Existence of Weak Solutions for an Unsteady Fluid–Plate Interaction Problem

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Abstract

The purpose of this work is to study the existence of solutions for an unsteady fluid–structure interaction problem. We consider a three–dimensional viscous incompressible fluid governed by the Navier-Stokes equations, interacting with an elastic plate in flexion located on one part of the fluid boundary. The fluid domain evolves according to the structure's displacement, itself resulting from the fluid force. We prove the existence of at least one weak solution as long as the structure does not touch the fixed part of the fluid boundary. The same result holds also for a 2D fluid interacting with a 1D membrane.

1 Introduction.

Many physical phenomena involve a fluid interacting with a moving or deformable structure. This kind of problems have a lot of important applications, for instance in aeroelasticity, biomechanics, hydroelasticity, sedimentation... From the mathematical point of view, they have been studied extensively over the last few years. In particular, the question of the existence of weak or strong solutions has been widely investigated in the case where the displacement of the structure is supposed to be large enough, and thus induces large variations of the fluid domain. In this case, we can

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not assume anymore that the fluid domain is fixed and independent of the structure displacement. For the case where the fluid domain is assumed to be fixed, we refer to [14], where existence of weak solutions is proven for the Navier–Stokes equations in a fixed domain coupled with linearized elasticity. When the fluid domain depends on time, we refer to [3], [4], [5], [6], [11], [12], [13], [17], [18], [19]. In all those studies, the fluid is described by the Navier–Stokes equations and the structure lies inside the fluid and is rigid or its behavior is described by a finite number of modal functions. In those cases, the displacement and the velocity of the moving structure are quite smooth. For existence results about a plate interacting with a linear compressible flow, we refer to [7], [8]. The problem we address here is the interaction between a three–dimensional viscous incompressible flow and an elastic plate in flexion. This kind of model can, for instance, be a first attempt to describe the behavior of the blood flow past arteries (see [15]). We prove that there exists at least one weak solution. Note that we can also treat the case of a two–dimensional fluid couped with an elastic membrane. In both cases the displacement of the structure has less regularity than the minimal regularity required in the previous studies. For a similar two–dimensional problem, an existence result of strong solutions locally in time for small data can be found in [2]. Regarding the steady state version of this problem we refer to [10] where existence of a unique regular solution is proven provided the data are small enough.

The main originality of this work is the low regularity of the structure motion that defines the evolution of the fluid domain and the fact that the global fluid–structure domain varies in time. To our knowledge this paper contains the first existence result of weak solutions for a three–dimensional incompressible fluid interacting with a two–dimensional structure.

1.1 Presentation of the problem

We assume that the fluid fills a three–dimensional cavity and interacts with a thin elastic structure, located on a part of the fluid domain boundary, the other part being rigid. For the sake of simplicity, we will assume that, at the reference state, the elastic part of the fluid boundary is located on $\omega \times \{1\}$, where ω denotes a Lipschitz domain of $\mathbb{R}^2$. At the initial state the fluid occupies the domain Ω_{η_0}:

$$\Omega_{\eta_0} = \left\{(x,y,z) \in \mathbb{R}^3,\ (x,y) \in \omega, 0 < z < 1 + \eta_0(x,y)\right\},$$

where η_0 is a given initial displacement of the elastic part. Note that we could also have considered the case of a fluid between two elastic plates and obtained the same kind of results. The rigid part of $\partial\Omega_{\eta_0}$ is denoted by Γ_0. For the structure part, we consider a linear two–dimensional elastic model. This type of reduced model is used when the thickness of the structure is small enough with respect to all other characteristic lengths. Here we consider a clamped plate and we neglect the longitudinal displacement. The equations describing the evolution of the transversal displacement η ($\eta = \eta(t,x,y) \in \mathbb{R}$) are:

$$\begin{cases} \partial_{tt}\eta + \Delta^2\eta + \mu\Delta^2\partial_t\eta = g + (T_f)_3 \text{ in } \omega, \\ \eta = \dfrac{\partial\eta}{\partial n} = 0 \text{ on } \partial\omega, \\ \eta(0) = \eta_0, \partial_t\eta(0) = \eta_1, \end{cases} \tag{1}$$

where g denotes the exterior forces applied on the plate and T_f the surface force exerted by the fluid on the structure. The definition of T_f will be precised later on. A "viscous" type term ($\mu\Delta^2\partial_t\eta$, $\mu > 0$) has been added to the usual plate equation. This additional term will ensure that the velocity of the structure is smooth enough. Instead of this additional viscous term, we could also have added $-\Delta\partial_{tt}\eta$, which models the inertia of rotation. For the sake of simplicity μ will be chosen to be equal to one. But all what follows holds for any $\mu > 0$. The domain occupied by the fluid at time t is denoted by $\Omega_\eta(t)$:

$$\Omega_\eta(t) = \left\{(x,y,z) \in \mathbb{R}^3\ ,\ (x,y) \in \omega,\ 0 < z < 1 + \eta(t,x,y)\right\} \tag{2}$$

The equations for the fluid part are

$$\begin{cases} \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega_\eta(t), \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega_\eta(t), \\ \mathbf{u}(t,\cdot) = 0 & \text{on } \Gamma_0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 & \text{in } \Omega_{\eta_0}, \end{cases} \tag{3}$$

where $\mathbf{u}$ denotes the fluid velocity, and p the pressure field. The bulk force $\mathbf{f}$ is given together with the initial velocity $\mathbf{u}_0$.

Since the fluid is viscous, it sticks to the structure and thus the velocities coincide (in a sense to be defined) at the interface:

$$\mathbf{u}(t,x,y,1+\eta(t,x,y)) = (0,0,\partial_t\eta(t,x,y))^T\ , \quad (x,y) \in \omega. \tag{4}$$

This condition, together with the incompressibility of the fluid leads to

$$\int_{\omega} \partial_t \eta = 0. \tag{5}$$

Condition (5) states that the global volume of the cavity is preserved. The surface force T_f exerted by the fluid on the plate can be defined by:

$$\int_{\omega} T_f \cdot \bar{\mathbf{v}} = \int_{\partial\Omega_\eta(t)\setminus\Gamma_0} (-2\nu D(\mathbf{u}) \cdot \mathbf{n}_t + p\mathbf{n}_t) \cdot \mathbf{v}, \qquad \forall \mathbf{v}, \tag{6}$$

where $D(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ is the strain tensor, $\mathbf{n}_t$ denotes the outer unit normal at $\omega_\eta(t) = \partial\Omega_\eta(t) \setminus \Gamma_0$, and $\bar{\mathbf{v}}(t,x,y) = \mathbf{v}(t,x,y,1+\eta(t,x,y))$, $\forall (x,y) \in \omega$. Note here that the pressure p is not defined up to a constant but is uniquely defined. Its average value ensures the global volume conservation of the fluid cavity. This average is in fact the Lagrange multiplier associated with the compatibility condition (5).

Remark 1 *At least formally we have that*

$$(2D(\mathbf{u}) \cdot \mathbf{n}_t)_3 = (\nabla \mathbf{u} \cdot \mathbf{n}_t)_3, \quad \text{on } \partial\Omega_\eta(t) \setminus \Gamma_0.$$

Indeed since $u_1(x,y,1+\eta(t,x,y)) = u_2(x,y,1+\eta(t,x,y)) = 0$, *on* ω *we deduce*

$$\partial_x u_1(x,y,1+\eta(t,x,y)) + \partial_x \eta(t,x,y) \partial_z u_1(x,y,1+\eta(t,x,y)) = 0,$$

and

$$\partial_y u_2(x,y,1+\eta(t,x,y)) + \partial_y \eta(t,x,y) \partial_z u_2(x,y,1+\eta(t,x,y)) = 0.$$

Adding these two equalities and using the fact that $\operatorname{div} \mathbf{u} = 0$, *we obtain that*

$$\begin{aligned} ((\nabla \mathbf{u})^T \cdot \mathbf{n}_t)_3 = \partial_z u_3(x,y,1+\eta(t,x,y)) - \partial_x \eta(t,x,y) \partial_z u_1(x,y,1+\eta(t,x,y)) \\ - \partial_y \eta(t,x,y) \partial_z u_2(x,y,1+\eta(t,x,y)) = 0. \end{aligned}$$

We are now able to derive, at least formally, energy estimates. This will lead to the definitions of the functional spaces we will use and to the definition of the weak formulation of the coupled system.

1.2 A priori Estimates

Let us multiply the Navier–Stokes equation by the velocity $\mathbf{u}$ and integrate over $\Omega_\eta(t)$. After integrating by parts and using the fact that $\operatorname{div}\mathbf{u} = 0$ we obtain:

$$\int_{\Omega_\eta(t)} \partial_t \mathbf{u}\cdot\mathbf{u} + \int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} \\ +2\nu\int_{\Omega_\eta(t)} |D(\mathbf{u})|^2 + \int_{\partial\Omega_\eta(t)\setminus\Gamma_0} (-2\nu D(\mathbf{u})\cdot\mathbf{n}_t + p\ \mathbf{n}_t)\cdot\mathbf{u} = \int_{\Omega_\eta(t)} \mathbf{f}\cdot\mathbf{u}.$$

We know that

$$\frac{d}{dt}\int_{\Omega_\eta(t)} k = \int_{\Omega_\eta(t)} \partial_t k + \int_{\partial\Omega_\eta(t)} k\mathbf{w}\cdot\mathbf{n}_t,$$

where $\mathbf{w}$ is the velocity of the boundary $\partial\Omega_\eta(t)$. The fluid domain moves with the fluid velocity $\mathbf{w} = \mathbf{u}$. We apply this identity to $k = |\mathbf{u}|^2$. We obtain, taking into account the fluid incompressibility,

$$\int_{\Omega_\eta(t)} \partial_t \mathbf{u}\cdot\mathbf{u} + \int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} = \frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)} |\mathbf{u}|^2.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)} |\mathbf{u}|^2 + 2\nu\int_{\Omega_\eta(t)} |D(\mathbf{u})|^2 \\ + \int_{\partial\Omega_\eta(t)\setminus\Gamma_0} (-2\nu D(\mathbf{u})\cdot\mathbf{n}_t + p\ \mathbf{n}_t)\cdot\mathbf{u} = \int_{\Omega_\eta(t)} \mathbf{f}\cdot\mathbf{u}. \quad (7)$$

Next, we multiply the plate equation by the structure velocity $\partial_t\eta$ and integrate over ω. After integrating by parts, we have:

$$\frac{1}{2}\frac{d}{dt}\int_\omega (\partial_t\eta)^2 + \frac{1}{2}\frac{d}{dt}\int_\omega (\Delta\eta)^2 + \int_\omega (\Delta\partial_t\eta)^2 = \int_\omega g\ \partial_t\eta + \int_\omega (T_f)_3\partial_t\eta. \quad (8)$$

Recalling the definition (6) of T_f and the equality of the velocities (4), and adding (7) and (8) the energy equality follows:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)} |\mathbf{u}|^2 + 2\nu\int_{\Omega_\eta(t)} |D(\mathbf{u})|^2 \\ +\frac{1}{2}\frac{d}{dt}\int_\omega (\partial_t\eta)^2 + \frac{1}{2}\frac{d}{dt}\int_\omega (\Delta\eta)^2 + \int_\omega (\Delta\partial_t\eta)^2 \\ = \int_{\Omega_\eta(t)} \mathbf{f}\cdot\mathbf{u} + \int_\omega g\ \partial_t\eta. \quad (9)$$

Hence, using Cauchy–Schwarz and Young's inequalities (remember that all this is formal):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)}|\mathbf{u}|^2+2\nu\int_{\Omega_\eta(t)}|D(\mathbf{u})|^2\\
&+\frac{1}{2}\frac{d}{dt}\int_\omega(\partial_t\eta)^2+\frac{1}{2}\frac{d}{dt}\int_\omega(\Delta\eta)^2+\int_\omega(\Delta\partial_t\eta)^2\\
&\le\frac{1}{2}\|\mathbf{f}\|^2_{L^2(\Omega_\eta(t))}+\frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega_\eta(t))}+\frac{1}{2}\|g\|^2_{L^2(\omega)}+\frac{1}{2}\|\partial_t\eta\|^2_{L^2(\omega)}
\end{aligned}$$

Gronwall's Lemma then implies that

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}(t,\cdot)\|^2_{L^2(\Omega_\eta(t))}+2\nu\int_0^t\|D(\mathbf{u})(s,\cdot)\|^2_{L^2(\Omega_\eta(s))}ds\\
&+\frac{1}{2}\|\partial_t\eta(t,\cdot)\|^2_{L^2(\omega)}+\frac{1}{2}\|\Delta\eta(t,\cdot)\|^2_{L^2(\omega)}+\int_0^t\|\Delta\partial_t\eta(s,\cdot)\|^2_{L^2(\omega)}ds\\
&\le e^t\left(\frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega_{\eta_0})}+\frac{1}{2}\|\eta_1\|^2_{L^2(\omega)}+\frac{1}{2}\|\Delta\eta_0\|^2_{L^2(\omega)}\right)\\
&+\frac{1}{2}\int_0^t\exp(t-s)\left(\|\mathbf{f}(s,\cdot)\|^2_{L^2(\Omega_\eta(s))}+\|g(s,\cdot)\|^2_{L^2(\omega)}\right)ds
\end{aligned}\tag{10}$$

Thus assuming that $\mathbf{f}\in L^2(0,T;L^2(\mathbb{R}^3))$, $g\in L^2(0,T;L^2(\omega))$, $\mathbf{u}_0\in L^2(\Omega(0))$, $\eta_0\in H_0^2(\omega)$, $\eta_1\in L^2(\omega)$,

$$\mathbf{u}\in L^\infty(0,T;L^2(\Omega_\eta(t))),\quad D(\mathbf{u})\in L^2(0,T;L^2(\Omega_\eta(t))),$$

and

$$\eta\in W^{1,\infty}(0,T;L^2(\omega))\cap H^1(0,T;H_0^2(\omega)).$$

Next we define

$$\overline{\mathbf{u}}=\left|\begin{array}{l}\mathbf{u}\text{ in }\Omega_\eta(t)\\ (0,0,\partial_t\eta)^T\text{ in }B\setminus\Omega_\eta(t),\end{array}\right.$$

where B is a Lipschitz continuous domain such that $\Omega_\eta(t)\subset B,\ \forall t\in[0,T]$. In view of the interface condition (4), $D(\overline{\mathbf{u}})$ belongs to $L^2(0,T;L^2(B))$. Thus applying Korn's inequality in B, $\overline{\mathbf{u}}\in L^2(0,T;H^1(B))$, and consequently $\mathbf{u}\in L^2(0,T;H^1(\Omega_\eta(t)))$.

Therefore, functional spaces of the type

$$L^p(0,T;L^q(\Omega_\delta(t)))\quad\text{and}\quad L^2(0,T;H^1(\Omega_\delta(t)))$$

will be needed, where $\Omega_\delta(t)$ is defined by

$$\Omega_\delta(t)=\left\{(x,y,z)\in\mathbb{R}^3,\ (x,y)\in\omega,\ 0<z<1+\delta(t,x,y)\right\},$$

with $\delta \in W^{1,\infty}(0,T;L^2(\omega)) \cap H^1(0,T;H_0^2(\omega))$.

Note that the functional space $H^1(0,T;H_0^2(\omega))$ is continuously embedded in $C^0([0,T]\times\overline{\omega})$. The regularity coming from the energy estimates therefore implies that $\Omega_\delta(t)$ is an open set but is not necessarily Lipschitz. Consequently, functional spaces in $\Omega_\delta(t)$ have to be carefully defined.

1.3 Preliminary definitions and properties

We now turn to the definition of some functional spaces. Let $T>0$ and δ belong to $C^0([0,T]\times\overline{\omega})$ such that for some positive M and α, $M \geq 1+\delta(t,x,y) \geq \alpha > 0$ for all $(t,x,y)\in[0,T]\times\overline{\omega}$, and such that $\delta=0$ on $\partial\omega$. The set $\Omega_\delta(t)$ defined by

$$\Omega_\delta(t) = \left\{(x,y,z)\in\mathbb{R}^3,\ (x,y)\in\omega,\ 0<z<1+\delta(t,x,y)\right\},$$

is an open subset of $\mathbb{R}^3$ for every $t\in[0,T]$ which is included in $B_M = \omega\times(0,M)$. Let $\widehat{\Omega}_\delta$ be the open domain of $\mathbb{R}^4$ defined by

$$\widehat{\Omega}_\delta = \bigcup_{t\in(0,T)} \{t\}\times\Omega_\delta(t).$$

We set $\widehat{B_M} = (0,T)\times B_M$. One can define in a standard way the spaces $L^p(\Omega_\delta(t))$, $H^1(\Omega_\delta(t))$, $H_0^1(\Omega_\delta(t))$, for every t, and $L^p(\widehat{\Omega}_\delta)$, $H^1(\widehat{\Omega}_\delta)$, $L^p(\widehat{B_M})$, $H^1(\widehat{B_M})$... The space $H^1_{0,\Gamma_0}(\Omega_\delta(t))$ will denote the subspace of $H_0^1(\Omega_\delta(t))$ of functions of zero trace on $\Gamma_0 = \omega\times\{0\}\cup\partial\omega\times[0,1]$.

We then define:

$$L^2(0,T;H^1(\Omega_\delta(t))) = \left\{v\in L^2(\widehat{\Omega}_\delta),\ \nabla v\in L^2(\widehat{\Omega}_\delta)\right\},$$

$$L^2(0,T;H_0^1(\Omega_\delta(t))) = \overline{\mathcal{D}(\widehat{\Omega}_\delta)}^{L^2(0,T;H^1(\Omega_\delta(t)))},$$

$$\mathcal{V}_\delta = \left\{\mathbf{v}\in C^1\left(\overline{\widehat{\Omega}_\delta}\right),\ \operatorname{div}\mathbf{v}=0,\ \mathbf{v}=0 \text{ on } (0,T)\times\Gamma_0\right\},$$

$$V_\delta = \overline{\mathcal{V}_\delta}^{L^2(0,T;H^1(\Omega_\delta(t)))},$$

and

$$L^\infty(0,T;L^2(\Omega_\delta(t))) = \left\{v\in L^2(\widehat{\Omega}_\delta),\ \sup\operatorname{ess}_{t\in(0,T)}\|v\|_{L^2(\Omega_\delta(t))} < +\infty\right\}.$$

Moreover we denote

$$V = \left\{ \mathbf{v} \in L^2(0,T;H^1(B_M)),\ \mathrm{div}\,\mathbf{v} = 0,\ \mathbf{v} = 0 \text{ on } (0,T) \times (\Gamma_0 \cup \Gamma_1) \right\},$$

where $\Gamma_1 = \partial\omega \times (1, M)$.

The space V_δ can be characterized as follows:

$$V_\delta = \left\{ \mathbf{v} \in L^2(0,T;H^1(\Omega_\delta(t))),\ \mathrm{div}\,\mathbf{v} = 0,\ \mathbf{v} = 0 \text{ on } (0,T) \times \Gamma_0 \right\}.$$

In the case of a Lipschitz or a star–shaped domain independent of time this follows from standard arguments (see [20] or [9]). In our case it can be proven using the fact that the domain $\Omega_\delta(t)$ is locally a subgraph. This property will be extensively used in all what follows.

Next we explain how the trace on $\partial\Omega_\delta(t)\backslash\Gamma_0$ makes sense. Let us consider the linear mapping $\gamma_{\delta(t)}$: $v \mapsto v(x, y, 1+\delta(t, x, y))$ defined for $v \in C^0(\overline{\Omega}_\delta(t))$.

Lemma 1 *For every $t \in [0,T]$, the mapping $\gamma_{\delta(t)}$ from $C^1(\overline{B_M})$ (resp. $C^1(\overline{\Omega}_\delta(t))$) in $C^0(\overline{\omega})$ can be extended by continuity to a linear mapping from $H^1(B_M)$ (resp. $H^1(\Omega_\delta(t))$) into $L^2(\omega)$.*

Proof: Take $v \in C^1(\overline{B_M})$ (resp. $C^1(\overline{\Omega}_\delta(t))$). Let s be a real number such that $0 \le s \le \alpha \le \min_{[0,T]\times\overline{\omega}}(1+\delta)$.

$$\begin{aligned}
&\|v(x,y,1+\delta(t,x,y))\|^2_{L^2(\omega)} = \int_\omega (v(x,y,1+\delta(t,x,y))^2 dxdy \\
&\le 2\int_\omega \left(\int_{1+\delta(t,x,y)-s}^{1+\delta(t,x,y)} \partial_z v(x,y,u)du \right)^2 dxdy \\
&\qquad\qquad +2\int_\omega (v(x,y,1+\delta(t,x,y)-s)^2 dxdy \\
&\le 2\int_\omega s \int_{1+\delta(t,x,y)-s}^{1+\delta(t,x,y)} (\partial_z v(x,y,u))^2 dudxdy \\
&\qquad\qquad +2\int_\omega (v(x,y,1+\delta(t,x,y)-s)^2 dxdy
\end{aligned}$$

Upon integrating for s from 0 to α, we get

$$\begin{aligned}
&\alpha\|v(x,y,1+\delta(t,x,y))\|^2_{L^2(\omega)} \le \\
&\alpha^2 \int_{\Omega_\delta(t)\backslash\Omega_{\delta-\alpha}(t)} (\partial_z v(x,y,z))^2 dxdydz + 2\int_{\Omega_\delta(t)\backslash\Omega_{\delta-\alpha}(t)} (v(x,y,z))^2 dxdydz.
\end{aligned}$$

The conclusion follows by density arguments, since $C^1(\overline{B_M})$ (respectively, $C^1(\overline{\Omega}_\delta(t))$) is dense in $H^1(B_M)$ (resp., $H^1(\Omega_\delta(t))$). For the density of $C^1(\overline{\Omega}_\delta(t))$ in $H^1(\Omega_\delta(t))$ we refer to [1, Thm 2, p. 54] (this theorem has to be slightly adapted to our case near $\{(x,y,1+\delta(t,x,y)),\ (x,y) \in \partial\omega\}$).

Corollary 1 *If* $v \in L^2(0,T;H^1(\Omega_\delta(t)))$ *then* $\gamma_{\delta(t)}(v) \in L^2(0,T;L^2(\omega))$.

Now we are going to give a characterization of $H_0^1(\Omega_\delta(t))$ with the help of the mapping $\gamma_{\delta(t)}$. An additional assumption on the boundary displacement δ is needed. δ is assumed to belong to $C^0([0,T];H^1(\omega))$ (this is not an optimal assumption).

Lemma 2 *Assuming that* $\delta \in C^0([0,T];C^0(\overline{\omega}) \cap H^1(\omega))$ *then*

$$H_0^1(\Omega_\delta(t)) = \left\{v \in H^1_{0,\Gamma_0}(\Omega_\delta(t)),\ \gamma_{\delta(t)}(v) = 0\right\}.$$

Proof: First suppose that $v \in H_0^1(\Omega_\delta(t))$. Since v can be approximated in $H_0^1(\Omega_\delta(t))$ by compactly supported functions v_ε, for which $\gamma_{\delta(t)}(v_\varepsilon) = 0$, and since the mapping $\gamma_\delta(t)$ is linear and continuous from $H_0^1(\Omega_\delta(t))$ into $L^2(\omega)$, we deduce that $\gamma_{\delta(t)}(v) = 0$.
Conversely, we have to prove that if $v \in H^1_{0,\Gamma_0}(\Omega_\delta(t))$ is such that $\gamma_{\delta(t)}(v) = 0$, then $v \in H_0^1(\Omega_\delta(t))$. We can prove by density arguments that the following Green formula is valid, for all $\delta \in C^0([0,T];C^0(\overline{\omega}) \cap H^1(\omega))$, $v \in H^1_{0,\Gamma_0}(\Omega_\delta(t))$, $\phi \in C^1(\overline{\Omega}_\delta(t))$:

$$-\int_{\Omega_\delta(t)} v\nabla\phi = \int_{\Omega_\delta(t)} \nabla v\phi + \int_\omega \gamma_{\delta(t)}(v)\gamma_{\delta(t)}(\phi)\begin{pmatrix} -\partial_x\delta \\ -\partial_y\delta \\ 1 \end{pmatrix}. \tag{11}$$

This formula implies that if $v \in H^1_{0,\Gamma_0}(\Omega_\delta(t))$ is such that $\gamma_{\delta(t)}(v) = 0$, the extended function $\overline{v}$ defined by

$$\overline{v} = \left| \begin{array}{l} v \text{ in } \Omega_\delta(t) \\ 0 \text{ in } B_M \setminus \Omega_\delta(t) \end{array} \right.$$

belongs to $H_0^1(B_M)$. Consequently, the function defined by $v_\sigma(x,y,z) = \overline{v}(x,y,\sigma z)$, $\sigma \geq 1$, is in $H_0^1(\Omega_\delta(t))$. Since it converges to v in $H^1(\Omega_\delta(t))$ as σ goes to 1, $v \in H_0^1(\Omega_\delta(t))$. Hence $H_0^1(\Omega_\delta(t))$ is the kernel of $\gamma_{\delta(t)}$ in $H^1_{0,\Gamma_0}(\Omega_\delta(t))$.

Corollary 2 *If* $v \in L^2(0,T;H^1_{0,\Gamma_0}(\Omega_\delta(t)))$ *and* $\gamma_{\delta(t)}(v) = 0$*, for a.e.* t*, then* $v \in L^2(0,T;H_0^1(\Omega_\delta(t)))$*, and the converse is also true.*

We now state a lemma that enables to extend a function $\mathbf{v} \in V_\delta$ such that $\gamma_{\delta(t)}(\mathbf{v}) = (0,0,b)^T,\ b \in L^2(0,T;H_0^1(\omega))$, the extension belonging to V.

Lemma 3 *We assume that* $\delta \in C^0([0,T]; C^0(\overline{\omega}) \cap H^1(\omega))$. *Let* $\mathbf{v} \in V_\delta$, *such that for a.e.* $t, \gamma_{\delta(t)}(\mathbf{v}) = (0,0,b)^T, b \in L^2(0,T; H_0^1(\omega))$. *The function defined by*

$$\overline{\mathbf{v}} = \left| \begin{array}{l} \mathbf{v} \ \textit{in} \ \widehat{\Omega}_\delta \\ (0,0,b)^T \ \textit{in} \ \widehat{B_M} \setminus \widehat{\Omega}_\delta \end{array} \right.$$

belongs to V, *and*

$$\|\overline{\mathbf{v}}\|_V \leq C(\|\mathbf{v}\|_{V_\delta} + \|b\|_{L^2(0,T;H^1(\omega))}),$$

where C *depends only on* M.

Proof: Using the same type of argument as in the proof of the previous lemma, it is easy to check that the extension $\overline{\mathbf{v}}$ is in $L^2(0,T; H^1(B_M))$, and that

$$\|\overline{\mathbf{v}}\|_V \leq \|\mathbf{v}\|_{V_\delta} + \|\mathbf{w}\|_{L^2(0,T;H^1(B_M \setminus \Omega_\delta(t)))},$$

where $\mathbf{w} = (0,0,b)^T$. But

$$\|\mathbf{w}\|^2_{L^2(0,T;H^1(B_M \setminus \Omega_\delta(t)))} \leq (M-\alpha)\|b\|^2_{L^2(0,T;H^1(\omega))} \leq M\|b\|^2_{L^2(0,T;H^1(\omega))},$$

thus the desired inequality holds. Moreover, since b depends only on t, x and y, the vector $(0,0,b)^T$ is divergence free. Thus $\operatorname{div} \overline{\mathbf{v}} = 0$ in $\widehat{B_M}$. Hence $\overline{\mathbf{v}} \in V$.

Remark 2 *If* $\mathbf{v} \in L^\infty(0,T; L^2(\Omega_\delta(t)))$ *and* $b \in L^\infty(0,T; L^2(\omega))$ *then also* $\overline{\mathbf{v}} \in L^\infty(0,T; L^2(B_M))$.

Next we build a lifting operator:

Lemma 4 *For every* $\phi \in H_0^1(\omega)$ *there exists* $w \in H^1_{0,\Gamma_0}(\Omega_\delta(t))$ *such that*

$$\gamma_{\delta(t)}(w) = \phi \quad \textit{and} \quad \|w\|_{H^1(\Omega_\delta(t))} \leq C_\alpha \|\phi\|_{H^1(\omega)}.$$

Proof: Let $\phi \in H_0^1(\omega)$. We consider

$$w = \left| \begin{array}{l} \phi \text{ in } \Omega_\delta(t) \setminus \mathcal{C}_\alpha \\ \mathcal{R}\tilde{\phi} \text{ in } \mathcal{C}_\alpha, \end{array} \right.$$

where $\mathcal{C}_\alpha = \omega \times (0,\alpha)$, $\tilde{\phi} = \frac{z}{\alpha}\phi$ is the extension of $\phi|_{\omega \times \{\alpha\}}$ by zero on $\Gamma_0^\alpha = \partial\mathcal{C}_\alpha \cap \Gamma_0$ and $\mathcal{R}$ is a continuous linear lifting operator from $H^{1/2}(\partial\mathcal{C}_\alpha)$ on $H^1(\mathcal{C}_\alpha)$. This function is well defined, since we have assumed that $1 + \delta(t,x,y) \geq \alpha \ \forall \ (t,x,y) \in (0,T) \times \omega$, and it has the required properties.

Remark 3 *The space*

$$\left\{\mathbf{v} \in V_\delta, \gamma_{\delta(t)}(\mathbf{v}) = (0,0,b)^T,\ \textit{for a.e. } t,\, b \in L^2(0,T;H_0^1(\omega))\right\},$$

is equal to the sum of the two following spaces :

$$\overline{\left\{\mathbf{v} \in \mathcal{D}(\widehat{\Omega}_\delta),\ \operatorname{div} \mathbf{v} = 0\right\}}^{L^2(0,T;H^1(\Omega_\delta(t))}\ ,$$

and

$$\left\{\mathbf{v},\ \mathbf{v} = \left| \begin{array}{l} (0,0,b)^T \ \textit{in } \Omega_\delta(t) \setminus \mathcal{C}_\alpha \\ \tilde{\mathcal{R}}(0,0,\frac{z}{\alpha}b)^T \ \textit{in } \mathcal{C}_\alpha\ , \end{array} \right. \textit{for a.e. } t,\ b \in L^2(0,T;H_0^1(\omega)), \int_\omega b = 0 \right\}$$

where $\tilde{\mathcal{R}}$ *is a continuous lifting operator from* $H^{1/2}(\partial\mathcal{C}_\alpha)$ *into* $H^1(\mathcal{C}_\alpha)$*, such that* $\operatorname{div}(\tilde{\mathcal{R}}\mathbf{v}) = 0$.

Now, for every function in $E_{\delta(t)} = \left\{\mathbf{v} \in L^2(\Omega_\delta(t))\ ,\ \operatorname{div} \mathbf{v} \in L^2(\Omega_\delta(t))\right\}$, we are going to give a weak sense to the normal trace on the moving boundary .

Lemma 5 *For every* t*, there exists a linear continuous operator* $\gamma^n_{\delta(t)}$ *from* $E_{\delta(t)}$ *to* $H^{-1}(\omega)$ *such that*

$$\gamma^n_{\delta(t)}(\mathbf{v}) = \mathbf{v}(t,x,y,1+\delta(t,x,y)) \cdot \mathbf{n}_t,\ \forall (x.y) \in \omega,\ \forall \mathbf{v} \in C^\infty(\overline{\Omega}_\delta(t)).$$

Proof:
We follow the same steps as for the standard case. The key points are the Stokes formula:

$$\int_\omega (\mathbf{v} \cdot \mathbf{n}_t)\gamma_{\delta(t)}(w) = \int_{\Omega_\delta(t)} \mathbf{v} \cdot \nabla w + \int_{\Omega_\delta(t)} w \operatorname{div} \mathbf{v},$$

$$\left| \int_{\Omega_\delta(t)} \mathbf{v} \cdot \nabla w + \int_{\Omega_\delta(t)} w \operatorname{div} \mathbf{v} \right| \leq \left(\|\mathbf{v}\|_{L^2(\Omega_\delta(t))} + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega_\delta(t))} \right) \|w\|_{H^1(\Omega_\delta(t))},$$

and the fact that for all $\phi \in H_0^1(\omega)$ there exists $w \in H^1(\Omega_\delta(t))$, defined by Lemma 4, such that $\gamma_{\delta(t)}(w) = \phi$.

Now we give an important lemma that justifies Remark 1 and Korn's inequality in the considered spaces.

Lemma 6 *For all* $\mathbf{u}$ *and* $\mathbf{v}$ *in*

$$\left\{\mathbf{v} \in V_\delta,\ \exists\, b \in L^2(0,T;H_0^1(\omega)),\ \gamma_{\delta(t)}(\mathbf{v}) = (0,0,b)^T \text{ for a.e. } t\right\},$$

we have

$$2\int_{\Omega_\delta(t)} D(\mathbf{u}) : D(\mathbf{v}) = \int_{\Omega_\delta(t)} \nabla\mathbf{u} : \nabla\mathbf{v}, \quad \text{for a.e. } t, \tag{12}$$

and consequently the following Korn's "equality" holds:

$$\sqrt{2}\|D(\mathbf{u})\|_{L^2(\Omega_{\delta(t)})} = \|\nabla\mathbf{u}\|_{L^2(\Omega_{\delta(t)})}, \text{ for a.e. } t. \tag{13}$$

Proof: For all such $\mathbf{u}$ and $\mathbf{v}$, we show

$$\int_{\Omega_\delta(t)} \nabla\mathbf{u} : (\nabla\mathbf{v})^T = 0.$$

We prove that this equality holds for smooth functions, the conclusion follows by standard density arguments. We have (using Einstein's convention on repeated indices)

$$\int_{\Omega_\delta(t)} \nabla\mathbf{u} : (\nabla\mathbf{v})^T = \int_\omega \gamma_{\delta(t)}(u_j)\gamma_{\delta(t)}(\partial_j v_i)n_i - \int_{\Omega_\delta(t)} \mathbf{u}\cdot\nabla(\operatorname{div}\mathbf{v}).$$

Since $\operatorname{div}\mathbf{v} = 0$ and since there exists a $b \in L^2(0,T;H_0^1(\omega))$ with $\gamma_{\delta(t)}(\mathbf{u}) = (0,0,b)^T$,

$$\int_{\Omega_\delta(t)} \nabla\mathbf{u} : (\nabla\mathbf{v})^T = \int_\omega b\,(\gamma^n_{\delta(t)}((\nabla\mathbf{v})^T))_3.$$

But if $\mathbf{v}$ is smooth, we have

$$(\gamma^n_{\delta(t)}((\nabla\mathbf{v})^T))_3 = 0.$$

Thus (12) holds true. Equality (13) follows by taking $\mathbf{v} = \mathbf{u}$ in (12).

Finally, we mention that Poincaré's inequality holds for the considered open sets:

Lemma 7 *Let* $v \in H^1_{0,\Gamma_0}(\Omega_\delta(t))$, *then*

$$\|v\|_{L^2(\Omega_\delta(t))} \le M\|\nabla v\|_{L^2(\Omega_\delta(t))}.$$

1.4 Weak formulation

Now that we have defined suitable functional spaces and given a sense to the continuity of the velocity on the elastic plate, we are able to define the weak formulation of the problem. Let $\eta_0 \in H_0^2(\omega)$, $(\mathbf{u}_0, \eta_1) \in L^2(\Omega_{\eta_0}) \times L^2(\omega)$ such that

$$\begin{aligned}
&\min_{\overline{\omega}}(1+\eta_0) > 0,\\
&\operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega_{\eta_0},\\
&\mathbf{u}_0 \cdot \mathbf{n} = 0 \text{ on } \Gamma_0,\\
&\gamma^n_{\eta_0}(\mathbf{u}_0) = (0,0,\eta_1)^T \cdot \mathbf{n}_0 \text{ on } \omega,\\
&\int_\omega \eta_1(x,y) = 0.
\end{aligned} \tag{14}$$

We shall say that $(\mathbf{u}, \eta)$ is a weak solution of (1), (3), (4), (6) on $[0,T)$ if

- $\mathbf{u} \in V_\eta \cap L^\infty(0,T; L^2(\Omega_\eta(t)))$, $\eta \in W^{1,\infty}(0,T; L^2(\omega)) \cap H^1(0,T; H_0^2(\omega))$,
- $\gamma_{\eta(t)}(\mathbf{u}) = (0,0,\partial_t \eta)^T$, for a.e. t,
- for all $(\phi, b) \in \mathcal{V}_\eta \times C^1([0,T]; H_0^2(\omega))$ such that $\phi(t,x,y,1+\eta(t,x,y)) = (0,0,b(t,x,y))^T$, $(t,x,y) \in [0,T] \times \omega$, we have for a.e. t

$$\begin{aligned}
&\int_{\Omega_\eta(t)} \mathbf{u}(t)\cdot\phi(t) - \int_0^t \int_{\Omega_\eta(s)} \mathbf{u}\cdot\partial_t\phi + 2\nu \int_0^t \int_{\Omega_\eta(s)} D(\mathbf{u}) : D(\phi)\\
&\quad + \int_0^t \int_{\Omega_\eta(s)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\phi - \int_0^t \int_\omega (\partial_t \eta)^2 b + \int_\omega \partial_t\eta(t)\, b(t)\\
&\quad - \int_0^t \int_\omega \partial_t\eta\, \partial_t b + \int_0^t \int_\omega \Delta\partial_t\eta\, \Delta b + \int_0^t \int_\omega \Delta\eta\, \Delta b\\
&\quad = \int_0^t \int_{\Omega_\eta(t)} \mathbf{f}\cdot\phi + \int_0^t \int_\omega g\; b + \int_{\Omega(0)} \mathbf{u}_0 \phi(0) + \int_\omega \eta_1 b(0)
\end{aligned} \tag{15}$$

Following Lemma 6 formulation (15) is equivalent to

$$\begin{aligned}
&\int_{\Omega_\eta(t)} \mathbf{u}(t)\cdot\phi(t) - \int_0^t \int_{\Omega_\eta(s)} \mathbf{u}\cdot\partial_t\phi + \nu \int_0^t \int_{\Omega_\eta(s)} \nabla\mathbf{u} : \nabla\phi\\
&\quad + \int_0^t \int_{\Omega_\eta(s)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\phi - \int_0^t \int_\omega (\partial_t \eta)^2 b + \int_\omega \partial_t\eta(t)\, b(t)\\
&\quad - \int_0^t \int_\omega \partial_t\eta\, \partial_t b + \int_0^t \int_\omega \Delta\partial_t\eta\, \Delta b + \int_0^t \int_\omega \Delta\eta\, \Delta b\\
&\quad = \int_0^t \int_{\Omega_\eta(t)} \mathbf{f}\cdot\phi + \int_0^t \int_\omega g\; b + \int_{\Omega(0)} \mathbf{u}_0 \phi(0) + \int_\omega \eta_1 b(0)
\end{aligned} \tag{16}$$

where we have just replaced $D(\mathbf{u}) : D(\phi)$ with $\nabla \mathbf{u} : \nabla \phi$.

Remark 4 *We remark that the convection term* $\int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\phi$ *can also be written as:*

$$\frac{1}{2}\int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\phi - \frac{1}{2}\int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\phi\cdot\mathbf{u} + \frac{1}{2}\int_\omega (\partial_t\eta)^2 b.$$

This remark will be essential in Section 3, as it will make much easier the construction of approximate solutions to our problem.

2 Main Result

The aim of the paper is to prove the existence of weak solutions as defined in the previous section. We prove that there exists at least one weak solution provided that the plate does not touch the bottom of the fluid cavity, in other words as long as $1+\eta(t,x,y) > 0$, $\forall (x,y) \in \omega$. We make the following hypotheses on the data (bulk forces, surface forces, initial velocities and initial displacement of the structure):

$$\begin{aligned} \mathbf{f} \in L^2((0,T)\times(0,M)),\ g \in L^2((0,T)\times\omega), \\ \mathbf{u}_0 \in L^2(\Omega_{\eta_0}),\ \eta_1 \in L^2(\omega),\ \eta_0 \in H_0^2(\omega), \end{aligned} \tag{17}$$

for all $T > 0$ and $M > 0$. We assume moreover that conditions (14) are satisfied. Then the following theorem holds true:

Theorem 1 *Under assumptions (14), (17), there exists* $T^* \in (0,+\infty]$ *and a weak solution* $(\mathbf{u},\eta)$ *of (1), (3), (4), (6) on* $[0,T]$, $T < T^*$. *This solution satisfies for all* $T < T^*$ *the following energy estimates:*

$$\begin{aligned} &\|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega_\eta(t)))} + \|\nabla\mathbf{u}\|_{L^2(0,T;L^2(\Omega_\eta(t)))} \\ &\qquad + \|\partial_t\eta\|_{L^\infty(0,T;L^2(\omega))} + \|\Delta\eta\|_{H^1(0,T;L^2(\omega))} \\ &\le C(T, \|\mathbf{u}_0\|_{L^2(\Omega_{\eta_0})}, \|\mathbf{f}\|_{L^2((0,T)\times\mathbb{R}^3)}, \|g\|_{L^2((0,T)\times\omega)}, \|\eta_0\|_{H_0^2(\omega)}, \|\eta_1\|_{L^2(\omega)}), \end{aligned}$$

where $C > 0$ *is nondecreasing with respect to its arguments. Moreover we have the following alternative*

- *either* $T^* = +\infty$,
- *or* $\lim_{t\to T^*} \min_{\overline{\omega}} (1+\eta) = 0$.

Several steps are needed in order to prove this theorem. In a first step (Section 3), approximate solutions $(\mathbf{u}_\varepsilon, \eta_\varepsilon)_{\varepsilon>0}$ are built by regularizing the convection velocities. Then compactness results (Section 4) allow to pass the limit in the equations as ε goes to zero (Section 5).

3 Approximate solutions

In this section we build a sequence of approximate solutions $(\mathbf{u}_\varepsilon, \eta_\varepsilon)_{\varepsilon>0}$. Let $\mathbf{u}_0^\varepsilon$, η_0^ε, η_1^ε be regularizations of the initial data such that div $\mathbf{u}_0^\varepsilon = 0$, $\mathbf{u}_0^\varepsilon(x,y,1+\eta_0^\varepsilon(x,y)) = (0,0,\eta_1^\varepsilon)^T$, $\mathbf{u}_0^\varepsilon = 0$ on Γ_0, $\int_\omega \eta_0^\varepsilon = \int_\omega \eta_0$ and such that

$$\rho_0^\varepsilon \mathbf{u}_0^\varepsilon \to \rho_0 \mathbf{u}_0 \text{ in } L^2(B_M),$$

$$\eta_1^\varepsilon \to \eta_1 \text{ in } L^2(\omega),$$

$$\eta_0^\varepsilon \to \eta_0 \text{ in } H_0^2(\omega),$$

as ε goes to zero, where ρ_0 (resp. ρ_0^ε) denotes the characteristic function of Ω_{η_0} (resp. $\Omega_{\eta_0^\varepsilon}$), and we adopt the convention that a product such as $\rho_0^\varepsilon \mathbf{u}_0^\varepsilon$ denotes the function which is equal to $\mathbf{u}_0^\varepsilon$ where $\rho_0^\varepsilon = 1$ and 0 where $\rho_0^\varepsilon = 0$.

Such sequences can be built as follows. First one builds $\eta_0^\varepsilon \in \mathcal{D}(\omega)$ satisfying $\int_\omega \eta_0^\varepsilon = \int_\omega \eta_0$ and $\eta_0^\varepsilon \to \eta_0$ in $H_0^2(\omega)$. For ε small enough, if $\min_{\overline{\omega}}(1+\eta_0) \geq 2\alpha > 0$ we have $\min_{\overline{\omega}}(1+\eta_0^\varepsilon) \geq 3\alpha/2$. Next we define $\overline{\mathbf{u}}_0$ by

$$\overline{\mathbf{u}}_0 = \left| \begin{array}{l} \mathbf{u}_0 \text{ in } \Omega_{\eta_0} \\ (0,0,\eta_1)^T \text{ in } \mathbf{B}_{M+1} \setminus \Omega_{\eta_0}. \end{array} \right.$$

Since $\gamma_{\eta_0}^n(\mathbf{u}_0) = (0,0,\eta_1)^T \cdot \mathbf{n}_0$, $\overline{\mathbf{u}}_0$ is divergence free. Let us consider

$$\mathbf{u}_0^\sigma(x,y,z) = (\sigma\overline{\mathbf{u}}_{01}(x,y,\sigma z), \sigma\overline{\mathbf{u}}_{02}(x,y,\sigma z), \overline{\mathbf{u}}_{03}(x,y,\sigma z)),\ \sigma \geq 1,$$

which is also divergence free. For $\sigma > 1$, $\mathbf{u}_0^\sigma = (0,0,\eta_1)^T$ in a neighborhood of $\{(x,y,1+\eta_0(x,y))\ ,\ (x,y) \in \omega\}$, thus in a neighborhood of $\{(x,y,1+\eta_0^\varepsilon(x,y))\ ,\ (x,y) \in \omega\}$ for ε small enough.

We regularize $\mathbf{u}_0^\sigma$ in a standard way: it gives the required approximations of $(\mathbf{u}_0, \eta_1)$. We shall prove that there exists a unique solution $(\mathbf{u}_\varepsilon, \eta_\varepsilon)$ in

$$(V_{\eta_\varepsilon^*} \cap L^\infty(0,T; L^2(\Omega_{\eta_\varepsilon^*}(t)))) \times (W^{1,\infty}(0,T; L^2(\omega)) \cap H^1(0,T; H_0^2(\omega)))$$

satisfying

- $\mathbf{u}_\varepsilon(t,x,y,1+\eta^*_\varepsilon(t,x,y)) = (0,0,\partial_t\eta_\varepsilon(t,x,y))^T$ on $\partial\omega$,
- $\partial_t\mathbf{u}_\varepsilon \in L^2(0,T;L^2(\Omega_{\eta^*_\varepsilon}(t)))$,
- $\partial_{tt}\eta_\varepsilon \in L^2(0,T;L^2(\omega))$,
- $\mathbf{u}_\varepsilon(0) = \mathbf{u}_0^\varepsilon$ and $\eta^\varepsilon(0) = \eta_0^\varepsilon$, $\partial_t\eta^\varepsilon(0) = \eta_1^\varepsilon$.
- and

$$\begin{aligned}
&\int_0^t\int_{\Omega_{\eta^*_\varepsilon}(s)} \partial_t\mathbf{u}_\varepsilon\cdot\phi_\varepsilon + \nu\int_0^t\int_{\Omega_{\eta^*_\varepsilon}(s)} \nabla\mathbf{u}_\varepsilon : \nabla\phi_\varepsilon \\
&+\frac{1}{2}\int_0^t\int_{\Omega_{\eta^*_\varepsilon}(s)} (\mathcal{R}_\varepsilon(\overline{\mathbf{u}}_\varepsilon)\cdot\nabla)\mathbf{u}_\varepsilon\cdot\phi_\varepsilon - \frac{1}{2}\int_0^t\int_{\Omega_{\eta^*_\varepsilon}(s)} (\mathcal{R}_\varepsilon(\overline{\mathbf{u}}_\varepsilon)\cdot\nabla)\phi_\varepsilon\cdot\mathbf{u}_\varepsilon \\
&\qquad + \frac{1}{2}\int_0^t\int_\omega \partial_t\eta_\varepsilon\partial_t\eta^*_\varepsilon b + \int_0^t\int_\omega \partial_{tt}\eta_\varepsilon\, b \\
&+\int_0^t\int_\omega \Delta\partial_t\eta_\varepsilon\,\Delta b + \int_0^t\int_\omega \Delta\eta_\varepsilon\,\Delta b = \int_0^t\int_{\Omega_{\eta^*_\varepsilon}(s)} \mathbf{f}\cdot\phi_\varepsilon + \int_0^t\int_\omega g\ b, \\
&\qquad \forall\phi_\varepsilon \in V_{\eta^*_\varepsilon},\ b\in L^2(0,T;H^2_0(\omega)) \text{ such that} \\
&\qquad \phi_\varepsilon(t,x,y,1+\eta^*_\varepsilon(t,x,y)) = (0,0,b(t,x,y))^T \text{ on } \omega. \qquad (18)
\end{aligned}$$

The superscript $*$ and $\mathcal{R}_\varepsilon$ denote regularization operators that will be made precise in the next Section 3.1. Thus, $\mathcal{R}_\varepsilon(\overline{\mathbf{u}}_\varepsilon)$ and η^*_ε are regularizations of $\overline{\mathbf{u}}_\varepsilon$ and η_ε, where $\overline{\mathbf{u}}_\varepsilon$ is the extension of $\mathbf{u}_\varepsilon$ defined in Lemma 3. Note that the convection terms have been rewritten according to Remark 4. A consequence of this reformulation is that the solution satisfies the same energy estimates as in the original problem. Indeed, choosing $(\mathbf{u}_\varepsilon, \partial_t\eta_\varepsilon)$ as a test function, we remark that:

$$\int_{\Omega_{\eta^*_\varepsilon}(t)} \partial_t\mathbf{u}_\varepsilon\cdot\mathbf{u}_\varepsilon + \frac{1}{2}\int_\omega (\partial_t\eta_\varepsilon)^2\partial_t\eta^*_\varepsilon = \frac{1}{2}\frac{d}{dt}\int_{\Omega_{\eta^*_\varepsilon}(t)} |\mathbf{u}_\varepsilon|^2,$$

since the moving boundary of the fluid domain $\Omega_{\eta^*_\varepsilon}(t)$ moves at the velocity $(0,0,\partial_t\eta^*_\varepsilon)^T$ and $\mathbf{u}_\varepsilon(t,x,y,1+\eta^*_\varepsilon(t,x,y)) = (0,0,\partial_t\eta_\varepsilon(t,x,y))^T$ on ω.

Thus we obtain:

$$\begin{aligned}
\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_{\eta^*_\varepsilon}(t)))} &+ \|\nabla\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega_{\eta^*_\varepsilon}(t)))} \\
&+ \|\partial_t\eta_\varepsilon\|_{L^\infty(0,T;L^2(\omega))} + \|\Delta\eta_\varepsilon\|_{H^1(0,T;L^2(\omega))} \le C, \quad (19)
\end{aligned}$$

where $C > 0$ depends only on the data and not on ε. Applying Lemma 7 leads to

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega_{\eta_\varepsilon^*}(t)))} \leq C, \tag{20}$$

where C depends on the data and on $\max_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon^*)$. But from (19) and since $H^1(0,T;H_0^2(\omega)) \hookrightarrow C^0([0,T]\times\overline{\omega})$, we deduce that $\min_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon)$ is bounded independently of ε and thus also $\min_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon^*)$. Hence C does not depend on ε.

Remark 5 *If the convection term had not been rewritten according to Remark 4 then we would have to impose that*

$$\mathcal{R}_\varepsilon(\overline{\mathbf{u}}_\varepsilon)(t,x,y,1+\eta_\varepsilon^*(t,x,y)) = (0,0,\partial_t\eta_\varepsilon^*(t,x,y))^T$$

on ω. In [6] the regularization operators were built in order to satisfy this condition. But we can not use the same kind of construction here because of the lack of regularity of the structure displacement.

This regularized problem is solved using a fixed point procedure. The proof of existence is split in several steps. In a first step, we linearize the weak formulation and prove that there exists a unique solution to this regularized – linearized problem thanks to a Galerkin method. Then additional regularity is derived that enables us to apply Schauder fixed point theorem and obtain the existence of $(\mathbf{u}_\varepsilon, \eta_\varepsilon)$.

3.1 Linearized approximate problem

We start by linearizing equation (18). Let us take $\delta \in H^1(0,T;C^0(\overline{\omega}) \cap H_0^1(\omega))$ on $\partial\omega$, $\delta|_{t=0} = \eta_0^\varepsilon$ and $M \geq 1+\delta(t,x,y) \geq \alpha > 0$, $\forall(t,x,y) \in [0,T]\times\overline{\omega}$. The constant M will be chosen later and α is taken such that $\min_{\overline{\omega}}(1+\eta_0) \geq 2\alpha > 0$. We take also $\mathbf{v} \in L^2(0,T;L^2(B_{2M}))$.

We consider $\delta_\varepsilon^* = \mathcal{N}_\varepsilon(\delta)$ and $\mathbf{v}_\varepsilon^* = \mathcal{R}_\varepsilon(\mathbf{v})$ space–time regularizations of δ and $\mathbf{v}$. We require that $\mathcal{N}_\varepsilon(\delta_\varepsilon)$ converges to δ in $C^0([0,T]\times\overline{\omega})$ whenever δ_ε converges to δ in $C^0([0,T]\times\overline{\omega})$, that $\partial_t\mathcal{N}_\varepsilon(\delta_\varepsilon)$ converges to $\partial_t\delta$ in $L^2(0,T;L^2(\omega))$ whenever $\partial_t\delta_\varepsilon$ converges to $\partial_t\delta$ in $L^2(0,T;L^2(\omega))$ and that $\mathcal{R}_\varepsilon(\mathbf{v}_\varepsilon)$ converges to $\mathbf{v}$ in $L^2(0,T;L^2(B_{2M}))$ whenever $\mathbf{v}_\varepsilon$ converges to $\mathbf{v}$ in $L^2(0,T;L^2(B_{2M}))$.

We build $\mathcal{N}_\varepsilon(\delta)$ as follows:

$$\mathcal{N}_\varepsilon(\delta) = S_\varepsilon(\delta - \delta|_{t=0}) + \eta_0^\varepsilon,$$

where S_ε denotes a space–time regularization such that $S_\varepsilon(b)|_{t=0} = 0$ whenever $b|_{t=0} = 0$. In particular $\mathcal{N}_\varepsilon(\delta)|_{t=0} = \eta_0^\varepsilon$. We can also suppose that $2M \geq 1 + \delta_\varepsilon^*(t,x,y) \geq \frac{\alpha}{2}$, $\forall (t,x,y) \in [0,T] \times \overline{\omega}$.

We want to solve the following problem: find $(\mathbf{u}_\varepsilon, \eta_\varepsilon)$ such that

- $\mathbf{u}_\varepsilon \in V_{\delta_\varepsilon^*} \cap L^\infty(0,T; L^2(\Omega_{\delta_\varepsilon^*}(t)))$
- $\eta_\varepsilon \in W^{1,\infty}(0,T; L^2(\omega)) \cap H^1(0,T; H_0^2(\omega))$
- $\mathbf{u}_\varepsilon(t,x,y,1+\delta_\varepsilon^*(t,x,y)) = (0,0,\partial_t \eta_\varepsilon(t,x,y))^T$ on $\partial\omega$,
- $\partial_t \mathbf{u}_\varepsilon \in L^2(0,T; L^2(\Omega_{\delta_\varepsilon^*}(t)))$,
- $\partial_{tt}\eta_\varepsilon \in L^2(0,T; L^2(\omega))$
- and

$$\int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} \partial_t \mathbf{u}_\varepsilon \cdot \phi_\varepsilon + \nu \int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} \nabla \mathbf{u}_\varepsilon : \nabla \phi_\varepsilon + \frac{1}{2} \int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} (\mathbf{v}_\varepsilon^* \cdot \nabla) \mathbf{u}_\varepsilon \cdot \phi_\varepsilon$$
$$- \frac{1}{2} \int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} (\mathbf{v}_\varepsilon^* \cdot \nabla) \phi_\varepsilon \cdot \mathbf{u}_\varepsilon + \frac{1}{2} \int_0^t \int_\omega \partial_t \eta_\varepsilon \partial_t \delta_\varepsilon^* b + \int_0^t \int_\omega \partial_{tt} \eta_\varepsilon \, b$$
$$+ \int_0^t \int_\omega \Delta \partial_t \eta_\varepsilon \, \Delta b + \int_0^t \int_\omega \Delta \eta_\varepsilon \, \Delta b = \int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} \mathbf{f} \cdot \phi_\varepsilon + \int_0^t \int_\omega g \; b,$$
$$\forall \phi_\varepsilon \in V_{\delta_\varepsilon^*}, \; b \in L^2(0,T; H_0^2(\omega)) \text{ s. t.}$$
$$\phi_\varepsilon(t,x,y,1+\delta_\varepsilon^*(t,x,y)) = (0,0,b(t,x,y))^T \text{ on } \omega, \qquad (21)$$

- $\mathbf{u}_\varepsilon(0) = \mathbf{u}_0^\varepsilon$, $\eta_\varepsilon(0) = \eta_0^\varepsilon$ and $\partial_t \eta_\varepsilon(0) = \eta_1^\varepsilon$.

In this problem the test functions do not depend on the solution. As previously, any solution of (21) satisfies energy estimates independent of ε.

3.1.1 Existence of solution: Galerkin method

We are going to discretize this problem using a Galerkin method. First we rewrite the equations in a "reference" configuration thanks to a change of variables, denoted by χ_ε. The reference configuration is the subdomain of $\mathbb{R}^3$ given by $\omega \times (0,1)$ and will be denoted by $\mathcal{C}$. We define χ_ε by

$$\chi_\varepsilon(t,x,y,z) = \left| \begin{array}{l} x \\ y \\ z(1+\delta_\varepsilon^*(t,x,y)) \end{array} \right. \qquad \forall (x,y,z) \in \mathcal{C}, t \in (0,T). \qquad (22)$$

The function χ_ε is smooth in space and time and $\chi_\varepsilon(t,\cdot)$ is a C^k–diffeomorphism that maps $\mathcal{C}$ into $\Omega_{\delta^*_\varepsilon(t)}$. We denote by $\underline{\mathbf{w}}_\varepsilon$ the time derivative of χ_ε, that is, $\underline{\mathbf{w}}_\varepsilon(t,x,y,z) = (0,0,z\partial_t\delta^*_\varepsilon)^T$. After this change of variables equations (21) become:

$$\begin{aligned}
&\int_0^t\int_{\mathcal{C}}\partial_t\underline{\mathbf{u}}_\varepsilon\cdot\underline{\phi}_\varepsilon J_\varepsilon+\nu\int_0^t\int_{\mathcal{C}}A_\varepsilon\nabla\underline{\mathbf{u}}_\varepsilon:\nabla\underline{\phi}_\varepsilon+\frac{1}{2}\int_0^t\int_{\mathcal{C}}(\underline{\mathbf{v}}^*_\varepsilon\cdot(B_\varepsilon\nabla))\underline{\mathbf{u}}_\varepsilon\cdot\underline{\phi}_\varepsilon\\
&-\frac{1}{2}\int_0^t\int_{\mathcal{C}}(\underline{\mathbf{v}}^*_\varepsilon\cdot(B_\varepsilon\nabla))\underline{\phi}_\varepsilon\cdot\underline{\mathbf{u}}_\varepsilon+\frac{1}{2}\int_0^t\int_\omega\partial_t\eta_\varepsilon\partial_t\delta^*_\varepsilon b-\int_0^t\int_{\mathcal{C}}(\underline{\mathbf{w}}_\varepsilon\cdot(B_\varepsilon\nabla))\underline{\mathbf{u}}_\varepsilon\cdot\underline{\phi}_\varepsilon\\
&+\int_0^t\int_\omega\partial_{tt}\eta_\varepsilon\, b+\int_0^t\int_\omega\Delta\partial_t\eta_\varepsilon\,\Delta b+\int_0^t\int_\omega\Delta\eta_\varepsilon\,\Delta b=\int_0^t\int_{\mathcal{C}}\underline{\mathbf{f}}\cdot\underline{\phi}_\varepsilon J_\varepsilon+\int_0^t\int_\omega g\, b,\\
&\qquad\forall\underline{\phi}_\varepsilon\in L^2(0,T;H^1_{0,\Gamma_0}(\mathcal{C})),\ b\in L^2(0,T;H^2_0(\omega))\ \text{s.t.}\\
&\qquad\quad\underline{\phi}_\varepsilon(t,x,y,1)=(0,0,b(t,x,y))^T\ \text{on}\ \omega,\ \operatorname{div}(B^T_\varepsilon\underline{\phi}_\varepsilon)=0\ \text{in}\ \mathcal{C}.\qquad(23)
\end{aligned}$$

Here $\underline{v}$ denotes the function v transported by the flow χ_ε, and A_ε, B_ε denote matrices appearing after the change of variables and depending on $\nabla\chi_\varepsilon$: $J_\varepsilon = \det\nabla\chi_\varepsilon = 1+\delta^*_\varepsilon$, $B_\varepsilon = \operatorname{cof}\nabla\chi_\varepsilon$, $A_\varepsilon = \dfrac{1}{J_\varepsilon}B^T_\varepsilon B_\varepsilon$. The equality of the velocities at the interface writes:

$$\underline{\mathbf{u}}_\varepsilon(t,x,y,1) = (0,0,\partial_t\eta_\varepsilon(t,x,y))^T,\ (x,y)\in\omega.$$

3.1.2 Construction of the Galerkin Basis

We generate a basis $(\psi^0_i)_{i\in\mathbb{N}}$ of the space $\{\mathbf{v}\in H^1_0(\mathcal{C}),\ \operatorname{div}\mathbf{v}=0 \text{ in } \mathcal{C}\}$ by considering all the eigenfunctions of the Stokes problem

$$\begin{aligned}
&-\Delta\psi^0_i+\nabla p^0_i=\mu^\varepsilon_i\psi^0_i\ \text{in}\ \mathcal{C},\\
&\operatorname{div}\psi^0_i=0\ \text{in}\ \mathcal{C},\\
&\psi^0_i=0\ \text{on}\ \partial\mathcal{C},
\end{aligned}$$

and we set $\phi^0_i = (B^T_\varepsilon)^{-1}\psi^0_i$. The family $(\phi^0_i)_{i\in\mathbb{N}}$ is a basis of the space $\{\mathbf{v}\in H^1_0(\mathcal{C}),\ \operatorname{div}(B^T_\varepsilon\mathbf{v}) = 0 \text{ in } \mathcal{C}\}$. The functions ϕ^0_i are smooth in time since B_ε is smooth in time.
We consider also $(\xi_i)_{i\in\mathbb{N}}$, a basis of the space $\{b\in H^2_0(\omega),\ \int_\omega b=0\}$, and we build functions $(\phi^{1,\varepsilon}_i)_{i\in\mathbb{N}}$ such that $\operatorname{div}(B^T_\varepsilon\phi^{1,\varepsilon}_i)=0$ and $\phi^{1,\varepsilon}_i(t,x,y,1) = (0,0,\xi_i(x,y))^T, \forall(x,y)\in\omega$. This can be done by solving a Stokes–like prob-

lem:

$$\begin{aligned}&-\Delta\phi_i^{1,\varepsilon} + (B_\varepsilon\cdot\nabla)p_i^{1,\varepsilon} = 0 \text{ in } \mathcal{C},\\ &\operatorname{div}(B_\varepsilon^T\phi_i^{1,\varepsilon}) = 0 \text{ in } \mathcal{C},\\ &\phi_i^{1,\varepsilon} = \left|\begin{array}{l} 0 \text{ on } \Gamma_0 \\ (0,0,\xi_i)^T \text{ on } \partial\mathcal{C}\setminus\Gamma_0\end{array}\right.\end{aligned}$$

The functions $\phi_i^{1,\varepsilon}$ are smooth in time since B_ε is smooth in time. We now look for $\eta_\varepsilon^n = \sum_{i=1}^n \beta_i(t)\xi_i + \eta_0^\varepsilon$ and $\underline{\mathbf{u}}_\varepsilon^{m,n} = \sum_{i=1}^m \alpha_i(t)\phi_i^{0,\varepsilon} + \sum_{j=1}^n \dot\beta_j(t)\phi_j^{1,\varepsilon}$ satisfying the discrete problem: $\forall 1\le i\le m$,

$$\begin{aligned}&\int_{\mathcal{C}} \partial_t\underline{\mathbf{u}}_\varepsilon^{m,n}\cdot\phi_i^{0,\varepsilon}J_\varepsilon + \nu\int_{\mathcal{C}} A_\varepsilon\nabla\underline{\mathbf{u}}_\varepsilon^{m,n}\cdot\nabla\phi_i^{0,\varepsilon} + \frac{1}{2}\int_{\mathcal{C}}(\underline{\mathbf{v}}_\varepsilon^*\cdot(B_\varepsilon\nabla))\underline{\mathbf{u}}_\varepsilon^{m,n}\cdot\phi_i^{0,\varepsilon}\\ &-\frac{1}{2}\int_{\mathcal{C}}(\underline{\mathbf{v}}_\varepsilon^*\cdot(B_\varepsilon\nabla))\phi_i^{0,\varepsilon}\cdot\underline{\mathbf{u}}_\varepsilon^{m,n} - \int_{\mathcal{C}}(\underline{\mathbf{w}}_\varepsilon\cdot(B_\varepsilon\nabla))\underline{\mathbf{u}}_\varepsilon^{m,n}\cdot\phi_i^{0,\varepsilon} = \int_{\mathcal{C}}\underline{\mathbf{f}}\cdot\phi_i^{0,\varepsilon}J_\varepsilon,\end{aligned} \tag{24}$$

and $\forall 1\le j\le n$,

$$\begin{aligned}&\int_{\mathcal{C}} \partial_t\underline{\mathbf{u}}_\varepsilon^{m,n}\cdot\phi_j^{1,\varepsilon}J_\varepsilon + \nu\int_{\mathcal{C}} A_\varepsilon\nabla\underline{\mathbf{u}}_\varepsilon^{m,n} : \nabla\phi_j^{1,\varepsilon} + \frac{1}{2}\int_{\mathcal{C}}(\underline{\mathbf{v}}_\varepsilon^*\cdot(B_\varepsilon\nabla))\underline{\mathbf{u}}_\varepsilon^{m,n}\cdot\phi_j^{1,\varepsilon}\\ &-\frac{1}{2}\int_{\mathcal{C}}(\underline{\mathbf{v}}_\varepsilon^*\cdot(B_\varepsilon\nabla))\phi_j^{1,\varepsilon}\cdot\underline{\mathbf{u}}_\varepsilon^{m,n} + \frac{1}{2}\int_\omega \partial_t\eta_\varepsilon^n\partial_t\delta_\varepsilon^*\xi_j - \int_{\mathcal{C}}(\underline{\mathbf{w}}_\varepsilon\cdot(B_\varepsilon\nabla))\underline{\mathbf{u}}_\varepsilon^{m,n}\cdot\phi_j^{1,\varepsilon}\\ &+\int_\omega\partial_{tt}\eta_\varepsilon^n\,\xi_j + \int_\omega\Delta\partial_t\eta_\varepsilon^n\,\Delta\xi_j + \int_\omega\Delta\eta_\varepsilon^n\,\Delta\xi_j = \int_{\mathcal{C}}\underline{\mathbf{f}}\cdot\phi_j^{1,\varepsilon}J_\varepsilon + \int_\omega g\;\xi_j,\end{aligned} \tag{25}$$

To those equations we add initial conditions $\beta_i(0)=0$, $\underline{\mathbf{u}}_\varepsilon^{m,n}(0) = \underline{\mathbf{u}}_{\varepsilon,0}^{n,m}$ and $\partial_t\eta_\varepsilon^n(0) = \eta_{\varepsilon,1}^n$ where $\underline{\mathbf{u}}_{\varepsilon,0}^{m,n}$ and $\eta_{\varepsilon,1}^n$ denote, respectively, the projections of $\underline{\mathbf{u}}_0^\varepsilon$ and η_1^ε onto the finite dimensional spaces span $(\phi_i^{0,\varepsilon},\phi_j^{1,\varepsilon})_{1\le i\le m,1\le j\le n}$ and span $(\xi_j)_{1\le j\le n}$. This is a second order system of linear ordinary differential equations that can be written as a first order system. The mass matrix is:

$$M_\varepsilon(t) + N = \begin{pmatrix}\int_{\mathcal{C}}\phi_i^{0,\varepsilon}\phi_j^{0,\varepsilon}J_\varepsilon & \int_{\mathcal{C}}\phi_i^{0,\varepsilon}\phi_j^{1,\varepsilon}J_\varepsilon\\ \int_{\mathcal{C}}\phi_j^{0,\varepsilon}\phi_i^{1,\varepsilon}J_\varepsilon & \int_{\mathcal{C}}\phi_i^{1,\varepsilon}\phi_j^{1,\varepsilon}J_\varepsilon\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \int_\omega\xi_i\xi_j\end{pmatrix}.$$

For all $t\in[0,T]$, the matrix $M_\varepsilon(t)$ is smooth and symmetric nonnegative and N is symmetric positive. Then system (24, 25) has a unique solution on $[0,T_{m,n}]$, for some $T_{m,n}>0$. To obtain energy estimates we multiply (24) by α_i and then add those equations for $i=1...m$, and multiply (25) by $\dot\beta_j$ and then add those equations for $j=1...n$. By adding those two contributions we obtain

$$\int_{\mathcal{C}} \partial_t \underline{\mathbf{u}}_\varepsilon^{m,n} \cdot \underline{\mathbf{u}}_\varepsilon^{m,n} J_\varepsilon + \nu \int_{\mathcal{C}} A_\varepsilon \nabla \underline{\mathbf{u}}_\varepsilon^{m,n} : \nabla \underline{\mathbf{u}}_\varepsilon^{m,n} + \frac{1}{2} \int_\omega \partial_t \eta_\varepsilon^n \partial_t \delta_\varepsilon^* \partial_t \eta_\varepsilon^n$$
$$- \int_{\mathcal{C}} (\underline{\mathbf{w}}_\varepsilon \cdot (B_\varepsilon \nabla)) \underline{\mathbf{u}}_\varepsilon^{m,n} \cdot \mathbf{u}_\varepsilon^{m,n} + \int_\omega \partial_{tt} \eta_\varepsilon^n \, \partial_t \eta_\varepsilon^n + \int_\omega \Delta \partial_t \eta_\varepsilon^n \, \Delta \partial_t \eta_\varepsilon^n$$
$$+ \int_\omega \Delta \eta_\varepsilon^n \, \Delta \partial_t \eta_\varepsilon^n = \int_{\mathcal{C}} \underline{\mathbf{f}} \cdot \phi^{1,\varepsilon} J_\varepsilon + \int_\omega g \, \partial_t \eta_\varepsilon^n.$$

But

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\underline{\mathbf{u}}_\varepsilon^{m,n}|^2 J_\varepsilon = \int_{\mathcal{C}} \partial_t \underline{\mathbf{u}}_\varepsilon^{m,n} \cdot \underline{\mathbf{u}}_\varepsilon^{m,n} J_\varepsilon + \frac{1}{2} \int_{\mathcal{C}} |\underline{\mathbf{u}}_\varepsilon^{m,n}|^2 \frac{d}{dt} J_\varepsilon.$$

We have

$$\frac{d}{dt} J_\varepsilon = \mathrm{div}\,(B_\varepsilon^T \underline{\mathbf{w}}_\varepsilon) J_\varepsilon,$$

and $\underline{\mathbf{w}}_\varepsilon(t,x,y,1) = (0,0,\partial_t \delta_\varepsilon^*(t,x,y))^T$, thus

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\underline{\mathbf{u}}_\varepsilon^{m,n}|^2 J_\varepsilon = \int_{\mathcal{C}} \partial_t \underline{\mathbf{u}}_\varepsilon^{m,n} \cdot \underline{\mathbf{u}}_\varepsilon^{m,n} J_\varepsilon + \frac{1}{2} \int_{\mathcal{C}} |\underline{\mathbf{u}}_\varepsilon^{m,n}|^2 \mathrm{div}\,(B_\varepsilon^T \underline{\mathbf{w}}_\varepsilon) J_\varepsilon$$
$$= \int_{\mathcal{C}} \partial_t \underline{\mathbf{u}}_\varepsilon^{m,n} \cdot \underline{\mathbf{u}}_\varepsilon^{m,n} J_\varepsilon + \frac{1}{2} \int_\omega \partial_t \eta_\varepsilon^n \partial_t \delta_\varepsilon^* \partial_t \eta_\varepsilon^n - \int_{\mathcal{C}} (\underline{\mathbf{w}}_\varepsilon \cdot (B_\varepsilon \nabla)) \underline{\mathbf{u}}_\varepsilon^{m,n} \cdot \mathbf{u}_\varepsilon^{m,n}.$$

Next, following the same lines as in Section 1.2, we obtain

$$\|\underline{\mathbf{u}}_\varepsilon^{m,n}\|_{L^\infty(0,T;L^2(\mathcal{C})))} + \|\nabla \underline{\mathbf{u}}_\varepsilon^{m,n}\|_{L^2(0,T;L^2(\mathcal{C}))}$$
$$+ \|\partial_t \eta_\varepsilon^n\|_{L^\infty(0,T;L^2(\omega))} + \|\Delta \eta_\varepsilon^n\|_{H^1(0,T;L^2(\omega))}$$
$$\leq C(T, \|\mathbf{u}_0\|_{L^2(\Omega_{\eta_0})}, \|\mathbf{f}\|_{L^2((0,T)\times\mathbb{R}^3)}, \|g\|_{L^2((0,T)\times\omega)},$$
$$\|\eta_0\|_{H_0^2(\omega)}, \|\eta_1\|_{L^2(\omega)}, \varepsilon, \alpha, M), \quad (26)$$

Note that these estimates are independent of (m,n). Thus $T_{m,n} = T$.

Remark 6 *The energy estimates on $\underline{\mathbf{u}}_\varepsilon^{m,n}$ and η_ε^n depend on ε (and on α and M) because of the change of variables we made. Nevertheless if we rewrite the equations on the deformed configuration, we obtain energy estimates independent of ε (and also independent of α and M):*

$$\|\mathbf{u}_\varepsilon^{m,n}\|_{L^\infty(0,T;L^2(\Omega_{\delta_\varepsilon^*}(t))))} + \|\nabla \mathbf{u}_\varepsilon^{m,n}\|_{L^2(0,T;L^2(\Omega_{\delta_\varepsilon^*}(t))}$$
$$+ \|\partial_t \eta_\varepsilon^n\|_{L^\infty(0,T;L^2(\omega))} + \|\Delta \eta_\varepsilon^n\|_{H^1(0,T;L^2(\omega))}$$
$$\leq C(T, \|\mathbf{u}_0\|_{L^2(\Omega_{\eta_0})}, \|\mathbf{f}\|_{L^2((0,T)\times\mathbb{R}^3)}, \|g\|_{L^2((0,T)\times\omega)},$$
$$\|\eta_0\|_{H_0^2(\omega)}, \|\eta_1\|_{L^2(\omega)}). \quad (27)$$

These estimates are obtained in the same way as in Section 1.2. This remark will be useful to pass to the limit as ε goes to zero.

3.1.3 Additional estimates

Here we want to derive additional estimates on the solution and more precisely we are going to prove that $\partial_t \underline{\mathbf{u}}_\varepsilon^{m,n}$ and $\partial_{tt}\eta_\varepsilon^n$ are bounded with respect to the data and ε, but independently of (m,n), in $L^2(0,T;L^2(\mathcal{C}))$ and $L^2(0,T;L^2(\omega))$ respectively. These estimates will, in particular, yield enough compactness to apply a fixed point procedure and prove the existence of a solution to the approximate problem (18).

Lemma 8 *There exists a constant $C>0$ depending on the data, α, M and on ε but independent of (m,n) such that*

$$\|\partial_t \underline{\mathbf{u}}_\varepsilon^{m,n}\|_{L^2(0,T;L^2(\mathcal{C}))} + \|\partial_{tt}\eta_\varepsilon^n\|_{L^2(0,T;L^2(\omega))} \leq C.$$

Remark 7 *From this estimate on $\partial_t \underline{\mathbf{u}}_\varepsilon^{m,n}$, one can deduce a bound of $\partial_t \mathbf{u}_\varepsilon^{m,n}$ in $L^2(0,T;L^2(\Omega_{\delta_\varepsilon^*}(t)))$ because*

$$\partial_t \underline{\mathbf{u}}_\varepsilon^{m,n}(t,\mathbf{x}) = \partial_t \mathbf{u}_\varepsilon^{m,n}(t,\chi_\varepsilon(t,\mathbf{x})) + (\underline{\mathbf{w}}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon^{m,n}(t,\chi_\varepsilon(t,\mathbf{x})).$$

Proof: We multiply (24) by $\dot{\alpha}_i$ and then sum up those equations for $i = 1\ldots m$, multiply (25) by $\ddot{\beta}_j$ and then add those equations for $j=1\ldots n$. It corresponds to taking $(\sum_{i=1}^m \dot{\alpha}_i\phi_i^{0,\varepsilon} + \sum_{j=1}^n \ddot{\beta}_j\phi_j^{1,\varepsilon}, \sum_{j=1}^n \ddot{\beta}_j\xi_j)$ as a test function. The second function is in fact $\partial_{tt}\eta_\varepsilon^n$, while the first is different from $\partial_t \underline{\mathbf{u}}_\varepsilon^{n,m}$ since the Galerkin functions $\phi_i^{0,\varepsilon}$ and $\phi_j^{1,\varepsilon}$ depend on time. The details of the calculations are given in the Appendix. These calculations are quite standard, but some particular attention has to be paid to the contributions of the time derivative of the basis functions and to the convection terms.

Thanks to those estimates we can pass to the limit in the discrete system as $m,n\to+\infty$. We obtain at least one solution of the approximate–linearized problem (21). This solution is unique and satisfies energy estimates and the additional estimates.

3.2 Fixed point procedure

Thus for any $(\delta,\mathbf{v})$ such that $\delta\in H^1(0,T;C^0(\overline{\omega})\cap H_0^1(\omega))$, $M\geq 1+\delta(t,x,y)\geq\alpha>0$, $\forall(t,x,y)\in[0,T]\times\overline{\omega}$ and $\mathbf{v}\in L^2(0,T;L^2(B_{2M}))$, we construct the solution $(\eta_\varepsilon,\mathbf{u}_\varepsilon)$ to the approximate–linearized problem (21).

This solution satisfies energy estimates independent of ε (see Remark 6):

$$\begin{aligned}
&\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_{\delta_\varepsilon^*}(t))))} + \|\nabla \mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega_{\delta_\varepsilon^*}(t)))} \\
&+ \|\partial_t \eta_\varepsilon\|_{L^\infty(0,T;L^2(\omega))} + \|\Delta \eta_\varepsilon\|_{H^1(0,T;L^2(\omega))} \\
&\leq C(T, \|\mathbf{u}_0\|_{L^2(\Omega_{\eta_0})}, \|\mathbf{f}\|_{L^2((0,T)\times\mathbb{R}^3)}, \|g\|_{L^2((0,T)\times\omega)}, \\
&\qquad \|\eta_0\|_{H_0^2(\omega)}, \|\eta_1\|_{L^2(\omega)}),
\end{aligned} \tag{28}$$

and additional estimates (see Lemma 8)

$$\|\partial_t \underline{\mathbf{u}}_\varepsilon\|_{L^2(0,T;L^2(\mathcal{C}))} + \|\partial_{tt}\eta_\varepsilon\|_{L^2(0,T;L^2(\omega))} \leq C_{\varepsilon,M,\alpha}. \tag{29}$$

In particular, η_ε belongs to $H^1(0,T;C^0(\overline{\omega}) \cap H_0^1(\omega))$ since $H^2(\omega) \hookrightarrow C^0(\overline{\omega})$. Moreover, by construction: $\eta_\varepsilon(0) = \eta_0^\varepsilon$. The function $\overline{\mathbf{u}}_\varepsilon$ defined in Lemma 3 by:

$$\overline{\mathbf{u}}_\varepsilon = \left| \begin{array}{l} \mathbf{u}_\varepsilon \text{ in } \Omega_{\delta_\varepsilon^*(t)} \\ (0,0,\partial_t \eta_\varepsilon)^T \text{ in } B_{2M} \setminus \Omega_{\delta_\varepsilon^*(t)}. \end{array} \right.$$

belongs to $L^2(0,T;L^2(B_{2M}))$. We set

$$Y \;=\; H^1(0,T;C^0(\overline{\omega}) \cap H_0^1(\omega)) \times L^2(0,T;L^2(B_{2M}))$$

and we consider the mapping F_ε :

$$\begin{array}{rlcl} F_\varepsilon : & B_M^{\alpha,\varepsilon} & \longrightarrow & Y \\ & (\delta, \mathbf{v}) & \longmapsto & (\eta_\varepsilon, \overline{\mathbf{u}}_\varepsilon), \end{array}$$

where

$$B_M^{\alpha,\varepsilon} = \Big\{ (\delta,\mathbf{v}) \in Y,\ \|(\delta,\mathbf{v})\|_Y \leq C_M,\ \alpha \leq 1 + \delta(t,x,y) \leq M,$$

$$\forall (t,x,y) \in [0,T] \times \overline{\omega},\ \delta|_{t=0} = \eta_0^\varepsilon \Big\}.$$

If we verify that, for each ε, F_ε has at least one fixed point, this will prove the existence of at least one solution of the approximate problem (18).

First let us check that $F_\varepsilon(B_M^{\alpha,\varepsilon}) \subset B_M^{\alpha,\varepsilon}$. If M is chosen sufficiently large with respect to the data then (28) leads to $\sup_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon) \leq M$. We have

$$\|\overline{\mathbf{u}}_\varepsilon\|_{L^2(0,T;L^2(B_{2M})} \leq \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega_{\delta_\varepsilon^*}(t)))} + 2M\|\partial_t \eta_\varepsilon\|_{L^2(0,T;L^2(\omega))}.$$

Thus C_M can be chosen large enough with respect to M and the data so $\|(\eta_\varepsilon, \overline{\mathbf{u}}_\varepsilon)\|_Y \leq C_M$. Moreover, we have to verify that there exists a time T such that $\min_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon) \geq \alpha > 0$. At this stage T could depend on ε. But since we want to pass to the limit as ε goes to zero, we shall check that T can be chosen to be independent of ε. We know that η_ε is bounded in $H^1(0,T;H_0^2(\omega))$ uniformly in ε. But $H^1(0,T;H_0^2(\omega)) \hookrightarrow C^{0,1/2}([0,T];C^{0,r}(\overline{\omega}))$, $\forall r < 1$ and $\min_{[0,T]\times\overline{\omega}}(1+\eta_0^\varepsilon) \geq 3\alpha/2 > 0$, thus there exists a time T, independent of ε, such that $\min_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon) \geq \alpha > 0$. Consequently $F_\varepsilon(B_M^{\alpha,\varepsilon}) \subset B_M^{\alpha,\varepsilon}$.

The second point to verify is that $F_\varepsilon({B_M}^{\alpha,\varepsilon})$ is relatively compact in Y. Let us consider a sequence $(\delta_n, \mathbf{v}_n)_{n\in\mathbb{N}}$ in ${B_M}^{\alpha,\varepsilon}$, and denote $(\eta_\varepsilon^n, \mathbf{u}_\varepsilon^n) = F_\varepsilon(\delta_n, \mathbf{v}_n)$. Thanks to estimates (28, 29), $\partial_t\eta_\varepsilon^n$ is bounded in $L^2(0,T;H_0^2(\omega))$ and $\partial_{tt}\eta_\varepsilon^n$ is bounded in $L^2(0,T;L^2(\omega))$ independently of n. Thus, applying Aubin's lemma and the fact that $H_0^2(\omega)$ is compactly embedded in $H_0^1(\omega)\cap C^0(\overline{\omega})$ (since $\omega \subset \mathbb{R}^2$), we obtain that $\partial_t\eta_\varepsilon^n$ is relatively compact in $L^2(0,T;H_0^1(\Omega)\cap C^0(\overline{\omega}))$. Concerning the fluid velocity, $\underline{\mathbf{u}}_\varepsilon^n$ is bounded in $L^2(0,T;H^1(\mathcal{C}))$ and $\partial_t\underline{\mathbf{u}}_\varepsilon^n$ is bounded in $L^2(0,T;L^2(\mathcal{C}))$ independently of n. Consequently $\underline{\mathbf{u}}_\varepsilon^n$ is relatively compact in $L^2(0,T;L^2(\mathcal{C}))$.

Thus, $\rho_\varepsilon^n\mathbf{u}_\varepsilon^n$ is relatively compact in $L^2(0,T;L^2(B_{2M}))$, where ρ_ε^n denotes the characteristic function of $\Omega_{\delta_{n,\varepsilon}^*}(t)$. Indeed, denoting by $\underline{\mathbf{u}}_\varepsilon$ the limit in $L^2(0,T;L^2(\mathcal{C}))$ of a subsequence of $(\underline{\mathbf{u}}_\varepsilon^n)_{n\in\mathbb{N}}$ then it is clear that

$$\|\rho_\varepsilon^n\mathbf{u}_\varepsilon^n - \rho_\varepsilon^n\underline{\mathbf{u}}_\varepsilon\circ(\chi_\varepsilon^n)^{-1}\|_{L^2(0,T;L^2(B_{2M}))} \leq C_\varepsilon\|\underline{\mathbf{u}}_\varepsilon^n - \underline{\mathbf{u}}_\varepsilon\|_{L^2(0,T;L^2(\mathcal{C}))},$$

where χ_ε^n is a flow associated with $\mathcal{N}_\varepsilon(\delta_n) = \delta_{n,\varepsilon}^*$ (see the construction introduced at the beginning of Section 3.1.1). Since $(\delta_n, \mathbf{v}_n) \in B_M^{\alpha,\varepsilon}$, a subsequence of $(\delta_n)_{n\in\mathbb{N}}$, which will be still denoted by $(\delta_n)_{n\in\mathbb{N}}$, converges weakly in $H^1(0,T;H^1(\omega))$. We denote by δ this limit and by χ_ε the flow associated with δ_ε^*. But $\rho_\varepsilon^n\underline{\mathbf{u}}_\varepsilon\circ(\chi_\varepsilon^n)^{-1} = (\rho_0^\varepsilon\underline{\mathbf{u}}_\varepsilon)\circ(\chi_\varepsilon^n)^{-1}$ converges to $(\rho_0^\varepsilon\underline{\mathbf{u}}_\varepsilon)\circ(\chi_\varepsilon)^{-1}$ as n goes to $+\infty$ since χ_ε^n converges to χ_ε in $C^1([0,T]\times\overline{\mathcal{C}})$ and $\chi_\varepsilon^n\circ(\chi_\varepsilon)^{-1}$ converges to the identity in C^1 as n goes to $+\infty$. Finally, since $\partial_t\eta_\varepsilon^n$ is relatively compact in $L^2(0,T;H_0^1(\Omega)\cap C^0(\overline{\omega}))$ and $\rho_\varepsilon^n\mathbf{u}_\varepsilon^n$ is relatively compact in $L^2(0,T;L^2(B_{2M}))$, $\overline{\mathbf{u}}_\varepsilon^n$ is relatively compact in $L^2(0,T;L^2(B_{2M}))$.

The last point to check is the continuity of the mapping F_ε for the strong topology of Y. Let us consider a sequence $(\delta_n, \mathbf{v}_n)_{n\in\mathbb{N}}$ such that $(\delta_n, \mathbf{v}_n) \in B_M^{\alpha,\varepsilon}$ and such that $(\delta_n, \mathbf{v}_n) \to (\delta, \mathbf{v})$ in Y as $n \to +\infty$. We have to prove that $F_\varepsilon(\delta_n, \mathbf{v}_n) = (\eta_\varepsilon^n, \overline{\mathbf{u}}_\varepsilon^n)$ converges to $F_\varepsilon(\delta, \mathbf{v}) = (\eta_\varepsilon, \overline{\mathbf{u}}_\varepsilon)$ in Y. We already know that a subsequence of $(\eta_\varepsilon^n, \overline{\mathbf{u}}_\varepsilon^n)_{n\in\mathbb{N}}$, still denoted by $(\eta_\varepsilon^n, \overline{\mathbf{u}}_\varepsilon^n)_{n\in\mathbb{N}}$, converges in Y, moreover $(\eta_\varepsilon^n, \overline{\mathbf{u}}_\varepsilon^n)$ converges weakly in $H^1(0,T;H_0^2(\omega)) \times L^2(0,T;H^1(B_{2M}))$ and $(\partial_{tt}\eta_\varepsilon^n, \rho_\varepsilon^n \partial_t \mathbf{u}_\varepsilon^n)$ converges weakly in $L^2(0,T;L^2(\omega))\times L^2(0,T;L^2(B_{2M}))$. We denote $(\tilde{\eta}_\varepsilon, \tilde{\overline{\mathbf{u}}}_\varepsilon)$ the limit of $(\eta_\varepsilon^n, \overline{\mathbf{u}}_\varepsilon^n)$. We have to verify that $(\tilde{\eta}_\varepsilon, \tilde{\overline{\mathbf{u}}}_\varepsilon) = F_\varepsilon(\delta, \mathbf{v})$. It is clear that $\tilde{\overline{\mathbf{u}}}_\varepsilon = (0,0,\partial_t\tilde{\eta}_\varepsilon)^T$ in $B_{2M}\setminus\Omega_{\delta_\varepsilon^*}(t)$, $\tilde{\overline{\mathbf{u}}}_\varepsilon(t,x,y,1+\delta_\varepsilon^*(t,x,y)) = (0,0,\partial_t\tilde{\eta}_\varepsilon(t,x,y))^T, \forall(x,y)\in\omega$ and $\operatorname{div}\tilde{\overline{\mathbf{u}}}_\varepsilon = 0$. Next we have to pass the limit as n goes to $+\infty$ in the weak formulation satisfied by $(\eta_\varepsilon^n, \mathbf{u}_\varepsilon^n)$ and we will conclude by an uniqueness argument. We have to pass to the limit as n goes to $+\infty$ in

$$\int_0^t\int_{\Omega_{\delta_{n,\varepsilon}^*}(s)} \partial_t\mathbf{u}_\varepsilon^n\cdot\phi_\varepsilon^n + \nu\int_0^t\int_{\Omega_{\delta_{n,\varepsilon}^*}(s)} \nabla\mathbf{u}_\varepsilon^n : \nabla\phi_\varepsilon^n + \frac{1}{2}\int_0^t\int_{\Omega_{\delta_{n,\varepsilon}^*}(s)} (\mathbf{v}_{n,\varepsilon}^*\cdot\nabla)\mathbf{u}_\varepsilon^n\cdot\phi_\varepsilon^n$$
$$-\frac{1}{2}\int_0^t\int_{\Omega_{\delta_{n,\varepsilon}^*}(s)} (\mathbf{v}_{n,\varepsilon}^*\cdot\nabla)\phi_\varepsilon^n\cdot\mathbf{u}_\varepsilon^n + \frac{1}{2}\int_0^t\int_\omega \partial_t\eta_\varepsilon^n\partial_t\delta_{n,\varepsilon}^* b + \int_0^t\int_\omega \partial_{tt}\eta_\varepsilon^n\, b$$
$$+\int_0^t\int_\omega \Delta\partial_t\eta_\varepsilon^n\,\Delta b + \int_0^t\int_\omega \Delta\eta_\varepsilon^n\,\Delta b = \int_0^t\int_{\Omega_{\delta_{n,\varepsilon}^*}(s)} \mathbf{f}\cdot\phi_\varepsilon^n + \int_0^t\int_\omega g\ b,$$
$$\forall\phi_\varepsilon^n\in V_{\delta_{n,\varepsilon}^*},\ b\in L^2(0,T;H_0^2(\omega)) \text{ s. t.}$$
$$\phi_\varepsilon(t,x,y,1+\delta_{n,\varepsilon}^*(t,x,y)) = (0,0,b(t,x,y))^T \text{ on } \omega,$$

Here the fluid test functions depend on n. Nevertheless we can restrict ourself to test functions which do not depend on n and that are admissible test functions for all n sufficiently large. Indeed, chose first

$$\phi_\varepsilon^0 \in \mathcal{D}(\widehat{\Omega}_{\delta_\varepsilon^*}), \qquad \operatorname{div}\phi_\varepsilon^0 = 0.$$

The pair $(\phi_\varepsilon^0, 0)$ is a pair of admissible test functions for n large enough since δ^n converges uniformly to δ. Next, for $b\in L^2(0,T;H_0^2(\omega))$ such that $\int_\omega b = 0$, we define:

$$\phi_\varepsilon^1(b) = \left|\begin{array}{l}(0,0,b)^T \text{ in } B_{2M}\setminus\mathcal{C}_{\alpha/2}\\ \mathcal{R}(0,0,b)^T \text{ in } \mathcal{C}_{\alpha/2},\end{array}\right.$$

where $\mathcal{R}$ denotes a linear continuous lifting such that $\operatorname{div}\mathcal{R}(0,0,b)^T = 0$ (one can for instance solve a Stokes problem in $\mathcal{C}_{\alpha/2}$). The pair $(\phi_\varepsilon^1(b), b)$ is a pair of admissible test functions for all n, since $2M \geq 1+\delta_{n,\varepsilon}^*(t,x,y) \geq \alpha/2$, $\forall(t,x,y)\in[0,T]\times\overline{\omega}$. Next, with this choice of test functions we can pass to the limit thanks to the weak convergence of $(\eta_\varepsilon^n, \mathbf{u}_\varepsilon^n)$. Then any test function

$\phi_\varepsilon \in V_{\delta_\varepsilon^*}$ satisfying $\phi_\varepsilon(t,x,y,1+\delta_\varepsilon^*(t,x,y)) = (0,0,b(t,x,y))^T$ on ω, is equal to $\phi_\varepsilon - \phi_\varepsilon^1(b) + \phi_\varepsilon^1(b)$, and $\phi_\varepsilon - \phi_\varepsilon^1(b)$ can be approximated by divergence free functions of $\mathcal{D}(\widehat{\Omega}_{\delta_\varepsilon^*})$. Thus we obtain that $(\tilde{\eta}_\varepsilon, \rho_\varepsilon \tilde{\overline{\mathbf{u}}}_\varepsilon)$ is the unique solution of the linearized approximate problem associated with $(\delta, \mathbf{v})$ and thus $(\tilde{\eta}_\varepsilon, \tilde{\overline{\mathbf{u}}}_\varepsilon)$ is equal to $F_\varepsilon(\delta,\mathbf{v})$ and all the sequence $(\eta_\varepsilon^n, \overline{\mathbf{u}}_\varepsilon^n)_{n\in\mathbb{N}}$ converges to $F_\varepsilon(\delta,\mathbf{v})$ in Y. Consequently F_ε is continuous for the strong topology of Y.

Applying Schauder's fixed point theorem, we obtain that F_ε has at least one fixed point. This proves the existence of at least one solution of the approximate problem. One can show that this solution exists as long as $\min_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon) > 0$.

Next we want to pass to the limit as ε goes to zero. To do this we need compactness on $\overline{\mathbf{u}}_\varepsilon$ in $L^2(0,T;L^2(B_{2M}))$ and $\partial_t\eta_\varepsilon$ in $L^2(0,T;L^2(\omega))$. Since the additional estimates we obtained depend on ε they can not be used to deduce the desired compactness. The next section is devoted to the derivation of suitable bounds on the solution, not depending on ε.

4 Compactness

In this section we prove strong convergence properties for the fluid and the structure velocities. We set $B = B_M$ and $B' = B_{2M}$. The solution we build verifies estimates (19) and (20). Moreover

$$\|\overline{\mathbf{u}}_\varepsilon\|_{L^2(0,T;H^1(B'))\cap L^\infty(0,T;L^2(B'))} \leq C. \tag{30}$$

But these estimates are not sufficient to obtain the desired convergence. Given any $h > 0$, we denote $g^-(t,\cdot) = g(t-h,\cdot)$ and $g^+(t,\cdot) = g(t+h,\cdot)$. We state the following lemma:

Lemma 9 *Let $T > 0$ such that $\min_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon) \geq \alpha > 0$. Then for all $h > 0$ small enough, we have*

$$\int_0^T \int_{B'} \rho_\varepsilon |\overline{\mathbf{u}}_\varepsilon - \overline{\mathbf{u}}_\varepsilon^-|^2 + \int_0^T \int_\omega (\partial_t\eta_\varepsilon - \partial_t\eta_\varepsilon^-)^2 \leq C\sqrt{h}, \tag{31}$$

and

$$\int_0^T \int_{B'} |\rho_\varepsilon \overline{\mathbf{u}}_\varepsilon - \rho_\varepsilon^- \overline{\mathbf{u}}_\varepsilon^-|^2 \leq C\sqrt{h}, \tag{32}$$

with η_ε extended by η_0^ε for $t < 0$ (hence $\partial_t\eta_\varepsilon$ extended by 0) and $\overline{\mathbf{u}}_\varepsilon$ extended by 0 for $t < 0$, and where ρ_ε denotes the characteristic function of $\Omega_{\eta_\varepsilon^*}(t)$. The constant C does not depend on ε.

Proof:
We first show that (31) implies (32). Indeed:

$$|\rho_\varepsilon \overline{\mathbf{u}}_\varepsilon - \rho_\varepsilon^- \overline{\mathbf{u}}_\varepsilon^-|^2 \leq C\left(\rho_\varepsilon |\overline{\mathbf{u}}_\varepsilon - \overline{\mathbf{u}}_\varepsilon^-|^2 + |\rho_\varepsilon - \rho_\varepsilon^-||\overline{\mathbf{u}}_\varepsilon^-|^2\right).$$

The estimate of the first contribution comes from (31). For the second contribution we have

$$\left|\int_0^T \int_{B'} |\rho_\varepsilon - \rho_\varepsilon^-||\overline{\mathbf{u}}_\varepsilon|^2\right| \leq \int_0^T \|\rho_\varepsilon - \rho_\varepsilon^-\|_{L^2(B')} \|\overline{\mathbf{u}}_\varepsilon\|_{L^4(B')}^2$$
$$\leq \left(\int_0^T \|\rho_\varepsilon - \rho_\varepsilon^-\|_{L^2(B')}^4\right)^{\frac{1}{4}} \left(\int_0^T \|\overline{\mathbf{u}}_\varepsilon\|_{L^4(B')}^{\frac{8}{3}}\right)^{\frac{3}{4}}.$$

Since $\overline{\mathbf{u}}_\varepsilon$ is bounded in $L^\infty(0,T;L^2(B')) \cap L^2(0,T;H^1(B'))$ independently of ε, by Sobolev's embedding and interpolation we find that $\overline{\mathbf{u}}_\varepsilon$ is bounded in $L^{\frac{8}{3}}(0,T;L^4(B'))$ independently of ε. Moreover, we recall that $\partial_t \eta_\varepsilon^*$ is bounded in $L^\infty(0,T;L^2(\omega))$ independently of ε, so that

$$\begin{aligned}
\int_{B'} |\rho_\varepsilon - \rho_\varepsilon^-|^2 &= \int_\omega |\eta_\varepsilon^* - (\eta_\varepsilon^*)^-| \\
&= \int_\omega \left|\int_{t-h}^t \partial_t \eta_\varepsilon^*(s)ds\right| \\
&\leq \int_\omega \int_{t-h}^t |\partial_t \eta_\varepsilon^*(s)|ds \\
&\leq Ch,
\end{aligned}$$

Therefore,

$$\left|\int_0^T \int_{B'} (\rho_\varepsilon - \rho_\varepsilon^-)|\overline{\mathbf{u}}_\varepsilon|^2\right| \leq C\sqrt{h}. \tag{33}$$

This shows (32).

To prove (31) we are going to make a suitable choice for the test functions in the weak formulation satisfied by $\mathbf{u}_\varepsilon$ and $\partial_t \eta_\varepsilon$. The same type of test functions were used in [16], where existence of weak solutions for the Navier–Stokes equations in a given time dependent domain is proven. The functions

$\mathbf{u}_\varepsilon$ and $\partial_t\eta_\varepsilon$ satisfy:

$$\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\partial_t\mathbf{u}_\varepsilon\cdot\phi_\varepsilon+\nu\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\nabla\mathbf{u}_\varepsilon:\nabla\phi_\varepsilon+\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}(\mathcal{R}_\varepsilon(\overline{\mathbf{u}}_\varepsilon)\cdot\nabla)\mathbf{u}_\varepsilon\cdot\phi_\varepsilon$$
$$-\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}(\mathcal{R}_\varepsilon(\overline{\mathbf{u}}_\varepsilon)\cdot\nabla)\phi_\varepsilon\cdot\mathbf{u}_\varepsilon+\frac{1}{2}\int_0^T\int_\omega\partial_t\eta_\varepsilon\partial_t\eta_\varepsilon^* b+\int_0^T\int_\omega\partial_{tt}\eta_\varepsilon\, b$$
$$+\int_0^T\int_\omega\Delta\partial_t\eta_\varepsilon\,\Delta b+\int_0^T\int_\omega\Delta\eta_\varepsilon\,\Delta b=\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{f}\cdot\phi_\varepsilon+\int_0^T\int_\omega g\ b,$$
$$\forall\phi_\varepsilon\in V_{\eta_\varepsilon^*},\ b\in L^2(0,T;H_0^2(\omega))\text{ s. t.}$$
$$\phi_\varepsilon(t,x,y,1+\eta_\varepsilon^*(t,x,y))=(0,0,b(t,x,y))^T\text{ on }\omega, \tag{34}$$

For $\sigma>1$ we define $\mathbf{v}_\sigma$ by

$$\mathbf{v}_\sigma(x,y,z)=(\sigma\mathbf{v}_1(x,y,\sigma z),\sigma\mathbf{v}_2(x,y,\sigma z),\mathbf{v}_3(x,y,\sigma z)).$$

If $\mathbf{v}$ is divergence free, $\mathbf{v}_\sigma$ is also divergence free.

We set

$$\phi_\varepsilon=\int_{t-h}^t(\overline{\mathbf{u}}_\varepsilon)_\sigma(s)ds,\ \ b=\int_{t-h}^t\partial_t\eta_\varepsilon(s)ds.$$

ϕ_ε belongs to $H^1(0,T;H^1(B'))$ and b belongs to $H^1(0,T;H_0^2(\omega))$. Remember that η_ε has been extended by η_0^ε for $t<0$ and $\overline{\mathbf{u}}_\varepsilon$ and $\partial_t\eta_\varepsilon$ extended by 0 for $t<0$. The function ϕ_ε is divergence free. Moreover since $\|\eta_\varepsilon\|_{H^1(0,T;H_0^2(\omega))}\le C$ and $H^1(0,T;H_0^2(\omega))\hookrightarrow C^{0,1/2}([0,T];C^0(\overline{\omega}))$, we have

$$\|\eta_\varepsilon-\eta_\varepsilon^-\|_{L^\infty((0,T)\times\overline{\omega})}\le C\sqrt{h},$$

and

$$\|\eta_\varepsilon^*-(\eta_\varepsilon^*)^-\|_{L^\infty((0,T)\times\overline{\omega})}\le C\sqrt{h}.$$

Thus, if σ is such as $\sigma\ge 1+\dfrac{2C}{\alpha}\sqrt{h}$, we have

$$\phi_\varepsilon(t,x,y,1+\eta_\varepsilon^*(t,x,y))=\left(0,0,\int_{t-h}^t\partial_t\eta_\varepsilon(s,x,y)ds\right)^T\text{ on }\omega.$$

Hence $\left(\int_{t-h}^t(\overline{\mathbf{u}}_\varepsilon)_\sigma(s)ds,\int_{t-h}^t\partial_t\eta_\varepsilon(s)ds\right)$ is a pair of admissible test functions. In the sequel we choose $\sigma=1+\dfrac{2C}{\alpha}\sqrt{h}$. We remark that

$$\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\partial_t\mathbf{u}_\varepsilon\cdot\left(\int_{t-h}^t(\overline{\mathbf{u}}_\varepsilon)_\sigma\right)=-\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{u}_\varepsilon\cdot((\overline{\mathbf{u}}_\varepsilon)_\sigma-(\overline{\mathbf{u}}_\varepsilon)_\sigma^-)$$
$$+\int_{\Omega_{\eta_\varepsilon^*}(T)}\mathbf{u}_\varepsilon(T)\cdot\left(\int_{T-h}^T(\overline{\mathbf{u}}_\varepsilon)_\sigma\right)-\int_0^T\int_\omega\partial_t\eta_\varepsilon\partial_t\eta_\varepsilon^*\left(\int_{t-h}^t\partial_t\eta_\varepsilon\right),$$

and

$$\int_0^T\int_\omega \partial_{tt}\eta_\varepsilon \int_{t-h}^t \partial_t\eta_\varepsilon \;=\; -\int_0^T \partial_t\eta_\varepsilon\partial_t(\eta_\varepsilon-\eta_\varepsilon^-)+\int_\omega \partial_t\eta_\varepsilon(T)\int_{T-h}^T \partial_t\eta_\varepsilon.$$

Hence, with the choice of our test functions (34) becomes:

$$\begin{aligned}
&-\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)} \mathbf{u}_\varepsilon\cdot((\overline{\mathbf{u}}_\varepsilon)_\sigma-(\overline{\mathbf{u}}_\varepsilon)_\sigma^-)+\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)} (\mathbf{u}_\varepsilon^*\cdot\nabla)\mathbf{u}_\varepsilon\cdot\left(\int_{t-h}^t(\overline{\mathbf{u}}_\varepsilon)_\sigma\right)\\
&-\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)} (\mathbf{u}_\varepsilon^*\cdot\nabla)\left(\int_{t-h}^t(\overline{\mathbf{u}}_\varepsilon)_\sigma\right)\cdot\mathbf{u}_\varepsilon+\nu\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\nabla\mathbf{u}_\varepsilon:\nabla\left(\int_{t-h}^t(\overline{\mathbf{u}}_\varepsilon)_\sigma\right)\\
&\qquad+\int_{\Omega_{\eta_\varepsilon^*}(T)}\mathbf{u}_\varepsilon(T)\cdot\left(\int_{T-h}^T(\overline{\mathbf{u}}_\varepsilon)_\sigma\right)-\int_0^T\int_\omega\partial_t\eta_\varepsilon\partial_t(\eta_\varepsilon-\eta_\varepsilon^-)\\
&\qquad-\frac{1}{2}\int_0^T\int_\omega\partial_t\eta_\varepsilon\partial_t\eta_\varepsilon^*(\eta_\varepsilon-\eta_\varepsilon^-)+\int_0^T\int_\omega\Delta\eta_\varepsilon\Delta(\eta_\varepsilon-\eta_\varepsilon^-)\\
&\qquad+\int_0^T\int_\omega\Delta\partial_t\eta_\varepsilon\Delta(\eta_\varepsilon-\eta_\varepsilon^-)+\int_\omega\partial_t\eta_\varepsilon(T)(\eta_\varepsilon(T)-\eta_\varepsilon^-(T))\\
&\qquad\qquad=\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{f}\cdot\left(\int_{t-h}^t(\overline{\mathbf{u}}_\varepsilon)_\sigma\right)+\int_0^T\int_\omega g(\eta_\varepsilon-\eta_\varepsilon^-)
\end{aligned}\tag{35}$$

The first term can be written as:

$$\begin{aligned}
&-\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{u}_\varepsilon\cdot((\overline{\mathbf{u}}_\varepsilon)_\sigma-(\overline{\mathbf{u}}_\varepsilon)_\sigma^-)\\
&=-\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{u}_\varepsilon\cdot(\overline{\mathbf{u}}_\varepsilon-\overline{\mathbf{u}}_\varepsilon^-)-\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{u}_\varepsilon\cdot(((\overline{\mathbf{u}}_\varepsilon)_\sigma-\overline{\mathbf{u}}_\varepsilon)-((\overline{\mathbf{u}}_\varepsilon)_\sigma^--\overline{\mathbf{u}}_\varepsilon^-)).
\end{aligned}$$

We have the estimate :

$$\begin{aligned}
&\left|\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{u}_\varepsilon\cdot(((\overline{\mathbf{u}}_\varepsilon)_\sigma-\overline{\mathbf{u}}_\varepsilon)-((\overline{\mathbf{u}}_\varepsilon)_\sigma^--\overline{\mathbf{u}}_\varepsilon^-))\right|\\
&\le 2\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega_{\eta_\varepsilon^*}(t))}\|(\overline{\mathbf{u}}_\varepsilon)_\sigma-\overline{\mathbf{u}}_\varepsilon)\|_{L^2(0,T;L^2(\Omega_{\eta_\varepsilon^*}(t))}\\
&\le C(\sigma-1)\le C\sqrt{h}
\end{aligned}$$

We set $I_1=\displaystyle\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{u}_\varepsilon\cdot(\overline{\mathbf{u}}_\varepsilon-\overline{\mathbf{u}}_\varepsilon^-)$.

$$\begin{aligned}
I_1 &= -\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}|\mathbf{u}_\varepsilon|^2+\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}|\overline{\mathbf{u}}_\varepsilon^-|^2-\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}|\overline{\mathbf{u}}_\varepsilon-\overline{\mathbf{u}}_\varepsilon^-|^2\\
&= -\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}|\mathbf{u}_\varepsilon|^2+\frac{1}{2}\int_0^{T-h}\int_{\Omega_{\eta_\varepsilon^*}(t+h)}|\overline{\mathbf{u}}_\varepsilon|^2-\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}|\overline{\mathbf{u}}_\varepsilon-\overline{\mathbf{u}}_\varepsilon^-|^2\\
&=\frac{1}{2}\int_0^{T-h}\int_{B'}(\rho_\varepsilon^+-\rho_\varepsilon)|\overline{\mathbf{u}}_\varepsilon|^2-\frac{1}{2}\int_{T-h}^T\int_{\Omega_{\eta_\varepsilon^*}(t)}|\mathbf{u}_\varepsilon|^2-\frac{1}{2}\int_0^T\int_{\Omega_{\eta_\varepsilon^*}(t)}|\overline{\mathbf{u}}_\varepsilon-\overline{\mathbf{u}}_\varepsilon^-|^2.
\end{aligned}$$

The same argument we used to prove (33) leads to

$$\int_0^T \int_{B'} |\rho_\varepsilon^+ - \rho_\varepsilon||\overline{\mathbf{u}}_\varepsilon|^2 \le C\sqrt{h}$$

This yields

$$I_1 \le C\sqrt{h} - \frac{1}{2}\int_0^T \int_{\Omega_{\eta_\varepsilon^*}(t)} |\overline{\mathbf{u}}_\varepsilon - \overline{\mathbf{u}}_\varepsilon^-|^2.$$

Now we take care of the convection terms:

$$I_2 = \frac{1}{2}\int_0^T \int_{\Omega_{\eta_\varepsilon^*}(t)} (\mathbf{u}_\varepsilon^* \cdot \nabla)\mathbf{u}_\varepsilon \cdot \left(\int_{t-h}^t (\overline{\mathbf{u}}_\varepsilon)_\sigma\right),$$

and

$$I_3 = -\frac{1}{2}\int_0^T \int_{\Omega_{\eta_\varepsilon^*}(t)} (\mathbf{u}_\varepsilon^* \cdot \nabla)\left(\int_{t-h}^t (\overline{\mathbf{u}}_\varepsilon)_\sigma\right) \cdot \mathbf{u}_\varepsilon.$$

We have

$$\begin{aligned} |I_2| &\le \frac{1}{2}\int_0^T \|\mathbf{u}_\varepsilon^*\|_{L^4(\Omega_{\eta_\varepsilon^*}(t))}\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_{\eta_\varepsilon^*}(t))}\int_{t-h}^t \|(\overline{\mathbf{u}}_\varepsilon)_\sigma\|_{L^4(\Omega_{\eta_\varepsilon^*}(t))} \\ &\le \frac{\sqrt{h}}{2}\int_0^T \|\mathbf{u}_\varepsilon^*\|_{L^4(\Omega_{\eta_\varepsilon^*}(t))}\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_{\eta_\varepsilon^*}(t))}\left(\int_{t-h}^t \|(\overline{\mathbf{u}}_\varepsilon)_\sigma\|^2_{L^4(\Omega_{\eta_\varepsilon^*}(t))}\right)^{\frac{1}{2}} \\ &\le C\sqrt{h}. \end{aligned}$$

$$\begin{aligned} |I_3| &\le \frac{1}{2}\int_0^T \|\mathbf{u}_\varepsilon^*\|_{L^4(\Omega_{\eta_\varepsilon^*}(t))}\|\mathbf{u}_\varepsilon\|_{L^4(\Omega_{\eta_\varepsilon^*}(t))}\int_{t-h}^t \|\nabla(\overline{\mathbf{u}}_\varepsilon)_\sigma\|_{L^2(\Omega_{\eta_\varepsilon^*}(t))} \\ &\le \frac{\sqrt{h}}{2}\int_0^T \|\mathbf{u}_\varepsilon^*\|_{L^4(\Omega_{\eta_\varepsilon^*}(t))}\|\mathbf{u}_\varepsilon\|_{L^4(\Omega_{\eta_\varepsilon^*}(t))}\left(\int_{t-h}^t \|(\overline{\mathbf{u}}_\varepsilon)_\sigma\|^2_{H^1(\Omega_{\eta_\varepsilon^*}(t))}\right)^{\frac{1}{2}} \\ &\le C\sqrt{h}. \end{aligned}$$

The next term to consider is $I_4 = \nu \int_0^T \int_{\Omega_{\eta_\varepsilon^*}(t)} \nabla \mathbf{u}_\varepsilon : \nabla\left(\int_{t-h}^t (\overline{\mathbf{u}}_\varepsilon)_\sigma\right)$.

$$\begin{aligned} |I_4| &\le \nu\int_0^T \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_{\eta_\varepsilon^*}(t))}\int_{t-h}^t \|\nabla(\overline{\mathbf{u}}_\varepsilon)_\sigma\|_{L^2(\Omega_{\eta_\varepsilon^*}(t))} \\ &\le \nu\sqrt{h}\int_0^T \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_{\eta_\varepsilon^*}(t))}\left(\int_{t-h}^t \|\nabla(\overline{\mathbf{u}}_\varepsilon)_\sigma\|^2_{L^2(\Omega_{\eta_\varepsilon^*}(t))}\right)^{\frac{1}{2}} \\ &\le C\sqrt{h}. \end{aligned}$$

The term $I_5 = \displaystyle\int_{\Omega_{\eta_\varepsilon^*}(T)} \mathbf{u}_\varepsilon(T)\cdot\left(\int_{T-h}^{T}(\overline{\mathbf{u}}_\varepsilon)_\sigma\right)$ can be estimated as follows:

$$\begin{aligned}
|I_5| &\le \|\mathbf{u}_\varepsilon(T)\|_{L^2(\Omega_{\eta_\varepsilon^*}(T))}\int_{T-h}^{T}\|(\overline{\mathbf{u}}_\varepsilon)_\sigma\|_{L^2(\Omega_{\eta_\varepsilon^*}(T))}\\
&\le \sqrt{h}\|\mathbf{u}_\varepsilon(T)\|_{L^2(\Omega_{\eta_\varepsilon^*}(T))}\left(\int_{T-h}^{T}\|(\overline{\mathbf{u}}_\varepsilon)_\sigma\|^2_{L^2(\Omega_{\eta_\varepsilon^*}(T))}\right)^{\frac{1}{2}}\\
&\le C\sqrt{h}.
\end{aligned}$$

We set $I_6 = -\displaystyle\int_0^T\int_\omega \partial_t\eta_\varepsilon\partial_t(\eta_\varepsilon-\eta_\varepsilon^-)$ We have

$$\begin{aligned}
I_6 &= -\frac{1}{2}\int_0^T\int_\omega(\partial_t\eta_\varepsilon)^2+\frac{1}{2}\int_0^T\int_\omega(\partial_t\eta_\varepsilon^-)^2-\frac{1}{2}\int_0^T\int_\omega(\partial_t\eta_\varepsilon-\partial_t\eta_\varepsilon^-)^2\\
&= -\frac{1}{2}\int_{T-h}^{T}\int_\omega(\partial_t\eta_\varepsilon)^2-\frac{1}{2}\int_0^T\int_\omega(\partial_t\eta_\varepsilon-\partial_t\eta_\varepsilon^-)^2\\
&\le -\frac{1}{2}\int_0^T\int_\omega(\partial_t\eta_\varepsilon-\partial_t\eta_\varepsilon^-)^2.
\end{aligned}$$

For the next term we have:

$$\begin{aligned}
|I_7| &= \left|\frac{1}{2}\int_0^T\int_\omega \partial_t\eta_\varepsilon\partial_t\eta_\varepsilon^*(\eta_\varepsilon-\eta_\varepsilon^-)\right|\\
&\le \frac{1}{2}\int_0^T\|\partial_t\eta_\varepsilon\|_{L^\infty(\omega)}\|\partial_t\eta_\varepsilon^*\|_{L^2(\omega)}\|\eta_\varepsilon-\eta_\varepsilon^-\|_{L^2(\omega)},
\end{aligned}$$

but

$$\|\eta_\varepsilon-\eta_\varepsilon^-\|_{L^2(\omega)}\le\int_{t-h}^{t}\|\partial_t\eta_\varepsilon\|_{L^2(\omega)}\le Ch,$$

consequently taking into account the energy estimates, we have:

$$|I_7|\le Ch.$$

The next term to consider is $I_8 = \displaystyle\int_0^T\int_\omega \Delta\eta_\varepsilon\Delta(\eta_\varepsilon-\eta_\varepsilon^-)$. It can be estimated as follows

$$\begin{aligned}
|I_8| &\le \int_0^T\|\Delta\eta_\varepsilon\|_{L^2(\omega)}\int_{t-h}^{t}\|\Delta\partial_t\eta_\varepsilon\|_{L^2(\omega)}\\
&\le \sqrt{h}\int_0^T\|\Delta\eta_\varepsilon\|_{L^2(\omega)}\left(\int_{t-h}^{t}\|\Delta\partial_t\eta_\varepsilon\|^2_{L^2(\omega)}\right)^{\frac{1}{2}}\\
&\le C\sqrt{h}.
\end{aligned}$$

Similarly the additional viscous term gives:

$$\begin{aligned}
|I_9| &= \left| \int_0^T \int_\omega \Delta \partial_t \eta_\varepsilon \Delta (\eta_\varepsilon - \eta_\varepsilon^-) \right| \\
&\leq \int_0^T \|\Delta \partial_t \eta_\varepsilon\|_{L^2(\omega)} \int_{t-h}^t \|\Delta \partial_t \eta_\varepsilon\|_{L^2(\omega)} \\
&\leq \sqrt{h} \int_0^T \|\Delta \partial_t \eta_\varepsilon\|_{L^2(\omega)} \left(\int_{t-h}^t \|\Delta \partial_t \eta_\varepsilon\|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \\
&\leq C\sqrt{h}.
\end{aligned}$$

We set $I_{10} = \displaystyle\int_\omega \partial_t \eta_\varepsilon(T)(\eta_\varepsilon(T) - \eta_\varepsilon^-(T))$ and find

$$\begin{aligned}
|I_{10}| &\leq \|\partial_t \eta_\varepsilon(T)\|_{L^2(\omega)} \int_{T-h}^T \|\partial_t \eta_\varepsilon\|_{L^2(\omega)} \\
&\leq Ch.
\end{aligned}$$

Next, $I_{11} = \displaystyle\int_0^T \int_{\Omega_{\eta_\varepsilon^*}(t)} \mathbf{f} \cdot \left(\int_{t-h}^t (\overline{\mathbf{u}}_\varepsilon) \right)$ and $I_{12} = \displaystyle\int_0^T \int_\omega g(\eta_\varepsilon - \eta_\varepsilon^-)$ can be estimated respectivelly by $C\sqrt{h}$ and Ch.

These estimates yield

$$\int_0^T \int_{\Omega_{\eta_\varepsilon^*}(t)} |\mathbf{u}_\varepsilon - \overline{\mathbf{u}}_\varepsilon^-|^2 + \int_0^T \int_\omega (\partial_t \eta_\varepsilon - \partial_t \eta_\varepsilon^-)^2 \leq C\sqrt{h}.$$

This ends the proof of the lemma.

Estimates (31, 32) prove that $\partial_t \eta_\varepsilon$ is relatively compact in $L^2(0,T;L^2(\omega))$ and that $\rho_\varepsilon \overline{\mathbf{u}}_\varepsilon$ is relatively compact in $L^2(0,T;L^2(B'))$. Consequently, $\overline{\mathbf{u}}_\varepsilon$ is relatively compact in $L^2(0,T;L^2(B'))$. This will enable us to pass to the limit in the equations as ε goes to zero.

5 Passage to the limit

We want to pass to the limit as ε goes to zero in the weak formulation satisfied by η_ε and $\mathbf{u}_\varepsilon$. We want also the equality between the structure velocity and the fluid velocity at the interface to hold in the limit.

Let $T > 0$ such that $\inf_\varepsilon \min_{[0,T]\times\overline{\omega}} (1 + \eta_\varepsilon) \geq \alpha > 0$. Let us denote by $(\eta, \tilde{\mathbf{u}})$ the limit of a subsequence of $(\eta_\varepsilon, \overline{\mathbf{u}}_\varepsilon)_{\varepsilon>0}$. We will denote any subsequence of $(\eta_\varepsilon, \overline{\mathbf{u}}_\varepsilon)_{\varepsilon>0}$ by $(\eta_\varepsilon, \overline{\mathbf{u}}_\varepsilon)_{\varepsilon>0}$. We have the following convergences as

ε goes to zero:

$$\begin{array}{lll}
\eta_\varepsilon & \rightarrow & \eta \text{ in } C^0([0,T]\times\overline{\omega}) \\
\eta_\varepsilon & \rightharpoonup & \eta \text{ in } H^1(0,T;H^2_0(\omega)) \\
\partial_t\eta_\varepsilon & \rightarrow & \partial_t\eta \text{ in } L^2(0,T;L^2(\omega)) \\
\overline{\mathbf{u}}_\varepsilon & \rightarrow & \tilde{\mathbf{u}} \text{ in } L^2(0,T;L^2(B')) \\
\rho_\varepsilon\mathbf{u}_\varepsilon & \rightarrow & \rho\tilde{\mathbf{u}} \text{ in } L^2(0,T;L^2(B')) \\
\overline{\mathbf{u}}_\varepsilon & \rightharpoonup & \tilde{\mathbf{u}} \text{ in } L^2(0,T;H^1_{0,\Gamma_0}(B')) \\
\eta^*_\varepsilon & \rightarrow & \eta \text{ in } C^0([0,T]\times\overline{\omega}) \\
\partial_t\eta^*_\varepsilon & \rightarrow & \partial_t\eta \text{ in } L^2(0,T;L^2(\omega)) \\
\mathbf{u}^*_\varepsilon & \rightarrow & \tilde{\mathbf{u}} \text{ in } L^2(0,T;L^2(B'))
\end{array} \tag{36}$$

Moreover $\rho_\varepsilon\nabla\mathbf{u}_\varepsilon$ tends to some $\mathbf{z}$ weakly in $L^2(0,T;L^2(B'))$ as ε goes to zero. It is easy to verify that, since $\eta^*_\varepsilon \rightarrow \eta$ in $C^0([0,T]\times\overline{\omega})$, $\mathbf{z}=0$ in $\widehat{B'}\setminus\widehat{\Omega}_{\eta^*_\varepsilon}$ and $\mathbf{z}|_{\widehat{\Omega}_\eta}=\nabla\tilde{\mathbf{u}}$. Thus,

$$\rho_\varepsilon\nabla\mathbf{u}_\varepsilon \rightharpoonup \rho\nabla\tilde{\mathbf{u}} \text{ in } L^2(0,T;L^2(B')) \tag{37}$$

First we take care of the equality

$$\mathbf{u}_\varepsilon(t,x,y,1+\eta^*_\varepsilon(t,x,y)) = (0,0,\partial_t\eta_\varepsilon(t,x,y))^T$$

on $(0,T)\times\omega$. The right hand side converges to $(0,0,\partial_t\eta)^T$ in $L^2(0,T;L^2(\omega))$. For the left hand side, we considere the function $\mathbf{w}_\varepsilon$ defined for a.e. t by

$$\mathbf{w}_\varepsilon = \left|\begin{array}{l} (0,0,\partial_t\eta_\varepsilon)^T \text{ in } B'\setminus C_{\alpha/2} \\ \mathcal{R}(0,0,\partial_t\eta_\varepsilon)^T \text{ in } C_{\alpha/2}, \end{array}\right.$$

where $\mathcal{R}$ is a linear continuous lifting operator from $H^{\frac{1}{2}}(\omega\times\{\alpha/2\})$ to $H^1_{0,\Gamma_0}(C_{\alpha/2})$ such that $\mathcal{R}(0,0,\partial_t\eta_\varepsilon)^T$ is divergence free. Then, the function $\overline{\mathbf{u}}_\varepsilon - \mathbf{w}_\varepsilon$ belongs to $L^2(0,T;H^1(B'))$ and is bounded in this space independently of ε. Thus, a subsequence of $(\overline{\mathbf{u}}_\varepsilon-\mathbf{w}_\varepsilon)_{\varepsilon>0}$ converges weakly in $L^2(0,T;H^1(B'))$ to $\mathbf{w}_0=\tilde{\mathbf{u}}-\mathbf{w}$ (with an obvious definition of $\mathbf{w}$). We have $\mathbf{w}_0=0$ on Γ_0 and in $\widehat{\Omega}_{\eta+\delta}$ for all $\delta>0$, since η^*_ε converges uniformly to η. Thus $\tilde{\mathbf{u}}=(0,0,\partial_t\eta)^T$ in $\widehat{B'}\setminus\widehat{\Omega}_\eta$ and $\mathbf{w}_0\in L^2(0,T;H^1_0(\Omega_\eta(t)))$. Lemma 2 implies that $\gamma_{\eta(t)}(\mathbf{w}_0)=0$, but $\gamma_{\eta(t)}(\mathbf{w})=(0,0,\partial_t\eta)^T$, hence

$$\gamma_{\eta(t)}(\tilde{\mathbf{u}})=(0,0,\partial_t\eta)^T.$$

Next we pass to the limit in the weak formulation:

$$\int_{\Omega_{\eta_\varepsilon^*}(t)} \mathbf{u}_\varepsilon(t)\cdot\phi_\varepsilon(t) - \int_0^t\int_{\Omega_{\eta_\varepsilon^*}(s)} \mathbf{u}_\varepsilon\cdot\partial_t\phi_\varepsilon + \nu\int_0^t\int_{\Omega_{\eta_\varepsilon^*}(s)} \nabla\mathbf{u}_\varepsilon:\nabla\phi_\varepsilon$$
$$+\frac{1}{2}\int_0^t\int_{\Omega_{\eta_\varepsilon^*}(s)}(\mathbf{u}_\varepsilon^*\cdot\nabla)\mathbf{u}_\varepsilon\cdot\phi_\varepsilon - \frac{1}{2}\int_0^t\int_{\Omega_{\eta_\varepsilon^*}(s)}(\mathbf{u}_\varepsilon^*\cdot\nabla)\mathbf{u}_\varepsilon\cdot\phi_\varepsilon$$
$$-\frac{1}{2}\int_0^t\int_\omega \partial_t\eta_\varepsilon\partial_t\eta_\varepsilon^* b + \int_\omega \partial_t\eta_\varepsilon(t)\,b(t) - \int_0^t\int_\omega \partial_t\eta_\varepsilon\,\partial_t b + \int_0^t\int_\omega \Delta\partial_t\eta_\varepsilon\,\Delta b$$
$$+\int_0^t\int_\omega \Delta\eta_\varepsilon\,\Delta b = \int_0^t\int_{\Omega_{\eta_\varepsilon^*}(t)}\mathbf{f}\cdot\phi_\varepsilon + \int_0^t\int_\omega g\ b + \int_{\Omega(0)}\mathbf{u}_0^\varepsilon\cdot\phi_\varepsilon(0) + \int_\omega \eta_1^\varepsilon b(0),$$

for a.e. t and for all $(\phi_\varepsilon, b)\in\mathcal{V}_{\eta_\varepsilon^*}\times C^1([0,T];H_0^2(\omega))$ such that $\phi_\varepsilon(t,x,y,1+\eta_\varepsilon(t,x,y)) = (0,0,b(t,x,y))^T$, $(t,x,y)\in[0,T]\times\omega$.

The fluid test functions *a priori* depend on ε. However, it is sufficient to consider test functions that do not depend on ε and that are admissible for ε small enough.

As before, we consider test functions of the form $(\phi^0, 0)$, with $\phi^0\in\mathcal{D}(\cup_{t\in[0,T]}\{t\}\times\Omega_\eta(t))$ and $\operatorname{div}\phi^0 = 0$. These test functions satisfy the property that $\phi^0(t,\cdot)\in\mathcal{D}(\Omega_\eta(t))$ for every t. Hence for ε small enough, $\phi^0\in\mathcal{V}_{\eta_\varepsilon^*}$, since η_ε^* converges uniformly to η as $\varepsilon\downarrow 0$.

The second pair of test functions we consider is (ϕ^1, b) where b belongs to $C^1([0,T];H_0^2(\omega))$, with $\int_\omega b = 0$ and for a.e. t:

$$\phi^1 = \left|\begin{array}{l}(0,0,b)^T \text{ in } B'\setminus C_{\alpha/2}\\ \mathcal{R}(0,0,b)^T \text{ in } C_{\alpha/2}.\end{array}\right.$$

Since $\min_{[0,T]\times\overline{\omega}}(1+\eta_\varepsilon^*)\geq\alpha/2$, (ϕ^1, b) is a pair of admissible test functions for all ε.

With both types of test functions, it is easy to pass to the limit in the weak formulation as ε goes to zero. We obtain the existence of a weak solution of (16) on $(0,T)$ satisfying energy estimates (10).

Eventually, we show that we can extend the solution as long as we have $\min_{[0,T]\times\overline{\omega}}(1+\eta)>0$. Let us build an increasing sequence of times $(T_k)_{k\geq1}$ as follows. First we choose a time $T_1>0$ such that there exists a weak solution up to T_1, with $m_1 = \min_{[0,T_1]\times\overline{\omega}}(1+\eta)>0$. Possibly changing slightly T_1, we may assume that $\partial_t\eta(T_1)\in L^2(\omega)$ and $\mathbf{u}(T_1)\in L^2(\Omega_\eta(T_1))$ (since this is true for almost every time).

Now, let $k\geq 1$ and assume that we have built a solution up to some time T_k, with $m_k = \min_{[0,T_k]\times\overline{\omega}}(1+\eta)>0$. Our construction allows us to build an extension of our solution, on some time interval starting from T_k. Thanks to the *a priori* energy estimate (10), we have for $s \geq T_k$

$$1+\eta(s) \geq 1+\eta(T_k)-(s-T_k)^{\frac{1}{2}}C(T_k,s) \geq m_k-(s-T_k)^{\frac{1}{2}}C(T_k,s)\,, \quad (38)$$

with

$$C(T_k,s)=\tilde{C}\Big(\|\mathbf{u}(T_k)\|_{L^2(\Omega_{\eta_0})}, \|\partial_t\eta(T_k)\|_{H_0^2(\omega)}, \int_{T_k}^{s}\exp(s-u)(\|\mathbf{f}\|_{L^2(\Omega_{\eta}(u))}(u)+\|g\|_{L^2(\omega)}(u))du\Big),$$

where $\tilde{C}$ is positive and nondecreasing with respect to its arguments, and $C(T_k,s) \leq C(0,s)$. This *a priori* estimate shows that if we let $\tau_k = \min\{1,(m_k/2C(T_k,T_k+1))^2\}$, we can build a solution starting from $\mathbf{u}(T_k)$, $\eta(T_k)$ and $\partial_t\eta(T_k)$ up to the time $T_k+\tau_k$ (this corresponds to choosing $\alpha = m_k/2$ in the construction of the solution). The time T_{k+1} is chosen close to $T_k+\tau_k$ (in $[T_k+\tau_k/2, T_k+\tau_k]$), in order to have also $\partial_t\eta(T_{k+1}) \in L^2(\omega)$ and $\mathbf{u}(T_{k+1}) \in L^2(\Omega_\eta(T_{k+1}))$.

We let $T^* = \sup_k T_k$. If $T^* < +\infty$, then $m^* = \min_{[0,T^*]\times\overline{\omega}}(1+\eta) = 0$. Otherwise, since $m_k \geq m^*$ for all k, $\tau_k \geq \min\{1,(m^*/2C(0,T^*))^2\} > 0$. But $T_{k+1}-T_k \geq \tau_k/2$ and goes to zero, which is a contradiction. This achieves the proof of the theorem.

6 Conclusion, remarks and extensions

We have proved above that there exists at least one weak solution of our coupled fluid–plate problem. A similar proof would imply the existence of a weak solution for a 2D fluid described by the Navier–Stokes equations coupled to a membrane with an additional viscous term. In this case the elastic energy is described by a second order operator and the structure equations are

$$\partial_{tt}\eta - \Delta\eta - \Delta\partial_t\eta = (T_f)_2 + g \text{ in } (0,1).$$

For the three–dimensional case, instead of the considered viscous additional term ($\Delta^2\partial_t\eta$), we could add other terms, such as $-\Delta\partial_t\eta$ or $-\Delta\partial_{tt}\eta$. To prove the existence of at least one weak solution for those coupled systems, one can first prove that there exists at least one solution for the fluid coupled with a structure satisfying

$$\partial_{tt}\eta + \Delta^2\eta + \mu\Delta^2\partial_t\eta + \left| \begin{array}{l} -\Delta\partial_t\eta \\ \text{or} \\ -\Delta\partial_{tt}\eta \end{array} \right. = (T_f)_3 + g \text{ in } \omega,$$

then send μ to 0. This is possible if estimates similar to those of in Lemma 9, and independent of μ are available. These estimates can be derived by cutting off high frequencies of the velocity field and using similarly the weak formulation tested against the low frequency part, which is smooth.

In all those cases the elastic displacement in not Lipschitz but the fluid domain is locally a subgraph. We remark that an additional term that regularizes the structure velocity is important at many steps: first to define the extension of a fluid velocity and next to obtain the estimates that give the compactness result.

Appendix: Proof of Lemma 8

Here we give the details of the calculations that lead to the estimate

$$\|\partial_t \underline{\mathbf{u}}_\varepsilon^{m,n}\|_{L^2(0,T;L^2(\mathcal{C}))} + \|\partial_{tt}\eta_\varepsilon^n\|_{L^2(0,T;L^2(\omega))} \leq C,$$

where C a strictly positive constant depending on the data, α, M and on ε but independent of (m,n).

We multiply (24) by $\dot{\alpha}_i$ and then add those equations for $i = 1 \ldots m$, and multiply (25) by $\ddot{\beta}_j$ and then add those equations for $j = 1 \ldots n$. It corresponds to taking $(\sum_{i=1}^m \dot{\alpha}_i \phi_i^{0,\varepsilon} + \sum_{j=1}^n \ddot{\beta}_j \phi_j^{1,\varepsilon}, \sum_{j=1}^n \ddot{\beta}_j \xi_j)$ as test functions. We obtain (we skip the indices ε, n, m in all the calculations that follow):

$$\int_{\mathcal{C}} |\sum_i \dot{\lambda}_i \phi_i|^2 J + \int_{\mathcal{C}} \sum_i \dot{\lambda}_i \phi_i \cdot \sum_i \lambda_i \partial_t \phi_i J + \nu \int_{\mathcal{C}} A \nabla \underline{\mathbf{u}} \nabla (\sum_i \dot{\lambda}_i \phi_i)$$
$$+ \frac{1}{2} \int_{\mathcal{C}} (\underline{\mathbf{v}}^* \cdot (B\nabla)) \underline{\mathbf{u}} \cdot \sum_i \dot{\lambda}_i \phi_i - \frac{1}{2} \int_{\mathcal{C}} (\underline{\mathbf{v}}^* \cdot (B\nabla)) \sum_i \dot{\lambda}_i \phi_i \cdot \underline{\mathbf{u}}$$
$$- \int_{\mathcal{C}} (\underline{\mathbf{w}} \cdot (B\nabla)) \underline{\mathbf{u}} \cdot \sum_i \dot{\lambda}_i \phi_i + \frac{1}{2} \int_\omega \partial_t \eta \partial_t \delta^* \partial_{tt} \eta + \int_\omega (\partial_{tt}\eta)^2$$
$$+ \int_\omega \Delta \partial_t \eta \Delta \partial_{tt} \eta + \int_\omega \Delta \eta \Delta \partial_{tt} \eta = \int_{\mathcal{C}} \underline{\mathbf{f}} \cdot \sum_i \dot{\lambda}_i \phi_i J + \int_\omega g \partial_{tt} \eta,$$

where

$$\lambda_i = \left| \begin{array}{l} \alpha_i, \ 1 \le i \le m \\ \dot{\beta}_{i-n}, \ m+1 \le i \le n+m \end{array} \right.$$

and

$$\phi_i = \left| \begin{array}{l} \phi_i^0, \ 1 \le i \le m \\ \phi_{i-n}^1, \ m+1 \le i \le n+m \end{array} \right. .$$

Thus,

$$\begin{aligned}
&\| \sum_i \dot{\lambda}_i \phi_i \sqrt{J} \|^2_{L^2(\mathcal{C})} + \frac{\nu}{2} \frac{d}{dt} \int_{\mathcal{C}} A \nabla \underline{\mathbf{u}} \nabla \underline{\mathbf{u}} + \| \partial_{tt} \eta \|^2_{L^2(\omega)} \\
&+ \frac{1}{2} \frac{d}{dt} \int_\omega (\Delta \partial_t \eta)^2 = - \int_{\mathcal{C}} \sum_i \dot{\lambda}_i \phi_i \sum_i \lambda_i \partial_t \phi_i J + \frac{\nu}{2} \int_{\mathcal{C}} \partial_t A \nabla \underline{\mathbf{u}} \nabla \underline{\mathbf{u}} \\
&+ \nu \int_{\mathcal{C}} A \nabla \underline{\mathbf{u}} \nabla \sum_i \lambda_i \partial_t \phi_i - \frac{1}{2} \int_{\mathcal{C}} (\underline{\mathbf{v}}^* \cdot (B \nabla)) \underline{\mathbf{u}} \cdot \sum_i \dot{\lambda}_i \phi_i \\
&+ \frac{1}{2} \int_{\mathcal{C}} (\underline{\mathbf{v}}^* \cdot (B \nabla)) \sum_i \dot{\lambda}_i \phi_i \cdot \underline{\mathbf{u}} + \int_{\mathcal{C}} (\underline{\mathbf{w}} \cdot (B \nabla)) \underline{\mathbf{u}} \cdot \sum_i \dot{\lambda}_i \phi_i \\
&- \frac{1}{2} \int_\omega \partial_t \eta \partial_t \delta^* \partial_{tt} \eta + \int_\omega (\Delta \partial_t \eta)^2 - \frac{d}{dt} \int_\omega \Delta \eta \Delta \partial_t \eta \\
&+ \int_{\mathcal{C}} \underline{\mathbf{f}} \cdot \sum_i \dot{\lambda}_i \phi_i J + \int_\omega g \partial_{tt} \eta.
\end{aligned}$$

Integrating in time we have:

$$\begin{aligned}
&\int_0^t \| \sum_i \dot{\lambda}_i \phi_i \sqrt{J} \|^2_{L^2(\mathcal{C})} + \frac{\nu}{2} \int_{\mathcal{C}} A(t) \nabla \underline{\mathbf{u}}(t) \nabla \underline{\mathbf{u}}(t) \\
&+ \int_0^t \| \partial_{tt} \eta \|^2_{L^2(\omega)} + \frac{1}{2} \| \Delta \partial_t \eta \|^2_{L^2(\omega)}(t) \\
&= \frac{\nu}{2} \int_{\mathcal{C}} A(0) \nabla \underline{\mathbf{u}}(0) \nabla \underline{\mathbf{u}}(0) + \frac{1}{2} \| \Delta \partial_t \eta \|^2_{L^2(\omega)}(0) + \int_\omega \Delta \eta(0) \Delta \partial_t \eta(0) \\
&- \int_0^t \int_{\mathcal{C}} \sum_i \dot{\lambda}_i \phi_i \sum_i \lambda_i \partial_t \phi_i J + \frac{\nu}{2} \int_0^t \int_{\mathcal{C}} \partial_t A \nabla \underline{\mathbf{u}} \nabla \underline{\mathbf{u}} \\
&+ \nu \int_0^t \int_{\mathcal{C}} A \nabla \underline{\mathbf{u}} \nabla \sum_i \lambda_i \partial_t \phi_i - \frac{1}{2} \int_0^t \int_{\mathcal{C}} (\underline{\mathbf{v}}^* \cdot (B \nabla)) \underline{\mathbf{u}} \cdot \sum_i \dot{\lambda}_i \phi_i \\
&+ \frac{1}{2} \int_0^t \int_{\mathcal{C}} (\underline{\mathbf{v}}^* \cdot (B \nabla)) \sum_i \dot{\lambda}_i \phi_i \cdot \underline{\mathbf{u}} + \int_0^t \int_{\mathcal{C}} (\underline{\mathbf{w}} \cdot (B \nabla)) \underline{\mathbf{u}} \cdot \sum_i \dot{\lambda}_i \phi_i \\
&- \frac{1}{2} \int_0^t \int_\omega \partial_t \eta \partial_t \delta^* \partial_{tt} \eta + \int_0^t \| \Delta \partial_t \eta \|^2_{L^2(\omega)} - \int_\omega \Delta \eta(t) \Delta \partial_t \eta(t) \\
&+ \int_0^t \int_{\mathcal{C}} \underline{\mathbf{f}} \cdot \sum_i \dot{\lambda}_i \phi_i J + \int_0^t \int_\omega g \partial_{tt} \eta.
\end{aligned}$$

The first three terms in the right hand side depend on the initial conditions, that are smooth. Thus we have eleven terms to estimate and we have to find bounds independent of (m,n). In all what follows C denotes a constant that depends on the data, M, α and ε but not on (m,n).

We have

$$|T_1| = \left| \int_0^t \int_{\mathcal{C}} \sum_i \dot{\lambda}_i \phi_i \sum_i \lambda_i \partial_t \phi_i J \right| \\ \leq C \| \sum_i \dot{\lambda}_i \phi_i \sqrt{J} \|_{L^2(0,t;L^2(\mathcal{C}))} \| \sum_i \lambda_i \partial_t \phi_i \|_{L^2(0,t;L^2(\mathcal{C}))}.$$

Hence we need to estimate $\| \sum_i \lambda_i \partial_t \phi_i \|_{L^2(0,t;L^2(\mathcal{C}))}$, and check that thanks to the previous estimates (26) on $\underline{\mathbf{u}}$ and η, this term is bounded independently of (m,n). We have (remember the definitions of the functions ϕ_i):

$$\begin{aligned} \sum_i \lambda_i \partial_t \phi_i &= \sum_{i=1}^m \alpha_i \partial_t B^{-T} \psi_i^0 + \sum_{j=1}^n \dot{\beta}_j \partial_t \phi_j^1 \\ &= \sum_{i=1}^m \alpha_i \partial_t B^{-T} B^T \phi_i^0 + \sum_{j=1}^n \dot{\beta}_j \partial_t \phi_j^1. \end{aligned}$$

Since $\partial_t B^{-T} B^T$ is bounded in L^∞ (the bound depending on ε),

$$\begin{aligned} \| \sum_{i=1}^m \alpha_i \partial_t B^{-T} B^T \phi_i^0 \|_{L^2(0,t;L^2(\mathcal{C}))} &\leq C \| \sum_{i=1}^m \alpha_i \phi_i^0 \|_{L^2(0,t;L^2(\mathcal{C}))} \\ &\leq C \| \underline{\mathbf{u}} \|_{L^2(0,t;L^2(\mathcal{C}))} \\ &\leq C. \end{aligned}$$

Next, we deal with $\sum_{j=1}^n \dot{\beta}_j \partial_t \phi_j^1$. Each $\partial_t \phi_j^1$ satisfies the following problem:

$$\begin{aligned} & -\Delta \partial_t \phi_j^1 + (B\nabla) \partial_t p_j^1 = (\partial_t B \nabla) p_j^1 \text{ in } \mathcal{C}, \\ & \operatorname{div}(B^T \partial_t \phi_j^1) = \operatorname{div}(\partial_t B^T \phi_j^1) \text{ in } \mathcal{C}, \\ & \partial_t \phi_j^1 = 0 \text{ on } \partial\mathcal{C}. \end{aligned}$$

By linearity, $\sum_{j=1}^n \dot{\beta}_j \partial_t \phi_j^1$ is solution to the same type of problem. Thus,

$$\| \sum_{j=1}^n \dot{\beta}_j \partial_t \phi_j^1 \|_{H^1(\mathcal{C})} \leq C \left(\| \sum_{j=1}^n \dot{\beta}_j p_j^1 \|_{L^2(\mathcal{C})} + \| \sum_{j=1}^n \dot{\beta}_j \phi_j^1 \|_{H^1(\mathcal{C})} \right).$$

But remembering the definition of ϕ_j^1 we have that

$$\| \sum_{j=1}^n \dot{\beta}_j p_j^1 \|_{L^2(\mathcal{C})} + \| \sum_{j=1}^n \dot{\beta}_j \phi_j^1 \|_{H^1(\mathcal{C})} \leq C \| \sum_{j=1}^n \dot{\beta}_j \xi_j \|_{H^{1/2}(\omega)} = C \| \partial_t \eta \|_{H^{1/2}(\omega)},$$

which leads to

$$\|\sum_{j=1}^{n}\dot{\beta}_j\partial_t\phi_j^1\|_{L^2(0,t;H^1(\mathcal{C}))}\leq C\|\partial_t\eta\|_{L^2(0,t;H^{1/2}(\omega))}. \tag{39}$$

Thus, this term is bounded independently of (m,n) since $\partial_t\eta$ can be estimated in $L^2(0,t;H_0^2(\omega))$ by a constant depending only on the data and ε (thanks to the term added to the standard plate equation). Finally

$$|T_1|\leq C+\frac{1}{8}\|\sum_i\dot{\lambda}_i\phi_i\|^2_{L^2(0,t;L^2(\mathcal{C}))}.$$

The second term $T_2=\dfrac{\nu}{2}\displaystyle\int_0^t\int_{\mathcal{C}}\partial_tA\nabla\underline{\mathbf{u}}\nabla\underline{\mathbf{u}}$ is bounded independently of (m,n) since $\underline{\mathbf{u}}$ is bounded in $L^2(0,t;H^1(\mathcal{C}))$ and ∂_tA is bounded in L^∞ independently of (m,n). The third term $T_3=\nu\displaystyle\int_0^t\int_{\mathcal{C}}A\nabla\underline{\mathbf{u}}\nabla\sum_i\lambda_i\partial_t\phi_i$ can be estimated by a constant independent of (m,n) thanks to the bound on A in L^∞, on $\underline{\mathbf{u}}$ in $L^2(0,t;H^1(\mathcal{C}))$ and on $\sum_i\lambda_i\partial_t\phi_i$ in $L^2(0,t;H^1(\mathcal{C}))$ (see (39)).

$$|T_4|=\left|-\frac{1}{2}\int_0^t\int_{\mathcal{C}}(\underline{\mathbf{v}}^*\cdot(B\nabla))\underline{\mathbf{u}}\cdot\sum_i\dot{\lambda}_i\phi_i\right|\leq C+\frac{1}{8}\|\sum_i\dot{\lambda}_i\phi_i\sqrt{J}\|^2_{L^2(0,t;L^2(\mathcal{C}))}.$$

We transform $T_5=\dfrac{1}{2}\displaystyle\int_0^t\int_{\mathcal{C}}(\underline{\mathbf{v}}^*\cdot(B\nabla))\sum_i\dot{\lambda}_i\phi_i\cdot\underline{\mathbf{u}}$ by integration by parts in space:

$$T_5=-\frac{1}{2}\int_0^t\int_{\mathcal{C}}\sum_{j=1}^{3}\partial_j((B^T\underline{\mathbf{v}}^*)_j\underline{\mathbf{u}})\cdot\sum_i\dot{\lambda}_i\phi_i+\frac{1}{2}\int_0^t\int_{\partial\mathcal{C}}(\underline{\mathbf{u}}\cdot\sum_i\dot{\lambda}_i\phi_i)(\underline{\mathbf{v}}^*\cdot B\mathbf{n}),$$

The first contribution can be estimated as follows

$$\left|\frac{1}{2}\int_0^t\int_{\mathcal{C}}\sum_{j=1}^{3}\partial_j((B^t\underline{\mathbf{v}}^*)_j\underline{\mathbf{u}})\cdot\sum_i\dot{\lambda}_i\phi_i\right|\leq C+\frac{1}{8}\|\sum_i\dot{\lambda}_i\phi_i\sqrt{J}\|^2_{L^2(0,t;L^2(\mathcal{C}))},$$

and the second one can be written:

$$\frac{1}{2}\int_0^t\int_{\partial\mathcal{C}}(\underline{\mathbf{u}}\cdot\sum_i\dot{\lambda}_i\phi_i)(\underline{\mathbf{v}}^*\cdot B\mathbf{n})=\frac{1}{2}\int_0^t\int_{\omega}\partial_t\eta\partial_{tt}\eta(\underline{\mathbf{v}}^*\cdot B\mathbf{n}),$$

which is less than

$$C+\frac{1}{6}\|\partial_{tt}\eta\|^2_{L^2(0,t;L^2(\omega))}.$$

Thus

$$|T_5| \le C + \frac{1}{8}\|\sum_i \dot{\lambda}_i \phi_i\|^2_{L^2(0,t;L^2(\mathcal{C}))} + \frac{1}{6}\|\partial_{tt}\eta\|^2_{L^2(0,t;L^2(\omega))}.$$

The next term is estimated as T_4:

$$|T_6| = \left| \int_0^t \int_{\mathcal{C}} (\underline{\mathbf{w}} \cdot (B\nabla)) \sum_i \dot{\lambda}_i \phi_i \cdot \underline{\mathbf{u}} \right| \le C + \frac{1}{8}\|\sum_i \dot{\lambda}_i \phi_i \sqrt{J}\|^2_{L^2(0,t;L^2(\mathcal{C}))}.$$

Next

$$|T_7| = |\frac{1}{2}\int_0^t \int_\omega \partial_t\eta \partial_t \delta^* \partial_{tt}\eta| \le C + \frac{1}{6}\|\partial_{tt}\eta\|^2_{L^2(0,t;L^2(\omega))}.$$

The term $T_8 = \displaystyle\int_0^t \|\Delta\partial_t\eta\|^2_{L^2(\omega)}$ is bounded by a constant C thanks to (26). This bound comes from the parabolic term that we added to the standard plate equation. This additional term is also useful to estimate the term $T_9 = -\displaystyle\int_\omega \Delta\eta(t)\Delta\partial_t\eta(t)$:

$$\begin{aligned} |T_9| &\le \frac{1}{4}\|\Delta\partial_t\eta\|^2_{L^2(\omega)}(t) + C\|\Delta\eta\|^2_{L^2(\omega)}(t) \\ &\le C + \frac{1}{4}\|\Delta\partial_t\eta\|^2_{L^2(\omega)}(t). \end{aligned}$$

Eventually

$$|T_{10}| = |\int_0^t \int_{\mathcal{C}} \underline{\mathbf{f}} \cdot \sum_i \dot{\lambda}_i \phi_i J| \le C + \frac{1}{8}\|\sum_i \dot{\lambda}_i \phi_i \sqrt{J}\|^2_{L^2(0,t;L^2(\mathcal{C}))},$$

and

$$|T_{11}| = |\int_0^t \int_\omega g\partial_{tt}\eta| \le C + \frac{1}{6}\|\partial_{tt}\eta\|^2_{L^2(0,t;L^2(\omega))}.$$

Consequently:

$$\frac{1}{2}\|\sum_i \dot{\lambda}_i \phi_i \sqrt{J}\|^2_{L^2(0,t;L^2(\mathcal{C}))} + \frac{1}{2}\|\partial_{tt}\eta\|^2_{L^2(0,t;L^2(\omega))} + \frac{1}{4}\|\Delta\partial_t\eta\|^2_{L^2(\omega)}(t) \le C. \tag{40}$$

Since $\partial_t\underline{\mathbf{u}} = \sum_i \dot{\lambda}_i\phi_i + \sum_i \lambda_i \partial_t\phi_i$ and $\sqrt{J} \ge \sqrt{\alpha/2}$, this gives the desired estimate.

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