The value of Markov chain games with lack of information on one side

Jérôme Renault

June 11, 2003

Abstract

We consider a two-player zero-sum game given by a Markov chain over a finite set of states $K$ and a family of zero-sum matrix games $(G^k)_{k \in K}$. The sequence of states follows the Markov chain. At the beginning of each stage, only player 1 is informed of the current state $k$, then the game $G^k$ is played, the actions played are observed by both players and the play proceeds to the next stage. We call such a game a Markov chain game with lack of information on one side. This model generalizes the model, introduced by Aumann and Maschler in the sixties, of zero-sum repeated games with lack of information on one side (which corresponds to a constant Markov chain). We generalize the proof of Aumann and Maschler and, from the definition and the study of appropriate “non revealing” auxiliary games with infinitely many stages, show the existence of the uniform value. An important difference with Aumann and Maschler’s model is that here, the notions for player 1 of using the information and revealing a relevant information are distinct.

Key words. Repeated games, incomplete information, Markov chain, stochastic games, lack of information on one side.

1 Introduction

Repeated games with incomplete information were introduced by Aumann and Maschler (1995 for a reedition of their work). Their simplest model is the one of two-player zero-sum repeated game with lack of information on one side and perfect observation: at the beginning of the play, a state of nature $k$ is chosen according to an initial probability $p$ over a finite set of states, and announced to player 1 only. This state of nature determines a finite zero-sum game $G^k$ which is then repeatedly played over and over, and after each stage the actions played are observed by both players. In this setup, Aumann and Maschler showed the existence of the value, and gave a famous characterization for it. This pioneering
work has been widely extended since the sixties, and a lot of important works are dedicated to this model, or close extensions of it, for example Aumann et al. (1995) or Mertens and Zamir (1971-1972) for the case of incomplete information on both sides, Kohlberg (1975) for an explicit construction of an optimal strategy for player 2, or Sorin (1983) Hart (1985) and Simon et al. (1995) for the non-zero sum case.

We only consider here two player zero-sum games and generalize the above model to the case where the state of nature is no longer fixed once and for all at the beginning of the game, but evolves according to a Markov chain, and at the beginning of each stage is observed by player 1 only. We call such games Markov chain games with lack of information on one side, since here also only player 2’s information is incomplete. Note that this model is a special case of stochastic game with incomplete information, which may fail to have a value (Sorin 1984, see also Rosenberg and Vieille 2000). Recently, Rosenberg et al. (2002) proved the existence of the value in a particular class of games generalizing Aumann and Maschler’s model. These games are defined via a collection of stochastic games and are neither more nor less general than the ones studied here.

In this paper, we generalize the proof of Aumann and Maschler and show the existence of the value in Markov chain games with lack of information on one side. However, several problems remain. Our expression of the value cannot be easily computed from the basic data of the game, and we present a very simple example where we can not compute it. About optimal strategies, we obtain the existence of an optimal strategy for player 2, but we still do not know if there exists such a strategy for player 1.

We now present the mains ideas of our proof, and first need to come back to the original proof of Aumann and Maschler. All games are indexed by the initial probability $p$ on $K$.

1) Aumann and Maschler defined, for each initial probability $p = (p^k)_{k \in K}$, a “non revealing” game corresponding to the game where player 1 does not use his information on the selected state, i.e. where he plays independently of the state. This game can be analysed as the repetition of the average matrix game $\sum_{k \in K} p^k G^k$. Consequently, its value, denoted by $u(p)$, is just the value of the average game and player 1 can guarantee this quantity by playing independently of the states. By standard properties of games with lack of information on one side, one can then deduce that player 1 can guarantee $cav u(p)$ in the original game, where $cav u$ stands for the concavification of $u$ on $\Delta(K)$.

2) Concerning what can be guaranteed by player 2, it is not difficult to show that he can guarantee the limit, as $N$ goes to infinity, of the value $v_N(p)$ of the $N$-stage game. The last point of Aumann and Maschler, is to show that $\lim_{N \to \infty} v_N(p) \leq cav u(p)$. This is done by considering, for a fixed strategy of player 1, the a posteriori of player 2 on the selected state of nature (i.e. after each stage, the conditional probability on the state depending on the previous actions of player 1). The sequence of a posteriori forms a martingale, and a classical bound on its $L_1$ variation gives the above inequality.
We now describe the main points of our proof. Denote by $\Gamma_\infty(p)$ the Markov chain game with initial probability $p$, and for each $N$ by $v_N(p)$ the value of the $N$-stage game with initial probability $p$.

One can show that here also, player 2 can guarantee the limit of $(v_N(p))_N$. The main conceptual difficulty is to define an appropriate version of the nonrevealing games. Imagine that at each stage the new state is chosen according to a probability distribution independent of the previous state. Then player 1 can use at each stage his knowledge of the current state, and player 2 will never learn something interesting. Hence in this case the information is “free” for player 1, since it will be generated anew. On the contrary in the Aumann and Maschler’s case, as soon as player 1 uses some piece of information, the martingale of a posteriori moves and a relevant information is revealed to player 2. In the general case, we will define and quantify the information of player 1 which is relevant to player 2 in the long run. The basic and main idea is that this information for player 2 can be described by the asymptotic behavior of the sequence of states.

If player 1 does not use his information on the states, the belief of player 2 on the current state of stage $n + 1$ is $pM^n$, where $M$ is the transition matrix of the Markov chain. An important property of finite Markov chain is that the limit behavior of the sequence $(pM^n)_n$ can always be approximated by a periodic sequence $(pB_0, pB_1, ..., pB_{L-1}, pB_0, pB_1, ...)$, $L$ being a common multiple of the periods of the recurrence classes of the Markov chain, and $B_0$ being a projection and stochastic matrix such that $M^{nL} \to_{n \to \infty} B_0$. The matrix $B_0$ is unique and takes into account the recurrence classes of $M$, but also the periodic aspect of these classes, and its existence is fundamental for our proof. In particular, if player 1 never uses his information the relevant information for player 2 can be summarized by $\lim_n pM^{nL} = pB_0$, since this characterizes the asymptotic periodic sequence. And if player 1 does not use his information after some stage $n$, the relevant information for player 2 about the limit behavior of the sequence of states is $q_n = p_n B_{-n}$, where $p_n$ is player 2’s belief, computed after stage $n$, on the state of stage $n + 1$, and $B_{-n} = \lim_t M^{tL-n}$.

Fix now any strategy of player 1. $(q_n)_n$ is a martingale representing player 2’s current relevant information over the limit behavior of the sequence of states. $q_n$ will simply be called player 2’s relevant information after stage $n$, and we define a non revealing strategy for player 1 at $p$ as a strategy such that this martingale is almost surely constant (and, for convenience, which plays independently of the actions of player 2). So a non revealing strategy of player 1 does not necessarily play independently of the states, as in Aumann and Maschler’s case. This is especially the case when the Markov chain has a unique recurrence class which is aperiodic (see Example A in section 3): if player 1 chooses his actions depending on the current state for a certain number of stages, and then play independently of the states, all the information obtained by player 2 will vanish since his belief on the current state will still converge to the unique invariant probability measure. For player 1, the two notions of not using the information and not revealing a relevant information are different here.
We define the non revealing game at $p \hat{\Gamma}_\infty(p)$ as the auxiliary game where player 1 is restricted to play a non revealing strategy at $p$. This auxiliary game has infinitely many stages, and can not be analysed as a repeated matrix game as in Aumann and Maschler’s case. Indeed, in the case of a recurrent aperiodic Markov chain, all strategies of player 1 are non revealing and the original game $\Gamma_\infty(p)$ and the non revealing game $\hat{\Gamma}_\infty(p)$ coincide. We will prove that in general the non revealing game $\hat{\Gamma}_\infty(p)$ has a value, via a careful study of the sets of non revealing strategies. We notably prove an appropriate version of the “splitting lemma” of Aumann and Maschler, and a recursive formula for the values $v_N(p)$ of the $N$-stages non revealing games. We then obtain that $\hat{\Gamma}_\infty(p)$ has a value $w(p)$ which satisfies: $w(p) = \inf_{N \geq 1} \hat{v}_{NL}(pB_0)$. Since player 1 can guarantee $w(p)$ in the non revealing game, a fortiori he can guarantee $w(p)$ in the original game $\Gamma_\infty(p)$, and one can show that player 1 can guarantee $\cav w(pB_0)$ in $\Gamma_\infty(p)$.

Our definitions of relevant information and non revealing games can only be justified if we link the values of the original $N$-stage games to the one of the non revealing game. Let’s consider a $LNT$-stage (original) game, with $N$ fixed and $T$ large, and fix any strategy of player 1. By considering, for each block $t = 1, \ldots, T$ of $LN$ stages a best reply of player 2, against an approximation of player 1’s strategy on this block as a non revealing one, in a corresponding $LN$-stage non revealing game, one can show from the bound on the $L^1$-variation of the martingale $(q_n)_n$ that $\lim_n v_N(p) \leq \cav \hat{v}_{NL}(pB_0)$. The rest is mainly technical. We show that $\cav w(pB_0) = \inf_{N \geq 1} \cav \hat{v}_{NL}(pB_0)$. Since in the original game with initial probability $p$, player 2 can guarantee $\lim_N v_N(p)$ and player 1 can guarantee $\cav w(pB_0)$, we are done and the value of $\Gamma_\infty(p)$ is $\cav w(pB_0) = \lim_N v_N(p)$.

The paper is organized as follows. Section 2 contains the model, and a few important examples are presented in section 3. In section 4 we introduce the projection matrix $B_0$ which allows to quantify player 2’s current knowledge on the limit behavior of the sequence of states. In section 5 we consider the original $N$-stage games and show that player 2 can guarantee $\lim_N v_N(p)$ in $\Gamma_\infty(p)$. Section 6 contains the definition and the study of the non revealing games. In section 7, we come back to the original game and prove that it has a value. We conclude with several remarks on the model and possible extensions.

2 The model

If $\mathcal{S}$ is a finite set, $|\mathcal{S}|$ denotes its cardinality and $\Delta(\mathcal{S})$ the set of probability distributions over $\mathcal{S}$. $\Delta(\mathcal{S})$ is viewed as a subset of $\mathbb{R}^\mathcal{S}$. For $q = (q_s)_{s \in \mathcal{S}}$ in $\mathbb{R}^\mathcal{S}$, we use $\|q\| = \sum_{s \in \mathcal{S}} |q_s|$.

We denote by $K = \{1, \ldots, |K|\}$ the set of states, by $I$ the set of actions of player 1 and by $J$ the set of actions of player 2. $K$, $I$ and $J$ are assumed to be finite and non empty. We have a family $(G^k)_{k \in K}$ of $|I| \times |J|$ payoff matrices for player 1, and a Markov chain on $K$, given by an initial probability $p$ on $K$ and a
transition matrix $M = (M_{kk'})_{(k,k') \in K \times K}$. All elements of $M$ are non negative and for each $k$ in $K$, $\sum_{k' \in K} M_{kk'} = 1$.

An element $q = (q^k)_{k \in K}$ in $\Delta(K)$ will also be represented by a row vector $q = (q^1, ..., q^{|K|})$, with $q^k \geq 0$ for each $k$ and $\sum_{k \in K} q^k = 1$. If the law of the state at some stage is $q$, the law of the state at the next stage is the product $qM$. For each $k$ in $K$, $\delta_k$ denotes the Dirac measure on $k$.

The play of the zero-sum game is as follows:

- at stage 1, $k_1$ is chosen according to $p$, and told to player 1 only. Players 1 and 2 independently choose an action in their own set of actions, $i_1 \in I$ and $j_1 \in J$ respectively. The stage payoff for player 1 is $G^{k_1}(i_1, j_1)$, $(i_1, j_1)$ is publicly announced, and the play proceeds to stage 2.

- at stage $n \geq 2$, $k_n$ is chosen according to $\delta_{k_{n-1}}M$, and told to player 1 only. The players independently choose an action in their own set of actions. If $i_n \in I$ and $j_n \in J$ are selected, the stage payoff for player 1 is $G^{k_n}(i_n, j_n)$. $(i_n, j_n)$ is publicly announced, and the play proceeds to the next stage.

Note that the payoffs are not announced after each stage. Players are assumed to have perfect recall, and the whole description of the game is public knowledge. We denote by $\Gamma_\infty(p)$ the game just defined.

A behavior strategy for player 1 is an element $\sigma = (\sigma_n)_{n \geq 1}$ where for each $n$ $\sigma_n$ is a mapping from the cartesian product $(K \times I \times J)^{n-1} \times K$ to $\Delta(I)$ giving the mixed action played by player 1 at stage $n$ depending on past and current states and past actions played. Since player 2 does not observe the states, a behavior strategy for him is an element $\tau = (\tau_n)_{n \geq 1}$ where for each $n$ $\tau_n$ is a mapping from $(I \times J)^{n-1}$ to $\Delta(J)$. Denote by $\Sigma$ and $\mathcal{T}$, respectively, the set of behavior strategies of player 1 and player 2. A strategy profile $(\sigma, \tau)$ induces a probability distribution over the set of plays $\Omega = (K \times I \times J)^\infty$, and we denote, for each positive $N$, the average expected payoff for player 1 induced by $\Gamma_\infty(p)$ at the first $N$ stages by:

$$\gamma_N^R(\sigma, \tau) = \mathbb{E}_{p,\sigma,\tau} \left( \frac{1}{N} \sum_{n=1}^{N} G^{k_n}(i_n, j_n) \right)$$

where $k_n$, $i_n$, $j_n$ respectively denote the state, action for player 1 and action for player 2 at stage $n$.

The $N$-stage game $\Gamma_N(p)$ is defined as the zero-sum game with strategy spaces $\Sigma$ and $\mathcal{T}$, and payoff function $\gamma_N^R$. By standard arguments, it has a value denoted by $v_N(p)$, and both players have optimal strategies. Concerning the infinite game $\Gamma_\infty(p)$, we will use the following standard notion of uniform value.

**Definition 2.1** Let $v$ be a real number.
- Player 1 can guarantee $v$ in $\Gamma_\infty(p)$ if:
  \[ \forall \varepsilon > 0 \, \exists \sigma \in \Sigma \, \exists N_0, \forall N \geq N_0 \, \forall \tau \in \mathcal{T}, \, \gamma_N^p(\sigma, \tau) \geq v - \varepsilon. \]
- Player 2 can guarantee $v$ in $\Gamma_\infty(p)$ if:
  \[ \forall \varepsilon > 0 \, \exists \tau \in \mathcal{T} \, \exists N_0, \forall N \geq N_0 \, \forall \sigma \in \Sigma, \, \gamma_N^p(\sigma, \tau) \leq v + \varepsilon. \]
- $v$ is the value of $\Gamma_\infty(p)$ if both players can guarantee $v$ in $\Gamma_\infty(p)$.

It is easy to see that if the value $v$ of $\Gamma_\infty(p)$ exists, it is necessarily unique and it is also the limit of the value of the $N$-stage games, as $N$ goes to infinity, and the limit of the value of the discounted games as the discount factor goes to zero (see Sorin 2002, lemma 3.1 p.27). In this case, a strategy $\sigma$ of player 1 is said to be optimal if $\forall \varepsilon > 0 \, \exists N_0, \forall N \geq N_0 \, \forall \tau \in \mathcal{T}, \, \gamma_N^p(\sigma, \tau) \geq v - \varepsilon$. Similarly, a strategy $\tau$ of player 2 is optimal if $\forall \varepsilon > 0 \, \exists N_0, \forall N \geq N_0 \, \forall \sigma \in \Sigma, \, \gamma_N^p(\sigma, \tau) \leq v + \varepsilon$.

In this paper we prove the following result.

**Theorem 2.2** $\Gamma_\infty(p)$ has a value, and player 2 has an optimal strategy.

We will use the following notations and definitions.

If $x = (x(i))_{i \in I} \in \Delta(I)$ and $y = (y(j))_{j \in J} \in \Delta(J)$, for each state $k$ we put $G^k(x, y) = \sum_{i \in I, j \in J} x(i) y(j) G^k(i, j)$. For each $p$ in $\Delta(K)$, $u(p)$ will denote the value of the non revealing game à la Aumann Maschler, i.e.:

$$u(p) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \sum_{k \in K} p^k G^k(x, y).$$

$C$ is an upper bound for all absolute values of payoffs, i.e. $C$ satisfies: $|G^k(i, j)| \leq C$ for all $k, i, j$.

If $f : \Delta(K) \rightarrow \mathbb{R}$ is bounded from above, cayf is the pointwise smallest concave function $g$ on $\Delta(K)$ satisfying $g(p) \geq f(p)$ for each $p$ in $\Delta(K)$.

We will also need to consider explicitly strategies in games with finitely many stages. For $N \geq 1$, $\Sigma_N$ will denote the set of $N$-stages behavior strategies of player 1. An element $\sigma$ in $\Sigma_N$ is just an element $(\sigma_n)_{n \in \{1, \ldots, N\}}$, where for each $n$, $\sigma_n$ is a mapping from $(K \times I \times J)^{n-1} \times K$ to $\Delta(I)$. Similarly, $\mathcal{T}_N$ will denote the set of $N$-stages strategies for player 2. The $N$-stage game $\Gamma_N(p)$ could equivalently (in terms of value) be defined as the zero-sum game with strategy spaces $\Sigma_N$ and $\mathcal{T}_N$, and payoff function $\gamma_N^p$.

A strategy $\sigma = (\sigma_n)_n$ of player 1 (in $\Sigma$ or in $\Sigma_N$) will be called independent of the actions of player 2 if for each $n$ and $(k_1, i_1, j_1, \ldots, k_{n-1}, i_{n-1}, j_{n-1}, k_n)$ in $(K \times I \times J)^{n-1} \times K$, $\sigma_n(k_1, i_1, j_1, \ldots, k_{n-1}, i_{n-1}, j_{n-1}, k_n)$ does not depend on $(j_1, \ldots, j_{n-1})$. We denote by $\Sigma^+_\Sigma$ (or $\Sigma^+_N$) the set of such strategies of player 1.

### 3 Examples

**Example A:** An irreducible aperiodic Markov chain
Consider the following transition matrix (there are two states): 
\[ M = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}. \]
This Markov chain has a unique recurrence class which is aperiodic, and has 
\[ p^* = (1/2, 1/2) \] as unique invariant measure. 
\((M^n)_n\) converges to 
\[ \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \]

hence for any \( p \) in \( \Delta(K) \), \((pM^n)_n\) converges to \( p^* \). This implies that if player 1 plays independently of the states, the belief of player 2 on the current state converges to \( p^* \).

For any \( N \) player 1 can play in \( \Gamma_{\infty}(p) \) as follows: (1) play for \( N \) stages an optimal strategy in \( \Gamma_N(p^*) \), then (2) play independently of the states for a fixed number of stages, and come back to (1). This shows that player 1 can guarantee \( \limsup_N v_N(p^*) \). Concerning player 2, for any \( N \) he can play as follows in \( \Gamma_{\infty}(p) \): play for \( N \) stages an optimal strategy in \( \Gamma_N(p^*) \), then forget everything and redo from start. This shows that player 2 can guarantee \( \inf_N v_N(p^*) \) in \( \Gamma_{\infty}(p) \). All this implies that \( \Gamma_{\infty}(p) \) has a value which is \( \inf_N v_N(p^*) \), hence independent of \( p \).

Example B: \( M \) is the identity matrix.
This is the standard case of repeated game with lack of information on one side. It is well known (see Aumann Maschler, 1995) that the value \( v(p) \) of \( \Gamma_{\infty}(p) \) exists and satisfies:
\[ v(p) = c a v u(p). \]

\( u(p) \) is the value of the non revealing game at \( p \), which corresponds in this case to the game where player 1 does not use his information on the states. The relevant information for player 2 is here the initial state, or equivalently the recurrence class of the Markov chain.

Example C: A periodic chain
Let \( K = \{a, b\} \), and \( M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).
The Markov chain has a unique recurrence class, which is periodic with period 2. Here the relevant information for player 2 is not the recurrence class, but if the sequence of states will be \( (a, b, a, b, a, b, ...) \), or \( (b, a, b, a, b, a, ...) \). By considering the play of stages by blocks of length 2, we can reduce the problem here to that of example B. The point is that \( M^2 \) is the identity matrix.

A last aspect of finite Markov chains is the possible existence of transient states. This does not play an important role here, because player 1 can always wait during the first stages until the current state is a recurrence one. We however present now an example with a transient state. This example, and our last example below, will be used later to illustrate several definitions.

Example D: \( K = \{a, b, c\} \), and \( M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \end{pmatrix} \).
Example E: $K = \{a, b, c\}$, and $M = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

4 Limit beliefs over the current state of an unobserved Markov chain

Consider the case where player 1 does not use his information on the states, i.e. he plays independently of the states. The belief of player 2 on the current state at stage $n + 1$ is then $pM^n$. This sequence of beliefs can always be approximated by a periodic sequence $(pB_0, pB_1, ..., pB_{L-1}, pB_0, pB_1, ...)$ that we define now.

Proposition 4.1 There exists a positive integer $L$ and stochastic matrices $B_0, B_1, ..., B_{L-1}$ such that:

$$\forall l \in \{0, ..., L - 1\}, \lim_{n \to \infty} M^{nL+l} = B_l.$$ 

The proof is omitted and can be easily deduced, e.g., from Gordon (1964), or Norris (Theorem 1.8.5, p.44, 1997). The only thing to show is the existence of some $L$ such that $(M^{nL})_n$ converges. From a linear algebra viewpoint, this is due to the fact that if $z$ is a complex eigenvalue of $M$ with $|z| \geq 1$, then there exists a positive integer $n$ such that $z^n = 1$. $L$ corresponds to a common multiple of all such $n$, or equivalently, to a common multiple of the periods of the recurrence classes of $M$.

As $(M^{nL})_n \to B_0$, $B_0$ is a projection matrix, that is $B_0^2 = B_0$. Notice that $B_0$ is necessarily unique. The set of invariant probability measures of $B_0$ will play an important role, hence we introduce the following notations.

Notations:

$$Q = \{q \in \Delta(K), qB_0 = q\} = \{pB_0, p \in \Delta(K)\}.$$ 

For each $q$ in $Q$, $A(q) = \{p \in \Delta(K), pB_0 = q\}$. If $q = qB_0$, then $qM^L = qB_0M^L = qB_0 = q$, hence it is easy to see that $Q$ also is the set of invariant probability measures of the matrix $M^L$. Notice also that $Q$ and each set $A(q)$ are polytopes.

It is convenient to define $B_n$ for each integer $n$. We put, for each (possibly negative) integer $n$, $B_n = B_l$, where $l \in \{0, ..., L - 1\}$ and $n - l$ is a multiple of $L$. We then have for each integer $n$, $M^{L+n} \to B_n$, and:

$$\forall p \in \mathbb{N}, \quad M^pB_n = B_nM^p = B_{n+p}.$$
5 \textit{N}-stage games

For each positive integer \( N \) and each \( p \) in \( \Delta(K) \), \( \Gamma_N(p) \) is the \( N \)-stage game with initial probability \( p \), and has a value \( v_N(p) \). The payoff function \( \gamma_N^p \) satisfies:

\[
\forall \sigma \in \Sigma, \forall \tau \in T, \quad \gamma_N^p(\sigma, \tau) = \sum_{k \in K} p^k \delta_k(\sigma, \tau).
\]

Consequently, \( v_N \) is, as a function of \( p \), Lipschitz with constant \( C \). Moreover it is concave, since \( \Gamma_N(p) \) is a game with incomplete information on one side with player 1 as the informed player (see for example Zamir, 1992). We now prove a recursive formulae for \( v_N \), and first need some notations.

Let \( p \) be in \( \Delta(K) \) representing player 2’s belief on the current state at some stage \( n \), this belief being computed before stage \( n \) is played. Consider for simplicity that player 1’s action at this stage will be chosen according to some \( x = (x_k)_{k \in K} \in \Delta(I)^K \), i.e. that player 1 plays, for each \( k \), according to \( x_k = (x_k(i))_{i \in I} \) if the current state is \( k \). The probability that player 1 plays at stage \( n \) some action \( i \) in \( I \) is:

\[
x(p)(i) = \sum_{k \in K} p^k x_k(i).
\]

For each \( i \) in \( I \), the conditional probability on the state of stage \( n \) given that player 1 has played \( i \) at this stage is denoted by \( \hat{p}(x, i) \in \Delta(K) \). We have:

\[
\hat{p}(x, i) = \left( \frac{p^k x_k(i)}{x(p)(i)} \right)_{k \in K}
\]

(if \( x(p)(i) = 0, \hat{p}(x, i) \) is defined arbitrarily in \( \Delta(K) \)). Obviously, \( \sum_{i \in I} x(p)(i)\hat{p}(x, i) = p \).

The expected stage payoff for player 1 is, if player 2 plays according to \( y \) in \( \Delta(J) \):

\[
G(p, x, y) = \sum_{k \in K} p^k G^k(x_k, y).
\]

We can now state the recursive formulae, where \( v_0 \) is defined arbitrarily.

\textbf{Proposition 5.1} For each \( n \geq 1 \) and \( p \) in \( \Delta(K) \),

\[
v_n(p) = \frac{1}{n} \max_{x \in \Delta(I)^K} \min_{y \in \Delta(J)} \left( G(p, x, y) + (n - 1) \sum_{i \in I} x(p)(i) v_{n-1}(\hat{p}(x, i) M) \right)
\]

\[
= \frac{1}{n} \min_{y \in \Delta(J)} \max_{x \in \Delta(I)^K} \left( G(p, x, y) + (n - 1) \sum_{i \in I} x(p)(i) v_{n-1}(\hat{p}(x, i) M) \right)
\]

In \( \Gamma_n(p) \), player 1 has an optimal strategy \( \sigma = (\sigma_n)_{n \geq 1} \) such that at each stage \( n' \), \( \sigma_{n'} \) only depends on player 1’s past actions \( i_1, \ldots, i_{n'-1} \) and on the current state \( k_{n'} \).
Proof: The proof of proposition 5.1 is very standard, and will be generalized in section 6 to non revealing games. It goes by induction on \( n \). The result holds for \( n = 1 \) by definition of \( \Gamma_1(p) \). Fix \( n \geq 2 \), and assume that the proposition holds for \( n - 1 \) and all \( p \). Fix \( p \) in \( \Delta(K) \) and consider the game \( \Gamma_n(p) \).

We define an auxiliary zero-sum game \( A_n(p) \) with strategy spaces \( \Delta(I)^K \) for player 1 and \( \Delta(J) \) for player 2, and payoff function for player 1 defined by:

\[
 f_n^p(x, y) = \frac{1}{n}(G(p, x, y) + (n-1) \sum_{i \in I} x(p)(i) v_{n-1}(\hat{p}(x,i)M)) \text{ for all } x \in \Delta(I)^K \text{ and } y \in \Delta(J).
\]

Consider \( x = \lambda x' + (1-\lambda)x'' \), with \( \lambda \in [0,1] \) and \( x', x'' \in \Delta(I)^K \). For each \( i \), we have \( x(p)(i) \hat{p}(x,i) = \lambda x'(p)(i) \hat{p}(x',i) + (1-\lambda)x''(p)(i) \hat{p}(x'',i) \), hence by concavity of \( v_{n-1} \),

\[
x(p)(i) v_{n-1}(\hat{p}(x,i)M) \geq \lambda x'(p)(i) v_{n-1}(\hat{p}(x',i)M) + (1-\lambda)x''(p)(i) v_{n-1}(\hat{p}(x'',i)M).
\]

This shows that \( f_n^p \) is concave in \( x \). As it is convex in \( y \) and continuous, by Sion’s theorem (see e.g. Sorin 2002, Appendix A), \( A_n(p) \) has a value that we denote by \( f_n(p) \).

We now formally prove that player 1 can guarantee \( f_n(p) \) in \( \Gamma_n(p) \). Let \( \sigma \) in \( \Sigma_n \) be as follows: - at stage 1, play some \( x^* \) optimal for player 1 in \( A_n(p) \). - if \( i \in I \) is the action played at stage 1, play from stage 2 to stage \( n \) an optimal strategy \( \sigma_i \) for player 1 in the game \( \Gamma_{n-1}(\hat{p}(x,i)M) \). Let \( \tau \) be in \( T \), and denote by \( \mathcal{Y} \) in \( \Delta(J) \) the mixed action played by \( \tau \) at stage 1 and for each \( (i, j) \in I \times J \), by \( \tau_{i,j} \) the strategy played by player 2 using \( \tau \) at stages 2,\(\ldots\),\( n \) if \( (i, j) \) is played at stage 1. We have:

\[
 \gamma^p_n(\sigma, \tau) = \frac{1}{n}(G(p, x^*, y) + (n-1) \mathbb{E}_{p, \sigma, \tau} \left( \frac{1}{n-1} \sum_{n'=2}^n G^{k_{n'}}(i_{n'}, j_{n'}) \right))
\]

and

\[
 \mathbb{E}_{p, \sigma, \tau} \left( \frac{1}{n-1} \sum_{n'=2}^n G^{k_{n'}}(i_{n'}, j_{n'}) \right) = \sum_{i, j, k} x^*(p)(i) y(j) \mathbb{P}_{p, \sigma, \tau}(k_2 = k | i_1 = i) \mathbb{E}_{p, \sigma, \tau} \left( \frac{1}{n-1} \sum_{n'=2}^n G^{k_{n'}}(i_{n'}, j_{n'}) \right) \mid k_2 = k, i_1 = i, j_1 = j
\]

Since \( \mathbb{P}_{p, \sigma, \tau}(k_2 = k | i_1 = i) \) is \( \hat{p}(x^*, i)M \) for all \( i \).

For each \( i \) and \( j \) we have by definition of \( \sigma_i \):

\[
 \gamma^p_n(\sigma_i, \tau_{i,j}) = \gamma^p_n(\sigma_i, \tau_{i,j}) \geq v_{n-1}(\hat{p}(x^*, i)M).
\]

Hence \( \gamma^p_n(\sigma, \tau) \geq f^p_n(x^*, y) \geq f_n(p) \), and \( v_n(p) \geq f_n(p) \).

It is similarly possible to show that player 2 can defend \( f_n(p) \) in \( \Gamma_n(p) \). Fix \( \sigma \) in \( \Sigma \), and denote by \( x \) in \( \Delta(I)^K \) the strategy induced by \( \sigma \) at stage 1. Define \( \tau \) as follows: - at stage 1, play \( y^* \) in \( \Delta(J) \) such that: \( G(p, x, y^*) = \min_{y \in \Delta(J)} G(p, x, y) \), -at stages 2 from \( n \), if \( i \) has been played at stage 1 by player 1, play \( \tau_i \) optimal in the game \( \Gamma_{n-1}(\hat{p}(x,i)M) \). Similar computations as before show that \( \gamma^p_n(\sigma, \tau) \leq f_n(p) \).

Finally \( v_n(p) \leq f_n(p) \), thus \( v_n(p) = f_n(p) \).

An optimal strategy for player 1 in \( \Gamma_n(p) \) can be constructed as follows: at each stage \( n' \), compute the belief \( \hat{p}' \) of player 2 on the current state (depending on the actions \( i_1, \ldots, i_{n'-1} \) previously played by player 1), and play according to
some \( x \) optimal in the zero-sum game \( A_{n-n'+1}(p') \). This strategy only depends on his own past actions \( i_1, \ldots, i_{n'-1} \) and on the current state \( k_{n'} \).

Remark 5.2 \( v_n(p) \) is also the value of the \( n \)-stage stochastic game where the set of states is \( \Delta(K) \), player 1’s set of actions is \( \Delta(I)^K \), player 2’s set of actions is \( \Delta(J) \), the stage payoff is given by \( (p, x, y) \mapsto G(p, x, y) \), and the transition between states is controled by player 1 only: if the state is \( p \), and player 1 plays \( x \), the new state is \( \tilde{p}(x, i)M \) with probability \( x(p)(i) \). In this stochastic game, player 1 has an optimal strategy which is a Markovian strategy: the action played at each stage only depends on the current state and on the stage number.

Corollary 5.3 \( \forall p \in \Delta(K), v_n(p) - v_{n-1}(pM) \longrightarrow_{n \to \infty} 0. \)

**Proof:** Since \( v_{n-1} \) is concave, we have for each \( p \): \( v_n(p) \leq \frac{1}{n}(v_1(p) + (n-1)v_{n-1}(pM)) \). On the other hand, player 1 may play independently of the state. By taking \( x \in \Delta(I)^K \) such that \( x^k = x^{k'} \) for all \( k \) and \( k' \), we get: \( v_n(p) \geq \frac{1}{n}(u(p) + (n-1)v_{n-1}(pM)) \). Since payoffs are uniformly bounded, \( v_n(p) - v_{n-1}(pM) \longrightarrow_{n \to \infty} 0. \)

The next proposition is inspired by example A. We first define a function \( v^* \) on \( \Delta(K) \) which will turn out to be the value of \( \Gamma_\infty(p) \).

Definition 5.4 \( \forall p \in \Delta(K), v^*(p) = \inf_{N \geq 1} v_{NL}(pB_0) \).

\( v^* \) is concave as an infimum of concave functions. For each \( p \), \( v^*(p) = v^*(pB_0) \), hence for each \( q \) in \( Q \) the restriction of \( v^* \) to \( A(q) \) is constant.

Proposition 5.5 For each \( p \) in \( \Delta(K) \), player 2 can guarantee \( v^*(p) \) in \( \Gamma_\infty(p) \).

**Proof:** Fix \( p \) in \( \Delta(K), N \geq 1 \), and let \( \tau_{NL} \) be an optimal strategy for player 2 in \( \Gamma_{NL}(pB_0) \).

We divide the set of stages \( \{1, 2, \ldots, n, \ldots\} \) into consecutive blocks \( B_1, B_2, \ldots, B^m, \ldots \) of length \( NL \). Define the strategy \( \tau \) of player 2 in \( \Gamma_\infty(p) \) as follows: at each block \( B^m \), play according to \( \tau_{NL} \) (and forget everything that has happened at previous blocks). For each \( m \), \( B^m \) begins at stage \( (m-1)NL + 1 \), and the (unconditional) probability on the state at this stage is \( pM^{(m-1)NL} \).

Since \( |\gamma^{pB_0}_{NL}(\sigma', \tau') - \gamma^{pM^{(m-1)NL}}_{NL}(\sigma', \tau')| \leq ||pB_0 - pM^{(m-1)NL}||C \) for each strategy pair \( (\sigma', \tau') \), we have that for any strategy \( \sigma \) in \( \Sigma \),

\[
E_{p, \sigma, \tau} \left( \frac{1}{NL} \sum_{n \in B^m} G^{k_n}(i_n, j_n) \right) \leq v_{NL}(pB_0) + C||pB_0 - pM^{(m-1)NL}||.
\]

As \( (M^{NL})_n \) converges to \( B_0 \), we obtain that: \( \forall \varepsilon > 0 \ \exists N_0 \ \forall N_1 \geq N_0, \forall \sigma \in \Sigma \ \gamma^{k_n}_{NL}(\sigma, \tau) \leq v_{NL}(pB_0) + \varepsilon \). Hence player 2 can guarantee \( v_{NL}(pB_0) \) in \( \Gamma_\infty(p) \).
Corollary 5.6 \((v_n)_{n \geq 1}\) uniformly converges to \(v^*\) on \(\Delta(K)\). \(v^*\) is Lipschitz with constant \(C\) and satisfies \(v^*(p) = v^*(pM)\) for each \(p\) in \(\Delta(K)\).

Proof: From the definition of guaranteeing, proposition 5.5 implies that for all \(p\) in \(\Delta(K)\):
\[
\limsup_n v_n(p) \leq \inf_{n \geq 1} v_{nL}(pB_0) \leq \liminf_n v_{nL}(pB_0)
\]
As \(B_0^2 = B_0\), \((v_{nL}(pB_0))_n\) converges to \(v^*(p)\). Since it is clear from the definition of \(\gamma^p_n\) that \(v_n(p) - v_{n+1}(p) \to_{n \to \infty} 0\), \((v_n(pB_0))_n\) converges to \(v^*(p)\).

Put \(\underline{v}(p) = \liminf_n v_n(p)\) and \(\overline{v}(p) = \limsup_n v_n(p)\). By corollary 5.3, \(\underline{v}(p) = v(pM)\) and \(\overline{v}(p) = \overline{v}(pM)\). Moreover \(\underline{v}\) and \(\overline{v}\) are Lipschitz with constant \(C\), hence continuous. Since \((M^nL)_n\) converges to \(B_0\), we have \(\underline{v}(p) = \underline{v}(pB_0) = v^*(p) = \overline{v}(pB_0) = \overline{v}(p)\). And \((v_n(p))_n\) converges to \(v^*(p)\).

Since \((v_n)_n\) is a sequence of continuous concave functions (pointwise) converging to \(v^*\) which is continuous and concave, and since \(\Delta(K)\) is a polytope, \((v_n)_n\) uniformly converges to \(v^*\) (see for example Mertens et al., part A, p.46, ex. 15).

\[\square\]

Remark 5.7 One can also show that there exists a strategy \(\tau\) of player 2 that guarantees \(v^*(p)\) in \(\Gamma_\infty(p)\), i.e. that satisfies: \(\forall \epsilon > 0 \exists N_0, \forall N \geq N_0 \forall \sigma \in \Sigma, \gamma_N^\infty(\sigma, \tau) \leq v^*(p) + \epsilon\). To construct such \(\tau\), modify slightly the strategy presented in proposition 5.5 as follows. Divide the set of stages into consecutive blocks \(B^1, ..., B^m, ...,\) where for each \(m\), \(B^m\) has cardinality \(mL\). At each block \(B^m\), play according to an optimal strategy for player 2 in \(\Gamma_mL(pB_0)\). As \(v_{mL}(pB_0) \to_{m \to \infty} v^*(p)\), \(\tau\) has the required property. Since we will finally show that \(v^*(p)\) is the value of \(\Gamma_\infty(p)\), this will imply that player 2 has an optimal strategy in \(\Gamma_\infty(p)\).

Remark 5.8 It is known (see e.g. Sorin 2002) that in Aumann and Maschler’s case, \((v_N(p))_{N \geq 1}\) is non-increasing. This property is no longer satisfied here, and in point 4 of the last section we give an example where \((v_{NL}(p))_{N \geq 1}\) is not non-increasing.

The rest of the paper is devoted to the proof that player 1 can guarantee \(v^*(p)\) in \(\Gamma_\infty(p)\).

6 Non revealing games

We introduce and study in this section appropriate notions of non revealing strategies for player 1 and non revealing games. They generalize the ones of Aumann and Maschler.

6.1 Non revealing strategies

The basic idea is that in \(\Gamma_\infty(p)\), the relevant information is only the asymptotic behavior of the sequences of states, hence the projection \(pB_0\). This is due to
the fact that both players can “wait” for a large number of stages and place themselves approximately in $\Gamma_\infty(pB_0)$.

Fix some strategy $\sigma$ of player 1 and some stage $n$, and consider the point of view of player 2 after stage $n$ has been played (player 1 using $\sigma$). Player 2 can compute his belief $p_n$ on the next state, i.e. on the state of stage $n + 1$. He can estimate the asymptotic behavior of this sequence of states as follows: for any $t \geq n + 1$, his belief on the state of stage $t$ is $q_{n,t} = p_n M^{t-(n+1)}$. This sequence of beliefs can be asymptotically approximated by a periodic sequence of period $L$. This sequence is characterized, for example, by $\lim_{t \to \infty} q_{n,tL+1}$, which is $p_n B_{-n}$. Hence the relevant information for player 2 after stage $n$ has been played can be summarized by $p_n B_{-n}$.

Consequently we will define non revealing strategies for player 1 as strategies such that the sequence $(p_n B_{-n})_n$ is (almost surely) constant. For convenience, and because player 1 does not need to use the actions played by his opponent, we will also require that a non revealing strategy for player 1 plays independently of the actions played by player 2. For any $n$ in $\mathbb{N}$, we denote by $\hat{H}_n$ the set of possible actions of player 1 up to stage $n$. $\hat{H}_n = \{(i_1, ..., i_n) \in I^n \} = I^n (H_0$ standing for a singleton $\{h_0\})$.

Fix $p$ in $\Delta(K)$, and assume that player 1 plays in $\Gamma_\infty(p)$ some strategy $\sigma$ in $\Sigma^+$. $(p, \sigma)$ induces a stochastic process $(k_1, i_1, k_2, i_2, ..., k_n, i_n, ...)$ over $(K \times I)^\infty$. For $n$ in $\mathbb{N}$ and $h_n = (i_1, ..., i_n)$ in $\hat{H}_n$, we denote by $p_n(p, \sigma)(h_n)$ the belief of player 2 on the state $k_{n+1}$ of stage $n+1$ knowing that player 1 plays $\sigma$ and $h_n$ has occurred (i.e. player 1 played $i_1$ at stage 1, ..., $i_n$ at stage $n$). $p_n(p, \sigma)(h_n)$ is just a conditional probability. Indeed, we simply have for each state $k$:

$$p_n^k(p, \sigma)(h_n) = \mathbb{P}_{p,\sigma}(k_{n+1} = k|h_n).$$

And we define the relevant information of player 2 after $h_n$ has been played as:

$$q_n(p, \sigma)(h_n) = p_n(p, \sigma)(h_n)B_{-n} \in \Delta(K).$$

$p_n(p, \sigma)(h_n)$ and $q_n(p, \sigma)(h_n)$ are defined arbitrarily in $\Delta(K)$ if $\mathbb{P}_{p,\sigma}(h_n) = 0$ (we will proceed similarly for all further conditional probabilities). $p_n(p, \sigma)$ and $q_n(p, \sigma)$ are random variables defined on the measurable space $(\Omega, \mathcal{H}_n)$, where $\mathcal{H}_n$ is the $\sigma$-algebra generated by the projection of any play to the first $n$ actions of player 1. $p_0(p, \sigma)$ is just $p$ and $q_0(p, \sigma)$ is $pB_0$.

**Lemma 6.1** For any $\sigma$ in $\Sigma^+$, $(q_n(p, \sigma))_{n \geq 0}$ is, with respect to $\mathbb{P}_{p,\sigma}$, a $(\mathcal{H}_n)_{n \geq 0}$ martingale.

**Proof:** Fix $n$ in $\mathbb{N}$ and $h_n = (i_1, ..., i_n) \in \hat{H}_n$. Let $i_{n+1}$ be in $I$ and put $h_{n+1} = (i_1, ..., i_{n+1})$. We denote the conditional probability on the state $k_{n+1}$ on stage $n + 1$ knowing that player 1 has played $h_{n+1}$ by $r_{n}(p, \sigma)(h_{n+1})$. We have

$$\mathbb{E}_{p,\sigma}(r_{n}(p, \sigma)|h_n) = p_n(p, \sigma)(h_n) \text{ and } p_{n+1}(p, \sigma)(h_{n+1}) = r_{n}(p, \sigma)(h_{n+1})M.$$
Consequently,

\[
\mathbb{E}_{p,\sigma}(q_{n+1}(p, \sigma)|h_n) = \mathbb{E}_{p,\sigma}(p_{n+1}(p, \sigma)B_{1-n}|h_n) \\
= \mathbb{E}_{p,\sigma}(r_n(p, \sigma)MB_{1-n}|h_n) \\
= p_n(p, \sigma)(h_n)B_{-n} \\
= q_n(p, \sigma)(h_n).
\]

\[\Box\]

**Definition 6.2** A strategy \(\sigma\) in \(\Sigma^+\) is called non revealing at \(p\) if for each \(n\) in \(\mathbb{N}\), \(q_{n+1}(p, \sigma) = q_n(p, \sigma)\) a.s. \(\mathbb{P}_{p,\sigma}\) a.s.

We can already notice the following point. Let \(\sigma\) be non revealing at \(p\), \(n\) be a multiple of \(L\) and \(h_n\) be in \(\hat{H}_n\) such that \(\mathbb{P}_{p,\sigma}(h_n) > 0\). Then \(p_n(p, \sigma)(h_n)B_{-n} = pB_0\), hence \(p_n(p, \sigma)(h_n)B_0 = p_n(p, \sigma)(h_n)B_{-n}M^n = pB_0M^n = pB_0\). Thus \(p_n(p, \sigma)(h_n)\) is non revealing at \(p\), \(h_n\) is in \(A(pB_0)\). Whenever player 1 uses a non revealing strategy at \(p\), the belief of player 2 on the current state essentially remains in the set \(A(pB_0)\). This will be used to study the values of the non revealing games.

We denote by \(\Sigma(p)\) the set of strategies of player 1 that are non revealing at \(p\). If \(\sigma = (\sigma_n)_{n \geq 1}\) in \(\Sigma^+\) is independent of the states (i.e. if \(\forall n \geq 1, \sigma_n\) only depends on the \((n - 1)\) first actions of player 1), then \(p_n(p, \sigma) = pM^n\) for each \(n\), hence \(q_n(p, \sigma) = q_0(p, \sigma)\). Thus \(\sigma\) is non revealing at \(p\), and \(\Sigma(p)\) contains all such strategies of player 1. We now give a characterization, which can be seen as an alternative definition, of non revealing strategies.

Fix \(\sigma\) in \(\Sigma^+\), let \(n\) in \(\mathbb{N}\) and \(h_n = (i_1, ..., i_n)\) in \(\hat{H}_n\) be such that \(\mathbb{P}_{p,\sigma}(h_n) > 0\). We denote by \(x(p, \sigma)(h_n)\) in \(\Delta(I)^K\) the expectation of player 1’s action at stage \(n + 1\) depending on the current state:

\[
\forall i \in I, \forall k \in K, x^k(p, \sigma)(h_n)(i) = \mathbb{P}_{p,\sigma}(i_{n+1} = i|k_{n+1} = k; h_n)
\]

Recall that if \(p \in \Delta(K)\) is player 2’s belief on the state at some stage \(n\), computed after stage \(n - 1\), and player 1 plays at this stage according to some \(x\) in \(\Delta(I)^K\), for each \(i\) in \(I\) \(\hat{p}(x, i)\) denotes the updated probability on the state of stage \(n\) given that player 1 has played \(i\) at this stage. Hence player 2’s information on the asymptotic behavior of the sequence of states can be described by \(pB_0\) before stage \(n\) is played, and by \(\hat{p}(x, i)B_0\) after \(i\) has been played at stage \(n\).

**Definition 6.3** For each probability \(p\) on \(K\), we put:

\(NR(p) = \{x \in \Delta(I)^K, \forall i \in I \text{ s.t. } x(p)(i) > 0, \hat{p}(x, i)B_0 = pB_0\}\).

It is plain that \(NR(p)\) can also be written as:

\(NR(p) = \{x \in \Delta(I)^K, \forall i \in I, (p^kx^k(i))_{k \in K}B_0 = x(p)(i)pB_0\}\).

\(NR(p)\) contains all \(x = (x^k)_{k \in K}\) with \(x^k = x^{k'}\) for all \(k\) and \(k'\), hence is non empty. It is clearly a convex compact subset of \(\Delta(I)^K\). We can now characterize non revealing strategies at \(p\).
Proposition 6.4 Let $\sigma$ be in $\Sigma^+$, and $p$ be in $\Delta(K)$.

$\sigma$ is non revealing at $p \iff \forall n \in \mathbb{N}, \forall h_n \in \hat{H}_n$ s.t. $P_{p,\sigma}(h_n) > 0$, $x(p,\sigma)(h_n) \in NR(p_n(p,\sigma)(h_n))$.

Proof: Fix $p$ and $\sigma$. For simplicity of notations, we omit to mention $(p,\sigma)$ when writing $x(p,\sigma)(h_n)$, $p_n(p,\sigma)(h_n)$ and $q_n(p,\sigma)(h_n)$.

Let $n$ be in $\mathbb{N}$, $h_n = (i_1, ..., i_n)$ be in $\hat{H}_n$ s.t. $P_{p,\sigma}(h_n) > 0$. Put for simplicity $x = x(p,\sigma)(h_n) \in \Delta(I)^K$, and $p_n = p_n(p,\sigma)(h_n) \in \Delta(K)$. For any $i$ in $I$, $P_{p,\sigma}(i_{n+1} = i | h_n) = \sum_{k \in K} P_{p,\sigma}(k_{n+1} = k | h_n)x^k(i) = \sum_{k \in K} p^k_{i}\Delta(I)^K$.

Fix $i_{n+1}$ in $I$, and let $h_{n+1}$ be $(i_1, ..., i_n, i_{n+1})$. If $P_{p,\sigma}(h_{n+1}) > 0$, we have: $(P_{p,\sigma}(k_{n+1} = k | h_{n+1}))_{k \in K} = \tilde{p}_n(x, i_{n+1})$. So $p_{n+1}(h_{n+1}) = \tilde{p}_n(x, i_{n+1})M$, and $q_{n+1}(h_{n+1}) = \tilde{p}_n(x, i_{n+1})B_n$.

We now prove the $\implies$ part of the proposition. Assume that $\sigma$ is non revealing at $p$. Then for each $n$ in $\mathbb{N}$, $h_n$ in $\hat{H}_n$ and $i_{n+1}$ in $I$ s.t. $P_{p,\sigma}(h_n) > 0$ and $x(h_n)(i_{n+1}) > 0$, we have $q_{n+1}(h_n, i_{n+1}) = q_n(h_n)$. Writing $p_n$ for $p_n(h_n)$, we have $p_n(x(h_n), i_{n+1})B_n = p_nB_n$, so $p_n(x(h_n), i_{n+1})B_0 = p_nB_0$. Hence $x(h_n) \in NR(p_n)$.

We conclude with the $\iff$ part. Assume that $\forall n \in \mathbb{N}, \forall h_n \in \hat{H}_n$ s.t. $P_{p,\sigma}(h_n) > 0$, $x(h_n) \in NR(p_n(h_n))$.

Fix $n$ in $\mathbb{N}$, and $h_{n+1} = (i_1, ..., i_{n+1})$ s.t. $P_{p,\sigma}(h_{n+1}) > 0$. Put $h_n = (i_1, ..., i_n)$. Then $q_{n+1}(h_{n+1}) = \tilde{p}_n(x(h_n), i_{n+1})B_n = \tilde{p}_n(x(h_n), i_{n+1})B_0M^l$, where $n + l$ is a multiple of $L$. By hypothesis, $\tilde{p}_n(x(h_n), i_{n+1})B_0 = p_n(h_n)B_0$, thus $q_{n+1}(h_{n+1}) = p_n(h_n)B_0M^l = p_n(h_n)B_l = q_n(h_n)$. $(q_n)_{n \geq 0}$ is constant $P_{p,\sigma}$-a.s.

In order to illustrate the previous notions, we come back to the examples of section 3.

In example A, $(M^n)_n$ converges to $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, which is $B_0$. For each $p$ in $\Delta(K)$, $pB_0 = (1/2, 1/2)$ does not depend on $p$, hence every strategy of player 1 in $\Sigma^+$ is non revealing at $p$. Player 1 can use his information on the current state without revealing to player 2 any information on the limit behavior of the sequence of states. Here $Q$ reduces to the singleton $\{(1/2, 1/2)\}$, and $B_0$ is the projection matrix on $Q$.

In examples B and C, $B_0$ is the identity matrix hence for each $p$, $NR(p) = \{x \in \Delta(I)^K, \forall (k, k') s.t. p^k > 0 \text{ and } p^{k'} > 0, x^k = x^{k'}\}$. A non revealing strategy for player 1 is, like in Aumann and Maschler case, a strategy in $\Sigma^+$ that plays at each stage independently of the current state (formally, that plays the same mixed action in all states having a positive probability to be the current one). Example B shows how our definition generalizes the one of Aumann and Maschler.

These first three examples indeed only show extreme cases where the set of non revealing strategies at $p$ is mainly independent of $p$. In case of example E,
one may obtain that $B_0 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and for each $p$ with full support, 

$$NR(p) = \left\{ x \in \Delta(I)^K, \forall i \in I, x(p)(i) = x^c(i) = \frac{p^{x^a(i)} + p^{b \cdot x^b(i)}}{p^a + p^b} \right\}.$$ 

Here, $Q = \{(p^k)_{k \in K} \in \Delta(K), p^a = p^b\}$ and $B_0$ corresponds to the orthogonal projection on $Q$.

In Example D we have two recurrence classes, and $c$ is a transient state. 

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2/3 & 1/3 & 0 \end{pmatrix},$$ and $Q = \{(p^k)_{k \in K} \in \Delta(K), p^c = 0\}$. For $p$ in $\Delta(K)$ with full support, 

$$NR(p) = \left\{ x \in \Delta(I)^K, \forall i \in I, \frac{p^{x^a(i)} + 2/3p^{x^c(i)}}{p^a + 2/3p^c} = \frac{p^{b \cdot x^b(i)} + 1/3p^{c \cdot x^c(i)}}{p^b + 1/3p^c} \right\}.$$ 

We now prove some elementary results on the structure of the set of nonrevealing strategies of player 1. We consider the next lemma as the appropriate analogous for Aumann and Machler’s splitting lemma. Note that since player 1 has perfect recall in the original game $\Gamma_\infty(p)$, by Kuhn’s theorem any strategy of player 1 in $\Sigma$ can be viewed as a mixed strategy, i.e. as a probability distribution over the set of pure strategies of player 1 in $\Gamma_\infty(p)$, and vice-versa.

**Lemma 6.5** Fix $q$ in $Q$, and $S$ a positive integer.

Consider $p = \sum_{s=1}^{S} \lambda_s p_s$, with $p_1, ..., p_S$ elements of $A(q)$, $\lambda_1, ..., \lambda_S$ non negative numbers such that $\sum_{s=1}^{S} \lambda_s = 1$, and for each $s$ let $\sigma_s$ be a non revealing strategy for player 1 at $p_s$. Then we have:

1. If $\sigma$ in $\Sigma^+$ satisfies $P_{p,\sigma} = \sum_{s=1}^{S} \lambda_s P_{p_s,\sigma_s}$, then $\sigma$ is non revealing at $p$.
2. There exists some strategy $\sigma$ non revealing at $p$ and such that $P_{p,\sigma} = \sum_{s=1}^{S} \lambda_s P_{p_s,\sigma_s}$.

**Proof:**

1. Let $n$ be in $N$ and $h_n$ be in $H_n$ such that $P_{p,\sigma}(h_n) > 0$. For each $k$ in $K$,

$$p^k_n(p,\sigma)(h_n) = \frac{P_{p,\sigma}(h_n, k_{n+1} = k)}{P_{p,\sigma}(h_n)} = \frac{\sum_{s=1}^{S} \lambda_s P_{p_s,\sigma_s}(h_n, k_{n+1} = k)}{\sum_{s=1}^{S} \lambda_s P_{p_s,\sigma_s}(h_n)} = \frac{\sum_{s=1}^{S} \lambda_s P_{p_s,\sigma_s}(h_n) p^k_s(p_s, \sigma_s)(h_n)}{\sum_{s=1}^{S} \lambda_s P_{p_s,\sigma_s}(h_n)}$$

Hence $p_n(p,\sigma)(h_n)$ is in the convex hull of $\{p_n(p_s, \sigma_s)(h_n), s = 1, ..., S\}$. Since for each $s$ in $S$, $q_n(p_s, \sigma_s)(h_n) = p_s B_0 = q$, we have $q_n(p, \sigma)(h_n) = q = p B_0$ and $\sigma$ is non revealing at $p$. 

16
We define $\sigma$ via the splitting procedure of Aumann and Maschler. To play according to $\sigma$, observe the first state $k$, then choose $s$ in $S$ with probability $\lambda_s p_k^s / p_k$. Finally play in the whole game according to $\sigma_s$.

$\sigma$ is actually defined as a mixture of behavior strategies, and by Kuhn’s theorem, $\sigma$ can be viewed as an element of $\Sigma$. Moreover $\lambda_s$ does not depend on the moves of player 2, hence can be considered as an element of $\Sigma^+$. Let now $A$ be any measurable subset of $(K \times I)^\infty$.

$$
\mathbb{P}_{p,\sigma}(A) = \sum_{k \in K} \mathbb{P}_{p,\sigma}(k_1 = k) \mathbb{P}_{p,\sigma}(A|k_1 = k) \\
= \sum_{k \in K} p_k^s \sum_{s \in S} \mathbb{P}_{p,\sigma}(s|k_1 = k) \mathbb{P}_{p,\sigma}(A|k_1 = k, s) \\
= \sum_{s \in S} \lambda_s \sum_{k \in K} p_s^k \mathbb{P}_{p,\sigma}(A) \\
= \sum_{s \in S} \lambda_s \mathbb{P}_{p,\sigma}(A)
$$

So $\mathbb{P}_{p,\sigma} = \sum_{s=1}^S \lambda_s \mathbb{P}_{p_s,\sigma}$, and (1) shows that $\sigma$ is non revealing at $p$. \hfill $\square$

In the following corollary, $\hat{\Sigma}(p)$ is viewed as a subset of the set of mixed strategies of player 1.

**Corollary 6.6** The set of non revealing strategies of player 1 at $p$ is convex.

**Proof:** Let $\sigma_1$ and $\sigma_2$ be in $\hat{\Sigma}(p)$, and let $\lambda$ be in $[0, 1]$. Define the strategy $\sigma$ of player 1 as follows: with probability $\lambda$, play $\sigma_1$ and with probability $(1 - \lambda)$, play $\sigma_2$. By point (1) of lemma 6.5, $\sigma$ is non revealing at $p$. \hfill $\square$

We will also need in subsection 6.3 more elaborate properties of non revealing strategies. We now define two types of continuation strategies for player 1. Fix $\sigma$ in $\Sigma^+$ and $p$ in $\Delta(K)$, and consider that player 1 uses $\sigma$ in the original game $\Gamma_\infty(p)$. Although what follows is conceptually simple, we need numerous notations to be precise.

We first define, for each $n$ in $\mathbb{N}$ and $h_n = (i_1, ..., i_n) \in I^n$ s.t. $\mathbb{P}_{p,\sigma}(h_n) > 0$, the “expected continuation strategy” $\sigma(p, h_n)$ of player 1 after stage $n$ and the play of $h_n$.

For every possible sequence of states up to stage $n$, $h^n = (k_1, ..., k_n)$ in $K^n$, we denote by $\omega(h_n, h^n)$ the finite history of states and actions of player 1 up to stage $n$: $\omega(h_n, h^n) = (k_1, i_1, ..., k_n, i_n)$. We denote by $\sigma(h_n, h^n)$ the behavior strategy in $\Sigma^+$ played by $\sigma$ after $\omega(h_n, h^n)$. We define $\sigma(p, h_n)$ in $\Sigma^+$ as follows, similarly to the splitting procedure (the tilde now denoting random variables to avoid confusion): if $k_1 = k$, then choose $h^n$ in $K^n$ according to $\mathbb{P}_{p,\sigma}(h^n|h_n, \tilde{k}_{n+1} = k)$ and play according to $\sigma(h_n, h^n)$ (if $\tilde{k}_1 = k$ s.t. $\mathbb{P}_{p,\sigma}(\tilde{k}_{n+1} = k; h_n) = 0$, just play arbitrarily).
$\sigma(p,h_n)$ has the following interesting property. For $m$ in $\mathbb{N}$, $(k_{n+1},i_{n+1},\ldots,k_{n+m},i_{n+m}) \in (K \times I)^m$, define the events:

\[
A = (\tilde{k}_{n+1} = k_{n+1}, \tilde{i}_{n+1} = i_{n+1}, \ldots, \tilde{k}_{n+m} = k_{n+m}, \tilde{i}_{n+m} = i_{n+m})
\]

and

\[
B = (k_1 = k_{n+1}, i_1 = i_{n+1}, \ldots, k_m = k_{n+m}, i_m = i_{n+m}).
\]

We have:

\[
\mathbb{P}_{p,\sigma}(A|h_n) = \sum_{k \in K} p^k_n(p,\sigma)(h_n) \mathbb{P}_{p,\sigma}(A|h_n, \tilde{k}_{n+1} = k)
\]

\[
= \sum_{k \in K} p^k_n(p,\sigma)(h_n) \sum_{h^n \in K^n} \mathbb{P}_{p,\sigma}(h^n|h_n, \tilde{k}_{n+1} = k) \mathbb{P}_{p,\sigma}(A|\omega(h_n,h^n), \tilde{k}_{n+1} = k)
\]

\[
= \sum_{k \in K} p^k_n(p,\sigma)(h_n) \sum_{h^n \in K^n} \mathbb{P}_{p,\sigma}(h^n|h_n, \tilde{k}_{n+1} = k) \mathbb{P}_{\delta^k,\sigma(h_n,h^n)}(B)
\]

\[
= \mathbb{P}_{p_n(p,\sigma)(h_n),\sigma(p,h_n)}(B)
\]

The following lemma expresses some kind of subgame property of non-revealing strategies.

**Lemma 6.7** Let $\sigma$ be in $\Sigma^+$, and $p$ be in $\Delta(K)$.

$\sigma \in \hat{\Sigma}(p) \iff \forall n \in \mathbb{N}, \forall h_n \in I^n \text{ s.t. } \mathbb{P}_{p,\sigma}(h_n) > 0, \sigma(p,h_n) \in \hat{\Sigma}(p_n(p,\sigma)(h_n))$

**Proof:**

$\iff$ just take $n = 0$.

$\implies$ Assume that $\sigma$ is in $\hat{\Sigma}(p)$, and let $n$ be in $\mathbb{N}$, $h_n = (i_1, \ldots, i_n) \in I^n$ be such that $\mathbb{P}_{p,\sigma}(h_n) > 0$. We have, by definition of $\hat{\Sigma}(p)$, $p_n(p,\sigma)(h_n)B_{-n} = pB_0$, and have to show that $\sigma(p,h_n) \in \hat{\Sigma}(p_n(p,\sigma)(h_n))$. We put for simplicity $\sigma' = \sigma(p,h_n)$ and $\rho' = p_n(p,\sigma)(h_n)$.

Fix $m$ in $\mathbb{N}$, and $h(m) = (i_{n+1}, \ldots, i_{n+m})$ in $I^m$. Put:

- $\omega = (i_1, \ldots, i_n, i_{n+1}, \ldots, i_{n+m})$.
- $A = (i_{n+1} = i_{n+1}, \ldots, i_{n+m} = i_{n+m})$
- $B = (i_1 = i_{n+1}, \ldots, i_m = i_{n+m})$.

Assume that $\mathbb{P}_{\rho',\sigma'}(B) > 0$. We have to show that $p_m(p',\sigma')(h(m))B_{-m} = p'B_0$.

$\mathbb{P}_{\rho,\sigma}(\omega) = \mathbb{P}_{\rho,\sigma}(h_n) \mathbb{P}_{\rho,\sigma}(A|h_n) = \mathbb{P}_{\rho,\sigma}(h_n) \mathbb{P}_{\rho',\sigma'}(B) > 0$. Since $\sigma$ is in $\hat{\Sigma}(p)$, $p_{n+m}(p,\sigma)(\omega)B_{-n-m} = pB_0$. For each $k$ in $K$,

\[
\mathbb{P}_{p,\sigma}(\omega, \tilde{k}_{n+m+1} = k) = \mathbb{P}_{p,\sigma}(h_n) \mathbb{P}_{p,\sigma}(A, \tilde{k}_{n+m+1} = k|h_n)
\]

\[
= \mathbb{P}_{p,\sigma}(h_n) \mathbb{P}_{\rho',\sigma'}(B, \tilde{k}_{n+m+1} = k)
\]

So we obtain:

\[
p^k_{p+m}(p,\sigma)(\omega) = \frac{\mathbb{P}_{\rho',\sigma'}(B, \tilde{k}_{n+m+1} = k)}{\mathbb{P}_{\rho',\sigma'}(B)} = p^k_m(p',\sigma')(h(m)).
\]

Hence $p_m(p',\sigma')(h(m))B_{-n-m} = p_{n+m}(p,\sigma)(\omega)B_{-n-m} = pB_0 = p'B_{-n}$. Multiplying by $M^n$ both sides, we obtain that $p_m(p',\sigma')(h(m))B_{-m} = p'B_0$.

\[\square\]
In the same spirit, we now define for each \( n \) in \( \mathbb{N} \) the expected strategy \( \sigma(p, n+) \) of player 1 in \( \Gamma_{\infty}(p) \) after stage \( n \). If \( k_1 = k \), choose \((h_n, h^n)\) in \( I^n \times K^n \) according to \( P_{p, \sigma}(\omega(h_n, h^n) | \mathbf{\tilde{k}}_{n+1} = k) \) and play according to \( \sigma(h_n, h^n) \) (if \( k_1 = k \) such that \( P_{p, \sigma}(\mathbf{\tilde{k}}_{n+1} = k) = 0 \), play arbitrarily).

Note that \( \sigma(p, 0+) \) is just \( \sigma \) (up to events with probability zero). For \( m \) in \( \mathbb{N} \) and \((k_{n+1}, i_{n+1}, \ldots, k_{n+m}, i_{n+m}) \in (K \times I)^m\), define as before:

\[
A = (\tilde{k}_{n+1} = k_{n+1}, \tilde{i}_{n+1} = i_{n+1}, \ldots, \tilde{k}_{n+m} = k_{n+m}, \tilde{i}_{n+m} = i_{n+m})
\]

and

\[
B = (k_1 = k_{n+1}, \tilde{k}_1 = i_{n+1}, \ldots, \tilde{k}_m = k_{n+m}, \tilde{i}_m = i_{n+m}).
\]

One has for each \( k \) in \( K \):

\[
P_{p, \sigma}(A | \tilde{k}_{n+1} = k) = \sum_{(h_n, h^n) \in I^n \times K^n} P_{p, \sigma}(\omega(h_n, h^n) | \tilde{k}_{n+1} = k) P_{p, \sigma}(A | \omega(h_n, h^n), \tilde{k}_{n+1} = k) = \sum_{(h_n, h^n) \in I^n \times K^n} P_{p, \sigma}(\omega(h_n, h^n) | \tilde{k}_{n+1} = k) P_{\tilde{k}, \sigma(h_n, h^n)}(B)
\]

Hence,

\[
P_{p, \sigma}(A) = \sum_{k \in K} P_{p, \sigma}(\tilde{k}_{n+1} = k) \sum_{(h_n, h^n) \in I^n \times K^n} P_{p, \sigma}(\omega(h_n, h^n) | \tilde{k}_{n+1} = k) P_{\tilde{k}, \sigma(h_n, h^n)}(B) = \sum_{k \in K} P_{p, \sigma}(\tilde{k}_{n+1} = k) P_{p, \sigma(p, n+)}(B) = P_{p, \sigma(p, n+)}(B)
\]

We then obtain that:

\[
P_{p, \sigma(p, n+)}(B) = P_{p, \sigma}(A) = \sum_{h_n \in I^n} P_{p, \sigma}(h_n) P_{p, \sigma}(A | h_n) = \sum_{h_n \in I^n} P_{p, \sigma}(h_n) P_{p, \sigma(p, n+)(h_n), \sigma(p, h_n)}(B)
\]

Consequently, \( P_{p, \sigma(p, n+)} = \sum_{h_n \in I^n} P_{p, \sigma(h_n)} P_{p, \sigma(p, n+)(h_n), \sigma(p, h_n)}. \)

We now have an analogous of lemma 6.7 for this type of continuation strategy.

**Lemma 6.8** Let \( \sigma \) be in \( \Sigma^+ \) and \( p \) be in \( \Delta(K) \).

\[
\sigma \in \Sigma(p) \iff \forall n \in \mathbb{N}, \sigma(p, n+) \in \Sigma(p M^n)
\]

**Proof:** \( \iff \) is clear. We prove the \( \Rightarrow \) part.

Let \( n \) be in \( \mathbb{N} \). Put \( p' = p M^n \), and \( \sigma' = \sigma(p, n+) \). We have:

\[
P_{p', \sigma'} = \sum_{h_n \in I^n} P_{p, \sigma}(h_n) P_{p, \sigma(p, n+)(h_n), \sigma(p, h_n)}, \quad p' = \sum_{h_n \in I^n} P_{p, \sigma}(h_n) p_{p, \sigma}(h_n).
\]

19
For each \( h_n \) such that \( P_{p,\sigma}(h_n) > 0 \), by the previous lemma \( \sigma(p, h_n) \in \hat{\Sigma}(p, \sigma)(h_n) \) and since \( \sigma \in \hat{\Sigma}(p), p_n(p, \sigma)(h_n) \in \mathcal{B} \) \(-\infty, \infty\). Hence \( p_n(p, \sigma)(h_n) \in \mathcal{B} \) \(-\infty, \infty\). By lemma 6.5, part (1), \( \sigma(p, n+) \in \hat{\Sigma}(pM^n) \).

Finally we also need to define explicitly \( N \)-stage non revealing strategies.

**Definition 6.9** If \( \sigma = (\sigma_n)_{n \geq 1} \) is in \( \hat{\Sigma}(p) \), the restriction of \( \sigma \) to the first \( N \)-stages, i.e. the strategy \( (\sigma_n)_{n=1,...,N} \), is called a \( N \)-stage non revealing strategy for player 1 at \( p \). We denote by \( \hat{\Sigma}_N(p) \) the set of such strategies.

### 6.2 Definition of the non revealing games

**Definition 6.10** For \( p \) in \( \Delta(K) \), the non revealing game at \( p \), denoted by \( \hat{\Gamma}_\infty(p) \), is the game obtained from \( \Gamma_\infty(p) \) by restricting player 1 to play a strategy in \( \hat{\Sigma}(p) \).

The notions of guaranteeing and value in \( \hat{\Gamma}_\infty(p) \) are defined as in definition 2.1: one just have to replace everywhere \( \Gamma_\infty(p) \) by \( \hat{\Gamma}_\infty(p) \) and \( \Sigma \) by \( \hat{\Sigma}(p) \). The \( N \)-stages non revealing games are defined as follows:

**Definition 6.11** For \( N \geq 1 \), the \( N \)-stage non revealing game at \( p \) is defined as the zero-sum game \( \hat{\Gamma}_N(p) \) with strategy spaces \( \hat{\Sigma}_N(p) \) for player 1, \( T_N \) for player 2 and with payoff function \( \gamma_N^p \) for player 1.

### 6.3 Value of the non revealing games

We first study the value of the \( N \)-stage games.

**Proposition 6.12** For each positive \( N \), and probability \( p \) on \( K \):

1. \( \hat{\Gamma}_N(p) \) has a value, denoted by \( \hat{v}_N(p) \) and both players have optimal strategies.
2. \( \hat{v}_N(p) \) is an upper semi-continuous (u.s.c.) mapping from \( \Delta(K) \) to \( \mathbb{R} \), and \( |\hat{v}_N(p)| \leq C \).
3. For each \( q \) in \( Q \), the restriction of \( \hat{v}_N \) to \( A(q) \) is concave.

**Proof:** \( \hat{\Gamma}_N(p) \) is the zero-sum game \( (\hat{\Sigma}_N(p), T_N, \gamma_N^p) \). In the original \( N \)-stage game \( (\Sigma_N, T_N, \gamma_N^p) \), the sets of mixed strategies of both players are the convex hull of a finite number of points, hence are convex subsets of some Euclidean space. By corollary 6.6, \( \hat{\Sigma}_N(p) \) is a convex subset of \( \Sigma_N \) (we always identify mixed and behavior strategies). Since \( \gamma_N^p \) is bilinear over \( \Sigma_N \times T_N \), if we show that \( \hat{\Sigma}_N(p) \) is closed in \( \Sigma_N \) we can apply Sion’s theorem to obtain that \( \hat{\Gamma}_N(p) \) has a value and both players have optimal strategies.

Put \( H = \bigcup_{n=0}^{N-1} \hat{H}_{n+1} \), and consider the mapping \( F \) from \( \Delta(K) \times \Sigma_N^+ \) to \( (\mathbb{R}^K)^H \) such that for each \( (p, \sigma) \) in \( \Delta(K) \times \Sigma_N^+ \), \( F(p, \sigma) \) is:

\[
F(p, \sigma) = \left( P_{p,\sigma}(h_{n+1}) \left( q_{h_{n+1}}^k(p, \sigma)(h_{n+1}) - q_{h_{n+1}}^k(p, \sigma)(i_1, ..., i_n) \right) \right)_{k \in K, n=0,...,N-1, h_{n+1}=(i_1,...,i_n) \in \hat{H}_{n+1}}.
\]
It is clear that we have \( F(p, \sigma) = 0 \) if and only if \( \sigma \in \Sigma_N(p) \).

For each \((p, \sigma)\), each \( n \) in \( \{0, \ldots, N-1\} \) and each \( h_{n+1} = (i_1, \ldots, i_{n+1}) \) in \( H_{n+1} \), we have:

\[
P_{p,\sigma}(h_{n+1}) \left( q_{n+1}(p, \sigma)(h_{n+1}) - q_n(p, \sigma)(h_n) \right) = P_{p,\sigma}(h_{n+1}) \left( (p_{n+1}(p, \sigma)(h_{n+1})B_{n+1} - p_n(p, \sigma)(h_n)B_n \right)
\]

\[
= (P_{p,\sigma}(h_{n+1}, k_{n+2} = k))_k B_{n+1} - (P_{p,\sigma}(h_n, k_{n+1} = k))_k B_n.
\]

(with an obvious convention if \( P_{p,\sigma}(h_n) = 0 \))

The mapping \(((p, \sigma) \mapsto P_{p,\sigma}(h))\) being continuous for any finite history \( h \) in \((K \times I)^N\), \( F \) is continuous.

We first obtain that \( \Sigma_N(p) \) is closed in \( \Sigma_N^+ \), hence is a compact subset of some Euclidean space. Thus (1) is proved by Sion’s theorem.

Secondly, the correspondence from \( \Delta(K) \) to \( \Sigma_N^+ \) which associates to each \( p \) the set \( \hat{\Sigma}_N(p) \) has a closed graph. So \( \hat{\Sigma}_N \) is upper semi continuous. \(|\hat{\Sigma}_N(p)| \leq C\) being obvious, (2) is proved.

Finally, (3) is just a consequence of the splitting lemma: with the same notations as lemma 6.5, choose for each \( s \) \( \sigma_s \) an optimal strategy of player 1 in \( \hat{\Sigma}_N(p_s) \). □

We can now write a recursive formulae for \( \hat{\Sigma}_n \) (\( \hat{\Sigma}_0 \) being defined arbitrarily).

**Proposition 6.13** For each \( n \geq 1 \) and \( p \) in \( \Delta(K) \),

\[
\hat{\Sigma}_n(p) = \frac{1}{n} \max_{x \in NR(p)} \min_{y \in \Delta(J)} \left( G(p, x, y) + (n - 1) \sum_{i \in I} x(p)(i) \hat{\Sigma}_{n-1}(\hat{p}(x, i)M) \right)
\]

\[
= \frac{1}{n} \max_{y \in \Delta(J)} \min_{x \in NR(p)} \left( G(p, x, y) + (n - 1) \sum_{i \in I} x(p)(i) \hat{\Sigma}_{n-1}(\hat{p}(x, i)M) \right)
\]

**Proof:** The proof is very similar to that of proposition 5.1. For \( n = 1 \), the result is clear since a strategy \( \sigma \) in \( \Sigma_1(p) \) can be seen as an element of \( NR(p) \) by proposition 6.4. Fix \( n \geq 2 \), and assume that the proposition holds for \( n - 1 \) for all \( p \). Fix \( p \) in \( \Delta(K) \).

Define the auxiliary zero-sum game \( \hat{A}_n(p) \) with strategy spaces \( NR(p) \) for player 1 and \( \Delta(J) \) for player 2, and payoff function for player 1 defined by:

\[
f_n(x, y) = \frac{1}{n} (G(p, x, y) + (n - 1) \sum_{i \in I} x(p)(i) \hat{\Sigma}_{n-1}(\hat{p}(x, i)M))
\]

for all \( x \) in \( NR(p) \) and \( y \) in \( \Delta(J) \). We are going to apply Sion’s theorem to prove that \( \hat{A}_n(p) \) has a value.

\( x \) being fixed, \( (y \mapsto f_n(x, y)) \) is affine hence convex and continuous. Consider \( x = \lambda x' + (1 - \lambda)x'' \), with \( \lambda \in [0, 1] \) and \( x', x'' \) in \( NR(p) \). For each \( i \) s.t. \( x'(p)(i) > 0 \) and \( x''(p)(i) > 0 \), we have \( \hat{p}(x, i)M = \frac{\lambda x'(p)(i)}{x'(p)(i)} \hat{p}(x', i)M + \frac{(1 - \lambda)x''(p)(i)}{x''(p)(i)} \hat{p}(x'', i)M \), and by definition of \( NR(p) \), \( \hat{p}(x', i)MB_0 = pB_0M = \hat{p}(x'', i)MB_0M \). Since \( B_0M = MB_0 \), we obtain that \( \hat{p}(x', i)M, \hat{p}(x'', i)M \) and \( \hat{p}(x, i)M \) all belong to \( A(pMB_0) \). By concavity of \( \hat{\Sigma}_{n-1} \) on \( A(pMB_0) \),

\[
x(p)(i) \hat{\Sigma}_{n-1}(\hat{p}(x', i)M) + \lambda x'(p)(i) \hat{\Sigma}_{n-1}(\hat{p}(x', i)M) + (1 - \lambda)x''(p)(i) \hat{\Sigma}_{n-1}(\hat{p}(x'', i)M).
\]
This proves that \( \hat{f}_n^p \) is concave in \( x \). Consider now a sequence \((x_t)\), of elements in \( NR(p) \) converging to some \( x \). For each \( i \) s.t. \( x(p)(i) > 0 \), \( \hat{p}(x_t, i) \rightarrow_{t \to \infty} \hat{p}(x, i) \) hence, since \( \hat{v}_{n-1} \) is u.s.c., \( \limsup_{t \to \infty} \sum_{i \in I} x(p_t)(i) \hat{v}_{n-1}(\hat{p}(x_t, i)M) \leq \sum_{i \in I} x(p)(i) \hat{v}_{n-1}(\hat{p}(x, i)M) \). This shows that \( y \) being fixed, \((x \rightarrow f_n^p(x, y)) \) is upper semi-continuous. By Sion’s theorem, \( \hat{A}_n(p) \) has a value that we denote by \( f_n(p) \).

We now show that \( f_n(p) \) is the value of \( \Gamma_n(p) \). As in the proof of proposition 5.1, we define \( \sigma \in \Sigma_N \) as follows: - at stage 1, play some \( x^* \) optimal for player 1 in \( \hat{A}_n(p) \). - from stage 2 to \( n \), if \( i \) in \( I \) is the action played at stage 1, play an optimal strategy \( \sigma_i \) for player 1 in the game \( \Gamma_{n-1}(\hat{p}(x, i)M) \). Using proposition 6.4, one can see that \( \sigma \) is non revealing at \( p \). As in the proof of proposition 5.1, \( \sigma \) guarantees \( f_n(p) \) in \( \Gamma_n(p) \). Similarly, we show that player 2 can defend \( f_n(p) \) in \( \Gamma_n(p) \). Fix \( \sigma \in \Sigma_N \), and denote by \( x \) in \( NR(p) \) the strategy induced by \( \sigma \) at stage 1. For each \( i \) in \( I \) s.t. \( x(p)(i) > 0 \), the strategy induced by \( \sigma \) at stage 2 to \( n \) if \( i \) has been played at stage 1 is in \( \bar{\Pi}_{n-1}(\hat{p}(x, i)M) \) by lemma 6.7. Consequently, the same strategy \( \tau \) as in the proof of proposition 5.1 satisfies \( \gamma_n^p(\sigma, \tau) \leq f_n(p) \). □

As for \( v_n \), we have for each \( n \geq 1 \) and \( p \in \Delta(K) \) that \( \hat{v}_n(p) \geq \frac{1}{n}(u(p) + (n-1)\hat{v}_{n-1}(pM)) \), and by concavity of \( \hat{v}_{n-1} \) on \( A(pMB_0) \), \( \hat{v}_n(p) \leq \frac{1}{n}(\hat{v}_1(p) + (n-1)\hat{v}_{n-1}(pM)) \). Hence \( \hat{v}_n(p) - \hat{v}_{n-1}(pM) \rightarrow_{n \to \infty} 0 \). As \( \hat{v}_n(p) = \max_{\sigma \in \Sigma(p)} \min_{\tau \in T} \gamma_n^p(\sigma, \tau) \) it is also clear that \( \hat{v}_n(p) - \hat{v}_{n-1}(p) \rightarrow_{n \to \infty} 0 \). Thus \( \hat{v}_n(p) - \hat{v}_n(pM) \rightarrow_{n \to \infty} 0 \).

A main interest of the recursive formulae for \( \hat{v}_n \) is the following corollary.

**Corollary 6.14** For each \( n \), \( \hat{v}_n \) is continuous.

**Proof:**

We first modify the recursive formulae by changing variables. For any \( p \) in \( \Delta(K) \) and \( x \) in \( NR(p) \), we put \( z(p, x) = (p^k x^k(i))_{k \in K, i \in I} \in (\mathbb{R}^K)_i \). Define \( Z \) as the correspondence from \( \Delta(K) \) to \((\mathbb{R}^K)^I \) which associates to each probability \( p \) the set \( \{z(p, x) \mid x \in NR(p)\} \).

For \( z = (z^k_{i})_{k \in K, i \in I} \) in the non-negative orthant \((\mathbb{R}^K)_+ \), we put for each \( i \) in \( I \), \( z(i) = \sum_{k \in K} z^k_{i} \) and \( z = (z^k_{i})_{k \in K, i \in I} \in \mathbb{R}^K \). \( \frac{z_i}{z(i)} \) is thus an element of \( \Delta(K) \) (defined arbitrarily if \( z(i) = 0 \)). Define finally \( H(z) = \min_{y \in \Delta(J)} \sum_{k \in K} \sum_{i \in I} z^k_{i} G_i^k(i, y) \). The recursive formulae becomes: \( \forall n \geq 1, \forall p \in \Delta(K), \)

\[
\hat{v}_n(p) = \frac{1}{n} \max_{z \in Z(p)} \left( H(z) + (n-1) \sum_{i \in I} z(i) \hat{v}_{n-1}(\frac{z_i}{z(i)}M) \right).
\]

We have \( Z(p) = \{z \in (\mathbb{R}^K)_+, \sum_{i \in I} z_i = p \text{ and } \forall i \in I, z_i B_0 = z(i) p B_0 \} \), hence \( Z(p) \) is a polytope. \( H \) is continuous, and if \( f : \Delta(K) \rightarrow \mathbb{R} \) is continuous, so is the mapping \( z \rightarrow \sum_{i \in I} z(i) f(\frac{z_i}{z(i)}M) \). If we prove that the correspondence \( Z \) is continuous (i.e. lower and upper semi-continuous) then by induction we obtain that \( \hat{v}_n \) is continuous for each \( n \). As it is clear that \( Z \) has a compact graph, we just have to show that \( Z \) is lower semi-continuous (l.s.c. for short).

22
Fix $p$ in $\Delta(K)$ and $z$ in $Z(p)$. We put $D = \{(k,i) \in K \times I, \, z(i)p^k - z^k_i > 0\}$. In order to prove that $Z$ is l.s.c., we will show that:

$$\forall p' \in \Delta(K), \, \exists z' \in Z(p') \text{ with } \|z - z'\| \leq \|p - p'\|(1 + A/B),$$

where $A = \sum_{k \in K, i \in I} |p^k z(i) - z^k_i|$ and $B = \min_{(k,i) \in D} z(i)p^k - z^k_i$, with the convention that $A/B = 0$ if $D = \emptyset$.

First define $z'$ in $(IR^K)^I$ s.t. for each $i$ and $k$, $z'^k_i = z^k_i + z(i)(p^k - p^k_k)$. We have $z'_i = p'$, for each $i$ in $I$, $z'(i) = z(i), (z'_i - z'(i)p')B_0 = 0$ and $\|z - z'\| = \|p - p'\|$. If $D = \emptyset$, $z' \in (IR^K)^I_+$ and we are done. Assume now that $z'$ is not in $(IR^K)^I_+$. Define $z'$ in $(IR^K)^I_+$ such that for each $i$ and $k$, $z'^k_i = p^k z(i)$. $z''$ belongs to $Z(p')$. Finally put, for every $\lambda$ in $[0, 1]$, $z(\lambda) = (1 - \lambda)z' + \lambda z''$. For each $\lambda$, $z(\lambda) \in Z(p')$ if and only if $z(\lambda)$ has non-negative coordinates.

Let $\lambda \in ]0, 1]$ be $\max_{(k,i) \in D} \frac{z(i)p^k - z^k_i - z(i)p^k_k}{z(i)p^k - z^k_i}$. A simple computation shows that $z(\lambda) \in (IR^K)^I_+$ and that $\lambda^* \leq \|p - p'\|/B$. Hence:

$$\|z(\lambda^*) - z\| = \sum_{k \in K} \sum_{i \in I} |z^k_i(\lambda^*) - z^k_i|$$

$$= \sum_{k \in K} \sum_{i \in I} |z(i)(p^k - p^k_k) + \lambda^* p^k z(i) - \lambda^* z^k_i|$$

$$\leq \sum_{k \in K} \sum_{i \in I} |z(i)| |p^k - p^k_k| + \lambda^* \sum_{k \in K} \sum_{i \in I} |p^k z(i) - z^k_i|$$

$$\leq \|p - p'\| + \|p - p'\| A/B$$

This proves that $Z$ is l.s.c., and we obtain that for each $n$, $\hat{v}_n$ is continuous.□

We can now study the non revealing games with infinitely many stages. We first define an auxiliary mapping from $\Delta(K)$ to $R$.

**Definition 6.15** $\forall p \in \Delta(K)$, $w(p) = \inf_{N \geq 1} \hat{v}_{NL}(pB_0)$.

As an infimum of continuous functions, $w$ is u.s.c. on $\Delta(K)$. For each $q$ in $Q$, it is constant on $A(q)$. $w(p)$ will turn out to be the value of the non revealing game $\hat{\Gamma}_\infty(p)$.

**Theorem 6.16** In any non revealing game $\hat{\Gamma}_\infty(p)$, player 2 can guarantee $w(p)$.

**Proof:** Let $N$ be a positive multiple of $L$, and define a strategy $\tau$ of player 2 as follows. Divide the set of stages into consecutive blocks $B^1, ..., B^m, ...$ of length $N$. For each positive $m$, $\tau$ plays on block $B^m$ an optimal strategy $\tau_m$ in $\hat{v}_N(pM^{(m-1)N})$, independently on what happened at previous blocks.

Fix now $\sigma$ a strategy for player 1 in $\hat{\Sigma}(p)$. At some block $B^m$, the expected payoff induced by $p$, $\sigma$ and $\tau$ at $B^m$ is:

$$E_{p,\sigma,\tau} \left( \frac{1}{N} \sum_{n \in B^m} G^k_n (i_n, j_n) \right)$$

Since $\sigma$ is independent of the moves of player 2, this payoff only depends on $\tau_m$ and on the stochastic process induced by $p$ and $\sigma$ over the states and actions of
player 1 at this block. But since $\mathbb{P}_{pM^{(m-1)}N, \sigma(p, (m-1)N+)}(B) = \mathbb{P}_{p, \sigma}(A)$ for any events $A$ and $B$ as before (see before lemma 6.8), we have that:

$$\mathbb{E}_{p, \sigma, \tau} \left( \frac{1}{N} \sum_{n \in B^m} G^{k_n}(i_n, j_n) \right) = \mathbb{E}_{pM^{(m-1)}N, \sigma(p, (m-1)N+), \tau_m} \left( \frac{1}{N} \sum_{n=1}^{N} G^{k_n}(i_n, j_n) \right)$$

By lemma 6.8, $\sigma(p, (m-1)N+)$ is in $\hat{\Sigma}(pM^{(m-1)}N)$, hence by definition of $\tau_m$, this expected payoff is at most $\hat{\nu}(pM^{(m-1)}N)$. Consequently, for each $M$:

$$\gamma^p_{NM}(\sigma, \tau) \leq \frac{1}{M} \sum_{m=1}^{M} \hat{\nu}(pM^{(m-1)}N),$$

and by concavity of $\hat{\nu}$ on $A(pB_0)$, we get:

$$\gamma^p_{NM}(\sigma, \tau) \leq \hat{\nu} \left( \frac{1}{M} \sum_{m=1}^{M} pM^{(m-1)}N \right).$$

$\hat{\nu}$ being continuous by corollary 6.14, the right-hand side converges to $\hat{\nu}(pB_0)$ as $M$ goes to infinity, and this convergence is uniform in $\sigma$. We have obtained:

$$\forall \varepsilon > 0 \exists T_0 \forall T \geq T_0 \forall \sigma \in \Sigma(p) \gamma^p_{N}(\sigma, \tau) \leq \hat{\nu}(pB_0) + \varepsilon.$$ 

Note that the previous proof also directly gives the following interesting property.

**Proposition 6.17** Let $p$ be in $\Delta(K)$, $M$ and $N$ be positive integers with $N$ a multiple of $L$. Then:

$$\hat{\nu}_{NM}(p) = \frac{1}{M} \sum_{m=1}^{M} \hat{\nu}(pM^{(m-1)}N).$$

For each $q$ in $Q$, we have $qM^L = q$, hence the previous proposition gives:

$$\forall N \geq 1, \forall M \geq 1, \hat{\nu}_{LM}(q) \leq \hat{\nu}_{LN}(q).$$

We obtain the following corollary, which will be used later in section 7:

**Corollary 6.18** For each $q$ in $Q$, $(\hat{\nu}_{L2N}(q))_{N \geq 1}$ is non-increasing.

We can now also prove the convergence of the value of the $N$-stage games.

**Corollary 6.19** For each $p$ in $\Delta(K)$, $\hat{\nu}(p) \longrightarrow_{N \to \infty} w(p)$.

**Proof**

For each $p$ in $\Delta(K)$, by theorem 6.16 player 2 can guarantee $w(p)$ in $\hat{\Gamma}(p)$, hence: $\limsup_N \hat{\nu}(p) \leq w(p)$.

Consequently, for each $q$ in $Q$:

$$\limsup_N \hat{\nu}(q) \leq \inf_N \hat{\nu}(q).$$

so $\hat{\nu}(q) \longrightarrow_{N \to \infty} w(q)$. As $\hat{\nu}(q) - \hat{\nu}(q-1) \longrightarrow_{n \to \infty} 0$, we obtain $\hat{\nu}(q) \longrightarrow_{N \to \infty} w(q)$.

Define now, for each $p$ in $\Delta(K)$, $\underline{w}(p) = \liminf_N \hat{\nu}(p)$. As for each $p$ in $\Delta(K)$, $\hat{\nu}(p) - \hat{\nu}(pM) \longrightarrow_{N \to \infty} 0$, we have for each $N$, $\underline{w}(p) = \underline{w}(pM) = \ldots = \underline{w}(pM^N).$
Fix finally $p$ in $\Delta(K)$, and put $q = pB_0 \in Q$. Since for each $N$ $\hat{v}_N$ is concave on $A(q)$ so is $\underline{v}$. Since $A(q)$ is a polytope, $\underline{v}$ is then necessarily l.s.c. on $A(q)$ (a reference for this is Mertens et al., part A, p.46, ex. 15). As $pM^{NL} \longrightarrow_{N \to \infty} q$, we get that $\underline{v}(p) \geq \underline{v}(q) = w(q)$.

Summing, we obtain that $\lim \sup_N \hat{v}_N(p) \leq w(p) = w(q) \leq \underline{v}(p)$, and the corollary is proved.

We can now conclude about the value of the non revealing games.

**Theorem 6.20** For each $p$ in $\Delta(K)$, the non revealing game $\hat{\Gamma}(p)$ has a value which is $w(p)$.

**Proof:** Fix $p$ in $\Delta(K)$, and put $q = pB_0$, and $w = w(p)$. By theorem 6.16, we just have to prove that player 1 can guarantee $w(p)$ in $\hat{\Gamma}(p)$.

$A(q)$ is a polytope and for each $N$ $\hat{v}_N$ is concave on $A(q)$, hence by the previous corollary we can obtain:

$$\forall \varepsilon > 0 \exists N_0 \forall N \geq N_0 \forall p' \in A(q) \hat{v}_N(p') \geq w - \varepsilon.$$

Fix now $\varepsilon > 0$, and let $N_0$ be as above. Divide the set of stages into consecutive blocks $B^1, \ldots, B^m, \ldots$ of length $N_0L$. We define $\sigma$ by induction on the blocks as follows:

- at block $B^1$, $\sigma$ plays an optimal strategy in $\hat{\Gamma}_{N_0L}(p)$.
- at block $B^m$, $\sigma$ is defined as follows: if $h_{(m-1)N_0L}$ in $\hat{H}_{(m-1)N_0L}$ has been played by player 1 at previous blocks, put $p_m = p_{(m-1)N_0L}(p, \sigma)(h_{(m-1)N_0L})$. Notice that $p_m$ only depends on the definition of $\sigma$ on previous blocks. $\sigma$ plays at block $B^m$ after $h_{(m-1)N_0L}$ an optimal strategy in $\hat{\Gamma}_{N_0L}(p_m)$.

By proposition 6.4, $\sigma$ is a non revealing strategy at $p$. And for each $m$ and $h_{(m-1)N_0L}$ such that $P_{p, \sigma}(h_{(m-1)N_0L}) > 0$, $p_{(m-1)N_0L}(p, \sigma)(h_{(m-1)N_0L})$ is in $A(q)$. Consequently, for any strategy $\tau$ of player 2, we have:

$$\forall m \geq 1, \ E_{p, \sigma, \tau} \left( \frac{1}{N_0L} \sum_{n \in B^m} C^{h_n}(i_n, j_n) \right) \geq w - \varepsilon.$$

Hence: $\exists N_1 \forall N \geq N_1 \forall \tau \in T, \ E_N(\sigma, \tau) \geq w - 2\varepsilon$. \qed

### 7 Value of the original Markov chain repeated game

We come back to the original game $\Gamma_\infty(p)$. We know by proposition 5.5 that player 2 can guarantee $v^*(p) = \inf_{N \geq 1} v_{NL}(pB_0)$ in this game. By theorem 6.20, player 1 can guarantee $w(p) = \inf_{N \geq 1} \hat{v}_{NL}(pB_0)$ in the non revealing game $\hat{\Gamma}_\infty(p)$. Thus a fortiori player 1 can guarantee $w(p)$ in $\Gamma_\infty(p)$. We are going to prove that $v^*(p) = \text{cav}w(pB_0)$, and that this is the value of $\Gamma_\infty(p)$.

We first investigate what can be guaranteed by player 1. For the moment, we define another mapping from $\Delta(K)$ to $\mathbb{R}$.
Definition 7.1 \( \forall p \in \Delta(K), \ w^*(p) = \text{cav}w(pB_0). \)

Notice that \( (w(p) = w(pB_0) \ \forall p) \) does not necessarily imply \( (\text{cav}w(p) = \text{cav}w(pB_0) \ \forall p) \). Recall that if \( f \) is an u.s.c. mapping from \( \Delta(K) \) to \( \mathbb{R} \), then \( \text{cav}f \) is continuous and for each \( p \) in \( \Delta(K) \), \( \text{cav}f(p) \) is:

\[
\max\{\sum_{s \in S} \lambda_s f(p_s), S \text{ finite set }, \forall s \in S \ \lambda_s \geq 0, p_s \in \Delta(K), \sum_{s \in S} \lambda_s = 1, \sum_{s \in S} \lambda_s p_s = p\}.
\]

Lemma 7.2 Let \( f \) be an u.s.c. mapping from \( \Delta(K) \) to \( \mathbb{R} \) such that for each \( p \) in \( \Delta(K) \), player 1 can guarantee \( f(p) \) in \( \Gamma_\infty(p) \). Then for each \( p \) in \( \Delta(K) \), player 1 can guarantee \( \text{cav}f(pB_0) \).

Proof: By the splitting procedure (see for example Zamir 1992), it is standard that for all \( p \) player 1 can guarantee \( \text{cav}f(p) \) in \( \Gamma_\infty(p) \). As \( \text{cav}f \) is continuous, to prove the lemma we show that if \( g: \Delta(K) \rightarrow \mathbb{R} \) is continuous and if for each \( p \) player 1 can guarantee \( g(p) \) in \( \Gamma_\infty(p) \), then for each \( p \) he can guarantee \( g(pB_0) \) in \( \Gamma_\infty(p) \).

Fix \( p \) in \( \Delta(K) \), and \( \varepsilon > 0 \). For \( N \geq 1 \), let \( \sigma_N \) in \( \Sigma \) be a strategy of player 1 such that: \( \exists T_0 \ \forall T \geq T_0 \ \forall \tau \in \mathcal{T} \ \gamma^{B^M_N}(\sigma_N, \tau) \geq g(pM^N) - \varepsilon \). Define \( \sigma(N) \) as follows: play arbitrarily independently of the states up to stage \( N \), then play according to \( \sigma_N \). It is clear that \( \sigma(N) \) guarantees \( g(pM^N) - \varepsilon \) in \( \Gamma_\infty(p) \). For \( N \) multiple of \( L \) and large enough, \( \sigma(N) \) thus guarantees \( g(pB_0) - 2\varepsilon \) in \( \Gamma_\infty(p) \). \( \square \)

As player 1 can guarantee \( w(p) \) in \( \Gamma_\infty(p) \) for each \( p \), the following corollary is immediate.

Corollary 7.3 In \( \Gamma_\infty(p) \), player 1 can guarantee \( w^*(p) \).

Up to this point, we know that for each \( p \) in \( \Delta(K) \), in the game \( \Gamma_\infty(p) \):

player 2 can guarantee \( v^*(p) = \lim_{n \to -\infty} v_n(p) = \inf_{N \geq 1} w_{NL}(pB_0) \).

player 1 can guarantee \( w^*(p) = \text{cav}w(pB_0) \),

where \( w(p) = \lim_{n \to -\infty} \hat{v}_n(p) = \inf_{N \geq 1} \hat{v}_{NL}(pB_0) \).

We finally show that \( v^*(p) = w^*(p) \), proving that \( \Gamma_\infty(p) \) has a value. This is done in two steps. First, we show that \( v^*(p) = \inf_{N \geq 1} \text{cav}\hat{v}_{NL}(pB_0) \). Then we will prove that \( \inf_{N \geq 1} \text{cav}\hat{v}_{NL}(pB_0) = \text{cav}w(pB_0) \).

The following proposition establishes the link between the limit of the values of the \( N \)-stage original games and the values of the \( N \)-stage non revealing games. Indeed, it justifies our definition of non revealing games.

Proposition 7.4 For \( p \) in \( \Delta(K) \) and \( N \geq 1 \), \( v^*(p) \leq \text{cav}\hat{v}_{NL}(pB_0) \).
We first roughly explain the ideas of the proof. Fix $N$ a multiple of $L$, we have to show that $v^* (p) \leq c a v \hat{v}_N (p B_0)$. Consider a strategy $\sigma$ of player 1 which is optimal in a $N T$-stage game $\Gamma_{NT} (p)$, with $T$ large. We construct a strategy $\tau$ for player 2 such that $\gamma^N_{NT} (\sigma, \tau) \leq f (N, T)$, where $f$ satisfies $\lim \sup_T f (N, T) \leq c a v \hat{v}_N (p B_0)$. $\sigma$ need not be non revealing at $p$, but since $(q_n (p, \sigma))_{n \geq 0}$ is a martingale, we know by the classical bound on its $L_1$ variation (see for example Zahir, 1992, p.122), that for each $M$, $\frac{1}{M} \sum_{n=0}^{M-1} E_{p, \sigma} (\| q_{n+1} (p, \sigma) - q_n (p, \sigma) \|) \leq \frac{1}{\sqrt{M}}$, hence is small when $M$ is large. The $N T$ stages will be viewed as $T$ blocks of length $N$. At the beginning of each block $m$, player 2 will compute his belief $p_m$ on the current state and the $N$-stage strategy $\sigma_m$ to be played at this block by player 1 using $\sigma$. $(p_m, \sigma_m)$ will be approximated by a non revealing pair $(\hat{p}_m, \hat{\sigma}_m)$, and player 2 will play at block $B_m$ a best response against $\hat{\sigma}_m$ in the non revealing game $\hat{\Gamma}_N (\hat{p}_m)$. Then player 1’s payoff at this block will be at most $\hat{v}_N (\hat{p}_m)$ plus an error term depending of $\| (p_m, \sigma_m) - (\hat{p}_m, \hat{\sigma}_m) \|$. By the previous bound, it will be possible to control the average error term and to show that it vanishes as $T$ goes to infinity. And we obtain the upper bound of $c a v \hat{v}_N (p B_0)$ by collecting the expected average of the non revealing $N$-stage values. We now formally prove proposition 7.4.

**Proof:**

(1) Fix a positive integer $N$, and $\varepsilon > 0$. As we will carefully use the usual bound on the $L_1$ variation of a martingale with values in $\Delta (K)$, we first need to be precise about the strategy spaces and the norms we will use. Recall that if $x = (x^s)_{s \in S}$ is an element of an Euclidean space $\mathbb{R}^S$, we use $\| x \| = \sum_{s \in S} | x^s |$.

It will be convenient to consider the following subset $\Sigma_N$ of $N$-stages behavior strategies of player 1. A strategy $\sigma = (\sigma_n)_{n=1, \ldots, N}$ in $\Sigma_N$ is in $\tilde{\Sigma}_N$ if at each stage $n$, $\sigma_n$ only depends on player 1’s past actions $i_1, \ldots, i_{n-1}$ and on the current state $k_n$. By proposition 5.1, for each $p$ player 1 has an optimal strategy in $\Gamma_N (p)$ which belongs to $\Sigma_N$.

We view $\Delta (I)$ as a subset of $\mathbb{R}^I$. An element $\sigma$ in $\Sigma_N$ can then be seen as a particular mapping from $\cup_{n=1}^N (K \times I \times J)^{n-1} \times K$ to $\mathbb{R}^I$, hence as an element of the Euclidean space $(\mathbb{R}^I)^{\cup_{n=1}^N (K \times I \times J)^{n-1} \times K}$. One can find a positive constant $C_1 \geq C$ such that for all $\sigma$ and $\sigma'$ in $\Sigma_N$, for each $p$ in $\Delta (K)$ and $\tau$ in $\Gamma_K$, we have $| \gamma^N_{\Sigma_N} (\sigma, \tau) - \gamma^N_{\Sigma_N} (\sigma', \tau) | \leq C_1 \| \sigma - \sigma' \|$. Notice that $C_1$ depends on $N$. Denote by $R$ the set $\Delta (K) \times \Sigma_N$. If $(p, \sigma)$ and $(p', \sigma')$ are elements of $R$, we use: $\| (p, \sigma) - (p', \sigma') \| = \| p - p' \| + \| \sigma - \sigma' \|$, where $\| p - p' \| = \sum_{k \in K} | p^k - p'^k |$.

As in the proof of proposition 6.12, we put $H = \cup_{n=0}^{N-1} \tilde{H}_{n+1}$. We consider the following mapping $F$ from $R$ to $(\mathbb{R}^K)^H$ such that for each $(p, \sigma)$ in $\Delta (K) \times \Sigma_N$, $F(p, \sigma)$ is:

$$(\mathbb{P}_{p, \sigma} (h_{n+1}) (q_{n+1} (p, \sigma) (h_{n+1}) - q_n (p, \sigma) (i_1, \ldots, i_n)))_{n=0, \ldots, N-1, h_{n+1} = (i_1, \ldots, i_{n+1}) \in H_{n+1}}.$$

For $(p, \sigma)$ in $R$, we have $F(p, \sigma) = 0$ if and only if $\sigma \in \Sigma_N (p)$, and we define $\hat{R} = \{(p, \sigma) \in R, F(p, \sigma) = 0\}$. $F$ is continuous as in the proof of proposition 6.12 and $\hat{R}$ is compact. It implies that there exists some positive constant $C_2$, depending on $N$ and $\varepsilon$, satisfying:
\[(\forall (p, \sigma) \in R \exists (\hat{p}, \hat{\sigma}) \in \hat{R} \text{ such that } \| (p, \sigma) - (\hat{p}, \hat{\sigma}) \| \leq \varepsilon + C_2 \| F(p, \sigma) \|). \]

Let now \( \sigma \) be in \( \hat{\Sigma}_N \), and \( p \) be in \( \Delta(K) \). \((q_n(p, \sigma))_{n \geq 1}\) is, with respect to \( \mathcal{P}_{p, \sigma} \), a martingale with values in \( \Delta(K) \). Consequently, we have the classical bound:

\[
\frac{1}{N} \sum_{n=0}^{N-1} \mathcal{E}_{p, \sigma}\left(\|q_{n+1}(p, \sigma) - q_n(p, \sigma)\|\right) \leq \frac{|K|}{\sqrt{N}}.
\]

And the point is that the left-hand quantity is just \( \frac{\mathcal{E}(p, \sigma)}{N} \).

(2) We now conclude the proof of proposition 7.4. Fix \( N \) a positive multiple of \( L \), \( \varepsilon > 0 \), and \( p \) in \( \Delta(K) \) in all what follows, and let \( C_1 = C_1(N) \) and \( C_2 = C_2(N, \varepsilon) \) be defined as in point (1). Let now \( T \) be a positive integer, and let \( \sigma \) in \( \hat{\Sigma}_{NT} \) be an optimal strategy for player 1 in \( \Gamma_{NT}(p) \).

We define \( \tau \) a \( NT \)-stage strategy of player 2 as follows. Divide the set of stages into consecutive blocks \( B^1, ..., B^T \) of length \( N \). For \( m \in \{1, ..., T\} \), \( \tau \) plays at block \( B^m = \{(m-1)N + 1, ..., mN\} \) as follows: if \( h_m = (i_1, ..., i_{(m-1)N}) \in \hat{H}_{(m-1)N} \) has been played by player 1 at previous blocks, let \( p_m(h_m) = p_{(m-1)N}(p, \sigma)(h_m) \) in \( \Delta(K) \) be player 2’s belief on the state of the beginning of block \( B^m \). Consider the strategy \( \sigma_m(h_m) \) played by player 1 using \( \sigma \) at block \( B^m \) after \( h_m \) has occurred. \( \sigma_m(h_m) \) belongs to \( \hat{\Sigma}_N \), so it is possible to choose some \( (\hat{p}_m(h_m), \hat{\sigma}_m(h_m)) \) in \( \hat{R} \) such that:

\[
\| (p_m(h_m), \sigma_m(h_m)) - (\hat{p}_m(h_m), \hat{\sigma}_m(h_m)) \| \leq \varepsilon + C_2 \| F(p_m(h_m), \sigma_m(h_m)) \|.
\]

At block \( B^m \) after \( h_m \), \( \tau \) plays a best response \( \tau_m(h_m) \) against \( \hat{\sigma}_m(h_m) \) in the non revealing game \( \hat{\Gamma}_N(\hat{p}_m(h_m)) \).

By definition of \( \sigma \), \( \gamma_{NT}^p(\sigma, \tau) \geq \gamma_{NT}(p) \). We now compute an upper bound for \( \gamma_{NT}^p(\sigma, \tau) \). Let \( m \) be in \( \{1, ..., T\} \) and \( h_m \) be in \( \hat{H}_{(m-1)N} \).

\[
\mathcal{E}_{p, \sigma, \tau}\left(\frac{1}{N} \sum_{n \in B^m} G^k_n(i_n, j_n)|h_m\right) = \gamma_{NT}^p(h_m)(\sigma_m(h_m), \tau_m(h_m))
\]

\[
\leq C \| p_m(h_m) - \hat{p}_m(h_m) \| + \gamma_{NT}^p(h_m)(\sigma_m(h_m), \tau_m(h_m))
\]

\[
\leq C \| p_m(h_m) - \hat{p}_m(h_m) \| + \gamma_{NT}^p(h_m)(\hat{\sigma}_m(h_m), \tau_m(h_m)) + C_1 \| (\sigma_m(h_m) - \hat{\sigma}_m(h_m)) \|
\]

\[
\leq C_1 \varepsilon + C_2 \| F(p_m(h_m), \sigma_m(h_m)) \| + \hat{\nu}_N(\hat{p}_m(h_m)),
\]

since \( \hat{\sigma}_m(h_m) \) is non revealing at \( \hat{p}_m(h_m) \) and \( \tau_m(h_m) \) is optimal in \( \hat{\Gamma}_N(\hat{p}_m(h_m)) \).

Consequently, if we condition on all possible histories \( h_m \) we obtain that at block \( B^m \) player 1’s payoff \( \mathcal{E}_{p, \sigma, \tau}\left(\frac{1}{N} \sum_{n \in B^m} G^k_n(i_n, j_n)\right) \) is at most:

\[
\sum_{h_m \in \hat{H}_{(m-1)N}} \mathcal{E}_{p, \sigma}(h_m)(C_1 \varepsilon + C_1 C_2 \| F(p_m(h_m), \sigma_m(h_m)) \| + \hat{\nu}_N(\hat{p}_m(h_m))).
\]

We have \( \sum_{h_m \in \hat{H}_{(m-1)N}} \mathcal{E}_{p, \sigma}(h_m)p_m(h_m) = pM^{(m-1)N} \), and we put \( \bar{p}_m = \sum_{h_m \in \hat{H}_{(m-1)N}} \mathcal{E}_{p, \sigma}(h_m)\hat{p}_m(h_m) \). Since \( cav\hat{\nu}_N \) is concave and above \( \hat{\nu}_N \) we obtain that \( \mathcal{E}_{p, \sigma, \tau}\left(\frac{1}{N} \sum_{n \in B^m} G^k_n(i_n, j_n)\right) \) is at most:

\[
C_1 \varepsilon + C_1 C_2 \sum_{h_m \in \hat{H}_{(m-1)N}} \mathcal{E}_{p, \sigma}(h_m)\| F(p_m(h_m), \sigma_m(h_m)) \| + cav\hat{\nu}_N(\bar{p}_m).
\]
Summing up over blocks, we get:

$$\gamma_{NT}^p(\sigma, \tau) \leq C_1 \varepsilon + C_1 C_2 \frac{1}{T} \sum_{m=1}^{T} \sum_{h_m \in H_{(m-1)N}} \mathbb{P}_{p, \sigma}(h_m) \|F(p_m(h_m), \sigma_m(h_m))\| + \frac{1}{T} \sum_{m=1}^{T} \text{cav} \hat{v}_N(\tilde{p}_m).$$

We now bound the second and third terms on the right-hand side. First, for each \(m\) and \(h_m\) in \(H_{(m-1)N}\) we have:

$$\|F(p_m(h_m), \sigma_m(h_m))\| = \sum_{n=0}^{N-1} \mathbb{E}_{p_m(h_m), \sigma_m(h_m)}(\|q_{n+1}(p_m(h_m), \sigma_m(h_m)) - q_n(p_m(h_m), \sigma_m(h_m))\|) = \sum_{n \in B^m} \mathbb{E}_{p, \sigma}(\|q_{n+1}(p, \sigma) - q_n(p, \sigma)\| \|h_m\|)$$

Thus we get:

$$\frac{1}{T} \sum_{m=1}^{T} \sum_{h_m \in H_{(m-1)N}} \mathbb{P}_{p, \sigma}(h_m) \|F(p_m(h_m), \sigma_m(h_m))\| = \frac{1}{T} \sum_{n=0}^{NT-1} \mathbb{E}_{p, \sigma}(\|q_{n+1}(p, \sigma) - q_n(p, \sigma)\|) \leq \frac{|K| \sqrt{N}}{\sqrt{T}}.$$  

Secondly, we have \(\frac{1}{T} \sum_{m=1}^{T} \text{cav} \hat{v}_N(\tilde{p}_m) \leq \text{cav} \hat{v}_N(\frac{1}{T} \sum_{m=1}^{T} \tilde{p}_m)\) and:

$$\|\frac{1}{T} \sum_{m=1}^{T} \tilde{p}_m - \frac{1}{T} \sum_{m=1}^{T} pM^{(m-1)N}\| = \|\frac{1}{T} \sum_{m=1}^{T} \sum_{h_m} \mathbb{P}_{p, \sigma}(h_m)(p_m(h_m) - \tilde{p}_m(h_m))\| \leq \frac{1}{T} \sum_{m=1}^{T} \sum_{h_m} \mathbb{P}_{p, \sigma}(h_m)(\varepsilon + C_2 \|F(p_m(h_m), \sigma_m(h_m))\|) \leq \varepsilon + C_2 \frac{|K| \sqrt{N}}{\sqrt{T}}.$$  

Summing up, we have obtained:

$$\gamma_{NT}^p(\sigma, \tau) \leq C_1 \varepsilon + C_1 C_2 \frac{|K| \sqrt{N}}{\sqrt{T}} + \text{cav} \hat{v}_N(\frac{1}{T} \sum_{m=1}^{T} \tilde{p}_m)$$

where \(\|\frac{1}{T} \sum_{m=1}^{T} \tilde{p}_m - \frac{1}{T} \sum_{m=1}^{T} pM^{(m-1)N}\| \leq \varepsilon + C_2 \frac{|K| \sqrt{N}}{\sqrt{T}}\).

cav \hat{v}_N being continuous, we then have that \(v_{NT}(p)\) is at most:

$$C_1 \varepsilon + C_1 C_2 \frac{|K| \sqrt{N}}{\sqrt{T}} + \max \left\{ \text{cav} \hat{v}_N(p'), p' \text{ s.t. } \|p' - \frac{1}{T} \sum_{m=1}^{T} pM^{(m-1)N}\| \leq \varepsilon + C_2 \frac{|K| \sqrt{N}}{\sqrt{T}} \right\}.$$  

Recall that \(N, \varepsilon, C_1\) and \(C_2\) are fixed. \(\frac{1}{T} \sum_{m=1}^{T} pM^{(m-1)N} \rightarrow_{T \to \infty} pB_0\), so one can find \(T_0\) such that for each \(T \geq T_0\):
\[ v_{NT}(p) \leq C_1 \varepsilon + \frac{C_1 C_2 |K|^{\sqrt{N}}}{\sqrt{N}} + \max\{\text{cav} \hat{v}_N(p'), p' \text{ s.t. } \|p' - pB_0\| \leq 3\varepsilon\}. \]

And we obtain:
\[ v^*(p) = \limsup_{v_{NT}(p)} v_{NT}(p) \leq C_1 \varepsilon + \max\{\text{cav} \hat{v}_N(p'), p' \text{ s.t. } \|p' - pB_0\| \leq 3\varepsilon\}. \]

This inequality holds for each \( \varepsilon > 0 \). Since \( C_1 \) does not depend on \( \varepsilon \) and \( \text{cav} \hat{v}_N \) is continuous, we finally have \( v^*(p) \leq \text{cav} \hat{v}_N(pB_0) \) as wanted. \( \Box \)

**Corollary 7.5** For each \( p \) in \( \Delta(K) \),
\[ v^*(p) = \inf_{N \geq 1} \text{cav} \hat{v}_N(pB_0) = \lim_{N \to \infty} \text{cav} \hat{v}_N(pB_0). \]

**proof:** By the previous proposition, for each \( p \) we have \( v^*(p) \leq \inf_{N \geq 1} \text{cav} \hat{v}_N(pB_0) \). For each \( N \), \( v_{NT} \) is concave and above \( \hat{v}_N \), hence we obtain: \( \inf_{N \geq 1} \text{cav} \hat{v}_N(pB_0) \geq v^*(p) = \lim_{N \to \infty} v_{NT}(pB_0) \geq \limsup_{N \to \infty} \text{cav} \hat{v}_N(pB_0) \). Thus \( \text{cav} \hat{v}_N(pB_0) \) converges to \( \inf_{N \geq 1} \text{cav} \hat{v}_N(pB_0) = v^*(p) \). \( \Box \)

It now just remains to show that \( w^*(p) = \inf_{N \geq 1} \text{cav} \hat{v}_N(pB_0) \). We start with a lemma.

**Lemma 7.6** Let \( (f_n)_{n \geq 1} \) be a non-increasing sequence of u.s.c. mappings from \( \Delta(K) \) to \( \mathbb{R} \) pointwise converging to some mapping \( f \) from \( \Delta(K) \) to \( \mathbb{R} \). Then the sequence \( \text{cav} f_n \) for \( n \geq 1 \) pointwise converges to \( \text{cav} f \).

**Proof:** Fix \( p \) in \( \Delta(K) \). It is plain that \( \lim_n \text{cav} f_n(p) \geq \text{cav} f(p) \). By Carathéodory’s theorem, one can take a fixed finite set \( S \) satisfying: for each \( n \geq 1 \), there exists \( (p^n_s, \lambda^n_s) \in S \) such that for each \( s \) in \( S \), \( p^n_s \in \Delta(K) \), \( \lambda^n_s \geq 0 \), \( \sum_{s \in S} \lambda^n_s = 1 \), \( \sum_{s \in S} \lambda^n_s p^n_s = p \) and \( \text{cav} f_n(p) = \sum_{s \in S} \lambda^n_s f_n(p^n_s) \). Taking converging subsequences, one can find an increasing mapping \( \psi \) from the set of positive integers to itself, \( (\lambda_s)_{s \in S} \) and \( (p_s)_{s \in S} \) s.t. for each \( s \) in \( S \), \( \lambda_s^{(n)} \to_{n \to \infty} \lambda_s \) and \( p_s^{(n)} \to_{n \to \infty} p_s \).

Then for each \( n_0 \geq 1 \) we have for each \( n \geq n_0 \):
\[ \text{cav} f_{\psi(n)}(p) = \sum_{s \in S} \lambda_s^{(n)} f_{\psi(n)}(p^n_s) \leq \sum_{s \in S} \lambda_s^{(n)} f_{n_0}(p^n_s) \]

So \( \inf_{n \geq 1} \text{cav} f_{\psi(n)}(p) \leq \sum_{s \in S} \lambda_s f_{n_0}(p_s) \), since \( f_{n_0} \) is u.s.c.

Hence \( \text{cav} f(p) \leq \inf_{n \geq 1} \text{cav} f_n(p) \leq \sum_{s \in S} \lambda_s f_{n_0}(p_s) \) for each \( n_0 \geq 1 \). Thus \( \text{cav} f(p) \leq \inf_{n \geq 1} \text{cav} f_n(p) \leq \sum_{s \in S} \lambda_s f(p_s) \leq \text{cav} f(p) \), and \( \inf_n \text{cav} f_n(p) = \text{cav} f(p) \). \( \Box \)

**Proposition 7.7** For each \( p \) in \( \Delta(K) \),
\[ w^*(p) = \inf_{N \geq 1} \text{cav} \hat{v}_N(pB_0). \]
\textbf{Proof:} As \( w(p) = \inf_{N \geq 1} \hat{v}^{NL}(pB_0) \), it is clear that \( w^*(p) \leq \inf_{N \geq 1} \text{cav} \hat{v}^{NL}(pB_0) \). We define, for each positive \( n \), two mappings \( f_n \) and \( u_n \) from \( \Delta(K) \) to \( \mathbb{R} \) such that for each \( p \), \( f_n(p) = \hat{v}^{NL}(pB_0) \) and \( u_n(p) = \hat{v}^{2L}(p) \). Notice that \( \text{cav} f_n(p) \) may not be \( \text{cav} \hat{v}^{2L}(pB_0) \).

1. For each \( n \), \( f_n \) is continuous and \( (f_n)_n \) pointwise converges to \( w \). By corollary 6.18, \( (f_n)_n \) is non-increasing so by lemma 7.6, we first obtain that for each \( p \) in \( \Delta(K) \), \( \text{cav} f_n(p) \rightarrow_{n \rightarrow \infty} \text{cav} w(p) \).

2. By proposition 6.17, we also have: \( \forall p \in \Delta(K), \forall N \geq 1, \forall T \geq 1: \)

\[
\hat{v}_{NL}(p) = \frac{1}{T} \sum_{t=1}^{T} \hat{v}_{NL}(pM^{(t-1)NL}).
\]

\( \hat{v}_{NL} \) being concave on \( A(pB_0) \), we obtain:

\[
\hat{v}_{NL}(p) \leq \hat{v}_{NL} \left( \frac{1}{T} \sum_{t=1}^{T} pM^{(t-1)NL} \right).
\]

Hence

\[
\hat{v}_{L2N2T}(p) \leq \hat{v}_{L2N} \left( \frac{1}{2T} \sum_{t=1}^{2T} pM^{(t-1)2NL} \right),
\]

\[
u_{N+T}(p) \leq u_N \left( \frac{1}{2T} \sum_{t=1}^{2T} pM^{(t-1)2NL} \right).
\]

Fix \( N \geq 1 \) and \( \varepsilon > 0 \). \( u_N \) being uniformly continuous, one can find \( T_0 \) such that: \( \forall T \geq T_0 \), \( \forall p \in \Delta(K), \) \( u_{N+T}(p) \leq u_N(pB_0) + \varepsilon = f_N(p) + \varepsilon \). Thus for each \( p \) in \( \Delta(K) \) and \( T \geq T_0 \), we have \( \text{cav} u_{N+T}(p) \leq \text{cav} f_N(p) + \varepsilon \), and so \( \text{cav} u_{N+T}(pB_0) \leq \text{cav} f_N(pB_0) + \varepsilon \). Hence:

\[
\text{cav} f_N(pB_0) + \varepsilon \geq \inf_T \text{cav} u_{N+T}(pB_0) \\
\geq \inf_T \text{cav} \hat{v}_{TL}(pB_0).
\]

We have obtained: \( \forall N \geq 1, \forall \varepsilon > 0, \forall p \in \Delta(K), \) \( \inf_T \text{cav} \hat{v}_{TL}(pB_0) \leq \text{cav} f_N(pB_0) + \varepsilon \). Thus \( \inf_T \text{cav} \hat{v}_{TL}(pB_0) \leq \text{cav} \hat{v}^{NL}(pB_0) \), and this last quantity is just \( \text{cav} w(pB_0) \) by point 1. Consequently, \( w^*(p) = \inf_{N \geq 1} \text{cav} \hat{v}^{NL}(pB_0). \)

It just remains to conclude.

\textbf{Proof of Theorem 2.2} By proposition 5.5, player 2 can guarantee \( v^*(p) \) in \( \Gamma_\infty(p) \). By corollary 7.3, player 1 can guarantee \( w^*(p) = \text{cav} w(pB_0) \) in \( \Gamma_\infty(p) \).

By corollary 7.5 and proposition 7.7, \( v^*(p) = \inf_{N \geq 1} \text{cav} \hat{v}^{NL}(pB_0) = w^*(p) \). Hence \( \Gamma_\infty(p) \) has a value which is:

\[
v^*(p) = w^*(p) = \text{cav} w(pB_0) = \inf_{N \geq 1} \text{cav} \hat{v}^{NL}(pB_0).
\]

And player 2 has an optimal strategy by remark 5.7. \( \square \)
8 Concluding Remarks

1. Observation of player 1
A) The fact that player 1 observes after each stage the action played by player 2 plays no role. Hence if player 1 does not observe these actions, or just imperfectly observes them, the game still has the same value.
B) Assume now that player 1 observes at the beginning of stage 1 the whole sequence of states \( k_1, k_2, ..., k_n, ... \). Then the \( N \)-stage values \( v_N(p) \) still satisfy the same recursive formulae given by proposition 5.1, and player 2 can still guarantee their limit as shown by proposition 5.5. Hence again this modified game has the same value (\textit{idem} if player 1 observes the sequence of states in any manner such that, for each \( n \), player 1 knows the state \( k_n \) before choosing his action of stage \( n \)).

2. Optimal strategy for player 1
By slightly improving the proof of theorem 6.20, one can show that player 1 has a strategy \( \sigma \) that guarantees \( w(p) \) in \( \Gamma_\infty(p) \) (just play by consecutive blocks \( B^m \) of length \( mL \), and at \( B^m \) play an optimal strategy in \( \hat{\Gamma}_{mL}(p_m) \), if \( p_m \) is the belief of player 2 on the current state at the beginning of the block). \( w \) being u.s.c., by the splitting procedure one can construct \( \sigma \) that guarantees \( \text{cav} w(p) \) in \( \Gamma_\infty(p) \). But it is unclear whether there exists \( \sigma \) that guarantees the value \( \text{cav} w(pB_0) \) for player 1, hence the existence of an optimal strategy for player 1 in \( \Gamma_\infty(p) \) is an open question.

3. Values of the \( N \)-stage games
A) The proof of proposition 7.4 gives no useful bound on the speed of convergence of \( (v_N(p))_N \) to \( v^*(p) \), contrary to the Aumann-Maschler case where this convergence is according to \( 1/\sqrt{N} \).
B) It is unclear whether \( \hat{v}_N \), even \( \hat{v}_1 \), is Lipschitz or not. A proof of this would simplify things and give insights for the previous points.

4. Computing the value
The value is not easy to compute, and the author does not know how to proceed, even in a simple challenging example such as: \( K = \{a, b\}, M = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \), \( G^a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( G^b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

Note that we have here: \( v_1(\delta_a) = 0 < v_2(\delta_a) = 1/2(0 + 1/3) = 1/6 \), hence \( (v_n(\delta_a))_n \) is not non-increasing.

5. Generalizations
An interesting generalization is the case where player 2 observes after each stage a signal depending on the action just played by player 1 and on the current state. Is it possible, as in the Aumann-Maschler’s case, to generalize to this setup the definition of non revealing strategies in order to get the existence of the value ?

Another generalization, focussing on the stochastic game aspect, is the case
where at each stage the current state is chosen according to a probability distribution depending on the previous state and on the actions just played by both players. However, it is known (Sorin 1984) that the value may fail to exist in this general case. But in the intermediate case where the transitions do not depend on player 2’s actions, what about the existence of the value?

**Acknowledgement**  The author thanks S. Sorin for explanations about the fact that player 2 can guarantee \( \lim_{n \to \infty} v_n(p) \) in \( \Gamma_\infty(p) \).

**References**


