

# ELEMENTARY NOTIONS OF LATTICE TRIGONOMETRY.

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ABSTRACT. In this paper we study properties of lattice trigonometric functions of lattice angles in lattice geometry. We introduce the definition of sums of lattice angles and establish a necessary and sufficient condition for three angles to be the angles of some lattice triangle in terms of lattice tangents. This condition is a version of the Euclidean condition: three angles are the angles of some triangle iff their sum equals  $\pi$ . Further we find the necessary and sufficient condition for an ordered  $n$ -tuple of angles to be the angles of some convex lattice polygon. In conclusion we show applications to theory of complex projective toric varieties, and a list of unsolved problems and questions.

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## INTRODUCTION.

0.1. **The goals of this paper and some background.** Consider a two-dimensional oriented real vector space and fix some full-rank lattice in it. A triangle or a polygon is said to be *lattice* if all its vertices belong to the lattice. The angles of any lattice triangle are said to be *lattice*.

In this paper we introduce and study *lattice trigonometric* functions of lattice angles. The lattice trigonometric functions are invariant under the action of the group of lattice-affine transformations (i.e. affine transformations preserving the lattice), like the ordinary trigonometric functions are invariant under the action of the group of Euclidean length preserving transformations of Euclidean space.

One of the initial goals of the present article is to make a complete description of lattice triangles up to the lattice-affine equivalence relation (see Theorem 2.2). The classification problem of convex lattice polygons becomes now classical. There is still no a good description of convex polygons. It is only known that the number of such polygons with lattice area bounded from above by  $n$  grows exponentially in  $n$ , while  $n$  tends to infinity (see the works of V. Arnold [2], and of I. Bárány and A. M. Vershik [3]).

We expand the geometric interpretation of ordinary continued fractions to define lattice sums of lattice angles and to establish relations on lattice tangents of lattice angles. Further, we describe lattice triangles in terms of *lattice sums* of lattice angles.

In present paper we also show a lattice version of the sine formula and introduce a relation between the lattice tangents for angles of lattice triangles and the numbers of lattice points on the edges of triangles (see Theorem 1.18). In addition, we make first steps in study of non-lattice angles with lattice vertices. We conclude the paper with applications to toric varieties and some unsolved problems.

The study of lattice angles is an imprescriptible part of modern lattice geometry. Invariants of lattice angles are used in the study of lattice convex polygons and polytopes. Such polygons and polytopes play the principal role in Klein's theory of multidimensional continued fractions (see, for example, the works of F. Klein [14], V. I. Arnold [1], E. Korkina [16], M. Kontsevich and Yu. Suhov [15], G. Lachaud [17], and the author [10]).

Lattice polygons and polytopes of the lattice geometry are in the limelight of complex projective toric varieties (see for more information the works of V. I. Danilov [4], G. Ewald [5], T. Oda [18], and W. Fulton [6]). To illustrate, we deduce (in Appendix A) from Theorem 2.2 the corresponding global relations on the toric singularities for projective toric varieties associated to integer-lattice triangles. We also show the following simple fact: for any collection with multiplicities of complex-two-dimensional toric algebraic singularities there exists a complex-two-dimensional toric projective variety with the given collection of toric singularities (this result seems to be classical, but it is missing in the literature).

The studies of lattice angles and measures related to them were started by A. G. Khovanskii, A. Pukhlikov in [12] and [13] in 1992. They introduced and investigated special additive polynomial measure for the expanded notion of polytopes. The relations between sum-formulas of lattice trigonometric functions and lattice angles in Khovanskii-Pukhlikov sense are unknown to the author.

**0.2. Some distinctions between lattice and Euclidean cases.** This paper is organized as follows. Lattice trigonometric functions and Euclidean trigonometric functions have much in common. For example, the values of lattice tangents and Euclidean tangents coincide in a special natural system of coordinates. Nevertheless, lattice geometry differs a lot from Euclidean geometry. We provide this with the following 4 examples.

1. The angles  $\angle ABC$  and  $\angle CBA$  are always congruent in Euclidean geometry, but not necessary lattice-congruent in lattice geometry.
2. In Euclidean geometry for any  $n \geq 3$  there exist a regular polygon with  $n$  vertices, and any two regular polygons with the same number of vertices are homothetic to each other. In lattice geometry there are only 6 non-homothetic regular lattice polygons: two triangles (distinguished by lattice tangents of angles), two quadrangles, and two octagons. (See a more detailed description in [11].)
3. Consider three Euclidean criterions of triangle congruence. Only the first criterion can be taken to the case of lattice geometry. The others two are false in lattice trigonometry. (We refer to Appendix B.)
4. There exist two non-congruent right angles in lattice geometry. (See Corollary 1.13.)

**0.3. Description of the paper.** This paper is organized as follows.

We start in Section 1 with some general notation of lattice geometry and ordinary continued fractions. We define ordinary lattice angles, and the functions lattice sine, tangent, and cosine on the set of ordinary lattice angles, and lattice arctangent for rationals greater than 1. Further we indicate their basic properties. We proceed with the geometrical interpretation of lattice tangents in terms of ordinary continued fractions. We also say a few words about relations of transpose and adjacent lattice angles. In conclusion of Section 1 we study of the basic properties of angles in lattice triangles.

In Section 2 we introduce the sum formula for the lattice tangents of ordinary lattice angles of lattice triangles. The sum formula is a lattice generalization of the following Euclidean statement: three angles are the angles of some triangle iff their sum equals  $\pi$ .

Further in Section 3 we introduce the notion of expanded lattice angles and their normal forms and give the definition of sums of expanded and ordinary lattice angles. For the definition of expanded lattice angles we expand the notion of sails in the sense of Klein: we define and study oriented broken lines on the unit distance from lattice points.

In Section 4 we finally prove the first statement of the theorem on sums of lattice tangents for ordinary lattice angles in lattice triangles. In this section we also describe some relations between continued fractions for lattice oriented broken lines and the lattice tangents for the corresponding expanded lattice angles. Further we give a necessary and sufficient condition for an ordered  $n$ -tuple of angles to be the angles of some convex lattice polygon.

In Section 5 we generalize the notions of ordinary and expanded lattice angles and their sums to the case of angles with lattice vertices but not necessary lattice rays. We find normal forms and extend the definition of lattice sums for a certain special case of such angles (we call them *irrational*).

We conclude this paper with three appendices. In Appendix A we describe applications to theory of complex projective toric varieties mentioned above. Further in Appendix B we give a list of unsolved problems and questions. Finally in Appendix C we formulate criterions of lattice congruence for lattice triangles. These criterions lead to the complete list of lattice triangles with small lattice area (not greater than 10).

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## 1. DEFINITIONS AND ELEMENTARY PROPERTIES OF LATTICE TRIGONOMETRIC FUNCTIONS.

We start the section with general definitions of lattice geometry and notions of ordinary continued fractions. We define the functions lattice sine, tangent, and cosine on the set of ordinary lattice angles and formulate their basic properties.

Then we describe a geometric interpretation of lattice trigonometric functions in terms of ordinary continued fractions associated with boundaries of convex hulls for the sets of lattice points contained in angles. We also say a few words about relations of transpose and adjacent lattice angles.

Further we introduce the sine formula for the lattice angles of lattice triangles. Finally, we show how to find the lattice tangents of all angles and the lattice lengths of all edges of any lattice triangle, if we know the lattice lengths of two edges and the lattice tangent of the angle between them.

**1.1. Preliminary notions and definitions.** By  $\gcd(n_1, \dots, n_k)$  and by  $\text{lcm}(n_1, \dots, n_k)$  we denote the greater common divisor and the less common multiple of the nonzero integers  $n_1, \dots, n_k$  respectively. Suppose that  $a, b$  be arbitrary integers, and  $c$  be an arbitrary positive integer. We write that  $a \equiv b \pmod{c}$  if the remainders of  $a$  and  $b$  modulo  $c$  coincide.

**1.1.1. Lattice notation.** Here we define the main objects of lattice geometry, their lattice characteristics, and the relation of lattice-congruence.

Consider a two-dimensional oriented real vector space and fix some lattice in it. A straight line is said to be *lattice* if it contains at least two distinct lattice points. A ray is said to be *lattice* if its vertex is lattice, and the straight line containing the ray is lattice. An angle (i.e.

the union of two rays with the common vertex) is said to be *ordinary lattice* if the rays defining it are lattice. A segment is called *lattice* if its endpoints are lattice points.

By a *convex polygon* we mean a convex hulls of a finite number of points that do not lie in a straight line. A straight line  $\pi$  is said to be *supporting* for a convex polygon  $P$ , if the intersections of  $P$  and  $\pi$  is not empty, and the whole polygon  $P$  is contained in one of the closed half-planes bounded by  $\pi$ . An intersection of any polygon  $P$  with its supporting hyperplane is called a *vertex* or an *edge* of the polygon if the dimension of intersection is zero, or one respectively.

A triangle (or convex polygon) is said to be *lattice* if all it's vertices are lattice points.

The affine transformation is called *lattice-affine* if it preserves the set of all lattice points. Consider two arbitrary (not necessary lattice in the above sense) sets. We say that these two sets are *lattice-congruent* to each other if there exist a lattice-affine transformation of  $\mathbb{R}^2$  taking the first set to the second.

A lattice triangle is said to be *simple* if the vectors corresponding to its edges generate the lattice.

**Definition 1.1.** The *lattice length* of a lattice segment  $AB$  is the ratio between the Euclidean length of  $AB$  and the length of the basic lattice vector for the straight line containing this segment. We denote the lattice length by  $l\ell(AB)$ .

By the (non-oriented) *lattice area* of the convex polygon  $P$  we will call the ratio of the Euclidean area of the polygon and the area of any lattice simple triangle, and denote it by  $IS(P)$ .

Any two rays (straight lines) are lattice-congruent to each other. Two lattice segments are lattice-congruent iff they have equal lattice lengths. The lattice area of the convex polygon is well-defined and is proportional to the Euclidean area of the polygon.

1.1.2. *Finite ordinary continued fractions.* For any finite sequence  $(a_0, a_1, \dots, a_n)$  where the elements  $a_1, \dots, a_n$  are positive integers and  $a_0$  is an arbitrary integer we associate the following rational number  $q$ :

$$q = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

This representation of the rational  $q$  is called an *ordinary continued fraction* for  $q$  and denoted by  $[a_0, a_1, \dots, a_n]$ . (In the literature is also in use the following notation:  $[a_0; a_1, \dots, a_n]$ .) An ordinary continued fraction  $[a_0, a_1, \dots, a_n]$  is said to be *odd* if  $n+1$  is odd, and *even* if  $n+1$  is even.

Note that if  $a_n \neq 1$  then  $[a_0, a_1, \dots, a_n] = [a_0, a_1, \dots, a_n - 1, 1]$ .

Now we formulate a classical theorem of ordinary continued fractions theory.

**Theorem 1.2.** *For any rational there exist exactly one odd ordinary continued fraction and exactly one even ordinary continued fraction.* □

1.2. **Definition of lattice trigonometric functions.** In this subsection we define the functions lattice sine, tangent, and cosine on the set of ordinary lattice angles and formulate their basic properties. We describe a geometric interpretation of lattice trigonometric functions in terms of ordinary continued fractions. Then we give the definitions of ordinary lattice angles

that are adjacent, transpose, and opposite interior to the given angles. We use the notions of adjacent and transpose ordinary lattice angles to define ordinary lattice right angles.

Let  $A$ ,  $B$ , and  $C$  be three lattice points that do not lie in the same straight line. We denote the ordinary lattice angle with the vertex at  $B$  and the rays  $BA$  and  $BC$  by  $\angle ABC$ .

One can choose any other lattice point  $B'$  in the open lattice ray  $AB$  (but not in  $AC$ ) and any lattice point  $C'$  in the open lattice ray  $AC$ . For us the ordinary lattice angle  $\angle AOB$  coincides with the ordinary lattice angle  $\angle A'O'B'$ . Further we denote this by  $\angle AOB = \angle A'O'B'$ .

**Definition 1.3.** Two ordinary lattice angles  $\angle AOB$  and  $\angle A'O'B'$  are said to be *lattice-congruent* if there exist a lattice-affine transformation which takes the point  $O$  to  $O'$  and the rays  $OA$  and  $OB$  to the rays  $O'A'$  and  $O'B'$  respectively. We denote this as follows:  $\angle AOB \cong \angle A'O'B'$ .

Here we note that the relation  $\angle AOB \cong \angle BOA$  holds only for special ordinary lattice angles. (See below in Subsubsection 1.2.4.)

1.2.1. *Definition of lattice sine, tangent, and cosine for an ordinary lattice angle.* Consider an arbitrary ordinary lattice angle  $\angle AOB$ . Let us associate a special basis to this angle. Denote by  $\bar{v}_1$  and by  $\bar{v}_2$  the lattice vectors generating the rays of the angle:

$$\bar{v}_1 = \frac{\overline{OA}}{\mathbb{ll}(OA)}, \quad \text{and} \quad \bar{v}_2 = \frac{\overline{OB}}{\mathbb{ll}(OB)}.$$

The set of lattice points on unit lattice distance from the lattice straight line  $OA$  coincides with the set of all lattice points of two lattice straight lines parallel to  $OA$ . Since the vectors  $\bar{v}_1$  and  $\bar{v}_2$  are linearly independent, the ray  $OB$  intersects exactly one of the above two lattice straight lines. Denote this straight line by  $l$ . The intersection point of the ray  $OB$  with the straight line  $l$  divides  $l$  onto two parts. Choose one of the parts which lies in the complement to the convex hull of the union of the rays  $OA$  and  $OB$ , and denote by  $D$  the lattice point closest to the intersection of the ray  $OB$  with the straight line  $l$  (see Figure 1).

Now we choose the vectors  $\bar{e}_1 = \bar{v}_1$  and  $\bar{e}_2 = \overline{OD}$ . These two vectors are linearly independent and generate the lattice. The basis  $(\bar{e}_1, \bar{e}_2)$  is said to be *associated* to the angle  $\angle AOB$ .

Since  $(\bar{e}_1, \bar{e}_2)$  is a basis, the vector  $\bar{v}_2$  has a unique representation of the form:

$$\bar{v}_2 = x_1 \bar{e}_1 + x_2 \bar{e}_2,$$

where  $x_1$  and  $x_2$  are some integers.

**Definition 1.4.** In the above notation, the coordinates  $x_2$  and  $x_1$  are said to be the *lattice sine* and the *lattice cosine* of the ordinary lattice angle  $\angle AOB$  respectively. The ratio of the lattice sine and the lattice cosine ( $x_2/x_1$ ) is said to be the *lattice tangent* of  $\angle AOB$ .

On Figure 1 we show an example of lattice angle with the lattice sine equals 7 and the lattice cosine equals 5.

Let us briefly enumerate some elementary properties of lattice trigonometric functions.

**Proposition 1.5. a).** *The lattice sine and cosine of any ordinary lattice angle are relatively-prime positive integers.*

**b).** *The values of lattice trigonometric functions for lattice-congruent ordinary lattice angles coincide.*

**c).** *The lattice sine of an ordinary lattice angle coincide with the index of the sublattice generated*

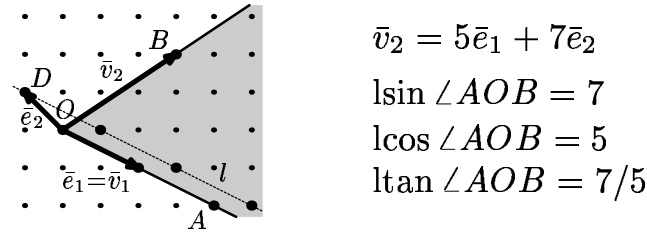


FIGURE 1. An ordinary lattice angle  $\angle AOB$  and its lattice trigonometric functions.

by all lattice vectors of two angle rays in the lattice.

d). For any ordinary lattice angle  $\alpha$  the following inequalities hold:

$$l\sin \alpha \geq l\cos \alpha, \quad \text{and} \quad l\tan \alpha \geq 1.$$

The equalities hold iff the lattice vectors of the angle rays generate the whole lattice.

e). **(Description of lattice angles.)** Two ordinary lattice angles  $\alpha$  and  $\beta$  are lattice-congruent iff  $l\tan \alpha = l\tan \beta$ . □

1.2.2. *Lattice arctangent.* Let us fix the origin  $O$  and a lattice basis  $\bar{e}_1$  and  $\bar{e}_2$ .

**Definition 1.6.** Consider an arbitrary rational  $p \geq 1$ . Let  $p = m/n$ , where  $m$  and  $n$  are positive integers. Suppose  $A = O + \bar{e}_1$ , and  $B = O + n\bar{e}_1 + m\bar{e}_2$ . The ordinary angle  $\angle AOB$  is said to be the *arctangent of  $p$  in the fixed basis* and denoted by  $larctan(p)$ .

The invariance of lattice tangents immediately implies the following properties.

**Proposition 1.7. a).** For any rational  $s \geq 1$ , we have:  $l\tan(larctan s) = s$ .

b). For any ordinary lattice angle  $\alpha$  the following holds:  $larctan(l\tan \alpha) \cong \alpha$ . □

1.2.3. *Lattice tangents, length-sine sequences, sails, and continued fractions.* Let us start with the notion of sails for the ordinary lattice angles. This notion is taken from theory of multidimensional continued fractions in the sense of Klein (see, for example, the works of F. Klein [14], and V. Arnold [1]).

Consider an ordinary lattice angle  $\angle AOB$ . Let also the vectors  $\overline{OA}$  and  $\overline{OB}$  be linearly independent, and of unit lattice length.

Denote the closed convex solid cone for the ordinary lattice angle  $\angle AOB$  by  $C(AOB)$ . The boundary of the convex hull of all lattice points of the cone  $C(AOB)$  except the origin is homeomorphic to the straight line. This boundary contains the points  $A$  and  $B$ . The closed part of this boundary contained between the points  $A$  and  $B$  is called the *sail* for the cone  $C(AOB)$ .

A lattice point of the sail is said to be a *vertex* of the sail if there is no lattice segment of the sail containing this point in the interior. The sail of the cone  $C(AOB)$  is a broken line with a finite number of vertices and without self intersections. Let us orient the sail in the direction from  $A$  to  $B$ , and denote the vertices of the sail by  $V_i$  (for  $0 \leq i \leq n$ ) according to the orientation of the sail (such that  $V_0 = A$ , and  $V_n = B$ ).

**Definition 1.8.** Let the vectors  $\overline{OA}$  and  $\overline{OB}$  of the ordinary lattice angle  $\angle AOB$  be linearly independent, and of unit lattice length. Let  $V_i$ , where  $0 \leq i \leq n$ , be the vertices of the

corresponding sail. The sequence of lattice lengths and sines

$$(\ell(V_0V_1), \text{lsin } \angle V_0V_1V_2, \ell(V_1V_2), \text{lsin } \angle V_1V_2V_3, \dots \\ \dots, \ell(V_{n-2}V_{n-1}), \text{lsin } \angle V_{n-2}V_{n-1}V_n, \ell(V_{n-1}V_n))$$

is called the *lattice length-sine* sequence for the ordinary lattice angle  $\angle AOB$ .

*Remark 1.9.* The elements of the lattice length-sine sequence for any ordinary lattice angle are positive integers. The lattice length-sine sequences of lattice-congruent ordinary lattice angles coincide.

**Theorem 1.10.** *Let  $(\ell(V_0V_1), \text{lsin } \angle V_0V_1V_2, \dots, \text{lsin } \angle V_{n-2}V_{n-1}V_n, \ell(V_{n-1}V_n))$  be the lattice length-sine sequence for the ordinary lattice angle  $\angle AOB$ . Then the lattice tangent of the ordinary lattice angle  $\angle AOB$  equals to the value of the following ordinary continued fraction*

$$[\ell(V_0V_1), \text{lsin } \angle V_0V_1V_2, \dots, \text{lsin } \angle V_{n-2}V_{n-1}V_n, \ell(V_{n-1}V_n)].$$

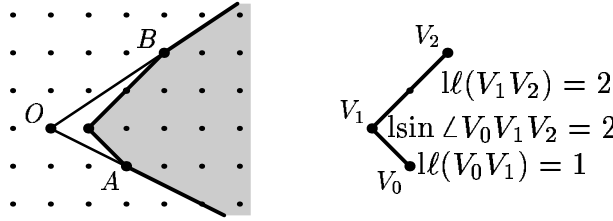


FIGURE 2.  $\text{ltan } \angle AOB = \frac{7}{5} = 1 + \frac{1}{2+1/2}$ .

On Figure 2 we show an example of an ordinary lattice angle with tangent equivalent to  $7/5$ .

Further in Theorem 3.5 we formulate and prove a general statement for generalized sails and signed length-sine sequences. In the proof of Theorem 3.5 we refer only on the preceding statements and definitions of Subsection 3.1, that are independent of the statements and theorems of all previous sections. For these reasons we skip now the proof of Theorem 1.10 (see also Remark 3.6).

1.2.4. *Adjacent, transpose, and opposite interior ordinary lattice angles.* First, we give the definition of the ordinary lattice angles transpose and adjacent to the given one.

**Definition 1.11.** An ordinary lattice angle  $\angle BOA$  is said to be *transpose* to the ordinary lattice angle  $\angle AOB$ . We denote it by  $(\angle AOB)^t$ .

An ordinary lattice angle  $\angle BOA'$  is said to be *adjacent* to an ordinary lattice angle  $\angle AOB$  if the points  $A$ ,  $O$ , and  $A'$  are contained in the same straight line, and the point  $O$  lies between  $A$  and  $A'$ . We denote the ordinary angle  $\angle BOA'$  by  $\pi - \angle AOB$ .

The ordinary lattice angle is said to be *right* if it is self-dual and lattice-congruent to the adjacent ordinary angle.

It immediately follows from the definition, that for any ordinary lattice angle  $\alpha$  the angles  $(\alpha^t)^t$  and  $\pi - (\pi - \alpha)$  are lattice-congruent to  $\alpha$ .

Further we will use the following notion. Suppose that some integers  $a$ ,  $b$  and  $c$ , where  $c \geq 1$ , satisfy the following:  $ab \equiv 1 \pmod{c}$ . Then we denote

$$a \equiv (b \pmod{c})^{-1}.$$



For the trigonometric functions of transpose and adjacent ordinary lattice angles the following relations are known.

**Theorem 1.12.** *Consider an ordinary lattice angle  $\alpha$ . If  $\alpha \cong \text{larctan}(1)$ , then*

$$\alpha^t \cong \pi - \alpha \cong \text{larctan}(1).$$

*Suppose now, that  $\alpha \not\cong \text{larctan}(1)$ , then*

$$\begin{aligned} \text{l sin}(\alpha^t) &= \text{l sin } \alpha, & \text{l cos}(\alpha^t) &\equiv (\text{l cos } \alpha \pmod{\text{l sin } \alpha})^{-1}; \\ \text{l sin}(\pi - \alpha) &= \text{l sin } \alpha, & \text{l cos}(\pi - \alpha) &\equiv (-\text{l cos } \alpha \pmod{\text{l sin } \alpha})^{-1}. \end{aligned}$$

*Note also that  $\pi - \alpha \cong \text{larctan}^t \left( \frac{\text{l tan } \alpha}{\text{l tan}(\alpha) - 1} \right)$ .* □

Theorem 1.12 (after applying Theorem 1.10) immediately reduces to the theorem of P. Popescu-Pampu. We refer the readers to his work [19] for the proofs.

1.2.5. *Right ordinary lattice angles.* It turns out that in lattice geometry there exist exactly two lattice non-equivalent right ordinary lattice angles.

**Corollary 1.13.** *Any ordinary lattice right angle is lattice-congruent to exactly one of the following two angles:  $\text{larctan}(1)$ , or  $\text{larctan}(2)$ .* □

**Definition 1.14.** Consider two lattice parallel distinct straight lines  $AB$  and  $CD$ , where  $A$ ,  $B$ ,  $C$ , and  $D$  are lattice points. Let the points  $A$  and  $D$  be in different open half-planes with respect to the straight line  $BC$ . Then the ordinary lattice angle  $\angle ABC$  is called *opposite interior* to the ordinary lattice angle  $\angle DCB$ .

Further we use the following proposition on opposite interior ordinary lattice angles.

**Proposition 1.15.** *Two opposite interior to each other ordinary lattice angles are lattice-congruent.* □

The proof is left for the reader as an exercise.

1.3. **Basic lattice trigonometry of lattice angles in lattice triangles.** In this subsection we introduce the sine formula for angles and edges of lattice triangles. Further we show how to find the lattice tangents of all angles and the lattice lengths of all edges of any lattice triangle, if the lattice lengths of two edges and the lattice tangent of the angle between them are given.

Let  $A$ ,  $B$ , and  $C$  be three distinct lattice points being not contained in the same straight line. We denote the lattice triangle with the vertices  $A$ ,  $B$ , and  $C$  by  $\triangle ABC$ .

**Definition 1.16.** The lattice triangles  $\triangle ABC$  and  $\triangle A'B'C'$  are said to be *lattice-congruent* if there exist a lattice-affine transformation which takes the point  $A$  to  $A'$ ,  $B$  to  $B'$ , and  $C$  to  $C'$  respectively. We denote:  $\triangle ABC \cong \triangle A'B'C'$ .

1.3.1. *The sine formula.* Let us introduce a lattice analog of the Euclidean sine formula for sines of angles and lengths of edges of triangles.

**Proposition 1.17. (The sine formula for lattice triangles.)** *The following holds for any lattice triangle  $\triangle ABC$ .*

$$\frac{\text{l}(AB)}{\text{l sin } \angle BCA} = \frac{\text{l}(BC)}{\text{l sin } \angle CAB} = \frac{\text{l}(CA)}{\text{l sin } \angle ABC} = \frac{\text{l}(AB) \text{l}(BC) \text{l}(CA)}{\text{lS}(\triangle ABC)}.$$

*Proof.* The statement of Proposition 1.17 follows directly from the definition of lattice sine. □

1.3.2. *On relation between lattice tangents of ordinary lattice angles and lattice lengths of lattice triangles.* Suppose that we know the lattice lengths of the edges  $AB$ ,  $AC$  and the lattice tangent of  $\angle BAC$  in the triangle  $\triangle ABC$ . Now we show how to restore the lattice length and the lattice tangents for the the remaining edge and ordinary angles of the triangle.

For the simplicity we fix some lattice basis and use the system of coordinates  $OXY$  corresponding to this basis (denoted  $(*, *)$ ).

**Theorem 1.18.** *Consider some triangle  $\triangle ABC$ . Let*

$$\ell(AB) = c, \quad \ell(AC) = b, \quad \text{and} \quad \angle CAB \cong \alpha.$$

*Then the ordinary angles  $\angle BCA$  and  $\angle ABC$  are defined in the following way.*

$$\angle BCA \cong \begin{cases} \text{larctan} \left( \pi - \frac{c \text{lsin } \alpha}{c \text{lcos } \alpha - b} \right) & \text{if } c \text{lcos } \alpha > b \\ \text{larctan}(1) & \text{if } c \text{lcos } \alpha = b \\ \text{larctan}^t \left( \frac{c \text{lsin } \alpha}{b - c \text{lcos } \alpha} \right) & \text{if } c \text{lcos } \alpha < b \end{cases},$$

$$\angle ABC \cong \begin{cases} \text{larctan}^t \left( \pi - \frac{b \text{lsin}(\alpha^t)}{b \text{lcos}(\alpha^t) - c} \right) & \text{if } b \text{lcos}(\alpha^t) > c \\ \text{larctan}(1) & \text{if } b \text{lcos}(\alpha^t) = c \\ \text{larctan} \left( \frac{b \text{lsin}(\alpha^t)}{c - b \text{lcos}(\alpha^t)} \right) & \text{if } b \text{lcos}(\alpha^t) < c \end{cases}.$$

*For the lattice length of the edge  $CB$  we have*

$$\frac{\ell(CB)}{\text{lsin } \alpha} = \frac{b}{\text{lsin } \angle ABC} = \frac{c}{\text{lsin } \angle BCA}.$$

*Proof.* Let  $\alpha \cong \text{larctan}(p/q)$ , where  $\gcd(p, q) = 1$ . Then  $\triangle CAB \cong \triangle DOE$  where  $D = (b, 0)$ ,  $O = (0, 0)$ , and  $E = (qc, pc)$ . Let us now find the ordinary lattice angle  $\angle DEO$ . Denote by  $Q$  the point  $(qc, 0)$ . If  $qc - b = 0$ , then  $\angle BCA = \angle DEO = \text{larctan } 1$ . If  $qc - b \neq 0$ , then we have

$$\angle QDE \cong \text{larctan} \left( \frac{cp}{cq - b} \right) \cong \text{larctan} \left( \frac{c \text{lsin } \alpha}{c \text{lcos } \alpha - b} \right).$$

The expression for  $\angle BCA$  follows directly from the above expression for  $\angle QDE$ , since  $\angle BCA \cong \angle QDE$ . (See Figure 3: here  $\ell(OD) = b$ ,  $\ell(OQ) = c \text{lcos } \alpha$ , and therefore  $\ell(DQ) = |c \text{lcos } \alpha - b|$ .)

To obtain the expression for  $\angle ABC$  we consider the triangle  $\triangle BAC$ . Calculate  $\angle CBA$  and then transpose all ordinary angles in the expression. Since

$$\begin{aligned} \text{IS}(ABC) &= \ell(AB) \ell(AC) \text{lsin } \angle CAB = \ell(BA) \ell(BC) \text{lsin } \angle BCA = \\ &= \ell(CB) \ell(CA) \text{lsin } \angle ABC, \end{aligned}$$

we have the last statement of the theorem. □

## 2. THEOREM ON SUM OF LATTICE TANGENTS FOR THE ORDINARY LATTICE ANGLES OF LATTICE TRIANGLES. PROOF OF ITS SECOND STATEMENT.

In this section we introduce the sum formula for the lattice tangents of ordinary lattice angles of lattice triangles. The sum formula is a lattice version of the following Euclidean statement: three angles are the angles of some triangle iff their sum equals  $\pi$ .

Throughout this section we fix some lattice basis and use the system of coordinates  $OXY$  corresponding to this basis.

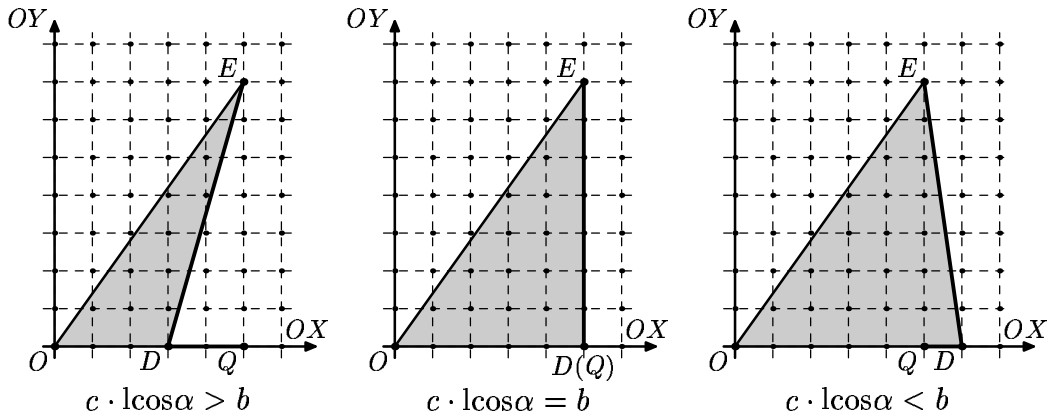


FIGURE 3. Three possible configuration of points  $O$ ,  $D$ , and  $Q$ .

**2.1. Finite continued fractions with not necessary positive elements.** We start this section with the commonly-used definition of finite continued fractions with not necessary positive elements. Let us expand the set of rationals  $\mathbb{Q}$  with the operations  $+$  and  $1/*$  on it with the element  $\infty$ . We pose  $q \pm \infty = \infty$ ,  $1/0 = \infty$ ,  $1/\infty = 0$  (we do not define  $\infty \pm \infty$  here). Denote this expansion by  $\overline{\mathbb{Q}}$ .

For any finite sequence of integers  $(a_0, a_1, \dots, a_n)$  we associate an element  $q$  of  $\overline{\mathbb{Q}}$ :

$$q = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

and denote it by  $]a_0, a_1, \dots, a_n[$ .

Let  $q_i$  be some rationals,  $i = 1, \dots, k$ . Suppose that the odd continued fraction for  $q_i$  is  $]a_{i,0}, a_{i,1}, \dots, a_{i,2n_i}[$  for  $i = 1, \dots, k$ . We denote by  $]q_1, q_2, \dots, q_n[$  the following number

$$]a_{1,0}, a_{1,1}, \dots, a_{1,2n_1}, a_{2,0}, a_{2,1}, \dots, a_{2,2n_2}, \dots, a_{k-1,0}, a_{k-1,1}, \dots, a_{k-1,2n_{k-1}}, a_{k,0}, a_{k,1}, \dots, a_{k,2n_k}[$$

**2.2. Formulation of the theorem and proof of its second statement.** In Euclidean geometry the sum of Euclidean angles of the triangle equals  $\pi$ . For any 3-tuple of angles with the sum equals  $\pi$  there exist a triangle with these angles. Two Euclidean triangles with the same angles are homothetic. Let us show one generalization of these statements to the case of lattice geometry.

We start this subsection with a definition of convex polygons  $n$ -tuple to other convex polygons.

Let  $n$  be an arbitrary positive integer, and  $A = (x, y)$  be an arbitrary lattice point. Denote by  $nA$  the point  $(nx, ny)$ .

**Definition 2.1.** Consider any convex polygon or broken line with vertices  $A_0, \dots, A_k$ . The polygon or broken line  $nA_0 \dots nA_k$  is called  $n$ -multiple (or multiple) to the given polygon or broken line.

**Theorem 2.2.** **On sum of lattice tangents of ordinary lattice angles of lattice triangles.**

**a).** Let  $(\alpha_1, \alpha_2, \alpha_3)$  be an ordered 3-tuple of ordinary lattice angles. There exist a triangle with three consecutive ordinary angles lattice-congruent to  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  iff there exist  $i \in \{1, 2, 3\}$  such that the angles  $\alpha = \alpha_i$ ,  $\beta = \alpha_{i+1(\text{mod } 3)}$ , and  $\gamma = \alpha_{i+2(\text{mod } 3)}$  satisfy the following conditions:

- i) the rational  $]\tan \alpha, -1, \tan \beta[$  is either negative or greater than  $\tan \alpha$ ;
- ii)  $]\tan \alpha, -1, \tan \beta, -1, \tan \gamma[ = 0$ .

**b).** Let the consecutive ordinary lattice angles of some triangle be  $\alpha$ ,  $\beta$ , and  $\gamma$ . Then this triangle is multiple to the triangle with vertices  $A_0 = (0, 0)$ ,  $B_0 = (\lambda_2 \cos \alpha, \lambda_2 \sin \alpha)$ , and  $C_0 = (\lambda_1, 0)$ , where

$$\lambda_1 = \frac{\text{lcm}(\sin \alpha, \sin \beta, \sin \gamma)}{\text{gcd}(\sin \alpha, \sin \gamma)}, \quad \text{and} \quad \lambda_2 = \frac{\text{lcm}(\sin \alpha, \sin \beta, \sin \gamma)}{\text{gcd}(\sin \alpha, \sin \beta)}.$$

*Remark 2.3.* Note that the statement of the Theorem 2.2a holds only for odd continued fractions for the tangents of the correspondent angles. We illustrate this with the following example. Consider a lattice triangle with the lattice area equals 7 and all angles lattice-congruent to  $\arctan 7/3$  (see on Figure 16). If we take the odd continued fractions  $7/3 = [2, 2, 1]$  for all lattice angles of the triangle, then we have

$$]2, 2, 1, -1, 2, 2, 1, -1, 2, 2, 1[ = 0.$$

If we take the even continued fractions  $7/3 = [2, 3]$  for all angles of the triangle, then we have

$$]2, 3, -1, 2, 3, -1, 2, 3[ = \frac{35}{13} \neq 0.$$

We prove the first statement of this theorem further in Subsections 4.2 and 4.3 after making definitions of expanded lattice angles and their sums, and introducing their properties.

*Proof of the second statement of Theorem 2.2.* Consider the triangle  $\triangle ABC$  with the ordinary lattice angles  $\alpha$ ,  $\beta$ , and  $\gamma$  (with vertices at  $A$ ,  $B$ , and  $C$  respectively). Suppose that for any  $k > 1$  and any lattice triangle  $\triangle KLM$  the triangle  $\triangle ABC$  is not lattice-congruent to the  $k$ -multiple of  $\triangle KLM$ . That is equivalent to the following

$$\text{gcd}(\ell(AB), \ell(BC), \ell(CA)) = 1.$$

Suppose that  $S$  is the lattice area of  $\triangle ABC$ . Then by the sine formula the following holds

$$\begin{cases} \ell(AB)\ell(AC) = S/\sin \alpha \\ \ell(BC)\ell(BA) = S/\sin \beta \\ \ell(CA)\ell(CB) = S/\sin \gamma \end{cases}.$$

Since  $\text{gcd}(\ell(AB), \ell(BC), \ell(CA)) = 1$ , we have  $\ell(AB) = \lambda_1$  and  $\ell(AC) = \lambda_2$ .

Therefore, the lattice triangle  $\triangle ABC$  is lattice-congruent to the lattice triangle  $\triangle A_0 B_0 C_0$  of the theorem.  $\square$

### 3. EXPANSION OF ORDINARY LATTICE ANGLES. THE NOTIONS OF SUMS FOR LATTICE ANGLES.

In this section we introduce the notion of expanded lattice angles and their normal forms and give the definition of sums of expanded lattice angles. From the definition of sums of expanded lattice angles we obtain a definition of sums of ordinary lattice angles. For the definition of expanded lattice angles we expand the notion of sails in the sense of Klein: we define and study oriented broken lines on the unit distance from lattice points.

Throughout this section we work in the oriented two-dimensional real vector space with a fixed lattice. We again fix some lattice (positively oriented) basis and use the system of coordinates  $OXY$  corresponding to this basis.

The lattice-affine transformation is said to be *proper* if it is orientation-preserving. We say that two sets are *proper lattice-congruent* to each other if there exist a proper lattice-affine transformation of  $\mathbb{R}^2$  taking the first set to the second.

**3.1. On a particular generalization of sails in the sense of Klein.** In this subsection we introduce the definition of an oriented broken lines on the unit lattice distance from a lattice point. This notion is a direct generalization of the notion of a sail in the sense of Klein (see page 7 for the definition of a sail). We expand the definition of lattice length-sine sequences and continued fractions to the case of these broken lines. We show that expanded lattice length-sine sequence for oriented broken lines uniquely identifies the proper lattice-congruence class of the corresponding broken line. Further, we study the geometrical interpretation of the corresponding continued fraction.

**3.1.1. Definition of a lattice signed length-sine sequence.** For the definition of expanded lattice angles we expand the definition of lattice length-sine sequence to the case of certain broken lines.

For the 3-tuples of lattice points  $A, B,$  and  $C$  we define the function  $\text{sgn}$  as follows:

$$\text{sgn}(ABC) = \begin{cases} +1, & \text{if the couple of vectors } \overline{BA} \text{ and } \overline{BC} \text{ defines the positive} \\ & \text{orientation.} \\ 0, & \text{if the points } A, B, \text{ and } C \text{ are contained in the same straight line.} \\ -1, & \text{if the couple of vectors } \overline{BA} \text{ and } \overline{BC} \text{ defines the negative} \\ & \text{orientation.} \end{cases}$$

We also denote by  $\text{sign} : \mathbb{R} \rightarrow \{-1, 0, 1\}$  the sign function over reals.

The segment  $AB$  is said to be *on the unit distance* from the point  $C$  if the lattice vectors of the segment  $AB$ , and the vector  $\overline{AC}$  generate the lattice.

A union of (ordered) lattice segments  $A_0A_1, A_1A_2, \dots, A_{n-1}A_n$  ( $n > 0$ ) is said to be a *lattice oriented broken line* and denoted by  $A_0A_1A_2 \dots A_n$  if any two consecutive segments are not contained in the same straight line. We also say that the lattice oriented broken line  $A_nA_{n-1}A_{n-2} \dots A_0$  is *inverse* to the lattice oriented broken line  $A_0A_1A_2 \dots A_n$ .

**Definition 3.1.** Consider a lattice oriented broken line and a lattice point in the complement to this line. The broken line is said to be *on the unit distance* from the point if all edges of the broken line are on the unit lattice distance from the point.

Let us now associate to any lattice oriented broken line on the unit distance from some point the following sequence of non-zero elements.

**Definition 3.2.** Let  $A_0A_1 \dots A_n$  be a lattice oriented broken line on the unit distance from some lattice point  $V$ . The sequence  $(a_0, \dots, a_{2n-2})$  defined as follows:

$$\begin{aligned} a_0 &= \text{sgn}(A_0VA_1) \ell(A_0A_1), \\ a_1 &= \text{sgn}(A_0VA_1) \text{sgn}(A_1VA_2) \text{sgn}(A_0A_1A_2) \text{lsin} \angle A_0A_1A_2, \\ a_2 &= \text{sgn}(A_1VA_2) \ell(A_1A_2), \\ &\dots \\ a_{2n-3} &= \text{sgn}(A_{n-2}VA_{n-1}) \text{sgn}(A_{n-1}VA_n) \text{sgn}(A_{n-2}A_{n-1}A_n) \text{lsin} \angle A_{n-2}A_{n-1}A_n, \\ a_{2n-2} &= \text{sgn}(A_{n-1}VA_n) \ell(A_{n-1}A_n), \end{aligned}$$

is called a *lattice signed length-sine* sequence for the lattice oriented broken line on the unit distance from the lattice point  $V$ .

The element of  $\mathbb{Q}$

$$]a_0, a_1, \dots, a_{2n-2}[$$

is called the *continued fraction for the broken line*  $A_0A_1 \dots A_n$ .

We show how to identify signs of elements of the lattice signed length-sine sequence for a lattice oriented broken line on the unit distance from the lattice point  $V$  on Figure 4.

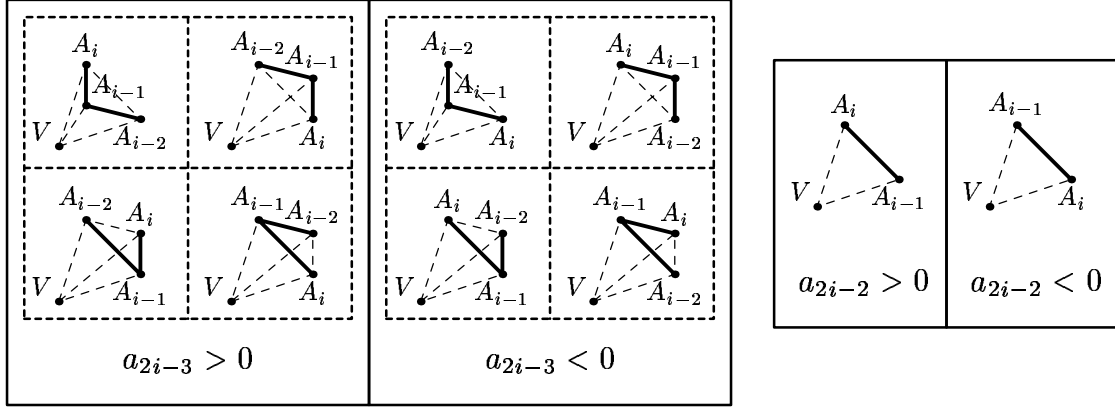


FIGURE 4. All possible (non-degenerate) proper affine decompositions for angles and segments of a signed length-sine sequence.

On Figure 5 we show an example of lattice oriented broken line on the unit distance from the lattice point  $V$  and the corresponding lattice signed length-sine sequence.

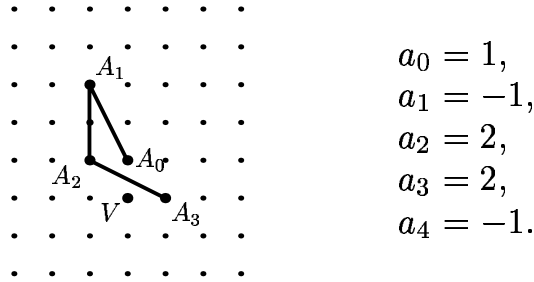


FIGURE 5. A lattice oriented broken line on the unit distance from the point  $V$  and the corresponding lattice signed length-sine sequence.

**Proposition 3.3.** *A lattice signed length-sine sequence for the given lattice oriented broken line and the lattice point is invariant under the group action of the proper lattice-affine transformations.*

*Proof.* The statement of the proposition holds, since the functions  $\text{sgn}$ ,  $l$ , and  $\text{lsin}$  are invariant under the group action of the proper lattice-affine transformations.  $\square$

3.1.2. *Proper lattice-congruence of lattice oriented broken lines on the unit distance from lattice points.* Let us formulate necessary and sufficient conditions for two lattice oriented broken lines on the unit distance from the same lattice point to be proper lattice-congruent.

**Theorem 3.4.** *The lattice signed length-sine sequences of two lattice oriented broken lines on the unit distance from lattice points  $V_1$  and  $V_2$  coincide iff there exist a proper lattice-affine transformation taking the point  $V_1$  to  $V_2$  and one lattice oriented broken line to the other.*

*Proof.* The lattice signed length-sine sequence for any lattice oriented broken line on the unit distance is uniquely defined, and by Proposition 3.3 is invariant under the group action of lattice-affine orientation preserving transformations. Therefore, the lattice signed length-sine sequences for two proper lattice-congruent lattice oriented broken lines coincide.

Suppose now that two lattice oriented broken lines  $A_0 \dots A_n$ , and  $B_0 \dots B_n$  on the unit distance from lattice points  $V_1$  and  $V_2$  respectively have the same lattice signed length-sine sequence  $(a_0, a_1, \dots, a_{2n-3}, a_{2n-2})$ . Let us prove that these broken lines are proper lattice-congruent. Without loose of generality we consider the point  $V_1$  at the origin  $O$ .

Let  $\xi$  be the proper lattice-affine transformation taking the point  $V_2$  to the point  $V_1 = O$ ,  $B_0$  to  $A_0$ , and the lattice straight line containing  $B_0B_1$  to the lattice straight line containing  $A_0A_1$ . Let us prove inductively that  $\xi(B_i) = A_i$ .

*Base of induction.* Since  $a_0 = b_0$ , we have

$$\text{sgn}(A_0OA_1)\ell(A_0A_1) = \text{sgn}(\xi(B_0)O\xi(B_1))\ell(\xi(B_0)\xi(B_1)).$$

Thus, the lattice segments  $A_0A_1$  and  $A_0\xi(B_1)$  are of the same lattice length and of the same direction. Therefore,  $\xi(B_1) = A_1$ .

*Step of induction.* Suppose, that  $\xi(B_i) = A_i$  holds for any nonnegative integer  $i \leq k$ , where  $k \geq 1$ . Let us prove, that  $\xi(B_{k+1}) = A_{k+1}$ . Denote by  $C_{k+1}$  the lattice point  $\xi(B_{k+1})$ . Let  $A_k = (q_k, p_k)$ . Denote by  $A'_k$  the closest lattice point of the segment  $A_{k-1}A_k$  to the vertex  $A_k$ . Suppose that  $A'_k = (q'_k, p'_k)$ . We know also

$$\begin{aligned} a_{2k-1} &= \text{sgn}(A_{k-1}OA_k) \text{sgn}(A_kOC_{k+1}) \text{sgn}(A_{k-1}A_kC_{k+1}) \text{lsin} \angle A_{k-1}A_kC_{k+1}, \\ a_{2k} &= \text{sgn}(A_kOC_{k+1}) \ell(A_kC_{k+1}). \end{aligned}$$

Let the coordinates of  $C_{k+1}$  be  $(x, y)$ . Since  $\ell(A_kC_{k+1}) = |a_{2k}|$  and the segment  $A_kC_{k+1}$  is on the unit lattice distance to the origin  $O$ , we have  $\text{IS}(\triangle OA_kC_{k+1}) = |a_{2k}|$ . Since the segment  $OA_k$  is of the unit lattice length, the coordinates of  $C_{k+1}$  satisfy the following equation:

$$|-p_kx + q_ky| = |a_{2k}|.$$

Since  $\text{sgn}(A_kOC_{k+1})\ell(A_kC_{k+1}) = \text{sign}(a_{2k})$ , we have

$$-p_kx + q_ky = a_{2k}.$$

Since  $\text{lsin} \angle A'_kA_kC_{k+1} = \text{lsin} \angle A_{k-1}A_kC_{k+1} = |a_{2k-1}|$ , and the lattice lengths of  $A_kC_{k+1}$ , and  $A'_kA_k$  are  $|a_{2k}|$  and 1 respectively, we have  $\text{IS}(\triangle A'_kA_kC_{k+1}) = |a_{2k-1}a_{2k}|$ . Therefore, the coordinates of  $C_{k+1}$  satisfy the following equation:

$$|-(p_k - p'_k)(x - q_k) + (q_k - q'_k)(y - p_k)| = |a_{2k-1}a_{2k}|.$$

Since

$$\begin{cases} \text{sgn}(A_{k-1}OA_k) \text{sgn}(A_kOC_{k+1}) \text{sgn}(A_{k-1}A_kC_{k+1}) &= \text{sign}(a_{2k-1}) \\ \text{sgn}(A_kOC_{k+1}) &= \text{sign}(a_{2k}) \end{cases},$$

we have

$$(p_k - p'_k)(x - q_k) - (q_k - q'_k)(y - p_k) = \operatorname{sgn}(A_{k-1}OA_k)a_{2k-1}a_{2k}.$$

We obtain the following:

$$\begin{cases} -p_kx + q_ky = a_{2k} \\ (p_k - p'_k)(x - q_k) - (q_k - q'_k)(y - p_k) = \operatorname{sgn}(A_{k-1}OA_k)a_{2k-1}a_{2k} \end{cases}.$$

Since

$$\left| \det \begin{pmatrix} -p_k & q_k \\ p'_k - p_k & q_k - q'_k \end{pmatrix} \right| = 1,$$

there exist and unique an integer solution for the system of equations for  $x$  and  $y$ . Hence, the points  $A_{k+1}$  and  $C_{k+1}$  have the same coordinates. Therefore,  $\xi(B_{k+1}) = A_{k+1}$ . We have proven the step of induction.

The proof of Theorem 3.4 is completed by induction.  $\square$

**3.1.3. Values of continued fractions for lattice oriented broken lines on the unit distance from the origin.** Now we show the relation between lattice oriented broken lines on the unit distance from the origin and the corresponding continued fractions for them.

**Theorem 3.5.** *Let  $A_0A_1 \dots A_n$  be a lattice oriented broken line on the unit distance from the origin  $O$ . Let also  $A_0 = (1, 0)$ ,  $A_1 = (1, a_0)$ ,  $A_n = (p, q)$ , where  $\gcd(p, q) = 1$ , and  $(a_0, a_1, \dots, a_{2n-2})$  be the corresponding lattice signed length-sine sequence. Then the following holds:*

$$\frac{q}{p} = ]a_0, a_1, \dots, a_{2n-2}].$$

*Proof.* To prove this theorem we use an induction on the number of edges of the broken lines.

*Base of induction.* Suppose that a lattice oriented broken line on the unit distance from the origin has a unique edge, and the corresponding sequence is  $(a_0)$ . Then  $A_1 = (1, a_0)$  by the assumptions of the theorem. Therefore, we have  $\frac{a_0}{1} = ]a_0[$ .

*Step of induction.* Suppose that the statement of the theorem is correct for any lattice oriented broken line on the unit distance from the origin with  $k$  edges. Let us prove the theorem for the arbitrary lattice oriented broken line on the unit distance from the origin with  $k+1$  edges (and satisfying the conditions of the theorem).

Let  $A_0 \dots A_{k+1}$  be a lattice oriented broken line on the unit distance from the origin with the lattice signed length-sine sequence  $(a_0, a_1, \dots, a_{2k-1}, a_{2k})$ . Let also

$$A_0 = (1, 0), \quad A_1 = (1, a_0), \quad \text{and} \quad A_{k+1} = (p, q).$$

Consider the lattice oriented broken line  $B_1 \dots B_{k+1}$  on the unit distance from the origin with a lattice signed length-sine sequence  $(a_2, a_3, \dots, a_{2k-2}, a_{2k})$ . Let also

$$B_1 = (1, 0), \quad B_2 = (1, a_2), \quad \text{and} \quad B_{k+1} = (p', q').$$

By the induction assumption we have

$$\frac{q'}{p'} = ]a_2, a_3, \dots, a_{2k}[.$$



We extend the lattice oriented broken line  $B_1 \dots B_{k+1}$  to the lattice oriented broken line  $B_0 B_1 \dots B_{k+1}$  on the unit distance from the origin, where  $B_0 = (1+a_0 a_1, -a_0)$ . Let the lattice signed length-sine sequence for this broken line be  $(b_0, b_1, \dots, b_{2k-1}, b_{2k})$ . Note that

$$\begin{aligned} b_0 &= \operatorname{sgn}(B_0 O B_1) \ell(B_0 B_1) = \operatorname{sign}(a_0) |a_0| = a_0, \\ b_1 &= \operatorname{sgn}(B_0 O B_1) \operatorname{sgn}(B_1 O B_2) \operatorname{sgn}(B_0 B_1 B_2) \operatorname{l} \sin \angle B_0 B_1 B_2 = \\ &\quad \operatorname{sign} a_0 \operatorname{sign} b_2 \operatorname{sign}(a_0 a_1 b_2) |a_1| = a_1, \\ b_l &= a_l, \quad \text{for } l = 2, \dots, 2k. \end{aligned}$$

Consider a proper lattice-linear transformation  $\xi$  that takes the point  $B_0$  to the point  $(1, 0)$ , and  $B_1$  to  $(1, a_0)$ . These two conditions uniquely defines  $\xi$ :

$$\xi = \begin{pmatrix} 1 & a_1 \\ a_0 & 1 + a_0 a_1 \end{pmatrix}.$$

Since  $B_{k+1} = (p', q')$ , we have  $\xi(B_{k+1}) = (p' + a_1 q', q' a_0 + p' + p' a_0 a_1)$ .

$$\frac{q' a_1 + p' + p' a_0 a_1}{p' + a_1 q'} = a_0 + \frac{1}{a_1 + q'/p'} = ]a_0, a_1, a_2, a_3, \dots, a_{2n}[.$$

Since, by Theorem 3.4 the lattice oriented broken lines  $B_0 B_1 \dots B_{k+1}$  and  $A_0 A_1 \dots A_{k+1}$  are lattice-linear equivalent,  $B_0 = A_0$ , and  $B_1 = A_1$ , these broken lines coincide. Therefore, for the coordinates  $(p, q)$  the following hold

$$\frac{q}{p} = \frac{q' a_0 + p' + p' a_0 a_1}{p' + a_1 q'} = ]a_0, a_1, a_2, a_3, \dots, a_{2k}[.$$

On Figure 6 we illustrate the step of induction with an example of lattice oriented broken line on the unit distance from the origin with the lattice signed length-sine sequence:  $(1, -1, 2, 2, -1)$ . We start (the left picture) from the broken line  $B_1 B_2 B_3$  with the lattice signed length-sine sequence:  $(2, 2, -1)$ . Then, (the picture in the middle) we expand the broken line  $B_1 B_2 B_3$  to the broken line  $B_0 B_1 B_2 B_3$  with the lattice signed length-sine sequence:  $(1, -1, 2, 2, -1)$ . Finally (the right picture) we apply a corresponding proper lattice-linear transformation  $\xi$  to achieve the resulting broken line  $A_0 A_1 A_2 A_3$ .

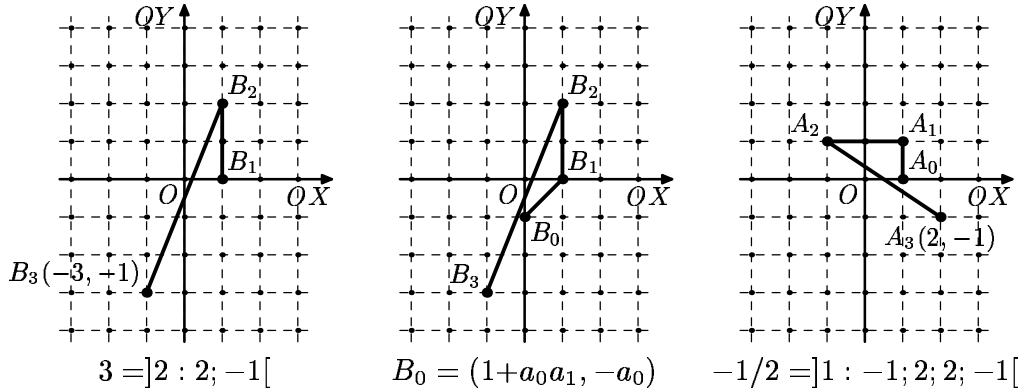


FIGURE 6. The case of lattice oriented broken line on the unit distance from the origin with lattice signed length-sine sequence:  $(1, -1, 2, 2, -1)$ .

We have proven the step of induction.

The proof of Theorem 3.5 is completed. □

*Remark 3.6.* Theorem 3.5 immediately implies the statement of Theorem 1.10. One should put the sail of an angle as an oriented-broken line  $A_0A_1 \dots A_n$ .

**3.2. Expanded lattice angles. Sums for ordinary and expanded lattice angles.** In this subsection we introduce the notion of expanded lattice angles and their normal forms. We use normal forms of expanded lattice angles to give the definition of sums of expanded lattice angles. From the definition of sums of expanded lattice angles we obtain a definition of sums of ordinary lattice angles.

**3.2.1. Equivalence classes of lattice oriented broken lines and the corresponding expanded lattice angles.** Let us give a definition of expanded lattice angles.

**Definition 3.7.** Two lattice oriented broken lines  $l_1$  and  $l_2$  on the unit distance from  $V$  are said to be *equivalent* if they have in common the first and the last vertices and the closed broken line generated by  $l_1$  and the inverse of  $l_2$  is homotopy equivalent to the point in  $\mathbb{R}^2 \setminus \{V\}$ .

An equivalence class of lattice oriented broken lines on the unit distance from  $V$  containing the broken line  $A_0A_1 \dots A_n$  is called the *expanded lattice angle for the equivalence class of  $A_0A_1 \dots A_n$  at the vertex  $V$*  (or, for short, *expanded lattice angle*) and denoted by  $\angle(V, A_0A_1 \dots A_n)$ .

We study the expanded lattice angles up to proper lattice-congruence (and not up to lattice-congruence).

**Definition 3.8.** Two expanded lattice angles  $\Phi_1$  and  $\Phi_2$  are said to be *proper lattice-congruent* iff there exist a proper lattice-affine transformation sending the class of lattice oriented broken lines corresponding to  $\Phi_1$  to the class of lattice oriented broken lines corresponding to  $\Phi_2$ . We denote this by  $\Phi_1 \cong \Phi_2$ .

**3.2.2. Revolution numbers for expanded lattice angles.** Let us describe one invariant of expanded lattice angles under the group action of the proper lattice-affine transformations.

Let  $r = \{V + \lambda \bar{v} \mid \lambda \geq 0\}$  be the oriented ray for an arbitrary vector  $\bar{v}$  with the vertex at  $V$ , and  $AB$  be an oriented (from  $A$  to  $B$ ) segment not contained in the ray  $r$ . Suppose also, that the vertex  $V$  of the ray  $r$  is not contained in the segment  $AB$ . We denote by  $\#(r, V, AB)$  the following number:

$$\#(r, V, AB) = \begin{cases} 0, & \text{if the segment } AB \text{ does not intersect the ray } r \\ \frac{1}{2} \operatorname{sgn}(A(A+\bar{v})B), & \text{if the segment } AB \text{ intersects the ray } r \text{ at } A \text{ or} \\ & \text{at } B \\ \operatorname{sgn}(A(A+\bar{v})B), & \text{if the segment } AB \text{ intersects the ray } r \text{ at the} \\ & \text{interior point of } AB \end{cases},$$

and call it the *intersection number* of the ray  $r$  and the segment  $AB$ .

**Definition 3.9.** Let  $A_0A_1 \dots A_n$  be some lattice oriented broken line, and let  $r$  be an oriented ray  $\{V + \lambda \bar{v} \mid \lambda \geq 0\}$ . Suppose that the ray  $r$  does not contain the edges of the broken line, and the broken line does not contain the point  $V$ . We call the number

$$\sum_{i=1}^n \#(r, V, A_{i-1}A_i)$$

the *intersection number* of the ray  $r$  and the lattice oriented broken line  $A_0A_1 \dots A_n$ , and denote it by  $\#(r, V, A_0A_1 \dots A_n)$ .

**Definition 3.10.** Consider an arbitrary expanded lattice angle  $\angle(V, A_0A_1 \dots A_n)$ . Denote the rays  $\{V + \lambda\overline{VA_0} \mid \lambda \geq 0\}$  and  $\{V - \lambda\overline{VA_0} \mid \lambda \geq 0\}$  by  $r_+$  and  $r_-$  respectively. The number

$$\frac{1}{2}(\#(r_+, V, A_0A_1 \dots A_n) + \#(r_-, V, A_0A_1 \dots A_n))$$

is called the *lattice revolution number* for the expanded lattice angle  $\angle(V, A_0A_1 \dots A_n)$ , and denoted by  $\#(\angle(V, A_0A_1 \dots A_n))$ .

Let us prove that the above definition is correct.

**Proposition 3.11.** *The revolution number of any expanded lattice angle is well-defined.*

*Proof.* Consider an arbitrary expanded lattice angle  $\angle(V, A_0A_1 \dots A_n)$ . Let

$$r_+ = \{V + \lambda\overline{VA_0} \mid \lambda \geq 0\} \quad \text{and} \quad r_- = \{V - \lambda\overline{VA_0} \mid \lambda \geq 0\}.$$

Since the lattice oriented broken line  $A_0A_1 \dots A_n$  is on the unit lattice distance from the point  $V$ , any segment of this broken line is on the unit lattice distance from  $V$ . Thus, the broken line does not contain  $V$ , and the rays  $r_+$  and  $r_-$  do not contain edges of the curve.

Suppose that

$$\angle(V, A_0A_1 \dots A_n) = \angle(V', A'_0A'_1 \dots A'_m).$$

This implies that  $V = V'$ ,  $A_0 = A'_0$ ,  $A_n = A'_m$ , and the broken line

$$A_0A_1 \dots A_nA'_{m-1} \dots A'_1A'_0$$

is homotopy equivalent to the point in  $\mathbb{R}^2 \setminus \{V\}$ . Thus,

$$\begin{aligned} & \#(\angle(V, A_0A_1 \dots A_n)) - \#(\angle(V', A'_0A'_1 \dots A'_m)) = \\ & \frac{1}{2}(\#(r_+, V, A_0A_1 \dots A_nA'_{m-1} \dots A'_1A'_0) + \#(r_-, V, A_0A_1 \dots A_nA'_{m-1} \dots A'_1A'_0)) = \\ & 0+0 = 0. \end{aligned}$$

Hence,

$$\#(\angle(V, A_0A_1 \dots A_n)) = \#(\angle(V', A'_0A'_1 \dots A'_m)).$$

Therefore, the revolution number of any expanded lattice angle is well-defined.  $\square$

**Proposition 3.12.** *The revolution number of expanded lattice angles is invariant under the group action of the proper lattice-affine transformations.*  $\square$

**3.2.3. Zero ordinary angles.** For the next theorem we will need to define zero ordinary angles and their trigonometric functions. Let  $A$ ,  $B$ , and  $C$  be three lattice points of the same lattice straight line. Suppose that  $B$  is distinct to  $A$  and  $C$  and the rays  $BA$  and  $BC$  coincide. We say that the ordinary lattice angle with the vertex at  $B$  and the rays  $BA$  and  $BC$  is *zero*. Suppose  $\angle ABC$  is zero, put by definition

$$\text{l sin}(\angle ABC) = 0, \quad \text{l cos}(\angle ABC) = 1, \quad \text{l tan}(\angle ABC) = 0.$$

Denote by  $\text{larctan}(0)$  the angle  $\angle AOA$  where  $A = (1, 0)$ , and  $O$  is the origin.

3.2.4. *On normal forms of expanded lattice angles.* Let us formulate and prove a theorem on normal forms of expanded lattice angles. We use the following notation.

By the sequence

$$((a_0, \dots, a_n) \times k\text{-times}, b_0, \dots, b_m),$$

where  $k \geq 0$ , we denote the following sequence:

$$\underbrace{(a_0, \dots, a_n, a_0, \dots, a_n, \dots, a_0, \dots, a_n, b_0, \dots, b_m)}_{k\text{-times}}.$$

**Definition 3.13. I).** Suppose  $O$  be the origin,  $A_0$  be the point  $(1, 0)$ . We say that the expanded lattice angle  $\angle(O, A_0)$  is *of the type I* and denote it by  $0\pi + \text{larctan}(0)$  (or  $0$ , for short). The empty sequence is said to be *characteristic* for the angle  $0\pi + \text{larctan}(0)$ .

Consider a lattice oriented broken line  $A_0A_1 \dots A_s$  on the unit distance from the origin  $O$ . Let also  $A_0$  be the point  $(1, 0)$ , and the point  $A_1$  be on the straight line  $x = 1$ . If the signed length-sine sequence of the expanded ordinary angle  $\Phi_0 = \angle(O, A_0A_1 \dots A_s)$  coincides with the following sequence (we call it *characteristic sequence* for the corresponding angle):

**II<sub>k</sub>**)  $((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1)$ , where  $k \geq 1$ , then we denote the angle  $\Phi_0$  by  $k\pi + \text{larctan}(0)$  (or  $k\pi$ , for short) and say that  $\Phi_0$  is *of the type II<sub>k</sub>*;

**III<sub>k</sub>**)  $((-1, 2, -1, 2) \times (k-1)\text{-times}, -1, 2, -1)$ , where  $k \geq 1$ , then we denote the angle  $\Phi_0$  by  $-k\pi + \text{larctan}(0)$  (or  $-k\pi$ , for short) and say that  $\Phi_0$  is *of the type III<sub>k</sub>*;

**IV<sub>k</sub>**)  $((1, -2, 1, -2) \times k\text{-times}, a_0, \dots, a_{2n})$ , where  $k \geq 0$ ,  $n \geq 0$ ,  $a_i > 0$ , for  $i = 0, \dots, 2n$ , then we denote the angle  $\Phi_0$  by  $k\pi + \text{larctan}([a_0, a_1, \dots, a_{2n}])$  and say that  $\Phi_0$  is *of the type IV<sub>k</sub>*;

**V<sub>k</sub>**)  $((-1, 2, -1, 2) \times k\text{-times}, a_0, \dots, a_{2n})$ , where  $k > 0$ ,  $n \geq 0$ ,  $a_i > 0$ , for  $i = 0, \dots, 2n$ , then we denote the angle  $\Phi_0$  by  $-k\pi + \text{larctan}([a_0, a_1, \dots, a_{2n}])$  and say that  $\Phi_0$  is *of the type V<sub>k</sub>*.

**Theorem 3.14.** *For any expanded lattice angle  $\Phi$  there exist and unique a type among the types I-V and a unique expanded lattice angle  $\Phi_0$  of that type such that  $\Phi_0$  is proper lattice congruent to  $\Phi$ .*

The expanded lattice angle  $\Phi_0$  is said to be *the normal form* for the expanded lattice angle  $\Phi$ .

For the proof of Theorem 3.14 we need the following lemma.

**Lemma 3.15.** *Let  $m, k \geq 1$ , and  $a_i > 0$  for  $i = 0, \dots, 2n$  be some integers.*

**a).** *Suppose the lattice signed length-sine sequences for the expanded lattice angles  $\Phi_1$  and  $\Phi_2$  are respectively*

$$\begin{aligned} &((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, -2, a_0, \dots, a_{2n}) \quad \text{and} \\ &((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, m, a_0, \dots, a_{2n}), \end{aligned}$$

*then  $\Phi_1$  is proper lattice-congruent to  $\Phi_2$ .*

**b).** *Suppose the lattice signed length-sine sequences for the expanded lattice angles  $\Phi_1$  and  $\Phi_2$  are respectively*

$$\begin{aligned} &((-1, 2, -1, 2) \times (k-1)\text{-times}, -1, 2, -1, m, a_0, \dots, a_{2n}) \quad \text{and} \\ &((-1, 2, -1, 2) \times (k-1)\text{-times}, -1, 2, -1, 2, a_0, \dots, a_{2n}), \end{aligned}$$

*then  $\Phi_1$  is proper lattice-congruent to  $\Phi_2$ .*

*Proof.* We prove the first statement of the lemma. Suppose that  $m$  is integer,  $k$  is positive integer, and  $a_i$  for  $i = 0, \dots, 2n$  are positive integers.

Let us construct the angle  $\Psi_1$  with vertex at the origin for the lattice oriented broken line  $A_0 \dots A_{2k+n+1}$ , corresponding to the lattice signed length-sine sequence

$$((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, -2, a_0, \dots, a_{2n}),$$

such that  $A_0 = (1, 0)$ ,  $A_1 = (1, 1)$ . Note that

$$\begin{cases} A_{2l} &= ((-1)^l, 0), & \text{for } l < k-1 \\ A_{2l+1} &= ((-1)^l, (-1)^l) & \text{for } l < k-1 \\ A_{2k} &= ((-1)^k, 0) \\ A_{2k+1} &= ((-1)^k, (-1)^k)a_0 \end{cases}.$$

Let us construct the angle  $\Psi_2$  with vertex at the origin for the lattice oriented broken line  $B_0 \dots B_{2k+n+1}$ , corresponding to the lattice signed length-sine sequence

$$((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, m, a_0, \dots, a_{2n}).$$

such that  $B_0 = (1, 0)$ ,  $B_1 = (-m-1, 1)$ . Note also that

$$\begin{cases} B_{2l} &= ((-1)^l, 0), & \text{for } l < k-1 \\ B_{2l+1} &= ((-1)^l(-m-1), (-1)^l) & \text{for } l < k-1 \\ B_{2k} &= ((-1)^k, 0) \\ B_{2k+1} &= ((-1)^k, (-1)^k)a_0 \end{cases}.$$

From the above we know, that the points  $A_{2k}$  and  $A_{2k+1}$  coincide with the points  $B_{2k}$  and  $B_{2k+1}$  respectively. Since the remaining parts of both lattice signed length-sine sequences (i. e.  $(a_0, \dots, a_{2n})$ ) coincide, the point  $A_l$  coincide with the point  $B_l$  for  $l > 2k$ .

Since the lattice oriented broken lines  $A_0 \dots A_{2k}$  and  $B_0 \dots B_{2k}$  are of the same equivalence class, and the point  $A_l$  coincide with the point  $B_l$  for  $l > 2k$ , we obtain

$$\Psi_1 = \angle(O, A_0 \dots A_{2k+n+1}) = \angle(O, B_0 \dots B_{2k+n+1}) = \Psi_2.$$

Therefore, by Theorem 3.4 we have the following:

$$\Phi_1 \hat{\cong} \Psi_1 = \Psi_2 \hat{\cong} \Phi_2.$$

This concludes the proof of Lemma 3.15a.

Since the proof of Lemma 3.15b almost completely repeats the proof of Lemma 3.15a, we omit the proof of Lemma 3.15b here.  $\square$

*Proof of Theorem 3.14.* First, we prove that any two distinct expanded lattice angles listed in Definition 3.13 are not proper lattice-congruent. Let us note that the revolution numbers of expanded lattice angles distinguish the types of the angles. The revolution number for the expanded lattice angle of the type **I** is 0. The revolution number for the expanded lattice angle of the type **II<sub>k</sub>** is  $1/2(k+1)$  where  $k \geq 0$ . The revolution number for the expanded lattice angle of the type **III<sub>k</sub>** is  $-1/2(k+1)$  where  $k \geq 0$ . The revolution number for the expanded lattice angles of the type **IV<sub>k</sub>** is  $1/4+1/2k$  where  $k \geq 0$ . The revolution number for the expanded lattice angles of the type **V<sub>k</sub>** is  $1/4-1/2k$  where  $k > 0$ .

So we have proven that two expanded lattice angles of different types are not proper lattice-congruent. For the types **I**, **II<sub>k</sub>**, and **III<sub>k</sub>** the proof is completed, since any such type consists of the unique expanded lattice angle.

Let us prove that normal forms of the same type **IV<sub>k</sub>** (or of the same type **V<sub>k</sub>**) are not proper lattice-congruent for any integer  $k \geq 0$  (or  $k > 0$ ). Consider an expanded lattice angle  $\Phi = k\pi + \text{larctan}([a_0, a_1, \dots, a_{2n}])$ . Suppose that a lattice oriented broken line  $A_0A_1 \dots A_m$  on

the unit distance from  $O$ , where  $m = 2|k|+n+1$  defines the angle  $\Phi$ . Suppose also that the signed lattice-sine sequence for this broken line is characteristic.

Suppose, that  $k$  is even, then the ordinary lattice angle  $\angle A_0OA_m$  is proper lattice-congruent to the ordinary lattice angle  $\text{larctan}([a_0, a_1, \dots, a_{2n}])$ . This angle is a proper lattice-affine invariant for the expanded lattice angle  $\Phi$ . This invariant distinguish the expanded lattice angles of type  $\mathbf{IV}_k$  (or  $\mathbf{V}_k$ ) with even  $k$ .

Suppose, that  $k$  is odd, then denote  $B = O+\overline{A_0O}$ . The ordinary lattice angle  $\angle BVA_m$  is proper lattice-congruent to the ordinary lattice angle  $\text{larctan}([a_0, a_1, \dots, a_{2n}])$ . This angle is a proper lattice-affine invariant for the expanded lattice angle  $\Phi$ . This invariant distinguish the expanded lattice angles of type  $\mathbf{IV}_k$  (or  $\mathbf{V}_k$ ) with odd  $k$ .

Therefore, the expanded lattice angles listed in Definition 3.13 are not proper lattice-congruent.

Now we prove that an arbitrary expanded lattice angle is proper lattice-congruent to one of the expanded lattice angles of the types  $\mathbf{I-V}$ .

Consider an arbitrary expanded lattice angle  $\angle(V, A_0A_1 \dots A_n)$  and denote it by  $\Phi$ . If  $\#(\Phi) = k/2$  for some integer  $k$ , then  $\Phi$  is proper lattice congruent to an angle of one of the types  $\mathbf{I-III}$ . Let  $\#(\Phi) = 1/4$ , then the expanded lattice angle  $\Phi$  is proper lattice-congruent to the expanded lattice angle defined by the sail of the ordinary lattice angle  $\angle A_0VA_n$  of the type  $\mathbf{IV}_0$ .

Suppose now, that  $\#(\Phi) = 1/4+k/2$  for some positive integer  $k$ , then one of its lattice signed length-sine sequence is of the following form:

$$((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, m, a_0, \dots, a_{2n}),$$

where  $a_i > 0$ , for  $i = 0, \dots, 2n$ . By Lemma 3.15 the expanded lattice angle defined by this sequence is proper lattice-congruent to an expanded lattice angle of the type  $\mathbf{IV}_k$  defined by the sequence

$$((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, -2, a_0, \dots, a_{2n}).$$

Finally, let  $\#(\Phi) = 1/4-k/2$  for some positive integer  $k$ , then one of its lattice signed length-sine sequence is of the following form:

$$((-1, 2, -1, 2) \times (k-1)\text{-times}, -1, 2, -1, m, a_0, \dots, a_{2n}),$$

where  $a_i > 0$ , for  $i = 0, \dots, 2n$ . By Lemma 3.15 the expanded lattice angle defined by this sequence is proper lattice-congruent to an expanded lattice angle of the type  $\mathbf{V}_k$  defined by the sequence

$$((-1, 2, -1, 2) \times (k-1)\text{-times}, -1, 2, -1, 2, a_0, \dots, a_{2n}).$$

This completes the proof of Theorem 3.14.  $\square$

Let us finally give the definition of trigonometric functions for the expanded lattice angles and describe some relations between ordinary and expanded lattice angles.

**Definition 3.16.** Consider an arbitrary expanded lattice angle  $\Phi$  with the normal form  $k\pi+\varphi$  for some ordinary (possible zero) lattice angle  $\varphi$  and for an integer  $k$ .

- a). The ordinary lattice angle  $\varphi$  is said to be *associated* with the expanded lattice angle  $\Phi$ .
- b). The numbers  $\text{ltan}(\varphi)$ ,  $\text{lsin}(\varphi)$ , and  $\text{lcos}(\varphi)$  are called the *lattice tangent*, the *lattice sine*, and the *lattice cosine* of the expanded lattice angle  $\Phi$ .

Since all sails for ordinary lattice angles are lattice oriented broken lines, the set of all ordinary angles is naturally embedded into the set of expanded lattice angles.

**Definition 3.17.** For any ordinary lattice angle  $\varphi$  the angle

$$0\pi + \text{larctan}(\text{l}\tan \varphi)$$

is said to be *corresponding* to the angle  $\varphi$  and denoted by  $\bar{\varphi}$ .

From Theorem 3.14 it follows that for any ordinary lattice angle  $\varphi$  there exists and unique an expanded lattice angle  $\bar{\varphi}$  corresponding to  $\varphi$ . Therefore, two ordinary lattice angles  $\varphi_1$  and  $\varphi_2$  are lattice-congruent iff the corresponding lattice angles  $\bar{\varphi}_1$  and  $\bar{\varphi}_2$  are proper lattice-congruent.

**3.2.5. Opposite expanded lattice angles. Sums of expanded lattice angles. Sums of ordinary lattice angles.** Consider an expanded lattice angle  $\Phi$  with the vertex  $V$  for some equivalence class of a given lattice oriented broken line. The expanded lattice angle  $\Psi$  with the vertex  $V$  for the equivalence class of the inverse lattice oriented broken line is called *opposite* to the given one and denoted by  $-\Phi$ .

**Proposition 3.18.** For any expanded lattice angle  $\Phi \hat{\cong} k\pi + \varphi$  we have:

$$-\Phi \hat{\cong} (-k - 1)\pi + (\pi - \varphi).$$

□

Let us introduce the definition of sums of ordinary and expanded lattice angles.

**Definition 3.19.** Consider arbitrary expanded lattice angles  $\Phi_i$ ,  $i = 1, \dots, l$ . Let the characteristic sequences for the normal forms of  $\Phi_i$  be  $(a_{0,i}, a_{1,i}, \dots, a_{2n_i,i})$  for  $i = 1, \dots, l$ . Let  $M = (m_1, \dots, m_{l-1})$  be some  $(l-1)$ -tuple of integers. The normal form of any expanded lattice angle, corresponding to the following lattice signed length-sine sequence

$$(a_{0,1}, a_{1,1}, \dots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \dots, a_{2n_2,2}, m_2, \dots, \\ \dots, m_{l-1}, a_{0,l}, a_{1,l}, \dots, a_{2n_l,l}),$$

is called the *M-sum of expanded lattice angles*  $\Phi_i$  ( $i = 1, \dots, l$ ) and denoted by

$$\sum_{M,i=1}^l \Phi_i, \quad \text{or equivalently by } \Phi_1 +_{m_1} \Phi_2 +_{m_2} \dots +_{m_{l-1}} \Phi_l.$$

**Proposition 3.20.** The *M-sum of expanded lattice angles*  $\Phi_i$  ( $i = 1, \dots, l$ ) is well-defined. □

Let us say a few words about properties of *M*-sums.

The *M*-sum of expanded lattice angles is non-associative. For example, let  $\Phi_1 \hat{\cong} \text{larctan } 2$ ,  $\Phi_2 \hat{\cong} \text{larctan}(3/2)$ , and  $\Phi_3 \hat{\cong} \text{larctan } 5$ . Then

$$\begin{aligned} \Phi_1 +_{-1} \Phi_2 +_{-1} \Phi_3 &= \pi + \text{larctan}(4), \\ \Phi_1 +_{-1} (\Phi_2 +_{-1} \Phi_3) &= 2\pi, \\ (\Phi_1 +_{-1} \Phi_2) +_{-1} \Phi_3 &= \text{larctan}(1). \end{aligned}$$

The *M*-sum of expanded lattice angles is non-commutative. For example, let  $\Phi_1 \hat{\cong} \text{larctan } 1$ , and  $\Phi_2 \hat{\cong} \text{larctan } 5/2$ . Then

$$\begin{aligned} \Phi_1 +_1 \Phi_2 &= \text{larctan}(12/7), \\ \Phi_2 +_1 \Phi_1 &= \text{larctan}(13/5). \end{aligned}$$

*Remark 3.21.* The *M*-sum of expanded lattice angles is naturally extended to the sum of classes of proper lattice-congruences of expanded lattice angles.

We conclude this section with the definition of sums of ordinary lattice angles.

**Definition 3.22.** Consider ordinary lattice angles  $\alpha_i$ , where  $i = 1, \dots, l$ . Let  $\bar{\alpha}_i$  be the corresponding expanded lattice angles for  $\alpha_i$ , and  $M = (m_1, \dots, m_{l-1})$  be some  $(l-1)$ -tuple of integers. The ordinary lattice angle  $\varphi$  associated with the expanded lattice angle

$$\Phi = \bar{\alpha}_1 + m_1 \bar{\alpha}_2 + m_2 \dots + m_{l-1} \bar{\alpha}_l.$$

is called the  $M$ -sum of ordinary lattice angles  $\alpha_i$  ( $i = 1, \dots, l$ ) and denoted by

$$\sum_{M, i=1}^l \alpha_i, \quad \text{or equivalently by} \quad \alpha_1 + m_1 \alpha_2 + m_2 \dots + m_{l-1} \alpha_l.$$

*Remark 3.23.* Note that the sum of ordinary lattice angles is naturally extended to the classes of lattice-congruences of lattice angles.

#### 4. RELATIONS BETWEEN EXPANDED LATTICE ANGLES AND ORDINARY LATTICE ANGLES. PROOF OF THE FIRST STATEMENT OF THEOREM 2.2.

In this section we show how to calculate the ordinary angle  $\varphi$  of the normal form: we describe relations between continued fractions for lattice oriented broken lines and the lattice tangents for the corresponding expanded lattice angles. Then we prove the first statement of the theorem on sums of lattice tangents for ordinary lattice angles in lattice triangles. Further we define lattice-acute-angled triangles. We conclude this section with a necessary and sufficient condition for an ordered  $n$ -tuple of angles to be the angles of some convex lattice polygon.

Throughout this section we again fix some lattice basis and use the system of coordinates  $OXY$  corresponding to this basis.

**4.1. On relations between continued fractions for lattice oriented broken lines and the lattice tangents of the corresponding expanded lattice angles.** For any real number  $r$  we denote by  $[r]$  the maximal integer not greater than  $r$ .

**Theorem 4.1.** Consider an expanded lattice angle  $\Phi = \angle(V, A_0 A_1 \dots A_n)$ . Suppose, that the normal form for  $\Phi$  is  $k\pi + \varphi$  for some integer  $k$  and an ordinary lattice angle  $\varphi$ . Let  $(a_0, a_1, \dots, a_{2n-2})$  be the lattice signed length-sine sequence for the lattice oriented broken line  $A_0 A_1 \dots A_n$ . Suppose that

$$]a_0, a_1, \dots, a_{2n-2}[ = q/p.$$

Then the following holds:

$$\varphi \cong \begin{cases} \text{larctan}(1), & \text{if } q/p = \infty, \\ \text{larctan}(q/p), & \text{if } q/p \geq 1, \\ \text{larctan}\left(\frac{|q|}{|p| - [(|p|-1)/|q|]|q|}\right), & \text{if } 0 < q/p < 1, \\ 0, & \text{if } q/p = 0, \\ \pi - \text{larctan}\left(\frac{|q|}{|p| - [(|p|-1)/|q|]|q|}\right), & \text{if } -1 < q/p < 0, \\ \pi - \text{larctan}(-q/p), & \text{if } q/p \leq -1. \end{cases}$$

*Proof.* Consider the following linear coordinates  $(*, *)'$  on the plane  $\mathbb{R}^2$ , associated with the lattice oriented broken line  $A_0 A_1 \dots A_n$  and the point  $V$ . Let the origin  $O'$  be at the vertex  $V$ ,  $(1, 0)' = A_0$ , and  $(1, 1)' = A_0 + \frac{1}{a_0} \text{sgn}(A_0 O' A_1) \overline{A_0 A_1}$ . The other coordinates are uniquely defined by linearity. We denote this system of coordinates by  $O' X' Y'$ .



The set of integer points for the coordinate system  $O'X'Y'$  coincides with the set of lattice points of  $\mathbb{R}^2$ . The basis of vectors  $(1, 0)'$  and  $(0, 1)'$  defines a positive orientation.

Suppose that the new coordinates of the point  $A_n$  are  $(p', q)'$ . Then by Theorem 3.5 we have  $q'/p' = q/p$ . This directly implies the statement of the theorem for the cases  $q' > p' > 0$ ,  $q'/p' = 0$ , and  $q'/p' = \infty$ .

Suppose now that  $p' > q' > 0$ . Consider the ordinary lattice angle  $\varphi = \angle A_0 P A_n$ . Let  $B_0 \dots B_m$  be the sail for it. The direct calculations show that the point

$$D = B_0 + \frac{\overline{B_0 B_1}}{\ell(B_0 B_1)}$$

coincides with the point  $(1 + \lfloor (p' - 1)/q' \rfloor, 1)$  in the system of coordinates  $O'X'Y'$ .

Consider the proper lattice-linear (in the coordinates  $O'X'Y'$ ) transformation  $\xi$  that takes the point  $A_0 = B_0$  to itself, and the point  $D$  to the point  $(1, 1)'$ . These conditions uniquely identify  $\xi$ .

$$\xi = \begin{pmatrix} 1 & -\lfloor (p' - 1)/q' \rfloor \\ 0 & 1 \end{pmatrix}$$

The transformation  $\xi$  takes the point  $A_n = B_m$  with the coordinates  $(p', q)'$  to the point with the coordinates  $(p' - \lfloor (p' - 1)/q' \rfloor q', q)'$ . Since  $q'/p' = q/p$ , we obtain the following

$$\varphi = \text{larctan} \left( \frac{q'}{p' - \lfloor (p' - 1)/q' \rfloor q'} \right) = \text{larctan} \left( \frac{q}{p - \lfloor (p - 1)/q \rfloor q} \right).$$

The proof for the case  $q' > 0$  and  $p' < 0$  repeats the described cases after taking to the consideration the adjacent angles.

Finally, the case of  $q' < 0$  repeats all previous cases by the central symmetry (centered at the point  $O'$ ) reasons.

This completes the proof of Theorem 4.1. □

**Corollary 4.2.** *The revolution number and the continued fraction for any lattice oriented broken line on the unit distance from the vertex uniquely define the proper lattice-congruence class of the corresponding expanded lattice angle.* □

**4.2. Proof of Theorem 2.2a: two preliminary lemmas.** We say that the lattice point  $P$  is on the lattice distance  $k$  from the lattice segment  $AB$  if the lattice vectors of the segment  $AB$  and the vector  $\overline{AP}$  generate a sublattice of the lattice of index  $k$ .

**Definition 4.3.** Consider a lattice triangle  $\triangle ABC$ . Denote the number of lattice points on the unit lattice distance from the segment  $AB$  and contained in the (closed) triangle  $\triangle ABC$  by  $\ell_1(AB; C)$  (see on Figure 7).

Note that all lattice points on the lattice unit distance from the segment  $AB$  in the (closed) lattice triangle  $\triangle ABC$  are contained in one straight line parallel to the straight line  $AB$ . Besides, the integer  $\ell_1(AB; C)$  is positive for any triangle  $\triangle ABC$ .

Now we prove the following lemma.

**Lemma 4.4.** *For any lattice triangle  $\triangle ABC$  the following holds*

$$\overline{ZCAB} +_{\ell(AB) - \ell_1(AB; C) - 1} \overline{ZABC} +_{\ell(BC) - \ell_1(BC; A) - 1} \overline{ZBCA} = \pi.$$

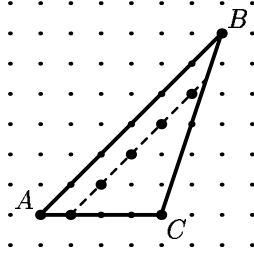


FIGURE 7. For the given triangle  $\triangle ABC$  we have  $l_1(AB; C) = 5$ .

*Proof.* Consider an arbitrary lattice triangle  $\triangle ABC$ . Suppose that the couple of vectors  $\overline{BA}$  and  $\overline{BC}$  defines the positive orientation of the plane (otherwise we apply to the triangle  $\triangle ABC$  some lattice-affine transformation changing the orientation and come to the same position). Denote (see Figure 9 below):

$$D = A + \overline{BC}, \quad \text{and} \quad E = A + \overline{AC}.$$

Since  $CADB$  is a parallelogram, the triangle  $\triangle BAD$  is proper lattice-congruent to the triangle  $\triangle ABC$ . Thus, the angle  $\angle BAD$  is proper lattice-congruent to the angle  $\angle ABC$ , and  $l_1(BA; D) = l_1(AB; C)$ . Since  $EABD$  is a parallelogram, the triangle  $\triangle AED$  is proper lattice-congruent to the triangle  $\triangle BAD$ , and hence is proper lattice-congruent to the triangle  $\triangle ABC$ . Thus,  $\angle DAE$  is proper lattice-congruent to  $\angle BCA$ , and  $l_1(DA; E) = l_1(BC; A)$ .

Let  $A_0 \dots A_n$  be the sail of  $\angle CAB$  with the corresponding lattice length-sine sequence  $(a_0, \dots, a_{2n-2})$ . Let  $B_0 B_1 \dots B_m$  be the sail of  $\angle BAD$  (where  $B_0 = A_n$ ) with the corresponding lattice length-sine sequence  $(b_0, \dots, b_{2m-2})$ . And let  $C_0 C_1 \dots C_l$  be the sail of  $\angle DAE$  (where  $C_0 = B_m$ ) with the corresponding lattice length-sine sequence  $(c_0, \dots, c_{2l-2})$ .

Consider now the lattice oriented broken line

$$A_0 \dots A_n B_0 B_1 \dots B_m C_0 C_1 \dots C_l.$$

The lattice oriented length-sine sequence for this broken line is

$$(a_0, \dots, a_{2n-2}, t, b_0, \dots, b_{2m-2}, u, c_0, \dots, c_{2l-2}).$$

By definition of the sum of expanded lattice angles this sequence defines the expanded lattice angle

$$\overline{ZCAB} +_t \overline{ZBAD} +_u \overline{ZDAE}.$$

By Theorem 4.1,

$$\overline{ZCAB} +_t \overline{ZBAD} +_u \overline{ZDAE} = \pi.$$

Let us find an integer  $t$ . Denote by  $A'_n$  the closest lattice point to the point  $A_n$  and distinct to  $A_n$  in the segment  $A_{n-1}A_n$ . Consider the set of lattice points on the unit distance from the segment  $AB$  and lying in the half-plane with the boundary straight line  $AB$  and containing the point  $D$ . This set coincides with the following set (See Figure 8):

$$\{A_{n,k} = A_n + \overline{A'_n A_n} + k \overline{AA_n} \mid k \in \mathbb{Z}\}.$$

Since  $A_{n,-2} = A + \overline{A'_n A}$ , the points  $A_{n,k}$  for  $k \leq -2$  are in the closed half-plane bounded by the straight line  $AC$  and not containing the point  $B$ .

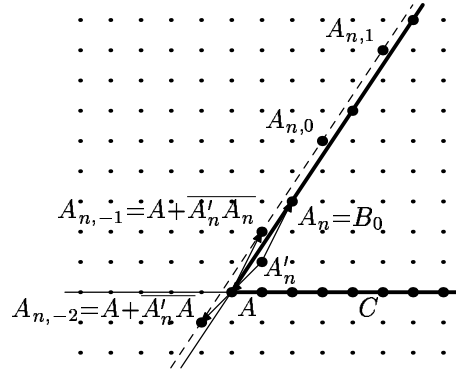


FIGURE 8. Lattice points  $A_{n,t}$ .

Since  $A_{n,-1} = A + \overline{A'_n A_n}$ , the points  $A_{n,k}$  for  $k \geq -1$  are in the open half-plane bounded by the straight line  $AC$  and containing the point  $B$ .

The intersection of the parallelogram  $AEDB$  and the open half-plane bounded by the straight line  $AC$  and containing the point  $B$  contains exactly  $\ell(AB)$  points of the described set: only the points  $A_{n,k}$  with  $-1 \leq k \leq \ell(AB) - 2$ .

Since the triangle  $\triangle BAD$  is proper lattice-congruent to  $\triangle ABC$ , the number of points  $A_{n,k}$  in the closed triangle  $\triangle BAD$  is  $\ell_1(AB; C)$ : the points  $A_{n,k}$  for

$$\ell(AB) - \ell_1(AB; C) - 1 \leq k \leq \ell(AB) - 2.$$

Denote the integer  $\ell(AB) - \ell_1(AB; C) - 1$  by  $k_0$ .

The point  $A_{n,k_0}$  is contained in the segment  $B_0B_1$  of the sail for the ordinary lattice angle  $\angle BAD$  (see Figure 9). Since the angles  $\angle BAD$  and  $\angle ABC$  are proper lattice-congruent, we have

$$\begin{aligned} t &= \text{sgn}(A_{n-1}AA_n) \text{sgn}(A_nAB_1) \text{sgn}(A_{n-1}A_nB_1) \text{lsin} \angle A_{n-1}A_nB_1 = \\ &= 1 \cdot 1 \cdot \text{sgn}(A_{n-1}A_nA_{n,k_0}) \text{lsin} \angle A_{n-1}A_nA_{n,k_0} = \\ &= \text{sign}(k_0) |k_0| = k_0 = \ell(AB) - \ell_1(AB; C) - 1. \end{aligned}$$

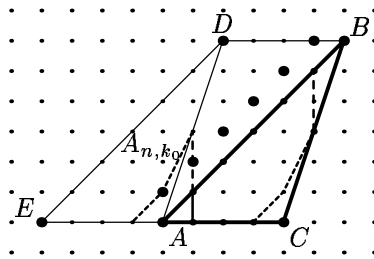


FIGURE 9. The point  $A_{n,k_0}$ .

Exactly by the same reasons,

$$u = \ell(DA) - \ell_1(DA; E) - 1 = \ell(BC) - \ell_1(BC; A) - 1.$$

Therefore,  $\overline{\angle CAB} +_{\ell(AB)-\ell_1(AB;C)-1} \overline{\angle ABC} +_{\ell(BC)-\ell_1(BC;A)-1} \overline{\angle BCA} = \pi$ .  $\square$

**Lemma 4.5.** *Let  $\alpha$ ,  $\beta$ , and  $\gamma$  be nonzero ordinary lattice angles. Suppose that  $\bar{\alpha} +_u \bar{\beta} +_v \bar{\gamma} = \pi$ , then there exist a triangle with three consecutive ordinary angles lattice-congruent to  $\alpha$ ,  $\beta$ , and  $\gamma$ .*

*Proof.* Denote by  $O$  the point  $(0, 0)$ , by  $A$  the point  $(1, 0)$ , and by  $D$  the point  $(-1, 0)$  in the fixed system of coordinates  $OXY$ .

Let us choose the points  $B = (p_1, q_1)$  and  $C = (p_2, q_2)$  with integers  $p_1, p_2$  and positive integers  $q_1, q_2$  such that

$$\angle AOB = \text{larctan}(\text{l}\tan \alpha), \quad \text{and} \quad \angle AOC = \overline{\angle AOB} +_u \bar{\beta}.$$

Thus the vectors  $\overline{OB}$  and  $\overline{OC}$  defines the positive orientation, and  $\angle BOC \cong \beta$ . Since

$$\bar{\alpha} +_u \bar{\beta} +_v \bar{\gamma} = \pi \quad \text{and} \quad \bar{\alpha} +_u \bar{\beta} \hat{\cong} \angle AOC,$$

the ordinary angle  $\angle COD$  is lattice-congruent to  $\gamma$ .

Denote by  $B'$  the point  $(p_1 q_2, q_1 q_2)$ , and by  $C'$  the point  $(p_2 q_1, q_1 q_2)$  and consider the triangle  $B'OC'$ . Since the ordinary angle  $\angle B'OC'$  coincides with the ordinary angle  $\angle BOC$ , we obtain

$$\angle B'OC' \cong \beta.$$

Since the ordinary angle  $\beta$  is nonzero, the points  $B'$  and  $C'$  are distinct and the straight line  $B'C'$  does not coincide with the straight line  $OA$ . Since the second coordinate of the both points  $B'$  and  $C'$  equal  $q_1 q_2$ , the straight line  $B'C'$  is parallel to the straight line  $OA$ . Thus, by Proposition 1.15 it follows that

$$\angle C'B'O \cong \angle AOB' = \angle AOB \cong \alpha, \quad \text{and} \quad \angle OC'B' \cong \angle C'OD = \angle COD \cong \gamma.$$

So, we have constructed the triangle  $\triangle B'OC'$  with three consecutive ordinary angles lattice-congruent to  $\alpha$ ,  $\beta$ , and  $\gamma$ .  $\square$

**4.3. Proof of Theorem 2.2a: conclusion of the proof.** Now we return to the proof of the first statement of the theorem on sums of lattice tangents for ordinary lattice angles in lattice triangles.

*Proof of Theorem 2.2a.* Let  $\alpha$ ,  $\beta$ , and  $\gamma$  be nonzero ordinary lattice angles satisfying the conditions *i*) and *ii*) of Theorem 2.2a.

The second condition

$$] \text{l}\tan(\alpha), -1, \text{l}\tan(\beta), -1, \text{l}\tan(\gamma) [= 0$$

implies that

$$\bar{\alpha} +_{-1} \bar{\beta} +_{-1} \bar{\gamma} = k\pi.$$

Since all three tangents are positive, we have  $k = 1$ , or  $k = 2$ .

Consider the first condition:  $] \text{l}\tan \alpha, -1, \text{l}\tan \beta [$  is either negative or greater than  $\text{l}\tan \alpha$ . It implies that  $\bar{\alpha} +_{-1} \bar{\beta} = 0\pi + \varphi$ , for some ordinary lattice angle  $\varphi$ , and hence  $k = 1$ .

Therefore, by Lemma 4.5 there exist a triangle with three consecutive ordinary lattice angles lattice-congruent to  $\alpha$ ,  $\beta$ , and  $\gamma$ .

Let us prove the converse. We prove that condition *ii*) of Theorem 2.2a holds by reductio ad absurdum. Suppose, that there exist a triangle  $\triangle ABC$  with consecutive ordinary angles

$\alpha = \angle CAB$ ,  $\beta = \angle ABC$ , and  $\gamma = \angle BCA$ , such that

$$\begin{cases} ]\operatorname{ltan}(\alpha), -1, \operatorname{ltan}(\beta), -1, \operatorname{ltan}(\gamma)[ \neq 0 \\ ]\operatorname{ltan}(\beta), -1, \operatorname{ltan}(\gamma), -1, \operatorname{ltan}(\alpha)[ \neq 0 \\ ]\operatorname{ltan}(\gamma), -1, \operatorname{ltan}(\alpha), -1, \operatorname{ltan}(\beta)[ \neq 0 \end{cases} .$$

These inequalities and Lemma 4.4 imply that at least two of the integers

$$\ell(AB) - \ell_1(AB; C) - 1, \quad \ell(BC) - \ell_1(BC; A) - 1, \quad \text{and} \quad \ell(CA) - \ell_1(CA; B) - 1$$

are nonnegative.

Without losses of generality we suppose that

$$\begin{cases} \ell(AB) - \ell_1(AB; C) - 1 \geq 0 \\ \ell(BC) - \ell_1(BC; A) - 1 \geq 0 \end{cases} .$$

Since all integers of the continued fraction

$$r = ]\operatorname{ltan}(\alpha), \ell(AB) - \ell_1(AB; C) - 1, \operatorname{ltan}(\beta), \ell(BC) - \ell_1(BC; A) - 1, \operatorname{ltan}(\gamma)[$$

are non-negative and the last one is positive, we obtain that  $r > 0$  (or  $r = \infty$ ). From the other hand, by Lemma 4.4 and by Theorem 4.1 we have that  $r = 0/-1 = 0$ . We come to the contradiction.

Now we prove that condition *i*) of Theorem 2.2a holds. Suppose that there exist a triangle  $\triangle ABC$  with consecutive ordinary angles  $\alpha = \angle CAB$ ,  $\beta = \angle ABC$ , and  $\gamma = \angle BCA$ , such that

$$]\operatorname{ltan}(\alpha), -1, \operatorname{ltan}(\beta), -1, \operatorname{ltan}(\gamma)[ = 0.$$

Since  $\bar{\alpha} + {}_{-1}\bar{\beta} + {}_{-1}\bar{\gamma} = \pi$ , we have  $\bar{\alpha} + {}_{-1}\bar{\beta} = 0\pi + \varphi$  for some ordinary lattice angle  $\varphi$ . Therefore, the first condition of the theorem holds.

This concludes the proof of Theorem 2.2.  $\square$

Let us give here the following natural definition.

**Definition 4.6.** The triangle  $\triangle ABC$  is said to be *lattice-acute-angled* if the integers

$$\ell(AB) - \ell_1(AB; C) - 1, \quad \ell(BC) - \ell_1(BC; A) - 1, \quad \text{and} \quad \ell(CA) - \ell_1(CA; B) - 1$$

are all equal to  $-1$ .

The triangle  $\triangle ABC$  is said *lattice right-angled* if one of these integers equals  $0$ .

The triangle  $\triangle ABC$  is said *lattice obtuse-angled* if one of these integers is positive.

Note that the property of the lattice triangle to be lattice acute-angled, right-angled, or obtuse-angled cannot be determined by one of the angles, unlike in Euclidean geometry. We will illustrate this with the following example. (Actually, there is no relation between right-angled triangles and right angles, defined before in Subsubsection 1.2.4.)

**Example 4.7.** On Figure 10 we show the lattice obtuse-angled triangle (on the figure to the left) with the lattice tangents of its ordinary angles equal to  $3/2$ ,  $8/3$ , and  $1$ . Nevertheless, each of these lattice angles is not supposed to be “lattice-obtuse”, since three lattice triangles (on the figure to the right) are all lattice acute-angled and contain ordinary lattice angles with the lattice tangents  $3/2$ ,  $8/3$ , and  $1$ .

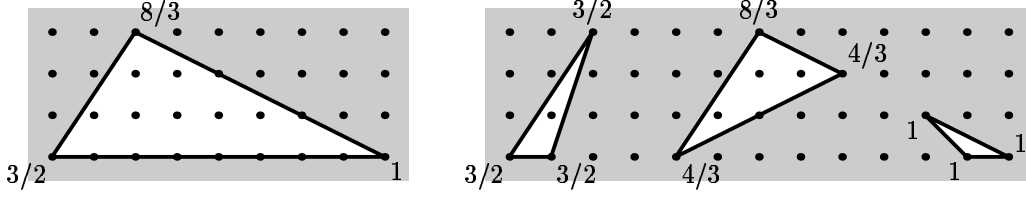


FIGURE 10. The triangle to the left is lattice obtuse-angled, three triangles to the right are all lattice acute-angled.

**4.4. Theorem on sum of lattice tangents for ordinary lattice angles of convex polygons.** A satisfactory description for lattice-congruence classes of lattice convex polygons has not been yet found. It is only known that the number of convex polygons with lattice area bounded from above by  $n$  grows exponentially in  $n$ , while  $n$  tends to infinity (see [2] and [3]). We conclude this section with the following theorem on necessary and sufficient condition for the lattice angles to be the angles of some convex lattice polygon.

**Theorem 4.8.** *Let  $\alpha_1, \dots, \alpha_n$  be an arbitrary ordered  $n$ -tuple of ordinary non-zero lattice angles. Then the following two conditions are equivalent:*

- there exist a convex  $n$ -vertex polygon with consecutive ordinary lattice angles lattice-congruent to the ordinary lattice angles  $\alpha_i$  for  $i = 1, \dots, n$ ;
- there exist a set of integers  $M = \{m_1, \dots, m_{n-1}\}$  such that

$$\sum_{M, i=1}^n \overline{\pi - \alpha_i} = 2\pi.$$

*Proof.* Consider an arbitrary  $n$ -tuple of ordinary lattice angles  $\alpha_i$ , here  $i = 1, \dots, n$ .

Suppose that there exist a convex polygon  $\overline{A_1 A_2 \dots A_n}$  with consecutive angles  $\alpha_i$  for  $i = 1, \dots, n$ . Let also the couple of vectors  $\overline{A_2 A_3}$  and  $\overline{A_2 A_1}$  defines the positive orientation of the plane (otherwise we apply to the polygon  $\overline{A_1 A_2 \dots A_n}$  some lattice-affine transformation changing the orientation and come to the initial position).

Let  $B_1 = O + \overline{A_n A_1}$ , and  $B_i = O + \overline{A_{i-1} A_i}$  for  $i = 2, \dots, n$ . We put by definition

$$\beta_i = \begin{cases} \angle B_i O B_{i+1}, & \text{if } i = 1, \dots, n-1 \\ \angle B_n O B_1, & \text{if } i = n \end{cases}.$$

Consider the union of the sails for all  $\beta_i$ . This lattice oriented broken line is of the class of the expanded lattice angle with the normal form  $2\pi + 0$ . The signed length-sine sequence for this broken line contains exactly  $n-1$  elements that are not contained in the length-sine sequences for the sails of  $\beta_i$ . Denote these numbers by  $m_1, \dots, m_{n-1}$ , and the set  $\{m_1, \dots, m_{n-1}\}$  by  $M$ . Then

$$\sum_{M, i=1}^n \overline{\beta_i} = 2\pi.$$

From the definition of  $\beta_i$  for  $i = 1, \dots, n$  it follows that  $\beta_i \cong \pi - \alpha_i$ . Therefore,

$$\sum_{M, i=1}^n \overline{\pi - \alpha_i} = 2\pi.$$

We complete the proof of the statement in one side.

Suppose now, that there exist a set of integers  $M = \{m_1, \dots, m_{n-1}\}$  such that

$$\sum_{M, i=1}^n \overline{\pi - \alpha_i} = 2\pi.$$

This implies that there exist lattice points  $B_1 = (1, 0)$ ,  $B_i = (x_i, y_i)$ , for  $i = 2, \dots, n-1$ , and  $B_n = (-1, 0)$  such that

$$\angle B_i O B_{i-1} \cong \pi - \alpha_{i-1}, \text{ for } i = 2, \dots, n, \text{ and } \angle B_1 O B_n \cong \pi - \alpha_n.$$

Denote by  $M$  the lattice point

$$O + \sum_{i=1}^n \overline{O B_i}.$$

Since all  $\alpha_i$  are non-zero, the angles  $\pi - \alpha_i$  are ordinary. Hence, the origin  $O$  is an interior point of the convex hull of the points  $B_i$  for  $i = 1, \dots, k$ . This implies that there exist two consecutive lattice points  $B_s$  and  $B_{s+1}$  (or  $B_n$  and  $B_1$ ), such that the lattice triangle  $\triangle B_s M B_{s+1}$  contains  $O$  and the edge  $B_s B_{s+1}$  does not contain  $O$ . Therefore,

$$O = \lambda_1 \overline{O M} + \lambda_2 \overline{O B_i} + \lambda_3 \overline{O B_{i+1}},$$

where  $\lambda_1$  is a positive integer, and  $\lambda_2$  and  $\lambda_3$  are nonnegative integers. So there exist positive integers  $a_i$ , where  $i = 1, \dots, n$ , such that

$$O = O + \sum_{i=1}^n (a_i \overline{O B_i}).$$

Put by definition  $A_0 = O$ , and  $A_i = A_{i-1} + a_i \overline{O B_i}$  for  $i = 2, \dots, n$ . The broken line  $A_0 A_1 \dots A_n$  is lattice and by the above it is closed (i. e.  $A_0 = A_n$ ). By construction, the ordinary lattice angle at the vertex  $A_i$  of the closed lattice broken line is proper lattice-congruent to  $\alpha_i$  ( $i = 1, \dots, n$ ). Since the integers  $a_i$  are positive for  $i = 1, \dots, n$  and the vectors  $\overline{O B_i}$  are all in the counterclockwise order, the broken line is a convex polygon.

The proof of Theorem 4.8 is completed. □

*Remark 4.9.* Theorem 4.8 generalizes the statement of Theorem 2.2a. Note that the direct generalization of Theorem 2.2b is false: the ordinary lattice angles do not uniquely determine the proper lattice-affine homothety types of convex polygons. See an example on Figure 11.



FIGURE 11. An example of different types of polygons with the proper lattice-congruent ordinary lattice angles.

## 5. ON LATTICE IRRATIONAL CASE.

The aim of this section is to generalize the notions of ordinary and expanded lattice angles and their sums to the case of angles with lattice vertices but not necessary lattice rays. We find normal forms and extend the definition of lattice sums for a certain special case of such angles.

**5.1. Infinite ordinary continued fractions.** We start with the standard definition of infinite ordinary continued fraction.

**Theorem 5.1.** *Consider a sequence  $(a_0, a_1, \dots, a_n, \dots)$  of positive integers. There exists the following limit:  $r = \lim_{k \rightarrow \infty} [a_0, a_1, \dots, a_k]$ .*  $\square$

This representation of  $r$  is called an (*infinite*) *ordinary continued fraction* for  $r$  and denoted by  $[a_0, a_1, \dots, a_n, \dots]$ .

**Theorem 5.2.** *For any irrational there exists and unique infinite ordinary continued fraction. Any rational does not have infinite ordinary continued fractions.*  $\square$

For the proofs of these theorems we refer to the book [7] by A. Ya. Hinchin.

**5.2. Ordinary lattice irrational angles.** Let  $A$ ,  $B$ , and  $C$  do not lie in the same straight line. Suppose also that  $B$  is lattice. We denote the angle with the vertex at  $B$  and the rays  $BA$  and  $BC$  by  $\angle ABC$ . If the open ray  $BA$  contains lattice points, and the open ray  $BC$  does not contain lattice points, then we say that the angle  $\angle ABC$  is ordinary lattice *R-irrational* angle. If the open ray  $BA$  does not contain lattice points, and the open ray  $BC$  contains lattice points, then we say that the angle  $\angle AOB$  is ordinary lattice *L-irrational* angle. If the union of open rays  $BA$  and  $BC$  does not contain lattice points, then we say that the angle  $\angle ABC$  is ordinary lattice *LR-irrational* angle. We also call R-irrational, L-irrational, and LR-irrational angles by *irrational* angles.

**Definition 5.3.** Two ordinary lattice irrational angles  $\angle AOB$  and  $\angle A'O'B'$  are said to be *lattice-congruent* if there exist a lattice-affine transformation which takes the vertex  $O$  to the vertex  $O'$  and the rays  $OA$  and  $OB$  to the rays  $O'A'$  and  $O'B'$  respectively. We denote this as follows:  $\angle AOB \cong \angle A'O'B'$ .

**5.3. Lattice-length sequences for ordinary lattice irrational angles.** In this subsection we generalize the notion of lattice-length sequences for the case of ordinary lattice irrational angles and study its elementary properties.

Consider an ordinary lattice angle  $\angle AOB$ . Let also the vectors  $\overline{OA}$  and  $\overline{OB}$  be linearly independent.

Denote the closed convex solid cone for the ordinary lattice irrational angle  $\angle AOB$  by  $C(AOB)$ . The boundary of the convex hull of all lattice points of the cone  $C(AOB)$  except the origin is homeomorphic to the straight line. The closure in the plane of the intersection of this boundary with the open cone  $AOB$  is called the *sail* for the cone  $C(AOB)$ . A lattice point of the sail is said to be a *vertex* of the sail if there is no lattice segment of the sail containing this point in the interior. The sail of the cone  $C(AOB)$  is a broken line with an infinite number of vertices and without self-intersections. We orient the sail in the direction from  $\overline{OA}$  to  $\overline{OB}$ . (For the definition of the sail and its higher dimensional generalization, see, for instance, the works [1], [16], and [10].)

In the case of ordinary lattice R-irrational angle we denote the vertices of the sail by  $V_i$ , for  $i \geq 0$ , according to the orientation of the sail (such that  $V_0$  is contained in the ray  $OA$ ). In the case of ordinary lattice L-irrational angle we denote the vertices of the sail by  $V_{-i}$ , for  $i \geq 0$ , according to the orientation of the sail (such that  $V_0$  is contained in the ray  $OB$ ). In the case of ordinary lattice LR-irrational angle we denote the vertices of the sail by  $V_{-i}$ , for  $i \in \mathbb{Z}$ , according to the orientation of the sail (such that  $V_0$  is an arbitrary vertex of the sail).



**Definition 5.4.** Suppose that the vectors  $\overline{OA}$  and  $\overline{OB}$  of an ordinary lattice irrational angle  $\angle AOB$  are linearly independent. Let  $V_i$  be the vertices of the corresponding sail. The sequence of lattice lengths and sines

$$\begin{aligned} &(\ell(V_0V_1), \text{lsin } \angle V_0V_1V_2, \ell(V_1V_2), \text{lsin } \angle V_1V_2V_3, \dots), \text{ or} \\ &(\dots, \text{lsin } \angle V_{-3}V_{-2}V_{-1}, \ell(V_{-2}V_{-1}), \text{lsin } \angle V_{-2}V_{-1}V_0, \ell(V_{-1}V_0)), \text{ or} \\ &(\dots, \text{lsin } \angle V_{-2}V_{-1}V_0, \ell(V_{-1}V_0), \text{lsin } \angle V_{-1}V_0V_1, \ell(V_0V_1), \dots) \end{aligned}$$

is called the lattice length-sine sequence for the ordinary lattice irrational angle  $\angle AOB$ , if this angle is R-irrational, L-irrational, or LR-irrational respectively.

**Proposition 5.5. a).** *The elements of the lattice length-sine sequence for any ordinary lattice irrational angle are positive integers.*

**b).** *The lattice length-sine sequences of lattice-congruent ordinary lattice irrational angles coincide.*  $\square$

**5.4. Lattice tangents for ordinary lattice R-irrational angles.** In this subsection we show, that the notion of lattice tangent is well-defined for the case of ordinary lattice R-irrational angles. We also formulate some basic properties of lattice tangent.

Let us generalize a notion of tangent to the case of R-irrational angles using the property of Theorem 1.10 for tangents of ordinary angles.

**Definition 5.6.** Let the vectors  $\overline{OA}$  and  $\overline{OB}$  of an ordinary lattice R-irrational angle  $\angle AOB$  be linearly independent. Suppose that  $V_i$  are the vertices of the corresponding sail. Let

$$(\ell(V_0V_1), \text{lsin } \angle V_0V_1V_2, \dots, \text{lsin } \angle V_{n-2}V_{n-1}V_n, \ell(V_{n-1}V_n), \dots)$$

be the lattice length-sine sequence for the ordinary lattice angle  $\angle AOB$ . The *lattice tangent* of the ordinary lattice R-irrational angle  $\angle AOB$  is the following number

$$[\ell(V_0V_1), \text{lsin } \angle V_0V_1V_2, \dots, \text{lsin } \angle V_{n-2}V_{n-1}V_n, \ell(V_{n-1}V_n), \dots].$$

We denote it by  $\text{ltan } \angle AOB$ . We say also that this number is the *continued fraction associated with the sail* of the ordinary lattice R-irrational angle  $\angle AOB$ .

**Proposition 5.7. a).** *For any ordinary lattice R-irrational angle  $\angle AOB$  with linearly independent vectors  $\overline{OA}$  and  $\overline{OB}$  the number  $\text{ltan } \angle AOB$  is irrational and greater than 1.*

**b).** *The values of the function  $\text{ltan}$  at two lattice-congruent ordinary lattice angles coincide.*  $\square$

**5.5. Lattice arctangent for ordinary lattice R-irrational angles.** We continue with the definition of lattice arctangents for ordinary lattice R-irrational angles and the main properties of these arctangents.

Consider the system of coordinates  $OXY$  on the space  $\mathbb{R}^2$  with the coordinates  $(x, y)$  and the origin  $O$ . We work with the integer lattice of  $OXY$ .

For any rationals  $p_1$  and  $p_2$  we denote by  $\alpha_{p_1, p_2}$  the angle with the vertex at the origin and two edges  $\{(x, p_i x) | x > 0\}$ , where  $i = 1, 2$ .

**Definition 5.8.** For any irrational real  $s > 1$ , the ordinary lattice angle  $\angle AOB$  with the vertex  $O$  at the origin,  $A = (1, 0)$ , and  $B = (1, s)$ , is called the *lattice arctangent* of  $s$  and denoted by  $\text{larctan } s$ .

**Theorem 5.9. a).** *For any irrational  $s$ , such that  $s > 1$ ,*

$$\text{ltan}(\text{larctan } s) = s.$$

b). For any ordinary lattice R-irrational angle  $\alpha$  the following holds:

$$\text{larctan}(\text{ltan } \alpha) \cong \alpha.$$

*Proof.* Let us prove the first statement of the theorem. Let  $s > 1$  be some irrational real. Suppose that the sail of the angle  $\text{larctan } s$  is the infinite broken line  $A_0A_1\dots$  and the corresponding ordinary continued fraction is  $[a_0, a_1, a_2, \dots]$ . Let also the coordinates of  $A_i$  be  $(x_i, y_i)$ .

We consider the ordinary lattice angles  $\alpha_i$ , corresponding to the broken lines  $A_0\dots A_i$ , for  $i > 0$ . Then,

$$\lim_{i \rightarrow \infty} (y_i/x_i) = s/1.$$

By Theorem 1.7 for any positive integer  $i$  the ordinary lattice angle  $\alpha_i$  coincides with  $\text{larctan}([a_0, a_1, \dots, a_{2i-2}])$ , and hence the coordinates  $(x_i, y_i)$  of  $A_i$  satisfy

$$y_i/x_i = [a_0, a_1, \dots, a_{2i-2}].$$

Therefore,

$$\lim_{i \rightarrow \infty} ([a_0, a_1, \dots, a_{2i-2}]) = s.$$

So, we obtain the first statement of the theorem:

$$\text{ltan}(\text{larctan } s) = s.$$

Now we prove the second statement. Consider an ordinary lattice R-irrational angle  $\alpha$ . Suppose that the sail of the angle  $\alpha$  is the infinite broken line  $A_0A_1\dots$

For any positive integer  $i$  we consider the ordinary angle  $\alpha_i$ , corresponding to the broken lines  $A_0\dots A_i$ .

Denote by  $C(\beta)$  the cone, corresponding to the ordinary lattice (possible irrational) angle  $\beta$ . Note that  $C(\beta')$  and  $C(\beta'')$  are lattice-congruent iff  $\beta \cong \beta'$ .

By Theorem 1.7 we have:

$$\text{larctan}(\text{ltan } \alpha_i) \cong \alpha_i.$$

Since for any positive integer  $n$  the following is true

$$\bigcup_{i=1}^n C(\alpha_i) \cong \bigcup_{i=1}^n C(\text{larctan}(\text{ltan } \alpha_i))$$

we obtain

$$C(\alpha) \cong \bigcup_{i=1}^{\infty} C(\alpha_i) \cong \bigcup_{i=1}^{\infty} C(\text{larctan}(\text{ltan } \alpha_i)) \cong C(\text{larctan}(\text{ltan } \alpha)).$$

Therefore,

$$\text{larctan}(\text{ltan } \alpha) \cong \alpha.$$

This concludes the proof of Theorem 5.9. □

There is a description of ordinary lattice R-irrational angles similar to the description of ordinary lattice angles (see Proposition 1.5e).

**Theorem 5.10. (Description of ordinary lattice R-irrational angles.)**

a). For any sequence of positive integers  $(a_0, a_1, a_2, \dots)$  there exist some ordinary lattice R-irrational angle  $\alpha$  such that  $\text{ltan } \alpha = [a_0, a_1, a_2, \dots]$ .

b). Two ordinary lattice R-irrational angles are lattice-congruent iff they have equal lattice tangents.

*Proof.* Theorem 5.9a implies the first statement of the theorem.

Let us prove the second statement. Suppose that the ordinary lattice R-irrational angles  $\alpha$  and  $\beta$  are lattice-congruent, then their sails are also lattice-congruent. Thus their lattice-sine sequences coincide. Therefore,  $\text{ltan } \alpha = \text{ltan } \beta$ .

Suppose now that the lattice tangents for two ordinary lattice R-irrational angles  $\alpha$  and  $\beta$  are equivalent. Since for any irrational real the corresponding continued fraction exists and is unique, the lattice length-sine sequences for the sails of  $\alpha$  and  $\beta$  coincide.

Let  $V_\alpha$  be the vertex of the angle  $\alpha$ , and  $A_0A_1 \dots$  be the sail for  $\alpha$ . For any positive integer  $i$  denote by  $\alpha_i$  the angle  $A_0V_\alpha A_i$  (with the sail  $A_0 \dots A_i$ ). Let  $V_\beta$  be the vertex of the angle  $\beta$ , and  $B_0B_1 \dots$  be the sail for  $\beta$ . For any positive integer  $i$  denote by  $\beta_i$  the angle  $B_0V_\beta B_i$  (with the sail  $B_0 \dots B_i$ ).

Since the ordinary lattice R-irrational angles  $\alpha$  and  $\beta$  have the same lattice length-sine sequences, for any integer  $i$  the ordinary lattice angles  $\alpha_i$  and  $\beta_i$  have the same lattice length-sine sequences, and, therefore,  $\alpha_i \cong \beta_i$ .

Consider the lattice-affine transformation  $\xi$  that takes the vertex  $V_\alpha$  to the vertex  $V_\beta$ , the lattice point  $A_0$  to the lattice point  $B_0$  and  $A_1$  to  $B_1$ . (Such transformation exist since  $\text{ll}(A_0A_1) = \text{ll}(B_0B_1)$ .)

Choose an arbitrary  $i \geq 1$ . Since

$$\xi(A_0) = B_0, \quad \xi(A_1) = B_1, \quad \text{and} \quad \alpha_i \cong \beta_i,$$

we have  $\xi(A_i) = \xi(B_i)$ . This implies that the lattice-affine transformation  $\xi$  takes the sail for the ordinary lattice R-irrational angle  $\alpha$  to the sail for the ordinary lattice R-irrational angle  $\beta$ . Therefore, the angles  $\alpha$  and  $\beta$  are lattice-congruent.  $\square$

**5.6. Lattice signed length-sine infinite sequences.** In this section we work in the oriented two-dimensional real vector space with the fixed lattice. As previously, we fix coordinates  $OXY$  on this space.

A union of (ordered) lattice segments  $\dots, A_{i-1}A_i, A_iA_{i+1}, A_{i+1}A_{i+2}, \dots$  infinite to the right (to the left, or both sides) is said to be a *lattice oriented R-infinite (L-infinite, or LR-infinite) broken line*, if any segment of the broken line is not of zero length, and any two consecutive segments are not contained in the same straight line. We denote this broken line by  $\dots A_{i-1}A_iA_{i+1}A_{i+2} \dots$ . We also say that the lattice oriented broken line  $\dots A_{i+2}A_{i+1}A_iA_{i-1} \dots$  is *inverse* to the broken line  $\dots A_{i-1}A_iA_{i+1}A_{i+2} \dots$ .

**Definition 5.11.** Consider a lattice infinite oriented broken line and a point not in this line. The broken line is said to be *on the unit distance* from the point if all edges of the broken line are on the unit lattice distance from the given point.

Now, let us now associate to any lattice infinite oriented broken line on the unit distance from some point the following sequence of non-zero elements.

**Definition 5.12.** Let  $\dots A_{i-1}A_iA_{i+1}A_{i+2} \dots$  be a lattice oriented infinite broken line on the unit distance from some lattice point  $V$ . Let

$$\begin{aligned} a_{2i-3} &= \text{sgn}(A_{i-2}VA_{i-1}) \text{sgn}(A_{i-1}VA_i) \text{sgn}(A_{i-2}A_{i-1}A_i) \text{lsin } \angle A_{i-2}A_{i-1}A_i, \\ a_{2i-2} &= \text{sgn}(A_{i-1}VA_i) \text{ll}(A_{i-1}A_i) \end{aligned}$$

for all possible indexes  $i$ . The sequence  $(\dots a_{2i-3}, a_{2i-2}, a_{2i-1} \dots)$  is called a *lattice signed length-sine* sequence for the lattice oriented infinite broken line on the unit distance from  $V$ .

**Proposition 5.13.** *A lattice signed length-sine sequence for the given lattice infinite oriented broken line and the point is invariant under the group action of proper lattice-affine transformations.*

*Proof.* The statement of the proposition holds, since the functions  $\text{sgn}$ ,  $\ell$ , and  $\text{lsin}$  are invariant under the group action of proper lattice-affine transformations.  $\square$

**5.7. Proper lattice-congruence of lattice oriented infinite broken lines on the unit distance from the lattice points.** Let us formulate a necessary and sufficient conditions for two lattice infinite oriented broken lines on the unit distance from two lattice points to be proper lattice-congruent.

**Theorem 5.14.** *The lattice signed length-sine sequences of two lattice infinite oriented broken lines on the unit distance from lattice points  $V_1$  and  $V_2$  coincide, iff there exist proper lattice-affine transformation taking the point  $V_1$  to  $V_2$  and one oriented broken line to the other.*

*Proof.* The lattice signed length-sine sequence for any lattice infinite oriented broken line on the unit distance is uniquely defined, and by Proposition 3.3 is invariant under the group action of proper lattice-affine transformations. Therefore, the lattice signed length-sine sequences for two proper lattice-congruent broken lines coincide.

Suppose now that we have two lattice oriented infinite broken lines  $\dots A_{i-1}A_iA_{i+1}\dots$ , and  $\dots B_{i-1}B_iB_{i+1}\dots$  on the unit distance from the points  $V_1$  and  $V_2$ , and with the same lattice signed length-sine sequences. Consider the lattice-affine transformation  $\xi$  that takes the point  $V_1$  to  $V_2$ ,  $A_i$  to  $B_i$ , and  $A_{i+1}$  to  $B_{i+1}$  for some integer  $i$ . Since  $\text{sgn}(A_iV_iA_{i+1}) = \text{sgn}(B_iV_iB_{i+1})$ , the lattice-affine transformation  $\xi$  is proper. By Theorem 3.4 the transformation  $\xi$  takes any finite oriented broken line  $A_sA_{s+1}\dots A_t$  containing the segment  $A_iA_{i+1}$  to the oriented broken line  $B_sB_{s+1}\dots B_t$ . Therefore, the transformation  $\xi$  takes the lattice oriented infinite broken lines  $\dots A_{i-1}A_iA_{i+1}\dots$  to the oriented broken line  $\dots B_{i-1}B_iB_{i+1}\dots$  and the lattice point  $V_1$  to the lattice point  $V_2$ .

This concludes the proof of Theorem 5.14.  $\square$

**5.8. Equivalence classes of almost positive lattice infinite oriented broken lines and corresponding expanded lattice infinite angles.** We start this subsection with the following general definition.

**Definition 5.15.** We say that the lattice infinite oriented broken line on the unit distance from some lattice point is *almost positive* if the elements of the corresponding lattice signed length-sine sequence are all positive, except for some finite number of elements.

Let  $l$  be the lattice (finite or infinite) oriented broken line  $\dots A_{n-1}A_n\dots A_mA_{m+1}\dots$ . Denote by  $l(-\infty, A_n)$  the broken line  $\dots A_{n-1}A_n$ . Denote by  $l(A_m, +\infty)$  the broken line  $A_mA_{m+1}\dots$ . Denote by  $l(A_n, A_m)$  the broken line  $A_n\dots A_m$ .

**Definition 5.16.** Two lattice oriented infinite broken lines  $l_1$  and  $l_2$  on unit distance from  $V$  are said to be *equivalent* if there exist two vertices  $W_{11}$  and  $W_{12}$  of the broken line  $l_1$  and two vertices  $W_{21}$  and  $W_{22}$  of the broken line  $l_2$  such that the following three conditions are satisfied:  
*i)* the broken line  $l_1(-\infty, W_{11})$  coincides with the broken line  $l_2(-\infty, W_{21})$ ;  
*ii)* the broken line  $l_1(W_{12}, +\infty)$  coincides with the broken line  $l_2(W_{22}, +\infty)$ ;  
*iii)* the closed broken line generated by  $l_1(W_{11}, W_{12})$  and the inverse of  $l_2(W_{21}, W_{22})$  is homotopy equivalent to the point on  $\mathbb{R}^2 \setminus \{V\}$ .

Now we give the definition of expanded lattice irrational angles.

**Definition 5.17.** An equivalence class of lattice R/L/LR-infinite oriented broken lines on unit distance from  $V$  containing the broken line  $l$  is called the *expanded lattice R/L/LR-infinite angle for the equivalence class of  $l$*  at the vertex  $V$  and denoted by  $\angle(V; l)$  (or, for short, *expanded lattice R/L/LR-infinite angle*).

**Definition 5.18.** Two expanded irrational lattice angles  $\Phi_1$  and  $\Phi_2$  are said to be *proper lattice-congruent* iff there exist a proper lattice-affine transformation sending the class of lattice oriented broken lines corresponding to  $\Phi_1$  to the class of lattice oriented broken lines corresponding to  $\Phi_2$ . We denote it by  $\Phi_1 \cong \Phi_2$ .

*Remark 5.19.* Since all sails for ordinary lattice irrational angles are lattice infinite oriented broken lines, the set of all ordinary lattice irrational angles is naturally embedded into the set of expanded lattice irrational angles. An ordinary lattice irrational angle with a sail  $S$  corresponds to the expanded lattice irrational angle with the equivalence class of the broken line  $S$ .

**5.9. Revolution number for expanded lattice L- and R-irrational angles.** Let us extend the revolution number to the case of almost positive infinite oriented broken lines.

**Definition 5.20.** Let  $\dots A_{i-1}A_iA_{i+1}\dots$  be some lattice R-, L- or LR-infinite almost positive oriented broken line, and  $r = \{V + \lambda\bar{v} | \lambda \geq 0\}$  be the oriented ray for an arbitrary vector  $\bar{v}$  with the vertex at  $V$ . Suppose that the ray  $r$  does not contain the edges of the broken line, and the broken line does not contain the vertex  $V$ . We call the number

$$\begin{aligned} \lim_{n \rightarrow +\infty} \#(r, V, A_0A_1 \dots A_n) & \quad \text{if the broken line is R-infinite,} \\ \lim_{n \rightarrow +\infty} \#(r, V, A_{-n} \dots A_{-1}A_0) & \quad \text{if the broken line is L-infinite,} \\ \lim_{n \rightarrow +\infty} \#(r, V, A_{-n}A_{-n+1} \dots A_n) & \quad \text{if the broken line is LR-infinite} \end{aligned}$$

the *intersection number* of the ray  $r$  and the lattice almost positive infinite oriented broken line  $\dots A_{i-1}A_iA_{i+1}\dots$  and denote it by  $\#(r, V, \dots A_{i-1}A_iA_{i+1}\dots)$ .

**Proposition 5.21.** *The intersection number of the ray  $r$  and an almost positive lattice infinite oriented broken line is well-defined.*

*Proof.* Consider an almost positive lattice infinite oriented broken line  $l$ . Let us show that the broken line  $l$  intersects the ray  $r$  only finitely many times.

By Definition 5.15 there exist vertices  $W_1$  and  $W_2$  of this broken line such that the signed lattice-sine sequence for the lattice oriented broken line  $l(-\infty, W_1)$  contains only positive elements, and the signed lattice-sine sequence for the oriented broken line  $l(W_2, +\infty)$  also contains only positive elements.

The positivity of lattice-sine sequences implies that the lattice oriented broken lines  $l(-\infty, W_1)$ , and  $l(W_2, +\infty)$  are the sails for some angles with the vertex  $V$ . Thus, these two broken lines intersect the ray  $r$  at most once each. Therefore, the broken line  $l$  intersects the ray  $r$  at most once at the part  $l(-\infty, W_1)$ , only a finite number times at the part  $l(W_1, W_2)$ , and at most once at the part  $l(W_2, +\infty)$ .

So, the lattice infinite oriented broken line  $l$  intersects the ray  $r$  only finitely many times, and, therefore, the corresponding intersection number is well-defined.  $\square$

Now we give a definition of the lattice revolution number for expanded lattice R-irrational and L-irrational angles.

**Definition 5.22. a).** Consider an arbitrary R-infinite (or L-infinite) expanded lattice angle  $\angle(V, l)$ , where  $V$  is some lattice point, and  $l$  is a lattice infinite oriented almost-positive broken line. Let  $A_0$  be the first (the last) vertex of  $l$ . Denote the rays  $\{V + \lambda \overline{VA_0} | \lambda \geq 0\}$  and  $\{V - \lambda \overline{VA_0} | \lambda \geq 0\}$  by  $r_+$  and  $r_-$  respectively. The following number

$$\frac{1}{2}(\#(r_+, V, l) + \#(r_-, V, l))$$

is called the *lattice revolution number* for the expanded lattice irrational angle  $\angle(V, l)$ , and denoted by  $\#(\angle(V, l))$ .

**Proposition 5.23.** *The revolution number of an R-irrational (or L-irrational) expanded lattice angle is well-defined.*

*Proof.* Consider an arbitrary expanded lattice R-irrational angle  $\angle(V, A_0A_1\dots)$ . Let

$$r_+ = \{V + \lambda \overline{VA_0} | \lambda \geq 0\} \quad \text{and} \quad r_- = \{V - \lambda \overline{VA_0} | \lambda \geq 0\}.$$

Since the lattice oriented broken line  $A_0A_1A_2\dots$  is on the unit lattice distance from the point  $V$ , any segment of this broken line is on the unit lattice distance from  $V$ . Thus, the broken line does not contain  $V$ , and the rays  $r_+$  and  $r_-$  do not contain edges of the broken line.

Suppose that

$$\angle V, A_0A_1A_2\dots = \angle V', A'_0A'_1A'_2\dots$$

This implies that  $V = V'$ ,  $A_0 = A'_0$ ,  $A_{n+k} = A'_{m+k}$  for some integers  $n$  and  $m$  and any non-negative integer  $k$ , and the broken lines  $A_0A_1\dots A_nA'_{m-1}\dots A'_1A'_0$  is homotopy equivalent to the point on  $\mathbb{R}^2 \setminus \{V\}$ . Thus,

$$\begin{aligned} \#(\angle V, A_0A_1\dots) - \#(\angle V', A'_0A'_1\dots) &= \\ \frac{1}{2}(\#(r_+, A_0A_1\dots A_nA'_{m-1}\dots A'_1A'_0) + \#(r_-, A_0A_1\dots A_nA'_{m-1}\dots A'_1A'_0)) &= \\ = 0+0 = 0. \end{aligned}$$

And hence

$$\#(\angle V, A_0A_1A_2\dots) = \#(\angle V', A'_0A'_1A'_2\dots).$$

Therefore, the revolution number of any expanded lattice R-irrational angle is well-defined.

The proof for L-irrational angles repeats the proof for R-irrational angles and is omitted here.  $\square$

**Proposition 5.24.** *The revolution number of expanded lattice R/L-irrational angles is invariant under the group action of the proper lattice-affine transformations.*  $\square$

**5.10. Normal forms of expanded lattice R- and L-irrational lattice angles.** In this subsection we formulate and prove a theorem on normal forms of expanded lattice R-irrational and L-irrational lattice angles.

For the theorems of this subsection we introduce the following notation. By the sequence

$$((a_0, \dots, a_n) \times k\text{-times}, b_0, b_1 \dots),$$

where  $k \geq 0$ , we denote the sequence:

$$\underbrace{(a_0, \dots, a_n, a_0, \dots, a_n, \dots, a_0, \dots, a_n)}_{k\text{-times}}, b_0, b_1, \dots).$$

By the sequence

$$(\dots, b_{-2}, b_{-1}, b_0, (a_0, \dots, a_n) \times k\text{-times}),$$

where  $k \geq 0$ , we denote the following sequence:

$$(\dots, b_{-2}, b_{-1}, b_0, \underbrace{a_0, \dots, a_n, a_0, \dots, a_n, \dots}_{k\text{-times}}, a_0, \dots, a_n).$$

We start with the case of expanded lattice R-irrational angles.

**Definition 5.25.** Consider a lattice R-infinite oriented broken line  $A_0A_1\dots$  on the unit distance from the origin  $O$ . Let also  $A_0$  be the point  $(1, 0)$ , and the point  $A_1$  be on the line  $x = 1$ . If the signed length-sine sequence of the expanded ordinary R-irrational angle  $\Phi_0 = \angle(O, A_0A_1\dots)$  coincides with the following sequence (we call it *characteristic sequence* for the corresponding angle):

**IV<sub>k</sub>**  $((1, -2, 1, -2) \times k\text{-times}, a_0, a_1, \dots)$ , where  $k \geq 0$ ,  $a_i > 0$ , for  $i \geq 0$ , then we denote the angle  $\Phi_0$  by  $k\pi + \text{larctan}([a_0, a_1, \dots])$  and say that  $\Phi_0$  is of the type **IV<sub>k</sub>**;

**V<sub>k</sub>**  $((-1, 2, -1, 2) \times k\text{-times}, a_0, a_1, \dots)$ , where  $k > 0$ ,  $a_i > 0$ , for  $i \geq 0$ , then we denote the angle  $\Phi_0$  by  $-k\pi + \text{larctan}([a_0, a_1, \dots])$  and say that  $\Phi_0$  is of the type **V<sub>k</sub>**.

**Theorem 5.26.** For any expanded lattice R-irrational angle  $\Phi$  there exist and unique a type among the types **IV-V** and a unique expanded lattice R-irrational angle  $\Phi$  of that type such that  $\Phi_0$  is proper lattice congruent to  $\Phi_0$ .

The expanded lattice R-irrational angle  $\Phi_0$  is said to be *the normal form* for the expanded lattice R-irrational angle  $\Phi$ .

*Proof.* First, we prove that any two distinct expanded lattice R-irrational angles listed in Definition 5.25 are not proper lattice-congruent. Let us note that the revolution numbers of expanded lattice angles distinguish the types of the angles. The revolution number for the expanded lattice angles of the type **IV<sub>k</sub>** is  $1/4 + 1/2k$  where  $k \geq 0$ . The revolution number for the expanded lattice angles of the type **V<sub>k</sub>** is  $1/4 - 1/2k$  where  $k > 0$ .

We now prove that the normal forms of the same type **IV<sub>k</sub>** (or **V<sub>k</sub>**) are not proper lattice-congruent for any integer  $k$ . Consider the expanded lattice R-infinite angle  $\Phi = k\pi + \text{larctan}([a_0, a_1, \dots])$ . Suppose that a lattice oriented broken line  $A_0A_1A_2\dots$  on the unit distance from  $O$  defines the angle  $\Phi$ . Let also that the signed lattice-sine sequence for this broken line be characteristic.

If  $k$  is even, then the ordinary lattice R-irrational angle with the sail  $A_{2k}A_{2k+1}\dots$  is proper lattice-congruent to the angle  $\text{larctan}([a_0, a_1, \dots])$ . This angle is a proper lattice-affine invariant for the expanded lattice R-irrational angle  $\Phi$  (since  $A_{2k} = A_0$ ). This invariant distinguish the expanded lattice R-irrational angles of type **IV<sub>k</sub>** (or **V<sub>k</sub>**) for even  $k$ .

If  $k$  is odd, then denote  $B_i = V + \overline{A_iV}$ . The ordinary lattice R-irrational angle with the sail  $B_{2k}B_{2k+1}\dots$  is proper lattice-congruent to the angle  $\text{larctan}([a_0, a_1, \dots])$ . This angle is a proper lattice-affine invariant of the expanded lattice R-irrational angle  $\Phi$  (since  $B_{2k} = V + \overline{A_0V}$ ). This invariant distinguish the expanded lattice R-irrational angles **IV<sub>k</sub>** (or **V<sub>k</sub>**) for odd  $k$ .

Therefore, the expanded lattice angles listed in Definition 5.25 are not proper lattice-congruent.

Secondly, we prove that an arbitrary expanded lattice R-irrational angle is proper lattice-congruent to some of the expanded lattice angles listed in Definition 5.25.

Consider an arbitrary expanded lattice R-irrational angle  $\Phi = \angle(V, A_0A_1\dots)$ . Suppose that  $\#(\Phi) = 1/4 + k/2$  for some non-negative integer  $k$ . By Proposition 5.21 there exist an integer positive number  $n_0$  such that the lattice oriented broken line  $A_{n_0}A_{n_0+1}\dots$  does not intersect

the rays  $r_+ = \{V + \lambda \overline{VA_0} \mid \lambda \geq 0\}$  and  $r_- = \{V - \lambda \overline{VA_0} \mid \lambda \geq 0\}$ , and the signed lattice length-sine sequence  $(a_{2n_0-2}, a_{2n_0-1}, \dots)$  for the oriented broken line  $A_{n_0}A_{n_0+1}\dots$  does not contain non-positive elements.

By Theorem 3.14 there exist integers  $k$  and  $m$ , and a lattice oriented broken line

$$A_0B_1B_2\dots B_{2k}B_{2k+1}\dots B_{2k+m}A_{n_0}$$

with lattice length-sine sequence of the form

$$((1, -2, 1, -2) \times k\text{-times}, b_0, b_1, \dots, b_{2m-2}),$$

where all  $b_i$  are positives.

Consider now the lattice oriented infinite broken line  $A_0B_1B_2\dots B_{2k+m-1}A_{n_0}A_{n_0+1}\dots$ . The length-sine sequence for this broken line is as follows

$$((1, -2, 1, -2) \times k\text{-times}, b_0, b_1, \dots, b_{2m-2}, v, a_{2n_0-2}a_{2n_0-1}, \dots),$$

where  $v$  is (not necessary positive) integer.

Note that the lattice oriented broken line  $A_0B_1B_2\dots B_{2k+m}A_{n_0}$  is a sail for the angle  $\angle A_0VA_{n_0}$  and the sequence  $A_{n_0}A_{n_0+1}\dots$  is a sail for some R-irrational angle (we denote it by  $\alpha$ ). Let  $H_1$  be the convex hull of all lattice points of the angle  $\angle A_0VA_{n_0}$  except the origin, and  $H_2$  be the convex hull of all lattice points of the angle  $\alpha$  except the origin. Note that  $H_1$  intersects  $H_2$  in the ray with the vertex at  $A_{n_0}$ .

The lattice oriented infinite broken line  $B_{2k}B_{2k+2}\dots B_{2k+m}A_{n_0}A_{n_0+1}\dots$  intersects the ray  $r_+$  in the unique point  $B_{2k}$  and does not intersect the ray  $r_-$ . Hence there exists a straight line  $l$  intersecting both boundaries of  $H_1$  and  $H_2$ , such that the open half-plane with the boundary straight line  $l$  containing the origin does not intersect the sets  $H_1$  and  $H_2$ .

Denote  $B_0 = A_0$  and  $B_{2k+m+1} = A_{n_0}$ . The intersection of the straight line  $l$  with  $H_1$  is either a point  $B_s$  (for  $2k \leq s \leq 2k+m+1$ ), or a boundary segment  $B_sB_{s+1}$  for some integer  $s$  satisfying  $2k \leq s \leq 2k+m$ . The intersection of  $l$  with  $H_2$  is either a point  $A_t$  for some integer  $t \geq n_0$ , or a boundary segment  $A_{t-1}A_t$  for some integer  $t > n_0$ .

Since the triangle  $\triangle VA_tB_s$  does not contain interior point of  $H_1$  and  $H_2$ , the lattice points of  $\triangle VA_tB_s$  distinct to  $B$  are on the segment  $A_tB_s$ . Hence, the segment  $A_tB_s$  is on unit lattice distance to the vertex  $V$ . Therefore, the lattice infinite oriented broken line

$$A_0B_1B_2\dots B_sA_tA_{t+1}\dots$$

is on lattice unit distance.

Since the lattice oriented broken line  $B_k\dots B_sA_tA_{t+1}\dots$  is convex, it is a sail for some lattice R-irrational angle. (Actually, the case  $B_s = A_t = A_{n_0}$  is also possible, then delete one of the copies of  $A_{n_0}$  from the sequence.) We denote this broken line by  $C_{2k+1}C_{2k+2}\dots$ . The corresponding signed lattice length-sine sequence is  $(c_{4k}, c_{4k+1}, c_{4k+2}, \dots)$ , where  $c_i > 0$  for  $i \geq 4k$ . Thus the signed lattice length-sine sequence for the lattice ordered broken line  $A_0B_1B_2\dots B_{2k}C_{2k+1}C_{2k+2}\dots$  is

$$((1, -2, 1, -2) \times (k-1)\text{-times}, 1, -2, 1, w, (c_{4k}, c_{4k+1}, c_{4k+2}, \dots)),$$

where  $w$  is an integer that is not necessary equivalent to  $-2$ .

Consider an expanded lattice angle  $\angle(V, A_0B_1B_2\dots B_{2k}C_{2k+1})$ . By Lemma 3.15 there exists a lattice oriented broken line  $C_0\dots C_{2k+1}$  with the vertices  $C_0 = A_0$  and  $C_{2k+1}$  of the same equivalence class, such that  $C_{2k} = B_{2k}$ , and the signed lattice length-sine sequence for it is

$$((1, -2, 1, -2) \times k\text{-times}, c_{4k}, c_{4k+1}).$$



Therefore, the lattice oriented R-infinite broken line  $C_0C_1\dots$  for the angle  $\angle(V, A_0A_1\dots)$  has the signed lattice length-sign sequence coinciding with the characteristic sequence for the angle  $k\pi + \text{larctan}([c_{4k}, c_{4k+1}, \dots])$ . Therefore,

$$\Phi \cong k\pi + \text{larctan}([c_{4k}, c_{4k+1}, \dots]).$$

This concludes the proof of the theorem for the case of nonnegative integer  $k$ .

The proof for the case of negative  $k$  repeats the proof for the nonnegative case and is omitted here.  $\square$

Let us give the definition of trigonometric functions for expanded lattice R-irrational angles.

**Definition 5.27.** Consider an arbitrary expanded lattice R-irrational angle  $\Phi$  with the normal form  $k\pi + \varphi$  for some integer  $k$ .

a). The ordinary lattice R-irrational angle  $\varphi$  is said to be *associated* with the expanded lattice R-irrational angle  $\Phi$ .

b). The number  $\text{ltan}(\varphi)$  is called the lattice *tangent* of the expanded lattice R-irrational angle  $\Phi$ .

We continue now with the case of expanded lattice L-irrational angles.

**Definition 5.28.** The expanded lattice irrational angle  $\angle(V, \dots A_{i+2}A_{i+1}A_i \dots)$  is said to be transpose to the expanded lattice irrational angle  $\angle(V, \dots A_iA_{i+1}A_{i+2} \dots)$  and denoted by  $(\angle(V, \dots A_iA_{i+1}A_{i+2} \dots))^t$ .

**Definition 5.29.** Consider a lattice L-infinite oriented broken line  $\dots A_{-1}A_0$  on the unit distance from the origin  $O$ . Let also  $A_0$  be the point  $(1, 0)$ , and the point  $A_{-1}$  be on the straight line  $x = 1$ . If the signed length-sine sequence of the expanded ordinary L-irrational angle  $\Phi_0 = \angle(O, \dots A_{-1}A_0)$  coincides with the following sequence (we call it *characteristic sequence* for the corresponding angle):

$\mathbf{IV}_k$ )  $(\dots, a_{-1}, a_0, (-2, 1, -2, 1) \times k\text{-times})$ , where  $k \geq 0$ ,  $a_i > 0$ , for  $i \leq 0$ , then we denote the angle  $\Phi_0$  by  $k\pi + \text{larctan}^t([a_0, a_{-1}, \dots])$  and say that  $\Phi_0$  is of the type  $\mathbf{IV}_k$ ;

$\mathbf{V}_k$ )  $(\dots, a_{-1}, a_0, (2, -1, 2, -1) \times k\text{-times})$ , where  $k > 0$ ,  $a_i > 0$ , for  $i \leq 0$ , then we denote the angle  $\Phi_0$  by  $-k\pi + \text{larctan}^t([a_0, a_{-1}, \dots])$  and say that  $\Phi_0$  is of the type  $\mathbf{V}_k$ .

**Theorem 5.30.** For any expanded lattice L-irrational angle  $\Phi$  there exist and unique a type among the types  $\mathbf{IV-V}$  and a unique expanded lattice L-irrational angle  $\Phi_0$  of that type such that  $\Phi$  is proper lattice congruent to  $\Phi_0$ .

The expanded lattice L-irrational angle  $\Phi_0$  is said to be *the normal form* for the expanded lattice L-irrational angle  $\Phi$ .

*Proof.* After transposing the set of all angles and change of the orientation of the plane the statement of Theorem 5.30 coincide with the statement of Theorem 5.26.  $\square$

5.11. **Sums of expanded lattice angles and expanded lattice irrational angles.** We conclude this section with a particular definitions of sums of ordinary lattice angles, and ordinary lattice R-irrational or/and L-irrational angles.

**Definition 5.31.** Consider expanded lattice angles  $\Phi_i$ , where  $i = 1, \dots, t$ , an expanded lattice R-irrational angle  $\Phi_r$ , and an expanded lattice L-irrational angle  $\Phi_l$ . Let the characteristic signed lattice lengths-sine sequences for the normal forms of the angles  $\Phi_i$  be  $(a_{0,i}, a_{1,i}, \dots, a_{2n_i,i})$ ; of

$\Phi_r$  be  $(a_{0,r}, a_{1,r}, \dots)$ , and of  $\Phi_l$  be  $(\dots, a_{-1,l}, a_{0,l})$ .

Let  $M_R = (m_1, \dots, m_{t-1}, m_r)$  be some  $t$ -tuple of integers. The normal form of any expanded lattice angle, corresponding to the following lattice signed length-sine sequence

$$\begin{aligned} & (a_{0,1}, a_{1,1}, \dots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \dots, a_{2n_2,2}, m_2, \dots \\ & \dots, m_{t-1}, a_{0,t}, a_{1,t}, \dots, a_{2n_t,t} m_r, a_{0,r}, a_{1,r}, \dots) \end{aligned}$$

is called the  $M_R$ -sum of expanded lattice angles  $\Phi_i$  ( $i = 1, \dots, t$ ) and  $\Phi_r$ .

Let  $M_L = (m_l, m_1, \dots, m_{t-1})$  be some  $t$ -tuple of integers. The normal form for any expanded lattice angle, corresponding to the following lattice signed length-sine sequence

$$\begin{aligned} & (\dots, a_{-1,l}, a_{0,l}, m_l, a_{0,1}, a_{1,1}, \dots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \dots, a_{2n_2,2}, m_2, \dots \\ & \dots, m_{t-1}, a_{0,t}, a_{1,t}, \dots, a_{2n_t,t}) \end{aligned}$$

is called the  $M_L$ -sum of expanded lattice angles  $\Phi_l$ , and  $\Phi_i$  ( $i = 1, \dots, t$ ).

Let  $M_{LR} = (m_l, m_1, \dots, m_{t-1}, m_r)$  be some  $(t+1)$ -tuple of integers. Any expanded lattice LR-irrational angle, corresponding to the following lattice signed length-sine sequence

$$\begin{aligned} & (\dots, a_{-1,l}, a_{0,l}, m_l, a_{0,1}, a_{1,1}, \dots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \dots, a_{2n_2,2}, m_2, \dots \\ & \dots, m_{t-1}, a_{0,t}, a_{1,t}, \dots, a_{2n_t,t} m_r, a_{0,r}, a_{1,r}, \dots) \end{aligned}$$

is called a  $M_{LR}$ -sum of expanded lattice angles  $\Phi_l$ ,  $\Phi_i$  ( $i = 1, \dots, t$ ) and  $\Phi_r$ .

#### APPENDIX A. ON GLOBAL RELATIONS ON ALGEBRAIC SINGULARITIES OF COMPLEX PROJECTIVE TORIC VARIETIES CORRESPONDING TO INTEGER-LATTICE TRIANGLES.

In this appendix we describe an application of theorems on sums of lattice tangents for the angles of lattice triangles and lattice convex polygons to theory of complex projective toric varieties. We refer the reader to the general definitions of theory of toric varieties to the works of V. I. Danilov [4], G. Ewald [5], W. Fulton [6], and T. Oda [18].

Let us briefly recall the definition of complex projective toric varieties associated to lattice convex polygons. Consider a lattice convex polygon  $P$  with vertices  $A_0, A_1, \dots, A_n$ . Let the intersection of this (closed) polygon with the lattice consists of the points  $B_i = (x_i, y_i)$  for  $i = 0, \dots, m$ . Let also  $B_i = A_i$  for  $i = 0, \dots, n$ . Denote by  $\Omega$  the following set in  $\mathbb{C}P^m$ :

$$\left\{ (t_1^{x_1} t_2^{y_1} t_3^{-x_1-y_1} : t_1^{x_2} t_2^{y_2} t_3^{-x_2-y_2} : \dots : t_1^{x_m} t_2^{y_m} t_3^{-x_m-y_m}) \mid t_1, t_2, t_3 \in \mathbb{C} \setminus \{0\} \right\}.$$

The closure of the set  $\Omega$  in the natural topology of  $\mathbb{C}P^m$  is called the *complex toric variety associated with the polygon  $P$*  and denoted by  $X_P$ .

For any  $i = 0, \dots, m$  we denote by  $\tilde{A}_i$  the point  $(0 : \dots : 0 : 1 : 0 : \dots : 0)$  where 1 stands on the  $(i+1)$ -th place.

From general theory it follows that:

**a)** the set  $X_P$  is a complex projective complex-two-dimensional variety with isolated algebraic singularities;

**b)** the complex toric projective variety contains the points  $\tilde{A}_i$  for  $i = 0, \dots, n$  (where  $n+1$  is the number of vertices of convex polygon);

**c)** the points of  $X_P \setminus \{\tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_n\}$  are non-singular;

**d)** the point  $\tilde{A}_i$  for any integer  $i$  satisfying  $0 \leq i \leq n$  is singular iff the corresponding ordinary lattice angle  $\alpha_i$  at the vertex  $A_i$  of the polygon  $P$  is not lattice-congruent to  $\text{larctan}(1)$ ;

e) the algebraic singularity at  $\tilde{A}_i$  for any integer  $i$  satisfying  $0 \leq i \leq n$  is uniquely determined by the lattice-affine type of the non-oriented sail of the lattice angle  $\alpha_i$ .

The algebraic singularity is said to be *toric* if there exists a projective toric variety with the given algebraic singularity.

Note that the lattice-affine classes of non-oriented sails for angles  $\alpha$  and  $\beta$  coincide iff  $\beta \cong \alpha$ , or  $\beta \cong \alpha^t$ . This allows us to associate to any complex-two-dimensional toric algebraic singularity, corresponding to the sail of the angle  $\alpha$ , the unordered couple of rationals  $(a, b)$ , where  $a = \text{ltan } \alpha$  and  $b = \text{ltan } \alpha^t$ .

*Remark A.1.* Note that the continued fraction for the sail  $\alpha$  is slightly different to the Hirzebruch-Jung continued fractions for toric singularities (see the works [9] by H.W.E. Jung, and [8] by F. Hirzebruch). The relations between these continued fractions is described in the paper [19] by P.Popescu-Pampu.

**Corollary A.2.** *Suppose, that we are given by three complex-two-dimensional toric singularities defined by couples of rationals  $(a_i, b_i)$  for  $i = 1, 2, 3$ . There exist a complex toric variety associated with some triangle with these three singularities iff there exist a permutation  $\sigma \in S_3$  and the rationals  $c_i$  from the sets  $\{a_i, b_i\}$  for  $i = 1, 2, 3$ , such that the following conditions hold:*

- i) the rational  $]c_{\sigma(1)}, -1, c_{\sigma(2)}[$  is either negative or greater than  $c_{\sigma(1)}$ ;
- ii)  $]c_{\sigma(1)}, -1, c_{\sigma(2)}, -1, c_{\sigma(3)}[ = 0$ .

We note again that we use odd continued fractions for  $c_1, c_2$ , and  $c_3$  in the statement of the above proposition (see Subsection 2.1 for the notation of continued fractions).

*Proof.* The proposition follows directly from Theorem 2.2a. □

**Proposition A.3.** *For any collection (with multiplicities) of complex-two-dimensional toric algebraic singularities there exist a complex-two-dimensional toric projective variety with exactly the given collection of toric singularities.*

For the proof of Proposition A.3 we need the following lemma.

**Lemma A.4.** *For any collection of ordinary lattice angles  $\alpha_i$  ( $i = 1, \dots, n$ ), there exist an integer  $k \geq n-1$  and a  $k$ -tuple of integers  $M = (m_1, \dots, m_k)$ , such that*

$$\overline{\alpha_1} + m_1 \dots + m_{n-1} \overline{\alpha_n} + m_n \text{larctan}(1) + m_{n+1} \dots + m_k \text{larctan}(1) = 2\pi.$$

*Proof.* Consider any collection of ordinary lattice angles  $\alpha_i$  ( $i = 1, \dots, n$ ) and denote

$$\Phi = \overline{\alpha_1} +_1 \overline{\alpha_2} +_1 \dots +_1 \overline{\alpha_n}.$$

There exist an oriented lattice broken line for the angle  $\Phi$  with the signed lattice-signed sequence with positive elements. Hence,  $\Phi \cong \varphi + 0\pi$ .

If  $\varphi \cong \text{larctan}(1)$ , we have

$$\Phi +_{-2} \text{larctan}(1) +_{-2} \text{larctan}(1) +_{-2} \text{larctan}(1) = 2\pi.$$

Then  $k = n + 2$ , and  $M = (1, \dots, 1, -2, -2, -2)$ .

Suppose now  $\varphi \not\cong \text{larctan}(1)$ , then the following holds

$$\overline{\varphi} +_{-1} \overline{\pi - \varphi} +_{-2} \text{larctan}(1) +_{-2} \text{larctan}(1) = 2\pi.$$

Consider the sail for the angle  $\pi - \varphi$ . Suppose the sequence of all its lattice points (not only vertices) is  $B_0, \dots, B_s$  (with the order coinciding with the order of the sail). Then we have

$$\angle B_i O B_{i+1} \cong \text{larctan}(1) \quad \text{for any } i = 1, \dots, s.$$

Denote by  $b_i$  the values of  $\text{lsin} \angle B_i O B_{i+1}$  for  $i = 1, \dots, s$ . Then we have

$$\begin{aligned} & \overline{\varphi} + {}_{-2} \text{larctan}(1) + {}_{-2} \text{larctan}(1) + {}_{-2} \text{larctan}(1) = \\ & \overline{\alpha_1} + {}_{+1} \overline{\alpha_2} + {}_{+1} \dots + {}_{+1} \overline{\alpha_n} + {}_{-1} \\ & \text{larctan}(1) + {}_{b_1} \text{larctan}(1) + {}_{b_2} \dots + {}_{b_s} \text{larctan}(1) + {}_{-2} \\ & \text{larctan}(1) + {}_{-2} \text{larctan}(1) + {}_{-2} \text{larctan}(1) = 2\pi. \end{aligned}$$

Therefore,  $k = n + s + 3$ , and

$$M = (\underbrace{1, 1, \dots, 1, 1}_{(n-1)\text{-times}}, -1, b_1, \dots, b_s, -2, -2, -2).$$

The proof of Lemma A.4 is completed.  $\square$

*Proof of the statement of the Proposition A.3.* Consider an arbitrary collection of two-dimensional toric algebraic singularities. Suppose that they are represented by ordinary lattice angles  $\alpha_i$  ( $i = 1, \dots, n$ ). By Lemma A.4 there exist an integer  $k \geq n - 1$  and a  $k$ -tuple of integers  $M = (m_1, \dots, m_k)$ , such that

$$\overline{(\pi - \alpha_1)} + {}_{m_1} \dots + {}_{m_{n-1}} \overline{(\pi - \alpha_n)} + {}_{m_n} \text{larctan}(1) + {}_{m_{n+1}} \dots + {}_{m_k} \text{larctan}(1) = 2\pi.$$

By Theorem 4.8 there exist a convex polygon  $P = A_0 \dots A_k$  with angles proper lattice-congruent to the ordinary lattice angles  $\alpha_i$  ( $i = 1, \dots, n$ ), and  $k - n + 1$  angles  $\text{larctan}(1)$ .

By the above, the toric variety  $X_P$  is nonsingular at points of  $P_X \setminus \{\tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_k\}$ . It is also nonsingular at the points  $\tilde{A}_i$  with the corresponding ordinary lattice angles lattice-congruent to  $\text{larctan}(1)$ . The collection of the toric singularities at the remaining points coincide with the given collection.

This concludes the proof of Proposition A.3.  $\square$

On Figure 12 we show an example of the polygon for a projective toric variety with the unique toric singularity, represented by the sail of  $\text{larctan}(7/5)$ . The ordinary lattice angle  $\alpha$  on the figure is proper lattice-congruent to  $\text{larctan}(7/5)$ , the angles  $\beta$  and  $\gamma$  are proper lattice-congruent to  $\text{larctan}(1)$ .

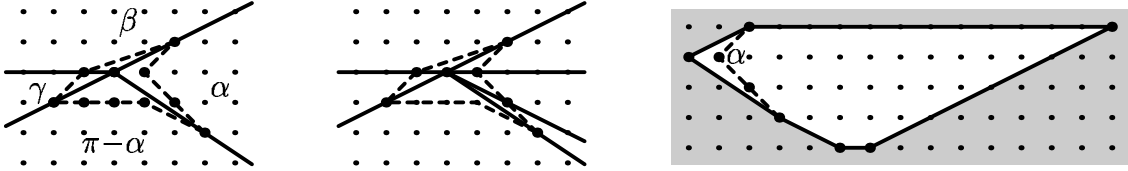


FIGURE 12. Constructing a polygon with all angles proper integer-congruent to  $\text{larctan}(1)$  except one angle that is proper integer-congruent to  $\text{larctan}(7/5)$ .

APPENDIX B. ON LATTICE-CONGRUENCE CRITERIONS FOR LATTICE TRIANGLES. EXAMPLES OF LATTICE TRIANGLES.

Here we discuss the lattice-congruence criterions for lattice triangles. By the first criterion of lattice-congruence for lattice triangles we obtain that the number of all lattice-congruence classes for lattice triangles with bounded lattice area is finite. Further we introduce the complete list of lattice-congruence classes for triangles with lattice area less then or equal to 10. There are exactly 33 corresponding lattice-congruence classes, shown on Figure 16.

**On criterions of lattice triangle lattice-congruence.** We start with the study of lattice analogs for the first, the second, and the third Euclidean criterions of triangle congruence.

**Statement B.1. (The first criterion of lattice triangle lattice-congruence.)** *Consider two lattice triangles  $\triangle ABC$  and  $\triangle A'B'C'$ . Suppose that the edge  $AB$  is lattice-congruent to the edge  $A'B'$ , the edge  $AC$  is lattice-congruent to the edge  $A'C'$ , and the ordinary angle  $\angle CAB$  is lattice-congruent to the ordinary angle  $\angle C'A'B'$ , then the triangle  $\triangle A'B'C'$  is lattice-congruent to the triangle  $\triangle ABC$ .  $\square$*

It turns out that the second and the third criterions taken from Euclidean geometry do not hold. The following two examples illustrate these phenomena.

**Example B.2.** The second criterion of triangle lattice-congruence does not hold in lattice geometry. On Figure 13 we show two lattice triangles  $\triangle ABC$  and  $\triangle A'B'C'$ . The edge  $AB$  is lattice-congruent to the edge  $A'B'$  (here  $\ell(A'B') = \ell(AB) = 4$ ). The ordinary angle  $\angle ABC$  is lattice-congruent to the ordinary angle  $\angle A'B'C'$  (since  $\angle ABC \cong \angle A'B'C' \cong \arctan(1)$ ), and the ordinary angle  $\angle CAB$  is lattice-congruent to the ordinary angle  $\angle C'A'B'$  (since  $\angle CAB \cong \angle C'A'B' \cong \arctan(1)$ ), The triangle  $\triangle A'B'C'$  is not lattice-congruent to the triangle  $\triangle ABC$ , since  $\text{IS}(\triangle ABC) = 4$  and  $\text{IS}(\triangle A'B'C') = 8$ .



FIGURE 13. The second criterion of triangle lattice-congruence does not hold.

**Example B.3.** The third criterion of triangle lattice-congruence does not hold in lattice geometry. On Figure 14 we show two lattice triangles  $\triangle ABC$  and  $\triangle A'B'C'$ . All edges of both triangles are lattice-congruent (of length one), but the triangles are not lattice-congruent, since  $\text{IS}(\triangle ABC) = 1$  and  $\text{IS}(\triangle A'B'C') = 3$ .

Instead of the second and the third criterions there exists the following additional criterion of lattice triangles lattice-congruence.

**Statement B.4. (An additional criterion of lattice triangle integer-congruence.)** *Consider two lattice triangles  $\triangle ABC$  and  $\triangle A'B'C'$  of the same lattice area. Suppose that the ordinary angle  $\angle ABC$  is lattice-congruent to the ordinary angle  $\angle A'B'C'$ , the ordinary angle  $\angle CAB$  is lattice-congruent to the ordinary angle  $\angle C'A'B'$ , the ordinary angle  $\angle BCA$  is lattice-congruent to the ordinary angle  $\angle B'C'A'$ , then the triangle  $\triangle A'B'C'$  is lattice-congruent to the triangle  $\triangle ABC$ .  $\square$*



FIGURE 14. The third criterion of triangle lattice-congruence does not hold.

In the following example we show that the additional criterion of lattice triangle lattice-congruence is not improvable.

**Example B.5.** On Figure 15 we show an example of two lattice non-equivalent triangles  $\triangle ABC$  and  $\triangle A'B'C'$  of the same lattice area equals 4 and the same ordinary lattice angles  $\angle ABC$ ,  $\angle CAB$ , and  $\angle A'B'C'$ ,  $\angle C'A'B'$  all lattice-equivalent to the angle  $\arctan(1)$ , but  $\triangle ABC \not\cong \triangle A'B'C'$ .

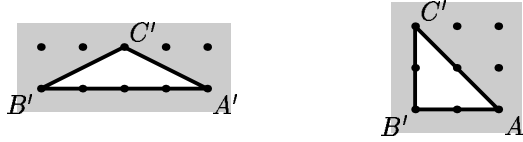


FIGURE 15. The additional criterion of lattice triangle lattice-congruence is not improvable.

**Examples of lattice triangles.** First, we define different types of lattice triangles.

**Definition B.6.** The lattice triangle  $\triangle ACB$  is called *dual* to the triangle  $\triangle ABC$ .

The lattice triangle is said to be *self-dual* if it is lattice-congruent to the dual triangle.

The lattice triangle is said to be *pseudo-isosceles* if it has at least two lattice-congruent angles.

The lattice triangle is said to be *lattice isosceles* if it is pseudo-isosceles and self-dual.

The lattice triangle is said to be *pseudo-regular* if all its ordinary angles are lattice-congruent.

The lattice triangle is said to be *lattice regular* if it is pseudo-regular and self-dual.

On Figure 16 we show the complete list of 33 triangles representing all lattice-congruence classes of lattice triangles with small lattice areas not greater than 10. We enumerate the vertices of the triangle in the clockwise way. Near each vertex of any triangle we write the tangent of the corresponding ordinary angle. Inside any triangle we write its area. We draw dual triangles on the same light gray area (if they are not self-dual). Lattice regular triangles are colored in dark grey, lattice isosceles but not lattice regular triangles are white, and the others are light grey.

#### APPENDIX C. SOME UNSOLVED QUESTION ON LATTICE TRIGONOMETRY.

We conclude this paper with a small collection of unsolved questions.

Let us start with some questions on elementary definitions of lattice trigonometry. In this paper we do not show any geometrical meaning of lattice cosine. Here arise the following question.

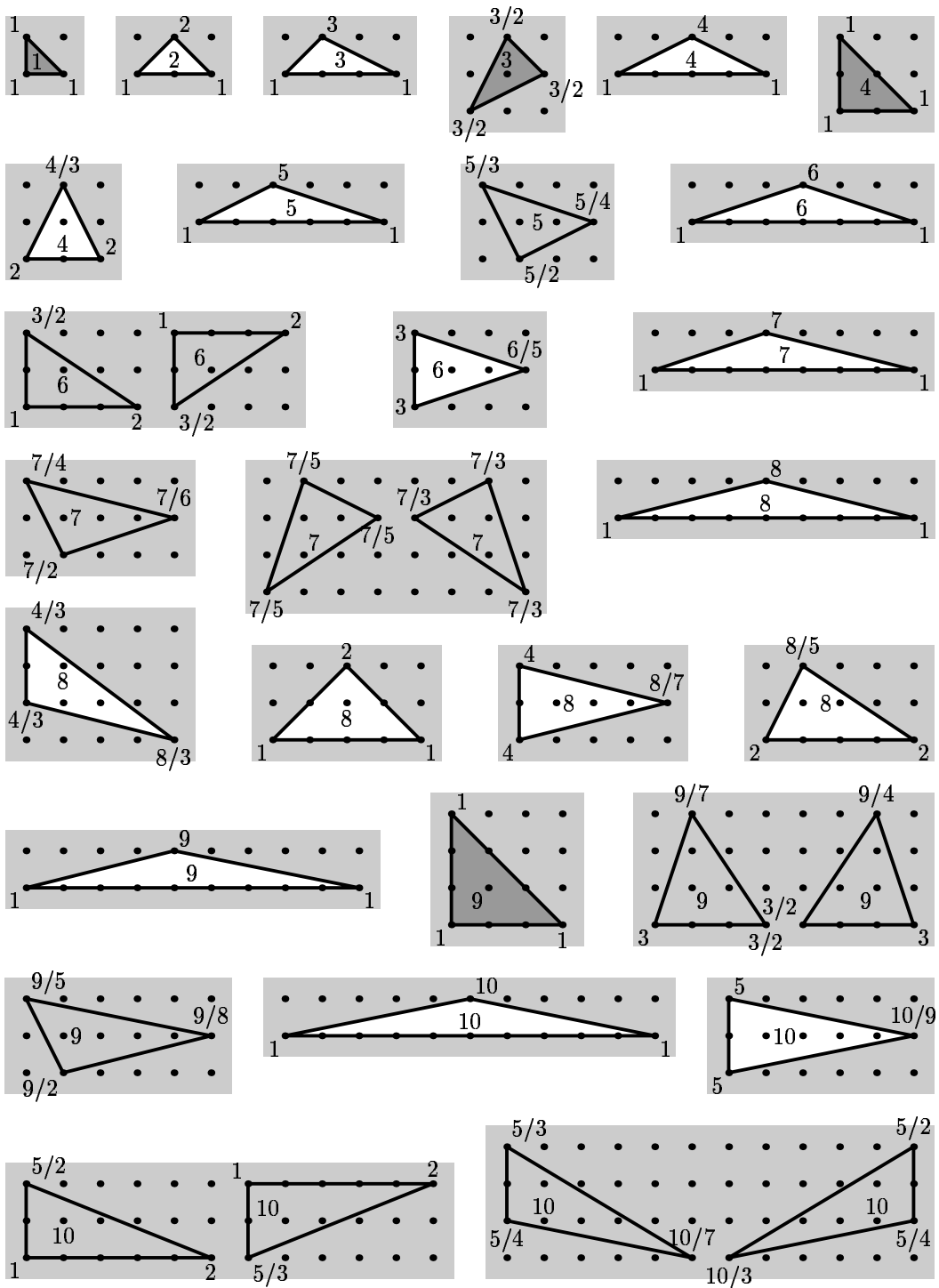


FIGURE 16. List of lattice triangles of lattice volume less than or equal 10.

**Problem 1.** Find a natural description of lattice cosine for ordinary lattice angles in terms of lattice invariants of the corresponding sublattices.

This problem seems to be close to the following one.

**Problem 2.** Does there exist a lattice analog of the cosine formula for the angles of triangles in Euclidean geometry?

Now we formulate some problems on definitions of lattice trigonometric functions for lattice irrational angles.

**Problem 3. a).** Find a natural definition of lattice tangents for lattice L-irrational angles, and lattice LR-irrational angles.

**b).** Find a natural definition of lattice sines and cosines for lattice irrational angles?

Let us continue with questions on lattice analogs of classical trigonometric formulas for trigonometric functions of angles of triangles in Euclidean geometry.

**Problem 4. a).** Knowing the lattice trigonometric functions for lattice angles  $\alpha$ ,  $\beta$  and integer  $n$ , find the explicit formula for the lattice trigonometric functions of the expanded lattice angle  $\overline{\alpha} + n\overline{\beta}$ .

**b).** Knowing the lattice trigonometric functions for a lattice angle  $\alpha$ , an integer  $m$ , and positive integer  $m$ , find the explicit formula for the lattice trigonometric functions of the expanded lattice angle

$$\sum_{M, i=1}^l \overline{\alpha}_i,$$

where  $M = (m, \dots, m)$  is an  $n$ -tuple.

The next problem is about a generalization of the statement of Theorem 2.2b to the case of  $n$  ordinary angles, that is important in toric geometry and theory of multidimensional continued fractions.

**Problem 5.** Find a necessary and sufficient conditions for the existence of an  $n$ -gon with the given ordered sequence of ordinary lattice angles  $(\alpha_1, \dots, \alpha_n)$  and the consistent sequence of lattice lengths of the edges  $(l_1, \dots, l_n)$  in terms of continued fractions for  $n \geq 4$ .

In Section 5, in particular, we introduced the definition of the sums of any expanded lattice L-irrational angle with any expanded lattice R-irrational angle.

**Problem 6.** Does there exist a natural definition of the sums of

- a)** any expanded lattice LR-irrational angle and any expanded lattice (irrational) angle;
- b)** any expanded lattice R-irrational angle and any expanded lattice (irrational) angle;
- c)** any expanded lattice (irrational) angle and any expanded lattice L-irrational angle?

We conclude this paper with the following problem being actual in the study of expanded lattice irrational angles.

**Problem 7.** Find an effective algorithm to verify whether two given almost-positive signed lattice length-sine sequences define lattice-congruent expanded lattice irrational angles, or not.



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