

Minimisation methods for quasi-linear problems, with an application to periodic water waves

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Abstract

Penalisation and minimisation methods are used to give an abstract “semi-global” result on the existence of non-trivial solutions of parameter-dependent quasi-linear differential equations in variational form. A consequence is a proof of existence, by infinite-dimensional variational means, of bifurcation points for quasi-linear equations which have a line of trivial solutions.

The approach is to penalise the functional twice. Minimisation gives the existence of critical points of the resulting problem and *a priori* estimates show that the critical points lie in a region unaffected by the leading penalisation. The other penalisation contributes to the value of the parameter.

As applications, we prove the existence of periodic water waves with and without, surface tension.

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1 Introduction

Using local finite-dimensional reduction followed by a constrained minimisation argument, Stuart [15], following Krasnosel'skii [12], gave a bifurcation theory for variational problems which was applied in [6] to Babenko's [2] quasi-linear equation for Stokes waves. But how to deal with such problems directly, using variational methods in infinite dimensions, remained unclear. In [8] we took a step in this direction by adapting some ideas of Turner [17] and the mountain-pass lemma in infinite dimensions to obtain an existence theory which was "semi-global," in the sense that parameter values were quantifiably not infinitesimally small, and finite-dimensional reduction was not involved.

Now, in Section 2, we present an abstract result that covers a general class of quasi-linear problems. In Section 3 we show that the capillary-gravity wave problem, with its curvature term that represents surface tension effects, is a special case. Our method should give explicit (and hopefully good) lower bounds on the size of the periodic capillary-gravity waves so obtained, but here the aim is merely to illustrate the generality of the abstract result. Moreover our abstract method could probably be enriched to encompass the various kinds of periodic capillary-gravity waves found in [10, 11, 14]. In Section 4, we apply the present method to the Stokes-wave problem (steady periodic water waves without surface tension). As in [8], we obtain the existence of a non-zero symmetric Stokes wave which is not a consequence of local existence theories [1, 6, 15]. In contrast to [8], we no longer appeal to the mountain-pass principle nor to Morse theory, and therefore the present proof is simpler.

Much more is known about steady water waves using global continuation methods [5, 6, 7], numerical investigations [9, 13] and computer assisted proofs [3, 4]. But the need for a global or "semi-global" variational theory of these and similarly degenerate variational problems, including Morse indices of solutions etc., remains our focus.

2 Abstract setting

Consider a real Hilbert space X_0 with inner-product $\langle \cdot, \cdot \rangle_0$ and norm $\| \cdot \|_0$, and suppose that A is a (possibly unbounded) positive-definite self-adjoint operator on X_0 such that $A^{-1} : X_0 \rightarrow X_0$ exists and is continuous. For $k \geq 1$

let X_k denote the domain of $A^{k/2}$, which is dense in X_0 . Then X_k is a Hilbert space with inner-product and norm defined by $\langle u, v \rangle_k = \langle A^{k/2}u, A^{k/2}v \rangle_0$ and $\|u\|_k = \|A^{k/2}u\|_0$ for $u, v \in X_k$, and

$$\|u\|_k \leq \|A^{-1/2}\| \|u\|_{k+1} \quad \text{for all } u \in X_{k+1}.$$

For $R_2 > 0$, let $U \subset X_2$ be the open ball $\{u \in X_2 : \|u\|_2 < R_2\}$ and suppose that $\mathcal{K}, \mathcal{L} \in C^1(U; \mathbb{R})$ are functionals. We are interested in the equation

$$\gamma \mathcal{K}'(w) + \mathcal{L}'(w) = 0, \quad w \in U \setminus \{0\}, \quad \gamma \geq \gamma_0 \geq 0, \quad (\star)$$

when the following inequalities hold for constants $C_1, C_2 > 0$ and for a continuous function $\psi : [0, \infty)^2 \rightarrow \mathbb{R}$, whose precise form depends on the problem:

$$\text{for } u \in U : \mathcal{K}(u) \geq C_1 \|u\|_1^2 \quad \text{and } \mathcal{K}(0) = 0, \quad (1a)$$

$$\text{for } u \in U : \mathcal{L}(u) \geq -C_2 \|u\|_1^2 \quad \text{and } \mathcal{L}(0) = 0, \quad (1b)$$

$$\text{for } u \in U \cap X_4 : \mathcal{K}'(u)Au \geq 0 \quad \text{and} \quad (1c)$$

$$\gamma_0 \mathcal{K}'(u)Au + \mathcal{L}'(u)Au \geq \psi(\|u\|_1, \|u\|_2). \quad (1d)$$

Observe that $\mathcal{K}'(0) = \mathcal{L}'(0) = 0$, so that 0 is a trivial solution of (\star) .

Roughly speaking, the function ψ takes positive and negative values, and will be such that $\psi(s, t) > 0$ when s is "not too large" and t is "not small" (see assumption (2) below). In the existence proof, we construct a functional \mathcal{J} on U , whose minimiser w satisfies $\gamma_0 \mathcal{K}'(w)Aw + \mathcal{L}'(w)Aw \leq 0$ with $s = \|w\|_1$ "not too large". This yields an upper bound, better than R_2 , on $t = \|w\|_2$ which ensures that w is solution of our problem. To verify the assumptions in practice, it will often be necessary to choose U small enough; whence the term "semi-global" in the abstract.

The next hypothesis is about weak solutions of a regularised problem: for all $\gamma \geq \gamma_0$, $\epsilon > 0$ and $w \in U$,

$$\text{if } \gamma \mathcal{K}'(w) + \mathcal{L}'(w) + \epsilon A^2 w = 0 \quad \text{in } X_2^*, \quad \text{then } w \in X_4. \quad (1e)$$

Finally we make the following assumptions:

$$\mathcal{K} \text{ and } \mathcal{L} \text{ are weakly lower semicontinuous on } U \subset X_2. \quad (1f)$$

$$\text{There exists } u_* \in U \text{ with } \gamma_0 \mathcal{K}(u_*) + \mathcal{L}(u_*) < 0 = \gamma_0 \mathcal{K}(0) + \mathcal{L}(0). \quad (1g)$$

Theorem 1. *Suppose that (1) holds and that*

$$\psi(s, R_2) > 0 \quad \text{for all } s \in [0, \sqrt{\mathcal{K}(u_*)/C_1}]. \quad (2)$$

Then there exists $w \in U \setminus \{0\}$ and $\gamma \geq \gamma_0$ such that

$$\begin{aligned} \gamma \mathcal{K}'(w) + \mathcal{L}'(w) &= 0, \\ \gamma_0 \mathcal{K}(w) + \mathcal{L}(w) &\leq \gamma_0 \mathcal{K}(u_*) + \mathcal{L}(u_*) < 0 \end{aligned} \quad (\star)$$

and, as a consequence of (1a) and (1b),

$$\|w\|_1^2 \geq \frac{\gamma_0 \mathcal{K}(u_*) + \mathcal{L}(u_*)}{\gamma_0 C_1 - C_2} > 0.$$

Remarks.

Note that $\gamma_0 C_1 - C_2 < 0$, otherwise u_ would not exist.*

A typical form of function ψ is $\alpha t^2 - \varphi(s, t)$, where $\alpha > 0$, φ is continuous from $[0, \infty)^2$ to \mathbb{R} , and $\varphi(0, R_2) = 0$. Then condition (2) becomes

$$\varphi(s, R_2) < \alpha R_2^2 \quad \text{for all } s \in [0, \sqrt{\mathcal{K}(u_*)/C_1}]. \quad (3a)$$

This condition is satisfied for $\mathcal{K}(u_)$ small enough, by continuity of φ at $(0, R_2)$. Note that α may depend on R_2 .*

In the examples of Sections 3 and 4 the function ψ is constructed in two steps. First we establish an inequality

$$\mathcal{K}'(u)Au \geq C_3 \|u\|_2^2 \quad \text{for all } u \in U \cap X_4, \quad (3b)$$

for some constant $C_3(R_2) > 0$. Then we find a function $\varphi(s, t)$ and a constant $C_4 \in \mathbb{R}$ such that

$$\mathcal{L}'(u)Au \geq -\varphi(\|u\|_1, \|u\|_2) - C_4 \|u\|_2^2 \quad \text{for all } u \in U \cap X_4. \quad (3c)$$

If $\gamma_0 > C_4/C_3$, the estimate (1d) will follow for $\psi(s, t) := \alpha t^2 - \varphi(s, t)$, with $\alpha := \gamma_0 C_3 - C_4 > 0$.

Proof of Theorem 1. Let $R_1, R_{\min} > 0$ be finite numbers such that

$$\mathcal{K}(u_*) < R_1^2, \quad \|u_*\|_2 \leq R_{\min} < R_2,$$

and

$$\psi(s, t) > 0 \quad \text{for all } s \in [0, R_1/\sqrt{C_1}] \quad \text{and } t \in [R_{\min}, R_2]. \quad (4)$$

These numbers exist, by assumption (2) and the uniform continuity of ψ on $[0, 2\sqrt{\mathcal{K}(u_*)/C_1}] \times [0, R_2]$. We define two smooth, non-decreasing *penalisation* functions $\rho_i : [0, R_i^2] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \rho_i(s) &\rightarrow \infty \text{ as } s \nearrow R_i^2, \quad i = 1, 2, \\ 0 \leq s \leq \mathcal{K}(u_*) &\Rightarrow \rho_1(s) = 0, \quad 0 \leq s \leq R_{\min}^2 \Rightarrow \rho_2(s) = 0, \end{aligned}$$

and consider the real-valued functional defined for $u \in U$ with $\mathcal{K}(u) < R_1^2$ by

$$\mathcal{J}(u) = \gamma_0 \mathcal{K}(u) + \mathcal{L}(u) + \rho_2(\|u\|_2^2) + \rho_1(\mathcal{K}(u)).$$

The indices denote the facts that ρ_2 and ρ_1 control, respectively, the norms in X_2 and X_1 (through the constant C_1). We refer to ρ_2 as the *leading penalisation*. Together they allow us to work in the domain $V := \{u \in U : \mathcal{K}(u) < R_1^2\}$. The inequalities (1a), (1d), (2) will ensure that critical points w of \mathcal{J} in this domain satisfy $\|w\|_2 \leq R_{\min}$ and are thus unaffected by the leading penalisation. At critical points the other penalisation may be non-zero, which leads to the loss of control of the value of the parameter γ in the statement of the theorem.

We must now find a critical point of \mathcal{J} , and a natural idea is to look for a minimiser. Note that \mathcal{J} is bounded from below on V , with $\mathcal{J}(u) \rightarrow \infty$ as $\|u\|_2 \nearrow R_2$ and $\mathcal{J}(u) \rightarrow \infty$ as $\mathcal{K}(u) \nearrow R_1^2$, by the existence of the constants C_1 and C_2 . From (1f) it follows that \mathcal{J} has a minimiser $w \in V$. We have $\mathcal{K}(w) < R_1^2$, $\|w\|_2 < R_2$, and w is a weak solution of the Euler equation

$$\{\gamma_0 + \rho_1'(\mathcal{K}(w))\} \mathcal{K}'(w) + \mathcal{L}'(w) + 2\rho_2'(\|w\|_2^2) A^2 w = 0. \quad (5)$$

Seeking a contradiction assume that $\epsilon := 2\rho_2'(\|w\|_2^2) > 0$. Then, by (1e), $w \in X_4$. Hence $Aw \in X_2$, and we have $\mathcal{J}'(w)Aw = 0$. From (1c) and (1d) it follows that

$$\psi(\|w\|_1, \|w\|_2) \leq -\rho_1'(\mathcal{K}(w)) \mathcal{K}'(w) Aw - 2\rho_2'(\|w\|_2^2) \|w\|_3^2 < 0.$$

Since $\mathcal{K}(w) < R_1^2$, (1a) gives the estimate $\|w\|_1 \leq R_1/\sqrt{C_1}$. Therefore, by (4), $\|w\|_2 < R_{\min}$. This shows that $\rho_2'(\|w\|_2^2) = 0$, which is the required contradiction.

Hence, we have proved that $w \in U$ satisfies (5), with $\rho_2'(\|w\|_2^2) = 0$. Therefore

$$\gamma \mathcal{K}'(w) + \mathcal{L}'(w) = 0 \text{ with } \gamma := \gamma_0 + \rho_1'(\mathcal{K}(w)).$$

Since w is a minimiser of \mathcal{J} and $\rho_1(\mathcal{K}(u_*)) = \rho_2(\|u_*\|_2^2) = 0$, we have

$$\gamma_0\mathcal{K}(w) + \mathcal{L}(w) \leq \mathcal{J}(w) \leq \mathcal{J}(u_*) = \gamma_0\mathcal{K}(u_*) + \mathcal{L}(u_*) < 0.$$

The critical point w is thus non-zero, and we have the estimate

$$(\gamma_0C_1 - C_2)\|w\|_1^2 \leq \gamma_0\mathcal{K}(u_*) + \mathcal{L}(u_*) < 0.$$

□

Additional hypotheses yield more information on the critical point w .

Theorem 2. *Suppose that (1) holds and*

$$\mathcal{K}'(u) + \mu Au \neq 0 \quad \text{for all } u \in U \setminus \{0\} \quad \text{and } \mu \geq 0. \quad (6)$$

Let the two constants $\underline{R}, \overline{R}$ satisfy $\|u_*\|_2 \leq \overline{R} < R_2$, $\underline{R} \geq \sqrt{\mathcal{K}(u_*)}$, and (instead of (2)) suppose that

$$\psi(s, \overline{R}) \geq 0 \quad \text{for all } s \in [0, \underline{R}/\sqrt{C_1}]. \quad (7)$$

Then there exists $w \in U \setminus \{0\}$ such that $\|w\|_2 \leq \overline{R}$, $\mathcal{K}(w) \leq \underline{R}^2$,

$$\gamma_0\mathcal{K}(w) + \mathcal{L}(w) = \min\{\gamma_0\mathcal{K}(u) + \mathcal{L}(u) : u \in U, \|u\|_2 \leq \overline{R}, \mathcal{K}(u) \leq \underline{R}^2\}$$

and

(i) if $\mathcal{K}(w) < \underline{R}^2$, then $\gamma_0\mathcal{K}'(w) + \mathcal{L}'(w) = 0$;

(ii) if $\mathcal{K}(w) = \underline{R}^2$, then $\gamma\mathcal{K}'(w) + \mathcal{L}'(w) = 0$ for some $\gamma \geq \gamma_0$.

Moreover, as a consequence of properties (1a), (1b), we have the lower estimate

$$\|w\|_1^2 \geq \frac{\gamma_0\mathcal{K}(u_*) + \mathcal{L}(u_*)}{\gamma_0C_1 - C_2} > 0.$$

Proof. By assumptions (1a) and (1b), the functional $\mathcal{I}(u) := \gamma_0\mathcal{K}(u) + \mathcal{L}(u)$ is bounded from below on the set $C := \{u \in X_2 : \mathcal{K}(u) \leq \underline{R}^2, \|u\|_2 \leq \overline{R}\}$. By assumption (1f), the set C is weakly closed in X_2 , and \mathcal{I} is weakly lower semicontinuous. So there exists a minimiser w of \mathcal{I} on C . Since $u_* \in C$,

$$\mathcal{I}(w) \leq \mathcal{I}(u_*) < 0.$$

Hence $(\gamma_0 C_1 - C_2) \|w\|_1^2 \leq \mathcal{I}(u_*) < 0$. By assumption (6) and the general theorem on Lagrange multipliers, w is a weak solution of

$$\epsilon Aw + \gamma \mathcal{K}'(w) + \mathcal{L}'(w) = 0$$

for some $\gamma \geq \gamma_0$ and $\epsilon \geq 0$ and

$$(\gamma - \gamma_0)(\underline{R}^2 - \mathcal{K}(w)) = 0 = \epsilon(\overline{R} - \|w\|_2).$$

It follows by contradiction, as in the proof of Theorem 1, that $\epsilon = 0$ (this follows from (1a), (1d), (7)). The alternative (i)-(ii) is thus satisfied. \square

3 Gravity-capillary water waves

Let $L_{2\pi}^2$ denote the usual real Banach space of 2π -periodic, real-valued, square-integrable measurable ‘functions’ on \mathbb{R} and let $L_{2\pi}^\infty$ denote the analogous space of essentially bounded functions. We denote by $C_{2\pi}^k$ (resp. $C_{2\pi}^\infty$), the space of 2π -periodic functions u which are k -times continuously differentiable (resp. infinitely differentiable).

With respect to the orthonormal basis $\{(2\pi)^{-\frac{1}{2}} e^{ikt} : k \in \mathbb{Z}\}$, let the Fourier coefficients of $u \in L_{2\pi}^2$ be denoted by \hat{u}_k , $k \in \mathbb{Z}$. Then $\hat{u}_{-n} = \overline{\hat{u}_n}$, since u is real, and $L_{2\pi}^2$ is a real Hilbert space with inner product

$$\langle u, v \rangle = \sum_{n \in \mathbb{Z}} \hat{u}_n \overline{\hat{v}_n},$$

For $u \in L_{2\pi}^2$ let

$$[u] = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t) dt = \frac{\hat{u}_0}{\sqrt{2\pi}}.$$

The fractional order Sobolev space $H_{2\pi}^s$ is the Hilbert space of functions $u \in L_{2\pi}^2$ with norm given by

$$\|u\|_s^2 = \hat{u}_0^2 + \sum_{k \in \mathbb{Z}} |k|^{2s} |\hat{u}_k|^2 < \infty. \quad (8)$$

Note that if $u \in C_{2\pi}^k$, $k \in \mathbb{N} \cup 0$, then

$$\|u\|_k^2 = \|u\|_{L_{2\pi}^2}^2 + \|u^{(k)}\|_{L_{2\pi}^2}^2,$$

where $u^{(k)}$ denotes the k^{th} derivative of u . The conjugation operation [18] on $L_{2\pi}^2$ is defined by

$$(\widehat{\mathcal{C}u})_0 = 0 \text{ and } (\widehat{\mathcal{C}u})_k = -i \operatorname{sgn}(k) \hat{u}_k \text{ for } k \in \mathbb{Z} \setminus \{0\}, \text{ when } u \in L_{2\pi}^2; \quad (9)$$

equivalently, $\mathcal{C}(\cos nt) = \sin nt$, $n \geq 0$ and $\mathcal{C}(\sin nt) = -\cos nt$, $n \geq 1$. Clearly $\mathcal{C} : L_{2\pi}^2 \rightarrow L_{2\pi}^2$ is a bounded linear operator. It is now clear that $u \mapsto \mathcal{C}u'$ is non-negative and symmetric in the sense that

$$0 \leq \langle u, \mathcal{C}v' \rangle = \langle \mathcal{C}u', v \rangle \text{ for all } u, v \in C_{2\pi}^\infty.$$

For any function $w \in H_{2\pi}^1$ with $[w] = 0$, writing $w + i\mathcal{C}w = \sqrt{\frac{2}{\pi}} \sum_{k>0} \hat{w}_k e^{ikt}$, one gets

$$\begin{aligned} \|w + i\mathcal{C}w\|_\infty &\leq \frac{1}{\sqrt{2\pi}} \sum_{k \neq 0} |\hat{w}_k| \leq \frac{1}{\sqrt{2\pi}} \left(\sum_{k \neq 0} k^2 |\hat{w}_k|^2 \right)^{1/2} \left(\sum_{k \neq 0} \frac{1}{k^2} \right)^{1/2} \\ &= \sqrt{\frac{\pi}{6}} \left(\sum_{k \neq 0} k^2 |\hat{w}_k|^2 \right)^{1/2} = \sqrt{\frac{\pi}{6}} \|w\|_1 = \sqrt{\frac{\pi}{6}} \|w'\|_{L_{2\pi}^2}. \end{aligned} \quad (10)$$

When surface-tension effects are included, the steady water-wave problem can be formulated as follows [6]: find w such that

$$\frac{\nu^2}{2} \{w'^2 + (1 + \mathcal{C}w')^2\}^{-1} + \lambda w - \beta \frac{(1 + \mathcal{C}w')w'' - w'(1 + \mathcal{C}w')'}{\{w'^2 + (1 + \mathcal{C}w')^2\}^{3/2}} = \frac{1}{2} \nu^2, \quad (11a)$$

$$w'^2 + (1 + \mathcal{C}w')^2 > 0, \quad w \in H_{2\pi}^2 \setminus \{0\}, \quad \lambda \geq 0, \quad \beta, \nu > 0. \quad (11b)$$

The parameters λ , β and ν^2 are dimensionless measures of gravity, the surface tension coefficient and the square of the wave velocity. Since we can divide (11a) by any one of these, there are effectively only two parameters in the problem.

The wave length has been chosen to be 2π , without loss of generality. Note that (11) is not a variational problem as it stands. However it is known [6] that (11a) is satisfied by any $w \in H_{2\pi}^2$ such that $w'^2 + (1 + \mathcal{C}w')^2 > 0$ and such that, almost everywhere,

$$\begin{aligned} 0 &= -\nu^2 \mathcal{C}w' + \lambda \{w + w\mathcal{C}w' + \mathcal{C}(ww')\} \\ &\quad - \beta \left\{ \frac{w'}{\sqrt{w'^2 + (1 + \mathcal{C}w')^2}} \right\}' + \beta \mathcal{C} \left\{ \frac{1 + \mathcal{C}w'}{\sqrt{w'^2 + (1 + \mathcal{C}w')^2}} \right\}'. \end{aligned} \quad (12)$$

Equation (12) is the Euler equation of the functional

$$J(w) = \int_{-\pi}^{\pi} \left\{ -\frac{1}{2}\nu^2 w \mathcal{C}w' + \frac{1}{2}\lambda w^2(1 + \mathcal{C}w') + \beta \sqrt{w'^2 + (1 + \mathcal{C}w')^2} - \beta(1 + \mathcal{C}w') \right\} dt.$$

For all w , the integral of the last term is $-2\beta\pi$ and it does not contribute to the variational principle (it is a null Lagrangian). It is included here only to ensure that the constant and linear parts of the integrand vanish when $w = 0$.

Observe that, when $\lambda = 0$, every constant function w is a solution of (12) and any translate of a solution is also a solution. These superfluous solutions complicate the problem unnecessarily and, to eliminate them, we work in the subspace X_2 of $H_{2\pi}^2$ consisting of even functions of zero mean with norm given by

$$\|w\|_{X_2}^2 = \int_{-\pi}^{\pi} |w''(t)|^2 dt = \|w\|_2^2.$$

The critical points of J under the constraint $[w] = 0$ satisfy (12) almost everywhere, but with 0 on the left-hand side replaced by the constant $\lambda[w\mathcal{C}w']$. So instead we consider the functional \tilde{J} defined on X_2 by

$$\tilde{J}(w) := J(w) - \frac{\lambda}{4\pi} \left\{ \int_{-\pi}^{\pi} w \mathcal{C}w' dt \right\}^2.$$

Critical points $w \in X_2$ of \tilde{J} satisfy

$$\begin{aligned} \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} w \mathcal{C}w' dt &= -\left(\nu^2 + \frac{\lambda}{\pi} \int_{-\pi}^{\pi} w \mathcal{C}w' dt \right) \mathcal{C}w' + \lambda(w + w \mathcal{C}w' + \mathcal{C}(ww')) \\ &+ \beta \left\{ \frac{-w'}{\sqrt{w'^2 + (1 + \mathcal{C}w')^2}} + \mathcal{C} \frac{1 + \mathcal{C}w'}{\sqrt{w'^2 + (1 + \mathcal{C}w')^2}} \right\}' \end{aligned} \quad (13)$$

and $\tilde{w} := w - [w\mathcal{C}w']$ satisfies (12), from which (11a) follows.

We now apply the abstract result of Section 1 to \tilde{J} . Henceforth, let $\nu = 1$ or, equivalently, divide (12), J , \tilde{J} and (13) by ν^2 and replace λ and β by λ/ν^2 and β/ν^2 . To put the functional \tilde{J} in the context of Section 2, let

$$\begin{aligned} X_0 &= \{w \in L_{2\pi}^2 : [w] = 0, w \text{ is even}\}, \\ Aw &= -w'', \\ X_k &= \{w \in H_{2\pi}^k : [w] = 0, w \text{ is even}\} \quad (k \geq 1). \end{aligned} \quad (14)$$

If $R_2 < \sqrt{6/\pi}$ then (10) implies that

$$\|w' + iCw'\|_\infty \leq \sqrt{\pi/6}R_2 < 1 \text{ when } \|w\|_{X_2} < R_2. \quad (15)$$

For $w \in U$, the ball of radius R_2 centred at the origin in X_2 , let

$$\begin{aligned} \mathcal{K}(w) &= \int_{-\pi}^{\pi} \sqrt{w'^2 + (1 + Cw')^2} - (1 + Cw') \, dt \\ &= \int_{-\pi}^{\pi} \frac{w'^2 dt}{|1 + Cw' - iw'| + (1 + Cw')}, \\ \mathcal{L}(w) &= -\frac{1}{2} \int_{-\pi}^{\pi} wCw' dt - \frac{\lambda}{4\pi} \left\{ \int_{-\pi}^{\pi} wCw' dt \right\}^2 + \frac{\lambda}{2} \int_{-\pi}^{\pi} w^2(1 + Cw') dt. \end{aligned}$$

With β in (13) represented from now on by γ , we check the hypotheses of Theorem 1 for $0 < R_2 < \sqrt{6/\pi}$ small enough.

Obviously, \mathcal{L} is of class C^1 on X_2 . Since (15) holds, we have

$$0 < 2(1 - \sqrt{\pi/6}R_2) \leq |1 + Cw' - iw'| + (1 + Cw') \leq 2(1 + \sqrt{\pi/6}R_2). \quad (16)$$

So \mathcal{K} is of class C^1 on U . Moreover, if we define

$$C_1(R_2) := \frac{1}{2(1 + \sqrt{\pi/6}R_2)}, \quad (17)$$

then

$$C_1 \|w'\|_{L^2_{2\pi}}^2 \leq \int_{-\pi}^{\pi} \frac{w'^2 dt}{|1 + Cw' - iw'| + (1 + Cw')} = \mathcal{K}(w). \quad (18)$$

So (1a) is satisfied.

Let us check (1b), for all $w \in U$ and $\lambda > 0$:

$$\begin{aligned} \mathcal{L}(w) &= \int_{-\pi}^{\pi} \left\{ -\frac{1}{2}wCw' + \frac{\lambda}{2}w^2(1 + Cw') \right\} dt - \frac{\lambda}{4\pi} \left\{ \int_{-\pi}^{\pi} wCw' dt \right\}^2 \\ &\geq -\frac{1}{2}\|w\|_1^2 + \frac{\lambda}{2}(1 - \sqrt{\pi/6}\|w\|_2)\|w\|_0^2 - \frac{\lambda}{4\pi}\|w\|_2^2\|w\|_0^2 \\ &\geq \frac{\lambda}{2}(1 - \sqrt{\pi/6}R_2 - \frac{1}{2\pi}R_2^2)\|w\|_0^2 - \frac{1}{2}\|w\|_1^2 \\ &\geq -\frac{1}{2}\|w\|_1^2 \end{aligned}$$

under the condition $\sqrt{\pi/6}R_2 + \frac{1}{2\pi}R_2^2 \leq 1$, which is satisfied, for instance, when $R_2 < 1$. So, under this restriction on R_2 , (1b) holds for $C_2 := 1/2$.

The following lemma shows the existence of $C_3(R_2) > 0$ satisfying (3b), for R_2 small enough.

Lemma 3. *If $w \in X_4$ is such that $\|w\|_{X_2} < R_2$ with $0 < R_2 < \sqrt{3/4\pi}$, then*

$$\mathcal{K}'(w)Aw \geq C_3(R_2)\|w\|_{X_2}^2$$

where

$$C_3(R_2) := \frac{1 - 2\sqrt{\pi/3}R_2}{(1 + \sqrt{\pi/6}R_2)^3}. \quad (19)$$

Proof. The product $\mathcal{K}'(w)Aw$ is the directional derivative of the functional \mathcal{K} at w in the direction $-w''$. It is also the derivative of the length of the parametrised curve $\{c(t) = (t + \mathcal{C}w(t), w(t)), 0 \leq t \leq 2\pi\}$ in the direction $\{\delta(t) = (-\mathcal{C}w''(t), -w''(t)), 0 \leq t \leq 2\pi\}$. Using this interpretation, one easily gets the formula

$$\begin{aligned} \mathcal{K}'(w)Aw &= \int_{-\pi}^{\pi} \frac{\{(1 + \mathcal{C}w')w'' - w'\mathcal{C}w''\}^2}{\{(1 + \mathcal{C}w')^2 + w'^2\}^{3/2}} dt \\ &= \int_{-\pi}^{\pi} \frac{|w'' + \operatorname{Re}\{(\mathcal{C}w' + iw')(w'' + i\mathcal{C}w'')\}|^2}{|(1 + \mathcal{C}w' + iw')|^3} dt \\ &\geq \int_{-\pi}^{\pi} \frac{|w''|^2 - 2|\mathcal{C}w' + iw'|\operatorname{Re}\{w'' + i\mathcal{C}w''\}|w''|}{(1 + \sqrt{\pi/6}R_2)^3} dt \\ &\geq \int_{-\pi}^{\pi} \frac{|w''|^2 - 2\sqrt{\pi/6}R_2\operatorname{Re}\{w'' + i\mathcal{C}w''\}|w''|}{(1 + \sqrt{\pi/6}R_2)^3} dt \\ &\geq \frac{1 - 2\sqrt{\pi/3}R_2}{(1 + \sqrt{\pi/6}R_2)^3} \int_{-\pi}^{\pi} |w''|^2 dt \end{aligned}$$

since (15) holds. □

We now build a function φ such that (3c) is true, with $C_4 := 0$. For $\lambda \geq 0$

and $w \in U \cap X_4$, we have

$$\begin{aligned}
\mathcal{L}'(w)(-w'') &= - \int_{-\pi}^{\pi} w' \mathcal{C} w'' dt - \frac{\lambda}{\pi} \left(\int_{-\pi}^{\pi} w \mathcal{C} w' dt \right) \left(\int_{-\pi}^{\pi} w' \mathcal{C} w'' dt \right) \\
&\quad - \lambda \int_{-\pi}^{\pi} (w + w \mathcal{C} w' + \mathcal{C}(w w')) w'' dt \\
&= - \int_{-\pi}^{\pi} w' \mathcal{C} w'' dt + \lambda \int_{-\pi}^{\pi} w'^2 dt - \lambda \int_{-\pi}^{\pi} w (w'' \mathcal{C} w' - w' \mathcal{C} w'') dt \\
&\quad + \frac{\lambda}{\pi} \left(\int_{-\pi}^{\pi} w \mathcal{C} w' dt \right) \left(\int_{-\pi}^{\pi} w' \mathcal{C} w'' dt \right) \\
&\geq - \|w\|_2 \|w\|_1 + \lambda \|w\|_1^2 - 2\lambda \|w\|_{\infty} \|w\|_2 \|w\|_1 - \frac{\lambda}{\pi} \|w\|_1^2 \|w\|_2^2 \\
&\geq -\varphi(\|w\|_1, \|w\|_2)
\end{aligned}$$

where

$$\varphi(s, t) := ts - \lambda(1 - \sqrt{2\pi/3}t - t^2/\pi)s^2.$$

Let $\gamma_0 > 0$ be given. Then, in (1d), we can choose

$$\psi(s, t) := \gamma_0 \mathcal{C}_3(R_2)t^2 - \varphi(s, t).$$

If $w \in U$ is a weak solution of $\tilde{\mathcal{J}}'(w) = 0$:

$$\begin{aligned}
\frac{\lambda}{2\pi} \int_{-\pi}^{\pi} w \mathcal{C} w' dt &= - \left(1 + \lambda \int_{-\pi}^{\pi} w \mathcal{C} w' dt / \pi \right) \mathcal{C} w' + \lambda (w + w \mathcal{C} w' + \mathcal{C}(w w')) \\
&\quad + \gamma \left\{ \frac{-w'}{\sqrt{w'^2 + (1 + \mathcal{C} w')^2}} + \mathcal{C} \frac{1 + \mathcal{C} w'}{\sqrt{w'^2 + (1 + \mathcal{C} w')^2}} \right\}' + \epsilon w^{iv}
\end{aligned}$$

with $\epsilon > 0$, a standard regularity argument shows that $w \in H_{2\pi}^4$ and thus hypothesis (1e) is verified. Hypothesis (1f) is a consequence of the compact Sobolev embedding $X_2 \subset C_{2\pi}^1$.

Theorem 4. *Let $\lambda \geq 0, \gamma_0 > 0, 0 < R_2 < \sqrt{3/4\pi}$, and suppose that there exists $u_* \in U$ such that*

$$\tilde{\mathcal{J}}(u_*) < 0, \tag{20}$$

$$\varphi(s, R_2) < \gamma_0 \mathcal{C}_3 R_2^2, \text{ for all } s \in [0, \sqrt{\mathcal{K}(u_*)/C_1}]. \tag{21}$$

Then there exists $w \in U \setminus \{0\}$ such that $\tilde{\mathcal{J}}(w) \leq \tilde{\mathcal{J}}(u_)$ and (13) holds with $\beta \geq \gamma_0$. Hence (12) and (11a) with $\nu = 1$ are satisfied if w is replaced by $\tilde{w} := w - [w \mathcal{C} w']$.*

Proof. This is a consequence of Theorem 1, since its hypotheses have been verified for this example. \square

Now we must confirm the existence of u_* satisfying (20) and (21). We treat two situations.

Situation 1: $\lambda \in (0, 1)$ is fixed, and we impose $\gamma_0 > 1 - \lambda$. For u_* , we try $u_*(t) = a(\cos t + k \cos 2t)$, where $a > 0$ and $k \in \mathbb{R}$ are small. Then $\|u_*\|_{L^2_{2\pi}}^2 = \pi a^2(1 + 4k^2)$, $\|u_*''\|_{L^2_{2\pi}}^2 = \pi a^2(1 + 16k^2)$ and

$$\int_{-\pi}^{\pi} u_* \mathcal{C}u_*' dt = \pi a^2(1 + 2k^2),$$

which yields

$$\tilde{\mathcal{J}}(u_*) = -\frac{\pi}{2}a^2\{1 + 2k^2 - \lambda(1 + k^2 + 2ak)\} - \frac{\lambda}{4\pi}\{\pi a^2(1 + 2k^2)\}^2 + \gamma_0 \mathcal{K}(u_*).$$

To evaluate $\tilde{\mathcal{J}}(u_*)$, we choose $k = pa$, $\gamma_0 - 1 + \lambda = Ba^2$, where $p, B \in \mathbb{R}$ are yet to be determined, and consider only the terms of order at most 4 in a . Recall that, for $|s| < 1$,

$$\sqrt{1+s} = 1 + \frac{1}{2}s - \frac{1}{8}s^2 + \frac{1}{16}s^3 - \frac{5}{128}s^4 + \dots$$

Hence

$$\begin{aligned} & \sqrt{u_*'^2 + (1 + \mathcal{C}u_*')^2} - (1 + \mathcal{C}u_*') = \sqrt{1 + 2\mathcal{C}u_*' + (\mathcal{C}u_*')^2 + (u_*')^2} - (1 + \mathcal{C}u_*') \\ &= \frac{1}{2}(u_*')^2 - \frac{1}{2}(u_*')^2 \mathcal{C}u_*' - \frac{1}{8}(u_*')^4 + \frac{1}{2}(u_*')^2 (\mathcal{C}u_*')^2 + \dots \\ &= \frac{1}{2}a^2 \left(\frac{1 - \cos 2t}{2} + 2k^2(1 - \cos 4t) + 2k(\cos t - \cos 3t) \right) \\ & \quad - \frac{1}{2}a^3 \left(\frac{\cos t}{2} - \frac{\cos t + \cos 3t}{4} + 2k \sin^2 2t + k \sin 4t \right) \\ & \quad - \frac{1}{8}a^4 \sin^2 t + \frac{5}{32}a^4 \sin^2 2t + \dots \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{J}}(u_*) &= -\frac{\pi}{2}a^2\{1 + 2k^2 - \lambda(1 + k^2 + 2ak)\} - \frac{\lambda}{4\pi}\{\pi a^2(1 + 2k^2)\}^2 \\ & \quad + \pi\gamma_0 \left\{ \frac{a^2}{2} + 2a^2k^2 - a^3k + \frac{a^4}{32} \right\} + \dots \\ &= \pi a^4 \left\{ \frac{B}{2} + (1 - (3/2)\lambda)p^2 + (2\lambda - 1)p + (1 - 9\lambda)/32 \right\} + \dots \end{aligned}$$

Our aim is to get $\tilde{\mathcal{J}}(u_*) < 0$ for small enough $a > 0$. If $\lambda \in [2/3, 1)$ we can choose

$$p = -\frac{1-9\lambda}{32(2\lambda-1)} - 1 \text{ and } B = 2\lambda - 1 > 0.$$

On the other hand if $\lambda \in (0, 2/3)$ then we can choose for p the value at which the following minimum is attained:

$$\min_{p \in \mathbb{R}} \left\{ (1 - (3/2)\lambda)p^2 + (2\lambda - 1)p + (1 - 9\lambda)/32 \right\} = \frac{-74\lambda^2 + 86\lambda - 28}{64(2 - 3\lambda)} < 0.$$

Hence we can choose in this case

$$p = -\frac{2\lambda - 1}{2 - 3\lambda} \text{ and } B = -\frac{-74\lambda^2 + 86\lambda - 28}{64(2 - 3\lambda)} > 0.$$

We can now check the other hypotheses of Theorem 4, for $\lambda > 0$ fixed. If R_2 is fixed small enough, then $\varphi(s, R_2) \leq R_2 s$ and $C_1 \geq 1/4$, $C_2 = 1/2$, $C_3 \geq 1/2$. If $a > 0$ is small, then $\|u_*\|_2^2 = \pi a^2(1 + 16k^2) < R_2^2$. Since $\mathcal{K}(u_*) = \frac{\pi a^2}{2} + O(a^4)$ and $\gamma_0 > 1 - \lambda$, it is clear that $R_2 \sqrt{\mathcal{K}(u_*)}/C_1 < \gamma_0 C_3 R_2^2$ for a small, and (21) is satisfied.

To sum up, we have proved the following result that is a particular case of those in [10, 11, 14]:

Theorem 5. *For all $\lambda \in (0, 1)$ and all $\delta > 0$, there exist $\beta > 1 - \lambda$ and $w \in U \setminus \{0\}$ such that $0 < \|w''\|_{L^2_{2\pi}} < \delta$ and (13) holds. Hence (12) and (11a) with $\nu = 1$ are satisfied if w is replaced by $\tilde{w} := w - [wCw']$.*

By refining the method, it should be possible to obtain explicit lower bounds on the size of w for not too small δ .

Situation 2: λ is fixed, and we take $\gamma_0 < 1 - \lambda$ in order to have good lower estimates on $\|w\|_1$. For u_* , we try $u_*(t) = a \cos t$. Then $\|u_*\|_{L^2_{2\pi}}^2 = \int_{-\pi}^{\pi} u_* C u_*' dt = \|u_*''\|_{L^2_{2\pi}}^2 = \pi a^2$ and

$$\mathcal{L}(u_*) = -\frac{\pi}{2} a^2 \{1 - \lambda\} - \frac{\lambda \pi a^4}{4}.$$

Moreover,

$$\mathcal{K}(u_*) \leq \int_{-\pi}^{\pi} \frac{(u_*')^2}{2(1-a)} dt = \frac{\pi a^2}{2(1-a)}.$$

In the sequel, we restrict our discussion to the case $\lambda = 0$, which corresponds to pure capillary waves (this choice leads to simpler calculations, but of course, we could also treat the case of a nonzero λ). Then

$$-(\gamma_0 \mathcal{K}(u_*) + \mathcal{L}(u_*)) \geq \frac{\pi}{2} a^2 \left(1 - \frac{\gamma_0}{1-a}\right)$$

and

$$b := \frac{\gamma_0 \mathcal{K}(u_*) + \mathcal{L}(u_*)}{\gamma_0 C_1 - C_2} \geq \frac{\pi}{1 - \frac{\gamma_0}{1 + \sqrt{\pi/6} R_2}} a^2 \left(1 - \frac{\gamma_0}{1-a}\right).$$

We want b to be positive (and not too small) under the constraint

$$\varphi(\sqrt{\mathcal{K}(u_*)/C_1}, R_2) < \gamma_0 C_3(R_2) R_2^2.$$

This constraint is satisfied if

$$\frac{a}{\sqrt{1-a}} < \frac{\gamma_0(1 - 2\sqrt{\pi/3} R_2)}{\sqrt{\pi}(1 + \sqrt{\pi/6} R_2)^{7/2}} R_2.$$

After some numerical investigation, one can take $R_2 := 0.16$, $\gamma_0 := 0.8$, $a := 0.0325$. Then the constraint is satisfied, and $b \geq 0.002029$, so there is a non-trivial solution with the lower bound $\|w'\|_{L^2_{2\pi}} \geq \sqrt{b} \geq 0.045$.

4 Stokes waves

In (11), (12), (13), J and \tilde{J} let $\beta = 0$ and $\lambda = 1$. Having done so, $w \in H^1_{2\pi}$ in (11a), (12) and we apply the abstract result of Section 1 with $\gamma := \nu^2$,

$$X_0 = \{w \in L^2_{2\pi} : [w] = 0, w \text{ is even}\},$$

$$Aw = \mathcal{C}w',$$

$$X_k = \{w \in H^{k/2}_{2\pi} : [w] = 0, w \text{ is even}\} \text{ for } k \in \{1, 2, 4\}.$$

The radius $R_2 > 0$ will be specified later. For $w \in U$ (the ball in X_2 of radius R_2 centred at the origin) let

$$\begin{aligned} \mathcal{K}(w) &= \int_{-\pi}^{\pi} w \mathcal{C}w' dt, \\ \mathcal{L}(w) &= \frac{1}{2\pi} \left\{ \int_{-\pi}^{\pi} w \mathcal{C}w' dt \right\}^2 - \int_{-\pi}^{\pi} w^2 (1 + \mathcal{C}w') dt. \end{aligned}$$

Assumption (1a) is satisfied for $C_1 := 1$, and in (3b) we can take $C_3 := 2$, since

$$\mathcal{K}'(w)\mathcal{C}w' = 2 \int_{-\pi}^{\pi} \mathcal{C}w'\mathcal{C}w' dt = 2\|w\|_2^2.$$

The following lemma from [16] is useful for finding the constant C_2 of (1b) and the function φ of (3c).

Lemma 6. *If $w \in H_{2\pi}^1$ and if $h \in C^\infty(\mathbb{R})$ is convex on the range of w , then*

$$h'(w(t))\mathcal{C}w'(t) - \mathcal{C}(h'(w)w')(t) \geq 0$$

almost everywhere and therefore

$$\int_{-\pi}^{\pi} h'(w(t))\mathcal{C}w'(t) dt \geq 0. \quad \square$$

Therefore

$$\int_{-\pi}^{\pi} w^2\mathcal{C}w' dt = \int_{-\pi}^{\pi} w(w\mathcal{C}w' - \mathcal{C}(ww')) dt + \frac{1}{2} \int_{-\pi}^{\pi} w^2\mathcal{C}w' dt$$

which, with Lemma 6 gives

$$\begin{aligned} \left| \int_{-\pi}^{\pi} w^2\mathcal{C}w' dt \right| &\leq 2\{\sup |w|\} \int_{-\pi}^{\pi} \{w\mathcal{C}w' - \mathcal{C}(ww')\} dt \\ &= 2\{\sup |w|\} \int_{-\pi}^{\pi} w\mathcal{C}w' dt. \end{aligned}$$

Thus

$$\mathcal{L}(w) \geq -(1 + 2\{\sup |w|\})\|w\|_1^2$$

and we can choose $C_2 := 1 + \sqrt{2\pi/3}R_2$ in (1b). We also have

$$\begin{aligned} \mathcal{L}'(w)\mathcal{C}w' &= \frac{2}{\pi}\mathcal{K}(w) \int_{-\pi}^{\pi} \mathcal{C}w'\mathcal{C}w' dt - 2 \int_{-\pi}^{\pi} w\mathcal{C}w' dt \\ &\quad - 2 \int_{-\pi}^{\pi} w\mathcal{C}w'\mathcal{C}w' dt - \int_{-\pi}^{\pi} \mathcal{C}(w^2)'\mathcal{C}w' dt \\ &\geq \frac{2}{\pi}\mathcal{K}(w) \int_{-\pi}^{\pi} \mathcal{C}w'\mathcal{C}w' dt - 2\mathcal{K}(w) \\ &\quad - 2\{\sup |w|\} \int_{-\pi}^{\pi} (\mathcal{C}w')^2 + (w')^2 dt \\ &\geq -\varphi(\|w\|_1, \|w\|_2) - C_4\|w\|_2^2 \end{aligned}$$

with $\varphi(s, t) := \frac{2}{\pi}s^2t^2 - 2s$ and $C_4 := 4\sqrt{\pi/6}R_2$. This gives (3c).

Now $A^2w = -w''$ and so, if $w \in H_{2\pi}^1$ is a weak solution of

$$-\frac{1}{2\pi} \int_{-\pi}^{\pi} w \mathcal{C}w' dt = \left(\nu^2 + \int_{-\pi}^{\pi} w \mathcal{C}w' dt / \pi \right) \mathcal{C}w' - w - w \mathcal{C}w' - \mathcal{C}(ww') + \epsilon A^2w$$

with $\epsilon > 0$, a standard regularity argument shows that $w \in W_{2\pi}^{2,2}$ and thus hypothesis (1e) is verified.

To check assumption (1f), note that if $w_n \rightharpoonup w$ weakly in $L_{2\pi}^2$, then $\mathcal{C}w'_n \rightharpoonup \mathcal{C}w'$ weakly in $H_{2\pi}^1$ and $w_n \rightarrow w$ uniformly on $[-\pi, \pi]$. It follows that $\mathcal{K}(w_n) \rightarrow \mathcal{K}(w)$, $\mathcal{L}(w_n) \rightarrow \mathcal{L}(w)$.

Theorem 7. *Let $\nu > 0$ and $R_2 < \nu^2\sqrt{3/(2\pi)}$. Assume that there exists $u_* \in U$ such that*

$$\nu^2\mathcal{K}(u_*) + \mathcal{L}(u_*) < 0$$

and

$$\frac{\mathcal{K}(u_*)}{\nu^2 - \sqrt{2\pi/3}R_2 + \pi^{-1}\mathcal{K}(u_*)} < R_2^2. \quad (22)$$

Then there exists $w \in U \setminus \{0\}$ and $\tilde{\nu} \geq \nu$ such that

$$\tilde{\nu}^2\mathcal{K}'(w) + \mathcal{L}'(w) = 0 \text{ and } \nu^2\mathcal{K}(w) + \mathcal{L}(w) \leq \nu^2\mathcal{K}(u_*) + \mathcal{L}(u_*).$$

Proof. This theorem follows from Theorem 1 and the particular form of ψ . Indeed, the inequality (22) is equivalent to (3a). \square

Let $u_*(t) = a(\cos t + k \cos 2t)$, where $a, k > 0$. Then $\|u\|_{L_{2\pi}^2}^2 = \pi a^2(1 + 4k^2)$,

$$\int_{-\pi}^{\pi} u_* \mathcal{C}u'_* dt = \pi a^2(1 + 2k^2) \quad \text{and} \quad \int_{-\pi}^{\pi} u_*^2 \mathcal{C}u'_* dt = 2\pi a^3 k.$$

Moreover, setting $r = \sqrt{\pi a^2(1 + 2k^2)}$, we get

$$\nu^2\mathcal{K}(u_*) + \mathcal{L}(u_*) = \pi a^2 \left\{ \nu^2 - 1 + (2\nu^2 - 1)k^2 - \frac{2rk}{\sqrt{\pi(1 + 2k^2)}} + \frac{r^2(1 + 2k^2)}{2\pi} \right\}.$$

All hypotheses of Theorem 7 are verified if

$$r \frac{\sqrt{1 + 4k^2}}{\sqrt{1 + 2k^2}} < R_2 < \frac{\sqrt{3}\nu^2}{\sqrt{2\pi}}, \quad (23)$$

$$R_2^2 > \frac{r^2}{\nu^2 - \sqrt{2\pi/3}R_2 + r^2/\pi}, \quad (24)$$

$$\nu^2 - 1 + (2\nu^2 - 1)k^2 - \frac{2rk}{\sqrt{\pi(1+2k^2)}} + \frac{r^2(1+2k^2)}{2\pi} < 0. \quad (25)$$

Then Theorem 4 provides us with a non-trivial solution w . Take $R_2 = 0.477$, $r = 0.28$, $k = 0.142$, $\nu^2 = 1/0.99$: conditions (23) to (25) above are fulfilled and we get a solution of (11a) with $\beta = 0$ and $\lambda = 1$ in which w and ν are replaced by

$$w_* = w - \frac{1}{2\pi} \int_{-\pi}^{\pi} wCw' dt$$

and $\nu_* \geq 0.99^{-1/2}$. Alternatively, we can fix $\nu = 1$ and let λ be the parameter, in which case the corresponding λ_* is in $(0, 0.99]$. The same result has been obtained in [8] via the mountain-pass theorem, but the present proof is simpler.

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