Monotonicity and Symmetry of positive solutions to nonlinear elliptic equations : local Moving Planes and Unique Continuation

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Abstract. We prove local properties of symmetry and monotonicity for nonnegative solutions of scalar field equations with nonlinearities which are not Lipschitz. Our main tools are a local Moving Planes method and a unique continuation argument which is connected with techniques used for proving the uniqueness of radially symmetric solutions.

1 Introduction

We consider the semilinear elliptic partial differential equation $\Delta u + f(u) = 0$, $x \in D \subset I\!R^N$ and its *ground states*, which are nonnegative solutions vanishing at the boundary. This equation, often called the *nonlinear scalar field equation*, plays an important role in various domains of Physics, Chemistry and Population Dynamics, and it is fundamental from the mathematical point of view.

Regarding ground states of scalar field equations there are two strongly related natural questions: symmetry and uniqueness. Coffman in [4] proved that $\Delta u - u + u^3 = 0$, $x \in I\!R^N$ has a unique radially symmetric ground state. Later Gidas, Ni and Nirenberg in [12] proved that all ground states are actually radially symmetric when f is the sum of a Lipschitz and of an increasing function.

Several authors continued the study of uniqueness of radially symmetric ground states, enlarging the class of nonlinearities considered. We have to mention the fundamental work of Peletier and Serrin [19] and the paper by Kwong [15] (see [6] for more complete references). More recently Franchi, Lanconelli and Serrin in [10] obtained uniqueness results for radially symmetric ground states in a rather surprising situation. In this paper, they assume that the nonlinearity f is locally Lipschitz continuous in (β, ∞) and merely continuous in $[0, \beta]$, where β is such that the primitive F of f such that $F(0) = 0$ is negative in the interval $(0, \beta)$ (growth and convexity assumptions of a sublinear nature were also considered). Later, Cortázar, Elgueta and Felmer in [7] studied the uniqueness problem for radially symmetric solutions in a case where the nonlinearity f has a superlinear nature, is locally Lipschitz continuous in $[\alpha, \infty)$ and merely continuous in $[0, \alpha]$, and has two zeroes, 0 and $\alpha > 0$.

In view of the uniqueness results in [10, 7] it is interesting to know if under this kind of hypotheses a symmetry result also holds for general ground states. Positive answers in this direction will give rise to actual uniqueness theorems for ground states.

The purpose of this paper is to analyze the symmetry and the related monotonicity question for nonlinearities that are continuous but not Lipschitz continuous in all their domain. The basic tools in the proof of our theorems are *a local* variant of the *Moving Planes Method* and the *Unique Continuation Principle*. These tools have a strong connection with the tools used for proving the uniqueness of positive radially symmetric solutions.

In proving symmetry results our strategy consists, as a first step, in finding what we call a γ-*core*, that is a subset of D on which the function u is symmetric with respect to a hyperplane orthogonal to the direction γ. Then by chosing appropriate directions γ in a second step we find a (radially symmetric) *core*, (*i.e.* a γ-core for any direction γ). Finally we show under some further regularity assumption on f that if D is a ball, the function u is actually radially symmetric, or has monotonicity properties in the other cases. This last step is performed with the aid of the Unique Continuation Principle in connection with a trick already used in [10] in studying the uniqueness of radially symmetric solutions. It introduces a new ingredient in the proof of symmetry theorems, that we believe could be of importance in obtaining further results: see [8] for the 2-dimensional case. Note that the Unique Continuation Principle had been already used in the study of symmetry by Lopes in [18], however its use was different in spirit.

More precisely we shall consider the following nonlinear elliptic problem

$$\Delta u + f(u) = 0, \quad u > 0 \qquad \text{in } D, \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial D, \tag{1.2}$$

assuming that f satisfies the assumptions:

(f1) $f : I\!R^+ \to I\!R$ is continuous,

(f2) For any $s \in [0, +\infty[$, there exists a constant $\eta > 0$ such that on $]s - \eta, s+\eta[\cap I\!R^+$ f is either strictly decreasing or it is the sum of a Lipschitz and of a nondecreasing function (in the latter case, we shall say that f is *Lipschitz + increasing* in a neighborhood of s in $I\!R^+$),

and that D is a domain in $I\!R^N$ with one of the two following properties (e_1 is given and we denote by x_1 the coordinate along this direction):

(B) Bounded case: D is x_1-convex and symmetric with respect to the hyperplane $T^{e_1} = \{x \in I\!R^N \ : \ x \cdot e_1 = 0\}$ and has the property

$$\forall\, \epsilon > 0 \quad \exists\, \eta > 0 \quad \text{such that} \qquad \forall\, \lambda > \epsilon$$

$$\nu \in S^{N-1},\ |\nu - e_1| < \eta \Longrightarrow \{x + 2(\lambda - x\cdot\nu)\nu \ : \ x \in D,\ x\cdot\nu > \lambda\} \subset D\ ,$$

(C) Case with "Cone property": there exists $\eta > 0$ such that for any $\lambda \in I\!R$, $\nu \in S^{N-1}$ such that $|\nu - e_1| < \eta$, the set $\{x \in D \ \ x\cdot\nu > \lambda\}$ is bounded and

$$D = \cup_{\substack{\nu \in S^{N-1} \\ |\nu - e_1| < \eta}} \{y - t\nu \ : \ y \in \partial D,\ t > 0\}\ .$$

Assumption (B) means that D is symmetric with respect to T^{e_1} and that for directions ν close to e_1, the image of the reflection by the hyperplane $\{x \in I\!R^N \ : \ x\cdot\nu = \lambda\}$ of $\{x \in D \ : \ x\cdot\nu > \lambda\}$ is contained in D provided $\lambda > \epsilon$, while assumption (C) essentially means that ∂D is the graph of a uniformly Lipschitz function of $x' = (x_2, x_3, ... \ x_N)$ which goes to $-\infty$ as $|x'| \to +\infty$.

In order to describe our results we need to introduce the following notion of local monotonicity.

Definition. A nonnegative function u is said to be *monotone up to cores* on $\tilde{D}$ in the direction e_1, where $\tilde{D} \subset D$ is a bounded subdomain, if there are nonnegative functions $\tilde{u}, u_1, ..., u_k$ defined on $\tilde{D}$ such that:

– $u|_{\tilde{D}} = \tilde{u} + \sum_{j=1}^{k} u_j$,

– the functions u_j have support in balls B_j intersecting $\tilde{D}$ and they are radially symmetric non increasing, with respect to the center of B_j, $j = 1, ...k$.

– if $B_i \cap B_j \neq \emptyset$, $i \neq j$, then either $B_i \subset B_j$ and u_j is constant on B_i or $B_j \subset B_i$ and u_i is constant on B_j

– $\tilde{u}$ is monotone non increasing on $\tilde{D}$ in the direction e_1, and it is constant on any B_j, $j = 1, ...k$.

Remark 1.1 See the beginning of Section 2 for a precise definition the notion of γ-*core* and (radially symmetric) *core*. See also Proposition 4.1 for an explanation why $u_j \geq 0$.

Now we can state our main result.

Theorem 1.1 *Assume that f satisfies (f1), (f2) and D satisfies (B) (resp. (C)). Let $u \in C^2(D) \cap C^0(\bar{D})$ be a positive solution of (1.1)-(1.2). Then u is monotone up to cores on $\tilde{D} = \{x \in D \ : \ x \cdot e_1 \geq 0\}$ in the direction e_1 (resp. on $\tilde{D} = D$). Moreover in case (B), with the above notations, $\tilde{u}$ is symmetric with respect to T^{e_1}.*

Actually, the result is a little bit stronger, and the monotonicity up to cores is true in any of the entering directions in case (C), or in any of the directions ν such that $|\nu - e_1| < \eta$, on the domain $\{x \in D \ : \ x \cdot \nu \geq \epsilon\}$ in case (B).

Under the following additional assumption

(f3) For any $u > 0$ such that $f(u) = 0$, $\liminf_{v \to u,\, v > u} \frac{f(v)}{v-u} > -\infty$,

we obtain monotonicity and symmetry results:

Theorem 1.2 *Assume that f satisfies (f1), (f2), (f3) and D satisfies either (B) or (C). Let $u \in C^2(D) \cap C^0(\bar{D})$ be a positive solution of (1.1)-(1.2). Then u is monotone strictly decreasing in any direction ν given by Conditions (B) or (C), on $\tilde{D}$ defined as in Theorem 1.1. In case (B), u is symmetric with respect to T^{e_1}.*

In the case of a ball, we recover a result proved by F. Brock [3] by the mean of the continuous Steiner symmetrization method: u is radially symmetric and $\frac{du}{dr}(r) < 0$ for any $r \in (0,1)$.

The use of the *Moving Planes Method* for proving symmetry results goes back to Alexandroff in the study of manifolds with constant mean curvature [1]. It was further developed by Serrin in [20] and by Gidas, Ni and Nirenberg in [11, 12] to study the symmetry of positive solutions of elliptic partial differential equations. In both of the above mentioned works the nonlinearity was assumed to be locally Lipschitz in $[0, +\infty)$ or at least the sum of a Lipschitz function and of a nondecreasing function. Several generalizations have been obtained since that fundamental work (see [17] for more complete references), in particular we mention that of Cortázar, Elgueta and Felmer [6], where some symmetry results are presented in case the nonlinearity looses the Lipschitz property at zero (see also [14]). In [6] this difficulty was overcome by assuming that f is decreasing in a neighborhood of $u = 0$. Our results are a generalization of Theorem 1.3 in [6]. Note that an independent work by Li and Ni [17] also introduces the decay condition on f near zero to overcome the lack of control on the decay of solutions when $f'(0) = 0$.

During the completion of this paper we learned the results obtained by F. Brock in [2, 3], where local symmetry results are proved only under the assumption of continuity for f, using a continuous Steiner symmetrization method. In the cases where Brock's result applies, our method is weaker. We think however that an extension of the Moving Planes Method to a general non-Lipschitz case is interesting in itself because it gives a local description of the mechanisms involved in monotonicity (symmetry) theorems. Moreover the Moving Planes method is constructive and robust. Moving Planes give monotonicity also in directions close to the direction of symmetry (this allows us to prove that when monotonicity holds up to cores, these cores are radially symmetric). The method provides monotonicity results in unbounded domains as well and can handle some nonlinear elliptic equations not in divergence form.

The paper is organized as follows. In Section 2 we introduce some definitions, develop a framework for a local Moving Planes Method and prove a crucial lemma that allows us to build the cores. In Section 3 we present the main proofs and give monotonicity and symmetry results using the Unique Continuation Principle. Section 4 is devoted to some extensions (weaker conditions on f, whole space results, fully nonlinear case). Some of the results of this paper were announced in [9].

2 A technical lemma

In this section we start setting up the basic notation and providing some definitions. Then we state and prove a lemma on how to build a *core* (see below the definition).

Consider a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of

$$\begin{aligned} &\Delta u + f(u) = 0\ , \quad u \geq 0 \quad \text{in } \Omega \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $I\!R^N$. In the following u is fixed and we define various types of sets according to their geometrical properties and to the behavior of u on them.

Definition: Given $\gamma \in S^{N-1}$, Ω is said to be a $\gamma -$ *core* of u if and only if

(i) There exists a real number λ_Ω such that Ω *and* u *are symmetric* with respect to the hyperplane $T_{\lambda_\Omega} = \{x \in I\!R^N \ : \ x \cdot \gamma = \lambda_\Omega\}$, that is, for any $x \in \Omega$ we have $x_{\lambda_\Omega} = x - 2(x \cdot \gamma - \lambda_\Omega)\gamma \in \Omega$ and $u(x_{\lambda_\Omega}) = u(x)$,

(ii) Ω is *convex in the* $\gamma -$ *direction* (or $\gamma -$ *convex*), that is, for any $x \in \Omega$, the set $\{t \in I\!R \ : \ (x + t\gamma) \in \Omega\}$ is an interval,

(iii) $\nabla u(x) \cdot \gamma \leq 0$ for any $x \in \Omega$ such that $x \cdot \gamma > \lambda_\Omega$.

Ω is said to be a (radially symmetric) *core* of u if it is a γ-core of u for any direction $\gamma \in S^{N-1}$.

We may observe that a core of u is a ball on which u is radially symmetric with respect to the center of the ball and non increasing along any radius.

Remark 2.1 In order to prove that a given set Ω is a core, it is sufficient (when u is continuous) to prove that it is a γ_i-core for N independent directions $\gamma_i \in S^{N-1}$, $i = 1, 2, \dots N$ such that the angle (γ_i, γ_j) is 2π-irrational for any (i, j) with $i \neq j$.

In dimension $N = 2$, if two lines make a 2π-irrational angle θ_0, then the composition of two orthogonal reflections with respect to each of these two lines gives a rotation of angle $\pm 2\theta_0$ which is 2π-irrational too. The set $\{n\theta_0\}_{n\in\mathbb{Z}}$ is dense in S^1 so that Ω is a disk and u is radially symmetric.

In dimension $N \geq 3$, let $x^0 \in \Omega$: $x^0 = \sum_{i=1}^N x_i^0 \gamma_i$, since the $\{\gamma_i\}_{i=1,2,\dots N}$ are linearly independent. Here we take the origin to be the unique point in the intersection of the hyperplanes T_{λ_i} associated to the directions γ_i. Next we consider the affine plane $\Pi_{1,N}(x^0) = \{x \in \mathbb{R}^N \ : \ x = x^0 + y, \ y \in \text{span}(\gamma_1, \gamma_N)\}$. By using the 2-dimensional argument given above we see that we can rotate x^0 in $\Pi_{1,N}(x^0)$ to obtain $x^1 = \sum_{i=1}^{N-1} x_i^1 \gamma_i$. Of course $|x^0| = |x^1|$. We can repeat this argument $N-1$ times until getting $x^{N-1} = x_1^{N-1}\gamma_1$, where $x_1^{N-1} = |x^0|$. Thus $u(x^0) = u(|x^0|\gamma_1)$, proving in this way that u is radially symmetric. Similarly, we conclude that Ω is a ball.

Now we set up some notational conventions. Whenever possible, given $\gamma \in S^{N-1}$ we will choose a system of coordinates so that $\gamma = e_1$. In that case we write x_1-*core* for a γ-core. Following the usual notation we consider $T_\lambda = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \ : \ x_1 = \lambda\}$ and $\Sigma_\lambda = \{x = (x_1, x') \in (\mathbb{R} \times \mathbb{R}^{N-1}) \ : \ x_1 > \lambda\}$. If $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$, then we write $x_\lambda = (2\lambda - x_1, x')$ and $u_\lambda(x) = u(x_\lambda)$ for any $x \in \mathbb{R}^N$ such that $x_\lambda \in \Omega$.

Definition: Let Ω be a non empty bounded domain in $\mathbb{R}^N$. We say that Ω satisfies *Property* $\mathcal{P}$ if and only if the following conditions are satisfied:

(i) Ω is symmetric with respect to the hyperplane $T = \{x = (x_1, x') \in$

$I\!R \times I\!R^{N-1} \ : \ x_1 = \lambda_\Omega\}$ for some $\lambda_\Omega \in I\!R$, that is, $(2\lambda_\Omega - x_1, x') \in \Omega$ for any $x = (x_1, x') \in I\!R \times I\!R^{N-1} \cap \Omega$,

(ii) Ω is x_1-convex,

(iii) There is a constant u_0 such that $u|_{\partial\Omega} \equiv u_0$ and $u > u_0$ on Ω,

(iv) There exists an $\epsilon > 0$ such that f is strictly decreasing on $[u_0, u_0 + \epsilon[$.

Remark 2.2 If Ω satisfies Property $\mathcal{P}$ then it is an x_1-core of u if and only if
1) $u_{\lambda_\Omega}(x) = u(x)$ for any $x \in \Omega$,
2) $\frac{\partial u}{\partial x_1}(x) \leq 0$ for any $x \in \overline{\Sigma}_{\lambda_\Omega} \cap \Omega = \{(x_1, x') \in I\!R \times I\!R^{N-1} \cap \Omega \ : \ x_1 \geq \lambda_\Omega\}$ if $\lambda_\Omega \geq 0$ (resp. for any λ_Ω) in case (B) (resp. in case (C)).

Lemma 2.1 *Under assumptions (f1) and (f2), if Ω is a non-empty subdomain of D satisfying Property $\mathcal{P}$, then there exists a $\overline{\lambda} \in I\!R$ and $\overline{x} \in \overline{\Sigma}_{\overline{\lambda}} \cap \Omega$ such that*

$$u_{\overline{\lambda}}(\overline{x}) = u(\overline{x}) \quad \textit{if} \quad \overline{x} \in \Sigma_{\overline{\lambda}}\,, \tag{2.1}$$

$$\frac{\partial u}{\partial x_1}(\overline{x}) = 0 \quad \textit{if} \quad \overline{x} \in T_{\overline{\lambda}}\,, \tag{2.2}$$

and

(i) one of the two following cases has to be fulfilled:
Case a: $\overline{\lambda} = \lambda_\Omega$ and $u_{\overline{\lambda}}(x) = u(x)$ for any $x \in \Omega$,
Case b: there exist $u_1 > u_0$ and $u_2 > u_1$, with $u(\overline{x}) \in]u_1, u_2[$, such that f is locally Lipschitz + increasing on $]u_1, u_2[$ (we assume in the following that $]u_1, u_2[$ is the maximal interval containing $u(\overline{x})$ in $]u_0, +\infty[$ and on which f is locally Lipschitz + increasing),

(ii) $\frac{\partial u}{\partial x_1} \leq 0$ on $\Omega \cap \Sigma_{\overline{\lambda}}$,

(iii) In Case b, let $\mathcal{C}$ be the connected component of $\{x \in \Omega \ : \ u_1 < u(x) < u_2\}$ containing $\overline{x}$, then we have $u_{\overline{\lambda}}(x) = u(x)$ for any $x \in \mathcal{C}$,

(iv) In Case b, let $\tilde{\mathcal{C}}$ be the x_1-convexified of $\mathcal{C}$, i.e. the set

$$\tilde{\mathcal{C}} = \{x \in \Omega \quad : \exists (y,z) \in \mathcal{C} \times \mathcal{C} \textit{ such that } z-y \textit{ is parallel to } x_1 \\ \textit{and } \exists t \in]0,1[\textit{ such that } x = ty + (1-t)z\} \ ,$$

and $\tilde{\Omega} = \{x \in \tilde{\mathcal{C}} \ : \ u(x) > u_2\}$. Then either $\tilde{\Omega} = \emptyset$ or $\tilde{\Omega}$ satisfies Property $\mathcal{P}$.

Remark 2.3 In Case a, the set Ω is an x_1-core of u. In Case b, if $\tilde{\Omega} = \emptyset$, then $\mathcal{C} = \tilde{\mathcal{C}}$ is an x_1-core of u. And if $\tilde{\Omega} \neq \emptyset$, let $\hat{\Omega} = \{x \in \Omega \ : \ u(x) > \tilde{u}_0\} \supset \tilde{\Omega}$, where

$$\tilde{u}_0 = \inf\{u \in [u_1, u_2] \ : \ f \text{ is strictly decreasing on } [u, u_2]\} \ .$$

If $\overline{u} = \max_{x \in \hat{\Omega}} u(x)$ is such that f is strictly decreasing on $[\tilde{u}_0, \overline{u}]$, then $\hat{\Omega}$ is an x_1-core of u. The proof is easy: on $\partial\hat{\Omega} \subset \mathcal{C}$, $u_{\overline{\lambda}} \equiv u$, $u_{\overline{\lambda}} \geq u$ in $\hat{\Omega}$ according to (ii) in Lemma 2.1 and since f is decreasing, $-\Delta(u_{\overline{\lambda}} - u) \leq 0$, which means $u_{\overline{\lambda}} \leq u$.

Proof of Lemma 2.1: The proof relies on the moving plane technique introduced by Alexandrov [1]. We will say that Ω satisfies the property Π_λ if

$$w_\lambda(x) = u_\lambda(x) - u(x) \geq 0 \quad \forall \, x \in \Omega \cap \Sigma_\lambda \ .$$

Let $\lambda^* = \sup\{x_1 \in I\!R \ : \ \exists x' \in I\!R^{N-1} \text{ such that } (x_1, x') \in \Omega \cap \Sigma_\lambda\}$ and $\overline{\lambda} = \inf\{\lambda \in [\lambda_\Omega, \lambda^*] \ : \ \forall \ \mu \in]\lambda, \lambda^*[\ \Pi_\mu \text{ is true }\}$. We will first see that $\overline{\lambda} < \lambda^*$. Assume indeed by contradiction that $\overline{\lambda} = \lambda^*$. Then there exists an increasing sequence $(\lambda_k)_{k \in I\!N}$ converging to λ^* such that

$$\forall \, k \in I\!N \quad \exists \, x_k \in \Sigma_{\lambda_k} \cap \Omega \quad w_{\lambda_k}(x_k) < 0 \ .$$

On $\partial(\Sigma_{\lambda_k} \cap \Omega)$, $w_{\lambda_k} \geq 0$, so that w_{λ_k} reaches its minimum value in a point of $\Sigma_{\lambda_k} \cap \Omega$ and we may assume that x_k realizes this minimum. Then we have

$$0 \geq -\Delta w_{\lambda_k}(x_k) = f(u_{\lambda_k}(x_k)) - f(u(x_k)) = f(u(x_k) + w_{\lambda_k}(x_k)) - f(u(x_k)) > 0,$$

for k large enough, since $u_0 < u(x_k) < u_0 + \epsilon$, a contradiction. Thus $\overline{\lambda} < \lambda^*$.

Assume now that $\overline{\lambda} > \lambda_\Omega$, where λ_Ω is defined in part (i) of Property $\mathcal{P}$. We recall that Ω is symmetric with respect to T_{λ_Ω}. We may again find an increasing sequence $(\lambda_k)_{k\in I\!N}$ converging to $\overline{\lambda}$ (with $\lambda_\Omega < \lambda_k < \overline{\lambda}$), and a sequence of points $(x_k)_{k\in I\!N}$ such that

$$x_k \in \Sigma_{\lambda_k} \cap \Omega\,, \quad w_{\lambda_k}(x_k) = \min_{x\in\Sigma_{\lambda_k}\cap\Omega} w_{\lambda_k}(x) < 0.$$

Here again, $x_k \notin \Omega_0 = \{x \in \Omega\ :\ f$ is strictly decreasing in a neighborhood of $u(x)$ in $[u_0, +\infty[\}$ because $-\Delta w_{\lambda_k}(x_k) \le 0$. Up to the extraction of a subsequence, we may assume

$$\lim_{k\to+\infty} x_k = \overline{x} \in \overline{\Omega \cap \Omega_0^c} \cap \overline{\Sigma}_{\overline{\lambda}}\,.$$

On one hand we have, $0 \ge u_{\overline{\lambda}}(\overline{x}) - u(\overline{x}) = \lim_{k\to+\infty} w_{\lambda_k}(x_k)$, and on the other hand $u_{\overline{\lambda}}(\overline{x}) \ge u(\overline{x})$ because of $\Pi_{\overline{\lambda}}$ ($\Pi_{\overline{\lambda}}$ is true since Π_λ is true for any $\lambda - \overline{\lambda} > 0$, small enough). Note that either $\overline{x} \in \Sigma_{\overline{\lambda}}$, or $\overline{x} \in T_{\overline{\lambda}}$. In the latter case, $\frac{\partial u}{\partial x_1}(\overline{x}) = -\frac{1}{2}\lim_{k\to+\infty}\frac{\partial w_{\lambda_k}}{\partial x_1}(x_k) = 0$.

Since f is strictly decreasing in $[u_0, u_0 + \epsilon[$ for some $\epsilon > 0$ (and since $u|_{\partial\Omega} = u_0$), again $\overline{x}$ cannot belong to $\partial\Omega$. Then $\overline{x} \in \Omega\backslash\Omega_0$ and so f is Lipschitz + increasing in a neighborhood of $u(\overline{x})$, which proves assertion (i) in Case b.

If $\overline{\lambda} = \lambda_\Omega$, we may exchange the direction x_1 and $-x_1$. We observe that Property $\mathcal{P}$ is invariant under the transformation $(x_1, x') \mapsto (2\lambda_\Omega - x_1, x')$. Then either we find a $\overline{\lambda} \ne \lambda_\Omega$ and we are back to the previous case, or we get $\overline{\lambda} = \lambda_\Omega$, which proves that $u_{\lambda_\Omega}(x) = u(x)$ for any $x \in \Omega$. This proves assertion (i) in Case a. At this point we have also completed (2.1) and (2.2).

Because of the definition of $\overline{\lambda}$, $\Pi_{\overline{\lambda}}$ is true and assertion (ii) follows. In fact, for any $\lambda \in [\overline{\lambda}, \lambda^*[$, for any $x = (\lambda, x') \in (T_\lambda \cap \Omega) \subset (\Sigma_{\overline{\lambda}} \cap \Omega)$,

$$0 \le n \cdot w_\lambda(\lambda + \frac{1}{n}, x') \to -2\frac{\partial u}{\partial x_1}(\lambda, x') \quad \text{as } n \to +\infty\,.$$

Assertion (iii) is obtained, in case $\overline{x} \in \Sigma_{\overline{\lambda}}$ as a consequence of the Maximum Principle applied to $w_{\overline{\lambda}}$. Note that $w_{\overline{\lambda}} \geq 0$ and $w_{\overline{\lambda}}(\overline{x}) = 0$ with $\overline{x} \in \mathcal{C}$. When $\overline{x} \in T_{\overline{\lambda}}$, assertion (iii) is a consequence of Hopf's Lemma, since in this case, $\frac{\partial w_{\overline{\lambda}}}{\partial x_1}(\overline{x}) = -2\frac{\partial u}{\partial x_1}(\overline{x}) = 0$.

To finish with the proof of Lemma 2.1, one has to check that in Case b $\tilde{\Omega}$ satisfies Property $\mathcal{P}$ if it is not empty. The symmetry and x_1-convexity follows from the definition of $\tilde{\Omega}$, and f is decreasing on a neighborhood of u_2 in $[u_2, +\infty[$ again because of assumption (f2). □

3 Proofs

First, let us give a precised version of Theorem 1.1 with a weaker assumption: we do not assume the strict positivity of u any more.

Theorem 3.1 *Assume that f satisfies (f1) and (f2). Let $u \in C^2(D) \cap C^0(\bar{D})$ be a solution of*

$$\Delta u + f(u) = 0, \; u \geq 0 \qquad \text{in } D\,, \tag{3.1}$$

$$u = 0 \qquad \text{on } \partial D\,. \tag{3.2}$$

If condition (B) is satisfied, there exists a finite number $\mathcal{N}$ of balls B_i ($i = 1, 2, \ldots \mathcal{N}$) contained in D such that there exists at least one $i_0 \in \{1, 2, \ldots \mathcal{N}\}$ satisfying

(i) for any $j = 1, 2, \ldots, \; i_0 - 1, i_0 + 1, \ldots \mathcal{N}$, if $B_j \cap B_{i_0} \neq \emptyset$, then $B_{i_0} \subset B_j$

(ii) $u|_{B_{i_0}}$ is radially symmetric and decreasing along any radius of B_{i_0},

(iii) if $\mathcal{N} > 1$, the C^2 function defined on D by

$$\tilde{u} = u \quad \text{in } D \backslash B_{i_0}$$
$$\tilde{u} = u|_{\partial B_{i_0}} = Const \quad \text{on } \partial B_{i_0}$$

is still a solution of equations (3.1)-(3.2) (and we can then iterate and apply again the above result to $\tilde{u}$ with the set of $\mathcal{N} - 1$ balls B_i, $i = 1, 2, 3, \ldots, \; i_0 - 1, i_0 + 1, \ldots \mathcal{N}$).

In case of Assumption (C), the same result is true except that $\mathcal{N}$ might be infinite. However, for any $\lambda \in I\!R$, $\mathcal{I}(\lambda) = \{j \in \mathcal{N} \ : \ B_j \cap \Sigma_\lambda \neq \emptyset\} = \{j_k(\lambda) \ : \ k = 1, 2, \ldots \mathcal{N}(\lambda)\}$ is finite and the same result as above applies to $u|_{D(\lambda)}$ where $D(\lambda) = (D \cap \Sigma_\lambda) \cup (\cup_{k=1}^{\mathcal{N}(\lambda)} B_{j_k(\lambda)})$.

On $D \setminus (\cup_{k=1}^{\mathcal{N}(\lambda)} B_{j_k(\lambda)} \cap \Sigma_0)$ in case (B), $D(\lambda) \setminus (\cup_{k=1}^{\mathcal{N}(\lambda)} B_{j_k(\lambda)} \cap \Sigma_\lambda)$ in case (C), u is monotone non increasing.

Remark 3.1 When D is a ball, the solution in Theorem 3.1 is locally radially symmetric, with a finite number of cores: there exists a finite partition in balls and annuli on which the solution is radially symmetric and (strictly) decreasing, and complementary domains on which the solution is constant. Because of Condition (f2), the number of possible cores is finite (Step 2 of the proof of Theorem 3.1).

If $u > 0$ is a solution of equations (3.1)-(3.2), which is not radially symmetric, there exists therefore a $\rho \in]0, 1[$ such that u is radially symmetric in the annulus $\{x \in I\!R^N \ : \ \rho < |x| < 1\}$ and $0 = \frac{d^2u}{dr^2}(\rho) = \frac{du}{dr}(\rho) = f(u(\rho))$. We can now notice that if Assumption (f3) is replaced by

(f3') For any $u > 0$ such that $f(u) = 0$, $\liminf_{v \to u,\, v < u} \frac{f(v)}{v-u} > -\infty$,

there exists a constant $C > 0$ such that for $u(\rho) - v > 0$ small enough, $\frac{f(v)}{u(\rho)-v} > -C$. Then Hopf's Lemma applied to $-\Delta(u(\rho)-u) - C(u(\rho)-u) < 0$ in $B((\rho+\epsilon)\frac{\overline{x}}{|\overline{x}|}, \epsilon)$ at $\overline{x}$, for any $\overline{x} \in B(0,1)$ such that $|\overline{x}| = \rho$ and $\epsilon > 0$ small enough, is in contradiction with $\nabla u(\overline{x}) = \frac{\overline{x}}{|\overline{x}|} \cdot \frac{du}{dr}(\rho) = 0$.

In the case of a ball, either (f3) or (f3') are therefore sufficient to prove the radial symmetry. This is probably true for more general domains although difficult to prove in full generality (see [8] for a proof in dimension $N = 2$).

Proof of Theorem 3.1: We first build an x_1-core, and then a radially symmetric core, which can be removed. By iteration and since the possible number of cores is finite in case (B), locally finite in case (C), we prove the theorem.

First step : *Building a x_1-core*

Let u be a solution of Equation (1.1) and define $\eta(u) = \sup\{\eta \geq 0 \ : \ u$ is either strictly decreasing or Lipschitz + increasing on $]u-\eta, u+\eta[\}$, $\overline{\eta} = \inf\{\eta(v) \ : \ v \in [0, \max_{x\in D} u(x)]\}$. If f satisfies (f2), then $\overline{\eta} > 0$.

We now want to apply Lemma 2.1: if f is strictly decreasing on $[0,\epsilon[$ for some $\epsilon > 0$ and if D is bounded (assumption (B)), we may take $\Omega = D$. If not, let us prove that $\Omega = \{x \in D \cap \Sigma_{\overline{\lambda}} \ : \ u(x) > u_*\}$, where $u_* = \inf\{u > 0 \ : \ f$ is not "Lipschitz + increasing" on $]u, u+\epsilon[$ for any $\epsilon > 0$ $\}$, satisfies property $\mathcal{P}$ of Lemma 2.1 for some well chosen $\overline{\lambda} > 0$.

Of course, if $u_* > \max_D u(x)$, the usual methods apply and the conclusions of Theorem 3.1 hold. The theorem is proved if $0 = \overline{\lambda} := \inf\{\lambda > 0 \ : \ u_\lambda \geq u$ in $D \cap \Sigma_\lambda\}$ in case of assumption (B) or if $\overline{\lambda} = -\infty$ in case (C). Assume then that $\overline{\lambda} > 0$ in case of assumption (B), $\overline{\lambda} > -\infty$ in case (C): there exists a sequence $(\lambda_k)_{k\in I\!N}$ with $\lambda_k < \overline{\lambda}$, $\lim_{k\to+\infty} \lambda_k = \overline{\lambda}$, and a sequence $(x_k)_{k\in I\!N}$ such that

$$u_{\lambda_k}(x_k) - u(x_k) = \min_{x\in\Sigma_{\lambda_k}\cap D} \Big(u_{\lambda_k}(x) - u(x)\Big) < 0 \ .$$

If $\overline{x} = \lim_{k\to+\infty} x_k$, then f has to be Lipschitz + increasing on $]u(\overline{x})-\overline{\eta}, u(\overline{x})+\overline{\eta}[$ for the same reason as in Lemma 2.1.

Exactly as in Lemma 2.1, $u_{\overline{\lambda}} \equiv u$ on the connected component of $\{x \in D \ : \ u_1 < u(x) < u_2\}$ where $]u_1, u_2[$ is the maximal interval on which u is Lipschitz + increasing and such that $u(\overline{x}) \in]u_1, u_2[$. $u_{\overline{\lambda}} \equiv u$ on D is impossible since $\overline{\lambda} > 0$ and $u > 0$ in D: $u_1 > 0$, and the set $\Omega \neq \emptyset$ satisfies property $\mathcal{P}$ because $\frac{\partial u}{\partial x_1} \leq 0$ on Ω.

If f is strictly decreasing on $[0,\epsilon[$ (and not Lipschitz + increasing on this interval) and if D satisfies condition (C), a similar argument allows us to extract a bounded domain having Property $\mathcal{P}$, unless u is monotone in the x_1-direction.

We may now apply Lemma 2.1 to $\Omega = \Omega_1$ and iterate n times to find a x_1-core, n being at most the integer part of $(\overline{\eta})^{-1} \cdot \max_{x\in D\cap\Sigma_{\overline{\lambda}}} u(x)$. Here

$\Sigma_{\overline{\lambda}}$ is the domain corresponding to the $\overline{\lambda}$ obtained at the first iteration. In the following, with the notations of Lemma 2.1, we note $\Omega_{k+1} = \tilde{\Omega}_k$ for $1 \leq k \leq n$.

Second step : *Building a radially symmetric core*

If Ω_n is the non-empty x_1-core given above, we may notice that u is constant on $\partial\Omega_n$ and strictly bigger than $u_{|\partial\Omega_n}$ in Ω_n: u reaches its maximum in Ω_n at some interior point $\overline{x}$. According to assumption (f2), again two cases are possible:

1) either there exists some $u \in]u_{|\partial\Omega_n}, u(\overline{x})]$ such that f is Lipschitz + increasing on $]u - \overline{\eta}, u + \overline{\eta}[$,

2) or not.

In case the first case, by construction of Ω_n, $u_{|\partial\Omega_n} < u - \overline{\eta} \leq u(\overline{x})$. In the second case, we may use the set $\hat{\Omega}_n$ defined as in Remark 2.3:

$$u(\overline{x}) > u_{|\partial\Omega_n} \geq u_{|\partial\hat{\Omega}_n} + \overline{\eta} \ .$$

In both cases, the method shows the existence of a x_1-core ω such that u reaches its maximum at some interior point $\overline{x}$ and

$$u(\overline{x}) = \max_{x \in \omega} u(x) > u_{|\partial\omega} + \overline{\eta} \ . \tag{3.3}$$

Let $M = \|\nabla u\|_{L^\infty(D)}$. Then $B(\overline{x}, \overline{r}) \subset \omega$ with $\overline{r} = \frac{\overline{\eta}}{M}$.

The number $\mathcal{N}$ of the connected components which are x_1-cores with Property (3.3) is finite and bounded by $\mathcal{N} \leq C\left(\frac{M}{\overline{\eta}}\right)^N$ for some constant C which depends on the volume of $D \cap \Sigma_{\overline{\lambda}}$ (where $\overline{\lambda}$ was defined in the first step of the proof).

Let us take $(N-1)\mathcal{N} + 1$ directions $\gamma_i \in S^{N-1}$, $i = 1, 2, \ldots\ (N-1)\mathcal{N} + 1$ satisfying the conditions defined by Assumptions (B) or (C), such that the angle (γ_i, γ_j) is 2π-irrational for any (i, j) with $i \neq j$, and such that any family of N such unit vectors generates $I\!R^N$. Then, applying the method of the first two steps successively to each of these directions (for each i, choose the direction x_1 as the one of γ_i), we find at least one core ω which is symmetric with respect to at least N directions. According to Remark 2.1, ω is a radially symmetric core (it is a ball) and $(x-\overline{x})\cdot\nabla u(x)\leq 0$ for any $x\in\omega$, where $\overline{x}$ is the center of ω.

Note that in case (B), either the maximum of the core obtained by iterating Lemma 2.1 belongs to the hyperplane T^{e_1} and the symmetry result inside the core follows, or not. In this last case, we may choose a direction ν close enough to e_1 so that it is also a ν-core which is reached by moving hyperplanes along the direction ν. $\square$

Remark 3.2 The assumption $u>0$ in Theorem 1.1 has been replaced by the weaker assumption $u\geq 0$ in Theorem 3.1. In this case any connected component of the support which is strictly included in D is a ball on which u is radially symmetric and decreasing up to cores. Note that this is possible only if f is not Lipschitz + increasing on a neighborhood of $u=0+$.

With this remark, the proof of Theorem 1.1 is straightforward.

Proof of Theorem 1.2: For simplicity, we assume that there is a (radially symmetric) core which reaches its maximum at $\overline{x}=0$ (this is easily achieved by the mean of a translation). We have to prove that the solution is radially symmetric under Assumption (f3). Our main tool will be a *Unique Continuation argument*. Let us define

$$\rho=\max\{r>0\ :\ B(0,r)\subset D \text{ and } u \text{ is radially symmetric in } B(0,r)\}.$$

If $\partial D\cap\partial B(0,\rho)\neq\emptyset$, then $B(0,\rho)=D$, otherwise, we would have a contradiction with the condition $u>0$ in D, since u is radially symmetric in $\overline{B(0,\rho)}$.

If $\partial D\cap\partial B(0,\rho)=\emptyset$, then there exists a sequence of points $(x_k)_{k\in I\!\!N}$ of D such that $(|x_k|)_{k\in I\!\!N}$ is decreasing and converges to ρ and such that $u(x_k)\neq u(R_k x_k)$, where R_k is the reflection with respect to some hyperplane containing the origin and defined by a direction ν close to e_1. Without loss of generality, we may assume that $x_k \to \overline{x}$ for some $\overline{x} \in \partial B(0,\rho)$ and $R_k \to \overline{R}$, where $\overline{R}$ is the reflection with respect to some fixed hyperplane. Here the convergence of the reflexion is in the sense that that the unit vectors corresponding to the hyperplanes converge.

Then, replacing $\overline{x}$ by $\overline{R}\overline{x}$ if necessary, one of the two following possibilities is fulfilled:
a) $\nabla u(\overline{x}) = 0$,
b) $\frac{\overline{x}}{|\overline{x}|} \cdot \nabla u(\overline{x}) < 0$, since $\nabla u(\overline{x}) = \frac{du}{dr}(\rho)\ \frac{\overline{x}}{|\overline{x}|}$.

If a) occurs, since $e_1 \cdot \nabla u(x) \leq 0$ in a neighborhood of ρe_1 (here we use the radial symmetry of u in $B(0,\rho)$), we can conclude that $\frac{d^2}{dt^2} u((\rho+t)e_1)_{|t=0} = 0$. Using the radial symmetry of u again, we find that

$$\Delta u(\overline{x}) = \left(\frac{d^2u}{dr^2} + \frac{N-1}{r}\frac{du}{dr}\right)_{|r=\rho} = 0$$

and then $f(u(\overline{x})) = 0$. But this is impossible since Assumption (f3) applies: there exists a constant $C > 0$ such that for $v - u(\overline{x}) > 0$ small enough, $\frac{f(v)}{v-u(\overline{x})} > -C$, and Hopf's applied to $-\Delta(u - u(\overline{x})) + C(u - u(\overline{x})) > 0$ in $B((\rho-\epsilon)\frac{\overline{x}}{|\overline{x}|}, \epsilon)$ at $\overline{x}$ for $\epsilon > 0$ small enough is in contradiction with: $\nabla u(\overline{x}) = \frac{\overline{x}}{|\overline{x}|}\frac{du}{dr}(\rho) = 0$.

We may now consider Case b). For notational convenience, we can perform a rotation $\overline{R}$ such that $\overline{x} = \rho \cdot e_1$ (the monotonicity with respect to e_1 is true at least locally because $\nabla(u(\overline{x}) \neq 0$: since the rest of the argument is local, we do not have to take care of the geometrical restrictions corresponding to Assumptions (B) or (C)). For some $\sigma > 0$ small enough, we have then $\frac{\partial u}{\partial x_1}(x) < 0$ for any $x \in B(\overline{x},\sigma)$. If we denote $\overline{u}(x) = u(\overline{R}x)$, we have that $\overline{u}$ provides another solution of

$$\Delta u + f(u) = 0\ , \quad x \in B(0,\rho)$$

such that $u \not\equiv \overline{u}$ in $B(\overline{x},\sigma)$, since for k is large enough, $x_k \in B(\overline{x},\sigma)$. We observe that, taking σ smaller if necessary, $\frac{\partial \overline{u}}{\partial x_1}(x) < 0$ for any $x \in B(\overline{x},\sigma)$.

Here we shall use a local argument which involves a local change of coordinates (this transformation is the extension to $N \geq 2$ of the one used in [8] in the case $N = 2$). By the Implicit Function Theorem, there exists a neighborhood $\mathcal{V}$ of $(u(\overline{x}), 0) \in I\!R \times I\!R^{N-1}$ and two functions v and $\overline{v}$ of class C^2 such that

$$
\begin{aligned}
t &= u(v(t,x'),x') \quad \text{and} \\
t &= \overline{u}(\overline{v}(t,x'),x') \qquad\qquad \forall\ (t,x') \in \mathcal{V}\ ,
\end{aligned}
$$

with $\frac{\partial v}{\partial t} \neq 0$ and $\frac{\partial \overline{v}}{\partial t} \neq 0$ in $\mathcal{V}$. After some computations, we find that the function v satisfies the quasilinear equation

$$
\Big(1+\sum_{i=2}^{N}(\frac{\partial v}{\partial x_i})^2\Big)\frac{\partial^2 v}{\partial t^2} - 2\frac{\partial v}{\partial t}\sum_{i=2}^{N}\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_i \partial t} + (\frac{\partial v}{\partial t})^2\sum_{i=2}^{N}\frac{\partial^2 v}{\partial x_i^2} = (\frac{\partial v}{\partial t})^3 f(t) \quad \text{in } \mathcal{V}\ .
$$

A similar equation is satisfied by the function $\overline{v}$. It is easy to see that these equations are elliptic in $\mathcal{V}$.

We may now consider the function $z(t,x') = v(t,x') - \overline{v}(t,x')$ that satisfies the equation

$$
\begin{aligned}
\Big(1+\sum_{i=2}^{N}(\frac{\partial v}{\partial x_i})^2\Big)\frac{\partial^2 z}{\partial t^2} \quad - \quad & 2\frac{\partial v}{\partial t}\sum_{i=2}^{N}\frac{\partial v}{\partial x_i}\frac{\partial^2 z}{\partial x_i \partial t} + (\frac{\partial v}{\partial t})^2\sum_{i=2}^{N}\frac{\partial^2 z}{\partial x_i^2} + \\
& b_1\frac{\partial z}{\partial t} + \sum_{i=2}^{N} b_i \frac{\partial z}{\partial x_i} = 0 \quad \text{in } \mathcal{V}\ ,
\end{aligned}
$$

where the coefficients b_i are given by

$$
\begin{aligned}
b_1(t,x') \quad = \quad & -2\sum_{i=2}^{N}\Big(\frac{\partial v}{\partial x_i}\frac{\partial^2 \overline{v}}{\partial x_i \partial t}\Big) + \Big(\frac{\partial v}{\partial t} + \frac{\partial \overline{v}}{\partial t}\Big)\Big(\sum_{i=2}^{N}\frac{\partial^2 \overline{v}}{\partial x_i^2}\Big) \\
& -f(t)\Big\{(\frac{\partial v}{\partial t})^2 + (\frac{\partial v}{\partial t}\cdot\frac{\partial \overline{v}}{\partial t}) + (\frac{\partial \overline{v}}{\partial t})^2\Big\}\ ,
\end{aligned}
$$

and

$$
b_i(t,x') = \frac{\partial^2 \overline{v}}{\partial t^2}(\frac{\partial v}{\partial x_i} + \frac{\partial \overline{v}}{\partial x_i}) - 2\frac{\partial^2 \overline{v}}{\partial x_i \partial t}\cdot\frac{\partial \overline{v}}{\partial t}, \quad i = 2, 3, \dots N.
$$

We observe that the coefficients of the second order term are all of class C^1, while the b_i's are of class C^0. Thus the equation satisfied by z has the Unique Continuation Property, see [13] for instance.

We conclude that since u and $\overline{u}$ coincide on the open set $B(0,\rho)\cap B(\overline{x},\sigma)$, the functions v and $\overline{v}$ coincide in the corresponding open set. Therefore $u\equiv\overline{u}$ in $B(\overline{x},\sigma)$, a contradiction.

The only possibility is therefore $B(0,\rho)=D$, which ends the proof of Theorem 1.2 if D is a ball, or if it is not, by contradiction with $u_{|\partial B(0,\rho)}=0$. $\square$

4 Further results

We will not try to give the most general possible results, but just quote some remarks and directions to which our results can be extended.

To start with, we may notice that our (local) symmetry results hold for nonnegative solutions of (1.1) and the solutions may eventually be identically equal to 0 on a non empty subdomain of D. We may also notice that on a (radially symmetric) core, the minimum of the function is reached on the boundary of the core. One may therefore wonder why a nonnegative solution of (1.1) when D is, for instance, a ball cannot have cores on which the solution reaches its minimum inside the core (if we forget the nonnegativity condition, such solutions are easy to build). The answer is given by the following result which is reproduced from [9].

Proposition 4.1 *Assume that f satisfies (f1). Let u be a solution of (1.1) on the unit ball $B(0,1)$ which is radially symmetric up to cores. Assume that $N\geq 2$. With the same notations as in Theorem 3.1, if $B_i\cap B_j=\emptyset$ for any $j\neq i$ (u is radially symmetric on B_i) and if $\underline{u}=\min_{x\in B_i}u(x)<\min_{x\in\partial B_i}u(x)$, then $\underline{u}<0$.*

Proof: For the completeness of the paper, we give the main argument of this result. Consider u^+ and u^- two solutions of $\frac{d^2u}{dr^2}+\frac{N-1}{r}\cdot\frac{du}{dr}'+f(u(r))=0$

defined respectively on the intervals $I^- =]r_1^-, r_0^-[$ and $I^+ =]r_0^+, r_1^+[$ (with $0 < r_1^- < r_0^- \leq r_0^+ < r_1^+$), such that $u(r_0^\pm) = a$, $\frac{du^\pm}{dr}(r_0^\pm) = 0$, $u^\pm(r) < a$ where $a > 0$ is chosen in order that $f(a) = 0$. u^- is increasing on I^- and u^+ is decreasing on I^+, at least as long as $\frac{du^\pm}{dr}$ does not vanish. According to the method introduced by L.A. Peletier and J. Serrin in [19], it is possible to extend these solutions uniquely if $\frac{du^\pm}{dr} \neq 0$. Eventually, decreasing r_1^- and increasing r_1^+, we may assume that I^- and I^+ are the maximal intervals in $\mathbb{R}^+$ on which the property is satisfied. Then for any $r \in I^-$, $\frac{du^-}{dr}(r) = 0$ is impossible unless $u^-(r) < \inf_{s\in I^+} u^+(s)$. The functions $r^\pm(t)$ are indeed such that $t = u^\pm(r^\pm(t))$ are solutions of $\frac{(r^\pm)''}{((r^\pm)')^3} = f(t) + (d-1)\frac{1}{r^\pm(r^\pm)'}$. Multiplying by $(r^\pm)'(t)$ and integrating between $u^+(r)$ and a, we obtain for any $r \in I^+$

$$\begin{aligned} 0 \leq \frac{1}{2}\Big(\frac{du^+}{dr}(r)\Big)^2 = \quad & \int_{u^+(r)}^a f(s)\ ds + (d-1)\int_{u^+(r)}^a \frac{ds}{r^+(s)(r^+)'(s)} \\ & < \int_{u^+(r)}^a f(s)\ ds + (d-1)\int_{u^+(r)}^a \frac{ds}{r^-(s)(r^-)'(s)} \\ & = \frac{1}{2}\Big(\frac{du^-}{dr}(r^-(u^+(r)))\Big)^2 . \end{aligned}$$

This computation is still valid if $\frac{du^-}{dr}(r_0^-) > 0$, $\frac{du^+}{dr}(r_0^+) = 0$ and one can easily extend the argument to the case where $\frac{du^+}{dr} \leq 0$ takes the value 0 in I^+ if we define r^+ by $r^+(t) = \inf\{s > r_0^+ \ : \ u^+(s) = t\}$.

Without loss of generality, we may assume that B_i is the unique core of u (if not, apply the procedure defined in Theorem 3.1). Up to a translation, we can then identify u^+ and u^- with $\tilde{u}$ and $u_{|\partial B_i}$ respectively and get $0 = u^+(1) < u^-(0) = \underline{u}$. □

To summarize, we could say that for $N \geq 2$, there are no cores "going down".

4.1 Without overlapping

In Assumption (f2), the condition that the range of u on which f is locally either Lipschitz + increasing, or strictly decreasing, is open means that there is always an overlapping of these conditions. This is actually crucial to prove that the number of cores is finite, in any bounded subdomain in D.

However, it is clear at least in the case of a ball (see [3, 8]) that the right condition to avoid the existence of cores is a condition – actually (f3) – on the regularity of f in a neighborhood of $\overline{u}$ whenever $f(\overline{u}) = 0$, $\overline{u} > 0$. Thus the overlapping is unnecessary, as we shall see on the following example. For simplicity, assume that $N = 2$ and replace (f2) and (f3) by the assumption

(f2')There exists an $a > 0$ such that f is strictly decreasing in $[0, a]$, f is locally Lipschitz + increasing on $[a, \infty)$ and $f(a) < 0$.

We assume here a strong regularity assumption, except that there is no more overlapping between the region where f is strictly decreasing and the region where f is Lipschitz + increasing. This condition could of course be extended to each point a such that $f(a) < 0$, f is strictly decreasing on a neighborhood of a_- and Lipschitz + increasing on a neighborhood of a_+, and even also to each point b such that $f(b) > 0$, f is strictly decreasing on a neighborhood of b_+ and Lipschitz + increasing on a neighborhood of b_-, as soon as one controls the number of possible cores. However a statement with such assumptions would be unnecessarily complicated. For simplicity again, we shall consider the case of a ball $B = B(0, 1)$.

Proposition 4.2 *Let $N = 2$. Assume that f satisfies (f1) and (f2') and consider a positive solution $u \in C^2(B) \cap C^0(\bar{B})$ of (1.1)-(1.2) on the unit ball $D = B$. Then u is radially symmetric and $\frac{du}{dr}(r) < 0$ for any $r \in (0, 1)$.*

Proof. We proceed exactly as in the proof of Lemma 2.1 with $\Omega = B$. Assume that $\overline{\lambda} > \lambda_\Omega$ and consider $\overline{x} \in B$ such that $u(\overline{x}) \geq a$ ($\overline{x}$ is the limit of $x_k \in \Sigma_{\lambda_k}$ such that $w_{\lambda_k}(x_k) < 0$, $\nabla w_{\lambda_k}(x_k) = 0$: $w_{\overline{\lambda}}(\overline{x}) = 0$ and $\nabla w_{\overline{\lambda}}(\overline{x}) = 0$. If $u(\overline{x}) > a$, the proof goes as before (the number of possible cores is finite because at a maximum, $-\Delta u = f(u) > 0$, so that we can give an estimate of $\mathcal{N}$ using the Lipschitz norm of u) and the only case one has to consider is the case $u(\overline{x}) = a$.

Let us prove that there is some $x \in \Sigma_{\overline{\lambda}}$ such that $u(x) > a$ (and therefore f is Lipschitz in a neighborhood of $u(x)$) and $w(x) = 0$.

We will distinguish two cases

– Case (1): $\overline{x} \in \Sigma_{\overline{\lambda}}$

– Case (2): $\overline{x} \in T_{\overline{\lambda}}$

Let us define first the set $\omega = \{x \in \Sigma_{\overline{\lambda}} \; : \; u(x) > a\}$. We may first notice that either $w \equiv 0$ on $\omega \cap \Sigma_{\overline{\lambda}}$ or $w > 0$ because of the Maximum Principle. Assume then that the second case is satisfied and look for a contradiction.

<u>*Case* (1) </u>: We have to prove that ω satisfies an interior sphere condition at $\overline{x}$.

a) if $\frac{\partial u}{\partial x_1}(\overline{x}) \neq 0$, $\partial\omega$ is locally of class C^2 in a neighborhood of $\overline{x}$: ω satisfies an interior sphere condition at $\overline{x}$ for some ball $B(\tilde{x}, \epsilon)$ of center $\tilde{x} \in \omega$ and radius $\epsilon > 0$, such that $(\tilde{x} - \overline{x}) \cdot e_1 \neq 0$, $(\tilde{x} - \overline{x}) \cdot \frac{\nabla u(\overline{x})}{|\nabla u(\overline{x})|} = \epsilon$, $|\tilde{x} - \overline{x}| = \epsilon$ and $u(x) > a$ for any $x \in B(\tilde{x}, \epsilon)$, provided $\epsilon > 0$ is small enough. Hopf's lemma applied to $w > 0$ in ω at $\overline{x}$ (see [11]) provides: $\nabla w_{\overline{\lambda}}(\overline{x}) \cdot (\tilde{x} - \overline{x}) < 0$, a contradiction with $\nabla w_{\overline{\lambda}}(\overline{x}) = \lim_{k \to +\infty} \nabla w_{\lambda_k}(x_k) = 0$.

b) if $\frac{\partial u}{\partial x_1}(\overline{x}) = 0$ and $\frac{\partial u}{\partial x_2}(\overline{x}) \neq 0$, we may again apply Hopf's lemma in the direction of x_2. Assume then that $\nabla u(\overline{x}) = 0$. The monotonicity of $x_1 \mapsto u(x_1, x')$ gives: $\frac{\partial^2 u}{\partial x_1 \partial x_2}(\overline{x}) = \frac{\partial^2 u}{\partial x_1^2}(\overline{x}) = 0$ because of the following Taylor development

$$u(x) - a = \sum_{\substack{i,j=1,2 \\ (i,j) \neq (1,1)}} \frac{\partial^2 u}{\partial x_i \partial x_j}(\overline{x}) \cdot (x - \overline{x})_i (x - \overline{x})_j + o(|x - \overline{x}|^2) \; .$$

Then, since $-\frac{\partial^2 u}{\partial x_2^2}(\overline{x}) = -\Delta u(\overline{x}) = f(a) < 0$, ω again clearly satisfies an interior sphere condition at $\overline{x}$. For the same reason as in case a), again we get a contradiction.

<u>*Case* (2) </u>: Assume now that $\overline{x} \in T_{\overline{\lambda}}$. Because of the definition of $\overline{x}$, $\frac{\partial u}{\partial x_1}(\overline{x}) = -\frac{1}{2} \lim_{k \to +\infty} e_1 \cdot \nabla w_{\lambda_k}(x_k) = 0$. If $\frac{\partial u}{\partial x_2} \neq 0$, we may apply Serrin's lemma (see for instance [11, 20]) :

Lemma 4.1 *Let $\mathcal{O}$ be a domain in $I\!R^N$ and assume that near $\overline{x} \in \mathcal{O}$ the boundary consists of two transversally intersecting hypersurfaces $\rho = 0$ and $\sigma = 0$. Suppose that $\rho, \sigma > 0$ in $\mathcal{O}$. Let $w > 0$ be a function in $C^2(\overline{\mathcal{O}})$ with $w > 0$ in $\mathcal{O}$, $w(\overline{x}) = 0$ satisfying the differential inequality $-\Delta w - c(x)w \geq 0$ for some function c in $L^\infty(\mathcal{O})$. Assume that $\sum_{i=1}^N \frac{\partial \rho}{\partial x_i}(\overline{x}) \cdot \frac{\partial \sigma}{\partial x_i}(\overline{x}) = 0$ and $D\left(\sum_{i=1}^N \frac{\partial \rho}{\partial x_i} \cdot \frac{\partial \sigma}{\partial x_i}\right)(\overline{x}) = 0$ for any derivative tangent at $\overline{x}$ to the submanifold $\{\rho = 0\} \cap \{\sigma = 0\}$. Then for any direction s which enters $\mathcal{O}$ at $\overline{x}$ transversally to each hypersurface, $\frac{\partial w}{\partial s} > 0$, $\frac{\partial^2 w}{\partial s^2} > 0$.*

But this is clearly in contradiction with $\nabla w_{\overline{\lambda}}(\overline{x}) = 0$.

If $\frac{\partial u}{\partial x_2} = 0$, we may still find a smaller cone $\mathcal{K}$ of summit $\overline{x}$ containing the direction $\pm x_2$ such that on $\omega \cap \mathcal{K}$, $w > 0$ and as in Case (1) we get in that way a contradiction with Serrin's lemma. □

4.2 Whole space results

In the case the domain D is the whole space $I\!R^N$, the method can still be adapted as soon as the moving plane technique can be started from $+\infty$ in any direction. One of the main features of the local moving planes method we develop in this paper is that we do not need to assume a strict positivity of w_λ as soon as f is strictly decreasing and can therefore handle in a unified framework the positive solutions as well as the nonnegative solutions that are compactly supported.

Theorem 4.1 *Assume f satisfies (f1)-(f2). Let u be a C^2 nonnegative solution of (1.1)-(1.2) satisfying $\lim_{|x| \to +\infty} u(x) = 0$. Then u is radially symmetric up to cores. If it is compactly supported, the support of u is a union of balls with disjoined interiors.*

Of course, with a further assumption on the positive critical levels of f, we may get a strict monotonicity on each component of the support (and a further assumption on the regularity of f at 0 would provide the result that the solution has to be radially symmetric, positive and strictly decreasing with respect to some point in $I\!R^N$):

Corollary 4.1 *Assume f satisfies (f1)-(f3). Let u be a C^2 nonnegative solution of (1.1)-(1.2) satisfying $\lim_{|x|\to+\infty} u(x) = 0$. Then any connected component of $\{x \in I\!R^N \ : \ u(x) > 0\}$ is a ball (or $I\!R^N$), and u restricted to each of these components is radially symmetric and strictly decreasing.*

4.3 Fully nonlinear case

It is possible to generalize the results given in Sections 1-3 for the Laplacian to more general fully nonlinear elliptic equations of the type

$$F\Big(u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\Big) = 0 \quad \text{with} \quad i, j = 1, 2, \dots N$$

when F is only continuous with respect to u, even in the case where the highest order part of the operator cannot be written in divergence form.

Let us consider the following assumptions, which are adapted from [16, 17],

$$\begin{aligned} F: \quad & I\!R^+ \times I\!R^N \times I\!R^{N\times N} \to I\!R \\ & (s, p, Q) \mapsto F(s, p, Q) \end{aligned}$$

has the following properties

(F1) F is continuous and C^1 with respect to $Q = (Q_{ij})_{i,j=1,2,\dots N}$,

(F2) F is either Lipschitz + increasing in s, or has the following strict decay property

$$F(u + w, p, Q + R) > F(u, p, Q) \tag{4.1}$$

for any $N \times N$ nonnegative symmetric matrix R and any $w < 0$, provided $(w, R) \neq (0, 0)$, (F3) For any (s,p,Q) such that $F(s,p,Q)=0$, there exists a neighborhood of (s,p,Q) in $\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^{N\times N}$ on which F is Lipschitz + increasing with respect to s,

(F4) For any $\xi\in\mathbb{R}^N$, $F_{p_ip_j}(s,p,Q)\xi_i\xi_j\geq\overline{\lambda}(s,p,Q)|\xi|^2$ for some $\overline{\lambda}(s,p,Q)$ which is uniformly positive.

(F5) F has the following symmetry property with respect to e_1:

$$F(s,(-p_1,p_2,\dots p_N),\tilde{Q})=F(s,p,Q)\ ,$$

$$\tilde{Q}=(Q_{11},-Q_{12},-Q_{13},\dots\ -Q_{1N},-Q_{21},Q_{22},Q_{23},\dots Q_{2N},\dots Q_{NN})\ ,$$

as well as for any direction $\gamma\in S^{N-1}$ such that $|\gamma-e_1|<\epsilon$ for some given $\epsilon>0$.

Theorem 4.2 *Assume that f satisfies (F1)-(F5) and D is x_1-convex, symmetric with respect to the hyperplane $\{x\in\mathbb{R}^N\ :\ x\cdot e_1=0\}$ and bounded. Let $u\in C^2(D)\cap C^0(\bar{D})$ be a positive solution of (1.1)-(1.2). Then u is monotone non increasing up to cores on $\tilde{D}=\{x\in D\ :\ x\cdot e_1\geq 0\}$ in the direction e_1.*

Of course, a similar monotonicity result holds for unbounded domains. Note that Assumption (F5) as in [16] is quite restrictive (see [8] for an example).

The proofs go exactly as for the Laplacian, but present purely computational technicalities that are unessential and will not be presented here. The main point is that one has to make sure that the local inversion theorem (unique continuation argument in the proof of Theorem 1.2) preserves the ellipticity of the operator.

Imposing a dependence in $|x|$ is easy and we can for instance state the following result. Consider in case (B)

$$\Delta u+f(|x|,u)=0,\ u>0\qquad\text{in } D,\tag{4.2}$$

$$u=0\qquad\text{on }\partial D.\tag{4.3}$$

Theorem 4.3 *Under the same assumptions as in Theorem 1.1, provided these assumptions on f are uniform in x, if D satisfies condition (B) and if f is monotone non increasing in $|x|$, then the same results hold for any solution of (4.2)-(4.3). If moreover $f(|x|, u)$ is strictly decreasing in $|x|$, then no cores may exist.*

Actually, to obtain the nonexistence of cores, it is sufficient to ask that $f(|x|, u)$ is strictly decreasing in $|x|$ for any u such that f is not strictly decreasing in u. In that case by Lemma 2.1, u would be symmetric with respect to some hyperplane $T_{\overline{\lambda}}$ in the range $u_1 < u(\overline{x}) < u_2$, in contradiction with the fact that

$$0 = \Delta u_{\overline{\lambda}} + f(u_{\overline{\lambda}}, |x_{\overline{\lambda}}|) = \Delta u + f(u, |x_{\overline{\lambda}}|) > \Delta u + f(u, |x|) = 0 \ ,$$

except if $\overline{\lambda} = 0$.

The only difficulty that may occur is the case $\lambda = \lambda_\Omega$ in Lemma 2.1, which can be solved by noticing first that if $\Omega = D$, $\lambda_\Omega = 0$, and then by applying the iteration method of the proof of Theorem 3.1 with care. Details are left to the reader. □

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