtained by exactly the same argument as step 1, by only assuming
\[
\min_{(a, e) \in A \times E} \left( -L^{n,e} \phi(t_0, x_0) - f(x_0, a) \right) < 0.
\]
When $(t_0, x_0) \in \{T\} \times \partial \mathcal{O}$, it follows from Proposition 2.1 and the fact that $\mathcal{A}_t \times \mathcal{E}_t \supset \mathcal{A}_T \times \mathcal{E}_T$ for all $t \leq T$ that $v$ is lower-semicontinuous at the points of $\{T\} \times \partial \mathcal{O}$. This leads clearly to $v_* \geq g$ on $\{T\} \times \partial \mathcal{O}$. \hfill \Box

4.2 The sub-solution property.

Let $(t_0, x_0) \in [t, T] \times \partial \mathcal{O}$ and $\varphi \in C^{1,2}([0, T] \times \partial \mathcal{O})$ such that
\[
\max(\text{strict})_{[0,T] \times \partial \mathcal{O}} v^* - \varphi = (v^* - \varphi)(t_0, x_0) = 0.
\]

1. We first consider the case where $(t_0, x_0) \in \{T\} \times \partial \mathcal{O}$.

We argue by contradiction and suppose that
\[
\min_{e \in E} \{ \min_{e \in E} \mathcal{H}^e \phi(t_0, x_0), (\varphi - g)(t_0, x_0) \} =: 2\epsilon > 0.
\]

Since $A$ is compact, after replacing $\varphi$ by $(t, x) \mapsto \varphi(t, x) + \sqrt{T - t + \epsilon}$, with $\epsilon > 0$ small, we can assume that $\lim_{t \rightarrow T} \partial t \varphi(t, x) = -\infty$. Then, there exists $\delta > 0$ such that
\[
\min_{e \in E} \{ \min_{(a, e) \in A \times E} \left( -L^{n,e} \varphi - f(\cdot, a) \right), \min_{e \in E} \varphi, \varphi - g \} \geq \epsilon \text{ on } \bar{B} \cap [T - \delta, T] \times \partial \mathcal{O}, \quad (4.3)
\]
where $B := [T - \delta, T) \times B(x_0, \delta)$.

It follows from the fact that $v^* - \varphi$ achieves a strict local maximum at $(t_0, x_0)$ and the fact that the domain $\mathcal{O}$ is bounded that
\[
\sup_{\partial \mathcal{O} B} (v^* - \varphi) =: -\eta < 0, \quad (4.4)
\]
where $\partial_B B := [T - \delta, T] \times (B(x_0, \delta) \cap \partial \mathcal{O}) \cup \{T\} \times B(x_0, \delta)$.

Let $(t_n, x_n)_n$ be a sequence in $[0, T) \times \mathcal{O}$ converging to $(t_0, x_0)$ such that
\[
v(t_n, x_n) \rightarrow v^*(t_0, x_0).
\]
4. THE PDE CHARACTERIZATION.

Obviously, we can assume that \((t_n, x_n) \in B\). Fix \((\varepsilon, \alpha) \in \mathcal{E}_t \times \mathcal{A}_t\) and define

\[
(X_n, L_n, \beta_n) := (X_{t_n, x_n}^{\varepsilon, \alpha}, L_{t_n, x_n}^{\varepsilon, \alpha}, \beta_{t_n, x_n}^{\varepsilon, \alpha}),
\]

together with \(\tau_n\) and \(\theta_n\), the first exit times of \((s, X_n(s))_{s \geq t_n}\) and \((X_n(s))_{s \geq t_n}\) from \(B\) and \(\mathcal{O}\), respectively.

Using Itô’s Lemma and (4.3), we deduce that

\[
\mathbb{E}[\beta_n(\tau_n)\varphi(\tau_n, X_n(\tau_n))] + \mathbb{E}[\beta_n(T)(g(X_n(T)) + \varepsilon)1_{\{\tau_n = T\}}] \leq \varphi(t_n, x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), \alpha_s)ds\right] - \mathbb{E}[(\varepsilon \wedge \eta)\beta_n(\tau_n)] - \mathbb{E}[\beta_n(\tau_n)L_n^c(\tau_n)].
\]

This, together with (4.4), implies that

\[
\mathbb{E}[\beta_n(\tau_n)[v^*, g](\tau_n, X_n(\tau_n))] \leq \varphi(t_n, x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), \alpha_s)ds\right] - \mathbb{E}[(\varepsilon \wedge \eta)1_{\tau_n \leq \theta_n} + e^{-cL_n(\tau_n)}(\varepsilon L_n^c(\tau_n) + \varepsilon \wedge \eta)1_{\tau_n > \theta_n}].
\]

Recalling that \(|\rho| \leq c\) for some \(c > 0\), we deduce that

\[
\mathbb{E}[\beta_n(\tau_n)[v^*, g](\tau_n, X_n(\tau_n))] \leq \varphi(t_n, x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), \alpha_s)ds\right] - \mathbb{E}[(\varepsilon \wedge \eta)1_{\tau_n \leq \theta_n} + e^{-cL_n(\tau_n)}(\varepsilon L_n^c(\tau_n) + \varepsilon \wedge \eta)1_{\tau_n > \theta_n}].
\]

Then,

\[
\mathbb{E}\left[\beta_n(\tau_n)[v^*, g](\tau_n, X_n(\tau_n)) + \int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), \alpha_s)ds\right] \leq \varphi(t_n, x_n) - \min\{\varepsilon, \eta, \nu\mathbb{E}[e^{-c\Sigma_{s \leq \tau_n}(L(s))}]\},
\]

where \(\nu := \inf_{k \geq 0}(e^{-ck}(\varepsilon k + \zeta \wedge \varepsilon))\). Note that the same argument as in the proof of section 4.1 shows that \(\mathbb{E}[e^{-c\Sigma_{s \leq \tau_n}(L(s))}] \geq \kappa > 0\), for some \(\kappa\) independent on \(n\) and \(\zeta\).

Using the fact that \(\lim_{n \to \infty}(v - \varphi)(t_n, x_n) = 0\) and (2.3), we may then find \(\eta' > 0\), which is independent on \(\varepsilon, \alpha\) and \(n\), such that

\[
v(t_n, x_n) - \eta' \geq \mathbb{E}\left[\beta_n(\tau_n)[v^*, g](\tau_n, X_n(\tau_n)) + \int_{t_n}^{\tau_n} \beta_n(s)f(X_n(s), \alpha_s)ds\right],
\]

for \(n\) large enough. This leads to a contradiction to (2.1) in Theorem 2.1. \(\square\)

2. The case where \((t_0, x_0) \in [0, T) \times \partial\mathcal{O}\) can be treated similarly by similar argument as in previous step, see (4.3). We can indeed use a localization in order to assume that \(\tau_n \leq T - \varepsilon\) for some \(\varepsilon > 0\). Therefore, we do not need to compare the values of \(v^*\) with \(g\).

The case \((t_0, x_0) \in [0, T] \times \mathcal{O}\) is also treated similarly by using a localization argument as in step 2 of the proof in section 4.1.
5 The comparison theorem.

We now prove a comparison result for $B_+\varphi \geq 0$ and $B_-\varphi \leq 0$ on $[0,T] \times \bar{O}$, which implies that $v$ is continuous on $[0,T] \times \bar{O}$, admits a continuous extension to $[0,T] \times \bar{O}$, and is the unique viscosity super-solution (resp. sub-solution) of $B_+\varphi \geq 0$ (resp. $B_-\varphi \leq 0$) on $[0,T] \times \bar{O}$.

We first introduce an equivalent definition of viscosity solutions, which eliminates the appearance of test function in the integral associated to the measure of jumps. Let us denote by $G_{a,e}$ the operator from $[0,T] \times \bar{O} \times \mathbb{R}^d \times \mathbb{M}^d$ to $\mathbb{R}$ parameterized by a smooth function $\varphi$ as

$$G_{a,e}(x,q,p,M;\varphi) := q + \langle b(x,a),p \rangle + f(x,a) + \frac{1}{2} \text{Trace} [\sigma(x,a)\sigma^*(x,a)M]$$

$$+ \int_{\mathbb{R}^d} [e^{-\rho(x,a)}\pi^{a,e}(z)(\pi^{a,e}_x(z)) - \varphi(x)]\bar{\mu}(dz)$$

so that

$$L^{a,e}(t,x) + f(x,a) = G_{a,e}(x,\partial_t\varphi(t,x), D\varphi(t,x), D^2\varphi(t,x); \varphi(t,\cdot)).$$

We also define $F_\pm$ as the operator associated to $B_\pm$ by the implicit relation :

$$F_\pm(x,\partial_t\varphi(t,x), D\varphi(t,x), D^2\varphi(t,x); \varphi(t,\cdot)) = B_\pm \varphi(t,x).$$

Note that, for $(a,e) \in A \times E$, $G_{a,e}$ is a continuous function satisfying the elliptical condition, i.e. it is non increasing with respect to $M \in \mathbb{M}^d$ and $I$. In view of Definition 4 in [2] and the fact that $A$ and $E$ are compact, we can provide the following equivalent definition of viscosity solutions :

**Definition 5.1** A lower semicontinuous (resp. upper semicontinuous) function $w$ on $[0,T] \times \bar{O}$ is a viscosity super-solution (resp. sub-solution) of (4.1) if, for $(t_0,x_0) \in [0,T] \times \bar{O}$, $(q_0,p_0,M_0) \in \tilde{P}_\varphi w(t,x)$ (resp. $\tilde{P^+}_\varphi w(t,x)$) and $\varphi \in C^{1,2}([0,T] \times \bar{O})$ so that

$(t_0,x_0)$ is a maximum (resp. minimum) point of $w - \varphi$, $w(t_0,x_0) = \varphi(t_0,x_0)$, and

$$q_0 = \partial_t \varphi(t_0,x_0), \quad p_0 = D\varphi(t_0,x_0), \quad M_0 \geq D^2\varphi(t_0,x_0) \quad (\text{resp. } M_0 \leq D^2\varphi(t_0,x_0)),$$

we have

$$F_+(x_0,q_0,p_0,M_0;w(t_0,\cdot)) \geq 0 \quad (\text{resp. } F_-(x_0,q_0,p_0,M_0;w(t_0,\cdot)) \leq 0).$$
5. THE COMPARISON THEOREM.

See [18] for the standard notations $\bar{P}_O^+$ and $\bar{P}_O^-$. Motivated by the comparison result of Proposition 3.4 in [8], we add some assumptions:

**G3.**

(i) There exists $b > 0$ such that

$$B(x - b\gamma(x,e), b) \cap O = \emptyset, \text{ for all } (x,e) \in \partial O \times E.$$  \hspace{1cm} (5.1)

(ii) There exists a $C^2(\bar{O})$ function $\tilde{h}$ such that

$$\langle \gamma(x,e), D\tilde{h}(x) \rangle \geq 1, \text{ for all } x \in \partial O \text{ and } e \in E.$$  \hspace{1cm} (5.2)

(iii) For all $x \in \partial O$, we have

$$\inf_{e \in E} \langle \gamma(x,e), \gamma(x,e_x) \rangle > 0,$$

where $e_x \in \arg\max \{\rho(x,e) : e \in E\}$. Note that G3(iii) is weaker than the corresponding assumption (3.17) in [8]. We now prove a comparison result:

**Theorem 5.1** Suppose that G3 holds. Let $V$ (resp. $U$) be a lower-semicontinuous (resp. upper-semicontinuous) locally bounded map on $[0,T] \times \bar{O}$. Assume that $V$ is a viscosity supersolution of $B_+ \varphi \geq 0$ on $[0,T] \times \bar{O}$ and $U$ is a viscosity subsolution of $B_- \varphi \leq 0$ on $[0,T] \times \bar{O}$. Then, $V \geq U$ on $[0,T] \times \bar{O}$.

**Proof.** Fix $\kappa > 0$ and set $\tilde{U}(t,x) := e^{\kappa t}U(t,x)$, $\tilde{V}(t,x) := e^{\kappa t}V(t,x)$, $\tilde{f}(t,x) := e^{\kappa t}f(t,x)$ and $\tilde{g}(t,x) := e^{\kappa t}g(x)$. Here, $\kappa$ is chosen such that

$$-\tilde{f}(\cdot, a) - \mathcal{L}^{a,e}H \geq 0, \text{ for all } (a,e) \in A \times E,$$  \hspace{1cm} (5.3)

where $H(t,x) := e^{-\kappa t - k(x)}$.

We argue by contradiction, and therefore assume that

$$\sup_{[0,T] \times O} \left(\tilde{U} - \tilde{V}\right) > 0.$$  \hspace{1cm} (5.4)

It follows from the fact that the domain $O$ is bounded that $\Phi^n := \tilde{U} - \tilde{V} - 2\eta H$ achieves its maximum at $(t^n, x^n)$ on $[0,T] \times \bar{O}$ and satisfies

$$\Phi^n(t^n, x^n) =: m > 0, \text{ for } \eta > 0 \text{ small enough}.$$  \hspace{1cm} (5.5)

1. We first study the case $U(t^n, x^n) \geq 0$, up to a subsequence. We define the function $\Psi^n_\eta$ on $[0,T] \times \bar{O}^2$ as

$$\Psi^n_\eta(t,x,y) := \Theta(t,x,y) - |x - x^n|^4 - |t - t^n|^2 - \frac{n}{2}|x - y|^2 - \rho(x^n, e^n)\tilde{U}(t^n, x^n)(\gamma(x^n, e^n), x - y),$$

where $\Theta(t,x,y)$ is a certain function defined in [8].
where
\[ \Theta(t, x, y) := \tilde{U}(t, x) - \tilde{V}(t, y) - \eta (H(t, x) + H(t, y)) , \]
and \( \varphi \in \text{argmin}\{\rho(x^n, e) : e \in E\} \).

Assume that \( \Psi_n^\eta \) achieves its maximum at some \( (t_n^n, x_n^n, y_n^n) \in [0, T] \times \overline{O}^2 \). The inequality \( \Psi_n^\eta(t_n^n, x_n^n, y_n^n) \geq \Psi_n^\eta(t, x_n^n, x^n) \) implies that
\[ \Theta(t_n^n, x_n^n, y_n^n) \geq \Theta(t_n^n, x_n^n) + \rho(x_n^n, \varphi_\eta) \tilde{U}(t_n^n, x_n^n)(\gamma(x_n^n, \varphi_\eta), x_n^n - y_n^n) \]
\[ + |x_n^n - x_n^n|^4 + |t_n^n - t_n^n|^2 + \frac{n}{2} |x_n^n - y_n^n|^2 . \]

We deduce that the term on the second line is bounded in \( n \) so that, up to a subsequence, \( x_n^n, y_n^n \rightarrow x^0 \in \overline{O} \) and \( t_n^n \rightarrow t^0 \in [0, T] \). Sending \( n \rightarrow \infty \) in the previous inequality and using the maximum property of \( \Phi^\eta \) at \( (t^n, x^n) \), we obtain
\[ 0 \geq \Phi^\eta(t^0, x^0) - \Phi^\eta(t^n, x^n) \]
\[ \geq \limsup_{n \rightarrow \infty} (|x_n^n - x_n^n|^4 + |t_n^n - t_n^n|^2 + \frac{n}{2} |x_n^n - y_n^n|^2) , \]
This, together with (5.5), implies that
(a) \( x_n^n, y_n^n \rightarrow x^n \) and \( t_n^n \rightarrow t^n \),
(b) \( |x_n^n - x_n^n|^4 + |t_n^n - t_n^n|^2 + \frac{n}{2} |x_n^n - y_n^n|^2 \rightarrow 0 \),
(c) \( \tilde{U}(t_n^n, x_n^n) - \tilde{V}(t_n^n, y_n^n) \rightarrow (\tilde{U} - \tilde{V}) (t^n, x^n) \geq m + 2\eta H(t^n, x^n) > 0 \).

In view of Ishii’s Lemma, see Theorem 8.3 in [18], we deduce that, for each \( \lambda > 0 \), there are real coefficients \( b_{1,n}^\eta, b_{2,n}^\eta \) and symmetric matrices \( A_n^\eta, \lambda_n^\eta \) such that
\[ (b_{1,n}^\eta, p_n^\eta, A_n^\eta, \lambda_n^\eta) \in \overline{P}_O^+ \tilde{U}(t_n^n, x_n^n) \quad \text{and} \quad (-b_{2,n}^\eta, q_n^\eta, \lambda_n^\eta) \in \overline{P}_O^- \tilde{V}(t_n^n, y_n^n) , \]
where
\[ p_n^\eta := 4|x_n^n - x_n^n|^2(x_n^n - x_n^n) + n(x_n^n - y_n^n) + \rho(x_n^n, \varphi_\eta) \tilde{U}(t_n^n, x_n^n)(\gamma(x_n^n, \varphi_\eta), x_n^n) + \eta DH(t_n^n, x_n^n) \]
\[ q_n^\eta := n(x_n^n - y_n^n) + \rho(x_n^n, \varphi_\eta) \tilde{U}(t_n^n, x_n^n)(\gamma(x_n^n, \varphi_\eta) - \eta DH(t_n^n, y_n^n) , \]
and \( A_n^\eta, \lambda_n^\eta \) satisfy
\[ \begin{cases} b_{1,n}^\eta + b_{2,n}^\eta = 2(t_n^n - t^n) - \kappa \eta (H(t_n^n, x_n^n) + H(t_n^n, y_n^n)) \\ \begin{pmatrix} A_n^\eta + B_n^\eta \\ -\lambda_n^\eta \end{pmatrix} \leq (A_n^\eta + B_n^\eta)^2 + \lambda (A_n^\eta + B_n^\eta) \end{cases} \]
with
\[ A_n^\eta := \eta \begin{pmatrix} D^2 H(t_n^n, x_n^n) & 0 \\ 0 & D^2 H(t_n^n, y_n^n) \end{pmatrix} + \begin{pmatrix} 12(x_n^n - x_n^n) \otimes (x_n^n - x_n^n) & 0 \\ 0 & 0 \end{pmatrix} , \]
\[ B_n^\eta := n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} , \]
and $I$ stands for the identical matrix with dimension $d \times d$.

1.1. We first suppose that, up to a subsequence, $x^n_n \in \partial \mathcal{O}$ for all $n$. Fix $e \in E$. It follows from (5.1) that

$$|x^n_n - b\gamma(x^n_n, e) - y^n_n|^2 \geq b^2.$$  

Since $|\gamma| = 1$, this implies that

$$2\langle \gamma(x^n_n, e), y^n_n - x^n_n \rangle \leq -b^{-1}|y^n_n - x^n_n|^2.$$  

(5.7)

Since $x^n_n \xrightarrow{n \to \infty} x^n$, we have

$$\rho(x^n_n, e)\hat{U}(t^n_n, x^n_n) - \langle \gamma(x^n_n, e), p^n_n \rangle = (\rho(x^n, e) - \rho(x^n, e_n))\hat{U}(t^n, x^n) + \rho(x^n, e_n)\hat{U}(t^n, x^n)(1 - \langle \gamma(x^n, e), \gamma(x^n, e_n) \rangle) + n\langle \gamma(x^n_n, e), y^n_n - x^n_n \rangle + \eta\langle \gamma(x^n_n, e), D\hat{h}(x^n_n) \rangle H(t^n_n, x^n_n) + \lambda_n,$$

where $\lambda_n$ is independent on $e$ and comes to $0$ when $n \to \infty$. This, together with (5.2), (b), (5.7) and the fact that $\langle \gamma(x^n, e), \gamma(x^n, e_n) \rangle \leq 1$ since $|\gamma| \leq 1$, implies that

$$\mathcal{H}^e(x^n_n, \hat{U}(t^n_n, x^n_n), p^n_n) > \eta H(t^n, x^n) > 0,$$

when $n$ is large enough.

Using similar argument as above, if, up to a subsequence, $y^n_n \in \partial \mathcal{O}$, we then have

$$\mathcal{H}^e(y^n_n, \tilde{V}(t^n_n, y^n_n), q^n_n) < -\eta H(t^n, x^n) < 0,$$

for all $e \in E$, when $n$ is large enough.

1.2. We now suppose that, up to a subsequence, $t^n_n = T$, for all $n \geq 1$. In view of step 1.1, we have $\hat{U}(T, x^n_n) \leq \bar{g}(T, x^n_n)$ and $\tilde{V}(T, y^n_n) \geq \bar{g}(T, y^n_n)$, for all $n \geq 1$. Then, passing to the limit, recalling (a) and the fact that $g$ is continuous, it implies that $\hat{U}(T, x^n) \leq \bar{g}(T, x^n) \leq \tilde{V}(T, x^n)$. This leads to a contradiction of (c).

1.3. It follows from 1.1 and 1.2 that $(t^n_n, x^n_n, y^n_n) \in [0, T) \times \mathcal{O}^2$ for all $n$, after possibly passing to a subsequence. Then, there exists $(a^n_n, e^n_n)_{n \geq 1} \subset A \times E$ such that

$$0 \geq \kappa\hat{U}(t^n_n, x^n_n) - t^n_n - b(x^n_n, a^n_n), p^n_n) - \frac{1}{2} \text{Trace} \left[ \sigma(x^n_n, a^n_n)\sigma^*(x^n_n, a^n_n) \mathcal{X}^{\eta, \lambda}_n \right]$$

$$- \int_{\mathbb{R}^d} \left[ e^{-\rho(x^n_n, e^n_n)} \hat{U}(t^n_n, x^n_n) \pi^{a^n_n, e^n_n}(z) \hat{U}(t^n_n, x^n_n) \right] \tilde{\mu}(dz) - \tilde{f}(x^n_n, a^n_n)$$

$$0 \leq \kappa\tilde{V}(t^n_n, y^n_n) + t^n_n - b(y^n_n, a^n_n), q^n_n) - \frac{1}{2} \text{Trace} \left[ \sigma(y^n_n, a^n_n)\sigma^*(y^n_n, a^n_n) \mathcal{Y}^{\eta, \lambda}_n \right]$$

$$- \int_{\mathbb{R}^d} \left[ e^{-\rho(y^n_n, e^n_n)} \pi^{a^n_n, e^n_n}(z) \tilde{V}(t^n_n, y^n_n) \right] \tilde{\mu}(dz) - \tilde{f}(y^n_n, a^n_n).$$

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It follows from (b), (5.6) and the Lipschitz continuity of our coefficients that

\[
\kappa [ \tilde{U}(t^n, x^n_t) - \tilde{V}(t^n, x^n_t) - 2\eta H(t^n, x^n)] + 2\eta [ \tilde{f}(x^n, a^n_n) + L^{a^n_n, \varepsilon^n_n} H(t^n, x^n)] + C(n, \lambda) \\
\leq \int_{\mathbb{R}^d} \left[ e^{-\rho(x^n, \varepsilon^n_n)^{a^n_n, e^n_n}(z)} (\tilde{U} - \tilde{V} - 2\eta H)(t^n, \eta^{a^n_n, \varepsilon^n_n}(z)) \right] \varepsilon(dz) \\
- \int_{\mathbb{R}^d} \left[ e^{-\rho(x^n, \varepsilon^n_n)^{a^n_n, e^n_n}(z)} (\tilde{U} - \tilde{V} - 2\eta H)(t^n, y^n) \right] \varepsilon(dz),
\]

where \( C(n, \lambda) \) goes to 0 when \( \lambda \to 0 \) and then \( n \to \infty \).

This, together with the fact that

\[
(\tilde{U} - \tilde{V} - 2\eta H) \leq (\tilde{U} - \tilde{V} - 2\eta H)(t^n, x^n) = m > 0,
\]

and \( \rho, \lambda \geq 0 \), implies that

\[
\kappa [ \tilde{U}(t^n, x^n_t) - \tilde{V}(t^n, y^n_t)) - 2\eta H(t^n, x^n)] \leq -2\eta [ \tilde{f}(x^n, a^n_n) + L^{a^n_n, \varepsilon^n_n} H(t^n, x^n)] + C(n, \lambda).
\]

Finally, using (c) and (5.3), we deduce by sending \( \lambda \to 0 \) and then \( n \to \infty \) that

\[
\kappa m \leq 0,
\]

which is the required contradiction.

2. The case where \( \tilde{U}(t^n, x^n) < 0 \), up to a subsequence, is treated similarly. The difference comes from the test function which is chosen as follows

\[
\Psi_n(t, x, y) := \Theta(t, x, y) - |x - x^n|^4 - |t - t^n|^2 - \frac{n}{2} |x - y|^2 \\
- \tilde{b}_n^{-1} \rho(x^n, \tilde{e}_n) \tilde{U}(t^n, x^n) (\gamma(x^n, \tilde{e}_n), x - y),
\]

where \( \tilde{e}_n = \tilde{e}_{x^n}, \tilde{b}_n \) and \( \tilde{e}_n \in E \) satisfy

\[
\min_{e \in E} \langle \gamma(x^n, e), \gamma(x^n, \tilde{e}_x) \rangle = \langle \gamma(x^n, \tilde{e}_n), \gamma(x^n, \tilde{e}_x) \rangle = \tilde{b}_n > 0.
\]

Those variables are well defined, when the condition (iii) of assumption G3 holds. Then, if, up to a subsequence, \( x^n_n \in \partial \mathcal{O} \) for all \( n \), we have

\[
\rho(x^n_n, e) \tilde{U}(t^n_n, x^n_n) - \langle \gamma(x^n_n, e), p^n_n \rangle \\
= (\rho(x^n, e) - \rho(x^n, \tilde{e}_n)) \tilde{U}(t^n, x^n) \\
+ \rho(x^n, \tilde{e}_n) \tilde{U}(t^n, x^n) (1 - \tilde{b}_n^{-1} \langle \gamma(x^n, e), \gamma(x^n, \tilde{e}_n) \rangle) \\
+ n \langle \gamma(x^n_n, e), y^n_n - x^n_n \rangle + \eta \langle \gamma(x^n_n, e), D\tilde{h}(x^n_n) \rangle H(t^n_n, x^n_n) + \lambda_n,
\]

where \( \lambda_n \) goes to 0 as \( n \to \infty \).

This, together with (a), (b), (5.2) and (5.7), implies that

\[
\mathcal{H}^a(x^n_n, \tilde{U}(t^n_n, x^n_n), p^n_n) > \eta H(t^n, x^n) > 0,
\]

when \( n \) is large enough. The other cases are treated similarly as in step 1 above. \( \square \)
Bibliographie


