The Simple Exclusion Process

Among the simplest and most widely studied interacting particle systems is the simple exclusion process. It represents the evolution of random walks on the lattice $\mathbb{Z}^d$ with a hard-core interaction which prevents more than one particle per site. To describe the dynamics, fix a probability measure $p(\cdot)$ on $\mathbb{Z}^d$ and distribute particles on the lattice in such a way that each site is occupied by at most one particle. Particles evolve on $\mathbb{Z}^d$ as random walks with translation-invariant transition probability $p(x, y) = p(y - x)$. Each time a particle tries to jump over a site already occupied, the jump is suppressed to respect the exclusion rule.

This informal description corresponds to a Markov process on $\mathcal{X}_d = \{0, 1\}^{\mathbb{Z}^d}$ endowed with the weak topology that makes it compact. The generator $L$ is given by

$$
(L f)(\eta) = \sum_{x, z \in \mathbb{Z}^d} \eta(x) [1 - \eta(x + z)] p(z) [f(\sigma^{x,z} \eta(x)) - f(\eta)].
$$

Here, $\eta$ stands for a configuration of $\mathcal{X}_d$ so that $\eta(x)$ is equal to 1 (resp. 0) if the site $x$ is occupied (resp. vacant) for the configuration $\eta$. $f$ is a cylinder function, which means that it depends on $\eta$ only through a finite number of coordinates, and $\sigma^{x,z} \eta$ is the configuration obtained from $\eta$ by interchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$
(\sigma^{x,z} \eta)(z) = \begin{cases} 
\eta(z) & \text{if } z \neq x, y, \\
\eta(y) & \text{if } z = x, \\
\eta(x) & \text{if } z = y.
\end{cases}
$$

The interpretation is clear. Between 0 and $dt$ each particle tries, independently from the others, to jump from $x$ to $x + z$ at rate $p(z)$. The jump is suppressed if it leads to an already occupied site. It is proved in Liggett (1985) that the operator $L$ defined above is a Markov pre-generator and that the space $C$ of cylinder functions, i.e., which depend on the configuration $\eta$ only through a finite number of variables $\eta(x)$, forms a core for $L$. 

The simple exclusion process is said to be symmetric if the transition probability is symmetric \( p(z) = p(-z) \), it is said to be mean zero if \( p(\cdot) \) is not symmetric but has zero average: \( \sum z p(z) = 0 \). All other cases are said to be asymmetric.

Fix a cylinder function \( V \). The purpose of this chapter is to obtain sufficient conditions which guarantee a central limit theorem for

\[
\frac{1}{\sqrt{t}} \int_0^t V(\eta_s) \, ds .
\]  

The chapter is divided as follows. In Section 1 we discuss the main properties of exclusion processes. In Section 4, we show that the \( L^2 \) space can be decomposed as a direct sum \( \oplus_{n \geq 0} G_n \) in a way which allows the generator of the exclusion process to be written as a sum of four pieces, two of them mapping \( G_n \) in \( G_n \), the other two mapping \( G_n \) in \( G_{n+1} \). We recover in this manner the set-up of the previous chapter. We also prove in this section necessary and sufficient conditions for a local function to belong to \( H_1 \). In Section 2, we show that the generator of the mean zero asymmetric exclusion process satisfies a sector condition and deduce from this result a central limit theorem for (0.3). In Section 2 we prove that the pieces of the generator which changes the degree of a function satisfy a graded sector condition with respect to the symmetric part of the generator for the asymmetric exclusion process in dimension \( d \geq 3 \). This estimate together with some bounds on the solution of the resolvent equation permits to recover a central limit theorem for (0.3) in this context. In both sections we follow the approach presented in the previous chapter.

**1 Exclusion Processes**

To avoid degeneracies, we assume that the transition probability \( p(\cdot) \) is irreducible in the sense that the set \( \{ x : p(x) > 0 \} \) generates \( \mathbb{Z}^d \): for any pair of sites \( x, y \) in \( \mathbb{Z}^d \), there exists \( M \geq 1 \) and a sequence \( x = x_0, \ldots, x_M = y \) such that \( p(x_i, x_{i+1}) + p(x_{i+1}, x_i) > 0 \) for \( 0 \leq i \leq M - 1 \). Sometimes, we renormalize the transition probability in a slightly different way (we assume \( \sum x p(x) = d \) instead of 1). Finally, we will always suppose that the transition probability is of finite range: there exists \( A_0 \) in \( \mathbb{N} \) such that \( p(z) = 0 \) for all sites \( z \) outside the cube \([-A_0, A_0]^d\).

Denote by \( \{ S(t), \ t \geq 0 \} \) the semigroup of the Markov process with generator \( L \) given by (0.1). We use the same notation for semigroups acting on continuous functions or on the space \( \mathcal{M}_1(\mathcal{X}_d) \) of probability measures on \( \mathcal{X}_d \).

Notice that the total number of particles is conserved by the dynamics. This conservation is reflected in the existence of a one-parameter family of invariant measures. For \( 0 \leq \alpha \leq 1 \), denote by \( \nu_\alpha \) the Bernoulli product measure of parameter \( \alpha \). This means that under \( \nu_\alpha \) the variables \( \{ \eta(x), \ x \in \mathbb{Z}^d \} \) are independent with marginals given by
\[ \nu_\alpha \{ \eta(x) = 1 \} = \alpha = 1 - \nu_\alpha \{ \eta(x) = 0 \} . \]

**Proposition 3.1.** The Bernoulli measures \( \{ \nu_\alpha, 0 \leq \alpha \leq 1 \} \) are invariant for simple exclusion processes.

*Proof.* By a simple change of variables, for any local functions \( f, g \) and any bond \( \{ x, y \} \),

\[
\int f(\sigma^x \eta) g(\eta(\eta(x)) \eta(1 - \eta(y))) \nu_\alpha (d\eta) = \int f(\eta) g(\sigma^y \eta) \eta(1 - \eta(x)) \nu_\alpha (d\eta) .
\]

(1.1)

This identity, the fact that \( 1 = \sum_{z \in \mathbb{Z}^d} p(z) = \sum_{z \in \mathbb{Z}^d} p(-z) \) and a change in the order of summation prove that

\[
\int L f \, d\nu_\alpha = 0
\]

for all cylinder functions \( f \). This proves the proposition. \( \square \)

Notice that the family of invariant measures \( \nu_\alpha \) is parameterised by the density, for

\[
E_{\nu_\alpha} [\eta(0)] = \nu_\alpha \{ \eta(0) = 1 \} = \alpha .
\]

By Schwarz inequality, for any cylinder function \( f \),

\[
[S(t)f(\eta)]^2 \leq S(t)f^2(\eta) .
\]

In particular, since \( \nu_\alpha \) is a stationary measure for any \( \alpha \in [0, 1] \),

\[
\int [S(t)f(\eta)]^2 \nu_\alpha (d\eta) \leq \int f(\eta)^2 \nu_\alpha (d\eta) .
\]

Therefore, the semigroup \( S(t) \) extends to a Markov semigroup on \( L^2(\nu_\alpha) \). Its generator \( L_{\nu_\alpha} \) is the closure of \( L \) in \( L^2(\nu_\alpha) \). Since the density \( \alpha \) remains fixed, to keep notation simple, we denote \( L_{\nu_\alpha} \) by \( L \) and by \( D(L) \) the domain of \( L \) in \( L^2(\nu_\alpha) \).

Denote by \( L^* \) the generator defined by (0.1) associated to the transition probability \( p^*(x) = p(-x) \). By (1.1), \( L^* \) is the adjoint of \( L \) in \( L^2(\nu_\alpha) \). In particular, in the symmetric case, \( L \) is self-adjoint with respect to each \( \nu_\alpha \) (i.e. \( \nu_\alpha \) is a reversible measure for the symmetric simple exclusion).

Denote by \( s(\cdot) \) (resp. \( a(\cdot) \)) the symmetric (resp. asymmetric) part of the transition probability \( p \):

\[
s(x) = (1/2) \{ p(x) + p(-x) \} , \quad a(x) = (1/2) \{ p(x) - p(-x) \} .
\]

We can decompose the operator \( L \) in a symmetric and antisymmetric part as \( L = S + A \) where, for a cylinder function \( f \),
\[ (Sf)(\eta) = \sum_{x,z \in \mathbb{Z}^d} \eta(x)(1 - \eta(x + z)) s(z) [f(\sigma^{x,x+z}\eta) - f(\eta)], \]

\[ (Af)(\eta) = \sum_{x,z \in \mathbb{Z}^d} \eta(x)(1 - \eta(x + z)) a(z) [f(\sigma^{x,x+z}\eta) - f(\eta)]. \]

Performing a change of variables \( x' = x + z, z' = -z \), we obtain

\[ (Sf)(\eta) = \sum_{x,z \in \mathbb{Z}^d} \eta(x + z)(1 - \eta(x)) s(z) [f(\sigma^{x,x+z}\eta) - f(\eta)] \]

because \( s(\cdot) \) is symmetric. Adding the two previous formulae for \( Sf \) and since \( \sigma^{x,x+z}\eta = \eta \) unless \( \eta(x)(1 - \eta(x + z)) + \eta(x + z)(1 - \eta(x)) = 1 \), we deduce the simpler form

\[ (Sf)(\eta) = (1/2) \sum_{x,z \in \mathbb{Z}^d} s(z) [f(\sigma^{x,x+z}\eta) - f(\eta)] \]

for the operator \( S \). In fact, \( S \) is the generator of the simple exclusion process with transition probability \( s(\cdot) \). Since \( s(\cdot) \) is symmetric, \( S \) is a self-adjoint operator on \( L^2(\nu_\alpha) \).

For a cylinder function \( f \), denote by \( D(f) \) the Dirichlet form of \( f \):

\[ D(f) = \langle f, (-Lf) \rangle_{\nu_\alpha} = \langle f, (-Sf) \rangle_{\nu_\alpha} = (1/4) \sum_{x,z \in \mathbb{Z}^d} s(z) \int [f(\sigma^{x,x+z}\eta) - f(\eta)]^2 \nu_\alpha(d\eta). \]

Here, \( \langle \cdot, \cdot \rangle_{\nu_\alpha} \) stands for the inner product in \( L^2(\nu_\alpha) \). This formula holds also for functions \( f \) in the domain \( D(L) \) of the generator, and the series defined on the right hand side converge absolutely. This can be proved by an approximation argument (cf. Lemma 4.4.3 in Liggett (1985)).

**Theorem 3.2.** For any \( \alpha \in [0,1] \), \( \nu_\alpha \) is ergodic for \( L \).

**Proof.** Let \( f \in L^2(\nu_\alpha) \) such that \( S(t)f = f \) for any \( t \geq 0 \). Then \( f \in D(L) \) and \( Lf = 0 \). Multiplying this last equation by \( f \) and integrating, by (1.2) we obtain

\[ \sum_{x,z} s(z) \int [f(\sigma^{x,x+z}\eta) - f(\eta)]^2 \nu_\alpha(d\eta) = 0. \]

Since \( |a(z)| \leq s(z) \), the support of \( p(\cdot) \) is a subset of the support of \( s(\cdot) \). This implies that the support of \( s(\cdot) \) generates \( \mathbb{Z}^d \). We deduce that for any \( x, y \in \mathbb{Z}^d \)

\[ f(\sigma^{x,y}\eta) = f(\eta) \quad \nu_\alpha - a.e. \]

By De Finetti’s theorem we conclude that \( f \) is constant \( \nu_\alpha \)-a.e.
2 Central Limit Theorems for Additive Functionals

Fix $0 < \alpha < 1$ and a zero-mean local function $V$. We prove in this section a central limit theorem for the additive functional $t^{-1/2} \int_0^t V(\eta_s) \, ds$ in the context of zero-mean asymmetric simple exclusion process:

**Theorem 3.3.** Assume that the jump rate $p$ has mean zero: $\sum_{x \in \mathbb{Z}^d} xp(x) = 0$. Fix $0 < \alpha < 1$ and consider a zero-mean local function $V$ in $\mathcal{H}_{-\alpha}(L, \nu_\alpha)$. Then,

$$\frac{1}{\sqrt{t}} \int_0^t V(\eta_s) \, ds$$

converges in distribution, as $t \uparrow \infty$, to a zero-mean Gaussian variable with variance $\sigma^2(V) = 2 \lim_{\lambda \to 0} \|f_\lambda\|_1^2$, where $f_\lambda$ is the solution of the resolvent equation (2.1). In the symmetric case $\sigma^2(V) = 2\|V\|_{-1}^2$.

Consider the resolvent equation

$$\lambda f_\lambda - Lf_\lambda = V.$$  \hfill (2.1)

We have seen in Chapter 2 that a central limit theorem follows from a uniform bound on the $H_{-1}$ norm of the solution of the resolvent equation:

$$\sup_{0 < \lambda < 1} \|L f_\lambda\|_{-1} < \infty.$$  \hfill (2.2)

Assume that the jump rate is symmetric: $p(-x) = p(x)$. We have seen at the beginning of this chapter that in this case the generator $L$ is self-adjoint in $L^2(\nu_\alpha)$. In particular, by Theorem 2.2.2 and Subsection 2.6.1, (2.2) holds and so does Theorem 3.3.

To compute the variance, recall from Proposition 2.2.6 and from (2.4.2) that $\sigma^2(V) = 2 \lim_{\lambda \to 0} \|f_\lambda\|_1^2 \leq 2\|V\|_{-1}^2$. On the other hand, by the first part of the proof of Lemma 2.2.9, $L f_\lambda$ converges weakly in $\mathcal{H}_{-1}$ to $-V$ as $\lambda \downarrow 0$. In particular, $\|V\|_{-1} \leq \lim \inf_{\lambda \to 0} \|L f_\lambda\|_{-1} = \lim \inf_{\lambda \to 0} \|f_\lambda\|_1$ because $L$ is symmetric. This proves that $\sigma^2(V) = \|V\|_{-1}^2$ as claimed in Theorem 3.3.

3 The Zero-mean asymmetric case

Assume now that the jump rate has mean zero: $\sum_{x \in \mathbb{Z}^d} xp(x) = 0$. In this case the generator $L$ satisfies a sector condition:

**Proposition 3.4.** Assume that the transition probability has mean zero: $\sum_{x \in \mathbb{Z}^d} xp(x) = 0$. There exists a finite constant $B$, depending only on $p(\cdot)$ such that

$$|\langle -L f, g \rangle| \leq B \langle -L f, f \rangle^{1/2} \langle -L g, g \rangle^{1/2}$$

for all cylinder functions $f$, $g$. 

Theorem 3.3 follows from this proposition, Theorem 2.2.2 and Subsection 2.6.2. By Proposition 2.2.6, $\sigma^2(V) = 2\lim_{\lambda \to 0} ||f_\lambda||^2$.

We now turn to the proof of Proposition 3.4, which relies on a decomposition of the generator in cycles, similar to the one presented in Section xxx for bistochastic centered random walks. The proof is divided in three steps. We first show in Lemma 3.5 that all zero-mean finite-range probability measures are convex combinations of cycle probability measures. Then, in Lemma 3.6, we prove that a sector condition holds for a finite convex combination of generators if it holds individually. Finally, in Lemma 3.8 we conclude the proof of Proposition 3.4 showing that a sector condition holds for an exclusion process associated to a cycle probability.

A cycle $C$ of length $n$ is a sequence of $n$ sites of $\mathbb{Z}^d$ starting and ending at the same point: $\{y_0, y_1, \ldots, y_{n-1}, y_n = y_0\}$. To a cycle $C$ of length $n$, we associate a zero-mean transition probability $p_C$ defined by

$$p_C(x) = \frac{1}{n} \sum_{j=0}^{n-1} 1\{x = y_{j+1} - y_j\}.$$ 

$p_C$ has mean zero since

$$\sum_{x \in \mathbb{Z}^d} x p_C(x) = \frac{1}{n} \sum_{x \in \mathbb{Z}^d} \sum_{j=0}^{n-1} x 1\{x = y_{j+1} - y_j\} = \frac{1}{n} \sum_{j=0}^{n-1} (y_{j+1} - y_j) = \frac{y_n - y_0}{n} = 0.$$

The cycle probability measures are the finite-range zero-mean probability measures on $\mathbb{Z}^d$ taking rational values. Indeed, fix such a probability measure $p$ and denote its support by $S = \{x_1, \ldots, x_n\}$. There exists a sufficiently large positive integer $M$ for which $p(x) = m(x)/M$, where $m(x)$ are positive integers. Consider the cycle $C = (0, x_1, 2x_1, \ldots, m(x_1)x_1, m(x_1)x_1 + x_2, \ldots, m(x_1)x_1 + m(x_2)x_2, \ldots, m(x_1)x_1 + \cdots + m(x_{n-1})x_{n-1} + (m(x_n) - 1)x_n, 0)$. It is easy to check that the probability measure associated to this cycle is $p$.

For a positive integer $m$, finite cycles $C = \{C_1, \ldots, C_m\}$ and a probability $w = \{w_1, \ldots, w_m\}$, let $p_{C, w}()$ be the probability on $\mathbb{Z}^d$ defined by

$$p_{C, w}(\cdot) = \sum_{k=1}^{m} w_k p_{C_k}(\cdot).$$

The next lemma states that all finite-range, zero-mean transition probabilities may be written as a convex combination of cycle probability measures.

**Lemma 3.5.** Fix a finite-range, zero-mean transition probability $p$ on $\mathbb{Z}^d$. There exists $m \geq 1$, finite cycles $C = \{C_1, \ldots, C_m\}$ and a probability measure $w = \{w_1, \ldots, w_m\}$ such that $p = p_{C, w}(\cdot)$. 


Proof. The proof is divided in three steps.

Claim 1: Fix a finite-range zero-mean transition probability $p(\cdot)$. Then, there exists a zero-mean probability $q(\cdot)$ taking only rational values and with the same support as $p(\cdot)$.

To prove this claim, denote by $S = \{x_1, \ldots, x_n\}$ the support of $p(\cdot)$ and by $v_1, \ldots, v_d$ the $n$-dimensional vectors $v_j = ((x_1, e_j), \ldots, (x_n, e_j))$. Here \{\(e_1, \ldots, e_d\}\} stands for the canonical basis of $\mathbb{R}^d$ and $\langle\cdot, \cdot\rangle$ for the usual inner product in $\mathbb{R}^d$. Since $p(\cdot)$ has mean zero, $p = (p(x_1), \ldots, p(x_n))$ is orthogonal to $v_1, \ldots, v_d$.

Denote by $V$ the vector space of the real numbers over the rationals. The real numbers $p(x_1), \ldots, p(x_n)$ can be decomposed in terms of $\mathbb{Q}$-linearly independent reals: There exists $1 \leq \ell \leq n$, $\mathbb{Q}$-linearly independent real numbers $r_1, \ldots, r_\ell$ and rational numbers \{\(q_k(x_1)\ldots q_k(x_n)\)}, $1 \leq k \leq \ell$, such that

$$p(x) = \sum_{k=1}^{\ell} r_k q_k(x)$$

for each $x$ in the support of $p(\cdot)$. Since $p(\cdot)$ has mean zero, by definition of $q_k(x)$,

$$0 = \sum_{x \in S} x \ p(x) = \sum_{x \in S} x \sum_{k=1}^{\ell} r_k q_k(x) = \sum_{k=1}^{\ell} r_k \sum_{x \in S} x \ q_k(x).$$

Since $r_k$ are $\mathbb{Q}$-linearly independent and since $x$ are integer-valued vectors, for $1 \leq k \leq \ell$,

$$\sum_{x \in S} x \ q_k(x) = 0.$$

One could think that the claim is proved, for $q_k(\cdot)$, $1 \leq k \leq \ell$, assume only rational values. However, nothing forbids $q_k(\cdot)$ to be negative preventing us to turn $q$ into a probability measure. We just know from the previous identity that the vectors \(q_k = (q_k(x_1), \ldots, q_k(x_n))\), $1 \leq k \leq \ell$, are orthogonal to $v_1, \ldots, v_d$, as well as any linear combination of them. Since $p(\cdot)$ is a probability measure and since $p = \sum_{1 \leq k \leq \ell} r_k q_k$, there is one linear combination of these vectors which is strictly positive in the sense that all its coordinates are strictly positive. Approximating $r_1, \ldots, r_\ell$ by rationals $m_1, \ldots, m_\ell$ we obtain a vector $\tilde{q} = \sum_{1 \leq k \leq \ell} m_k q_k$ with rational, strictly positive entries and orthogonal to $v_1, \ldots, v_d$ because each vector $q_k$ is orthogonal. Thus a multiple $q'$ of $\tilde{q}$ is a probability measure taking only rational values and such that

$$\sum_{x \in S} x \ q'(x) = 0$$

because $q'$ is orthogonal to $v_1, \ldots, v_d$. This proves the claim.

Claim 2: Every finite-range zero-mean transition probability $p(\cdot)$ can be decomposed as $wp(\cdot) + (1-w)q(\cdot)$ for some $0 < w \leq 1$, where $\tilde{p}(\cdot)$ is a probability
measure whose support is strictly contained in the support of \( p(\cdot) \) and \( q(\cdot) \) is a probability measure taking only rational values and whose support is equal to the support of \( p(\cdot) \).

Fix such probability measure \( p(\cdot) \) and recall that we denote its support by \( S \). By Step 1, there exists a transition probability \( q(\cdot) \) taking only rational values and with the same support as \( p(\cdot) \). Let

\[
w = \min_{x \in S} \frac{p(x)}{q(x)}.
\]

Since both \( p \) and \( q \) are probability measures, \( w \leq 1 \). If \( w = 1 \), \( p = q \) and the claim is proved. If \( p < 1 \), let \( \tilde{p} \) be defined by

\[
\tilde{p}(x) = \frac{1}{1 - w} \{ p(x) - wq(x) \}.
\]

By definition of \( w \), \( \tilde{p} \) is a probability measure whose support is strictly contained in the support of \( p(\cdot) \). This proves the claim.

**Claim 3:** Every finite-range zero-mean transition probability \( p(\cdot) \) assuming only rational values is a cycle measure. This has been shown right after the definition of cycle probability.

We may now conclude the proof of the lemma. By Claims 2 and 3, every finite-range zero-mean transition probability \( p(\cdot) \) can be written as

\[
\mathbb{P} \sum_{j=1}^{m} w_j L_{C_j} \quad \text{for a positive integer} \ m, \ \text{a probability} \ \{w_1, \ldots, w_m\} \ \text{and cycles} \ \{C_1, \ldots, C_m\}.
\]

Let \( L \) be a generator which can be decomposed as a convex combination of two generators, \( L = \theta L_1 + (1 - \theta)L_2 \) for some \( 0 < \theta < 1 \). The next result states that \( L \) satisfies a sector condition if both \( L_1, L_2 \) do.

**Lemma 3.6.** Assume that there exists finite constants \( B_1, B_2 \) such that

\[
(LCf)(\eta) = \sum_{x, z \in \mathbb{Z}^d} \eta(x)[1 - \eta(x + z)] p_C(z) [f(\sigma^{x,x+z}\eta) - f(\eta)] .
\]
3 The Zero-mean asymmetric case

\[ |\langle -L_j f, g \rangle| \leq B_j (\langle -L_j f, f \rangle)^{1/2} (\langle -L_j g, g \rangle)^{1/2} \]

for \( j = 1, 2 \) and all functions \( f, g \) in a common core \( \mathcal{C} \) for \( L, L_1, L_2 \). Then,

\[ |\langle -L f, g \rangle| \leq B (\langle -L f, f \rangle)^{1/2} (\langle -L g, g \rangle)^{1/2} \]

for \( B = \theta B_1 + (1 - \theta) B_2 \) and all functions \( f, g \) in \( \mathcal{C} \).

**Proof.** Fix two functions \( f, g \) in \( \mathcal{C} \). By the triangle inequality and by assumption,

\[ |\langle -L f, g \rangle| \leq \theta |\langle -L_1 f, g \rangle| + (1 - \theta) |\langle -L_2 f, g \rangle| \]

\[ \leq \theta B_1 (\langle -L_1 f, f \rangle)^{1/2} (\langle -L_1 g, g \rangle)^{1/2} \]

\[ + (1 - \theta) B_2 (\langle -L_2 f, f \rangle)^{1/2} (\langle -L_2 g, g \rangle)^{1/2}. \]

Since \( L_1, L_2 \) are generators, \( \langle -L_j f, f \rangle \) are positive so that \( \langle -L_1 f, f \rangle, \langle -L_2 f, f \rangle \) are bounded above by \( \langle -L f, f \rangle \). The previous expression is thus less than or equal to

\[ \theta B_1 + (1 - \theta) B_2 \]

\[ \langle -L f, f \rangle^{1/2} (\langle -L g, g \rangle)^{1/2} \]

which concludes the proof of the lemma. \( \square \)

In view of the previous two lemmas, Proposition 3.4 follows from a sector condition for generators associated to cycles. This estimate is based on the following general statement.

**Lemma 3.7.** Fix a probability space \((\Omega, \mathcal{F}, \mu)\) and consider a family \( T_1, \ldots, T_n \) of measure-preserving bijections. Assume that \( T_1 \circ \cdots \circ T_n = I \), where \( I \) is the identity and consider the generator \( L \) defined by

\[ (Lf)(\eta) = \frac{1}{n} \sum_{j=1}^{n} [f(T_j \eta) - f(\eta)]. \]

For all functions \( f, g \) in a core of \( L \)

\[ |\langle L f, g \rangle| \leq 4n (\langle -L f, f \rangle)^{1/2} (\langle -L g, g \rangle)^{1/2}. \]

**Proof.** We first compute the Dirichlet form \( D(f) = \langle -L f, f \rangle \) associated to the generator \( L \). Since the operators \( T_1, \ldots, T_n \) are measure-preserving, an elementary computation shows that \( L^* \), the adjoint of \( L \) with respect to \( \mu \), is given by

\[ (L^* f)(\eta) = \frac{1}{n} \sum_{j=1}^{n} [f(T_j^{-1} \eta) - f(\eta)]. \]

In particular, an elementary computation shows that
\begin{align*}
\langle -L f, f \rangle &= \frac{1}{4n} \sum_{j=1}^{n} \int \{ f(T_j \eta) - f(\eta) \}^2 \mu(d\eta) \\
&\quad + \frac{1}{4n} \sum_{j=1}^{n} \int \{ f(T_j^{-1} \eta) - f(\eta) \}^2 \mu(d\eta).
\end{align*}

We may turn now to the proof. Fix two functions $f, g$ in the core of the generator $L$. Since the operators are measure-preserving, after a change of variables $T_j = T_j + 1$, we obtain that

$$\langle L f, g \rangle = \frac{1}{n} \sum_{j=1}^{n} \int \{ f(T_j \circ \cdots \circ T_n \eta) - f(T_{j+1} \circ \cdots \circ T_n \eta) \} g(T_{j+1} \circ \cdots \circ T_n \eta) \mu(d\eta).$$

Since $T_1 \circ \cdots \circ T_n = I$, $\sum_{j=1}^{n} f(T_j \circ \cdots \circ T_n \eta) - f(T_{j+1} \circ \cdots \circ T_n \eta) = f(T_1 \circ \cdots \circ T_n \eta) - f(\eta) = 0$. The previous expression is therefore equal to

$$\frac{1}{n} \sum_{j=1}^{n} \int \{ f(T_j \circ \cdots \circ T_n \eta) - f(T_{j+1} \circ \cdots \circ T_n \eta) \} \times \{ g(T_{j+1} \circ \cdots \circ T_n \eta) - g(\eta) \} \mu(d\eta).$$

Applying Schwarz inequality, we obtain that this expression is less than or equal to

$$\frac{1}{2An} \sum_{j=1}^{n} \int \{ f(T_j \circ \cdots \circ T_n \eta) - f(T_{j+1} \circ \cdots \circ T_n \eta) \}^2 \mu(d\eta)$$

$$+ \frac{A}{2} \sum_{j=1}^{n} \sum_{k=j+1}^{n} \int \{ g(T_k \circ \cdots \circ T_n \eta) - g(T_{k+1} \circ \cdots \circ T_n \eta) \}^2 \mu(d\eta)$$

for every $A > 0$. In the last step we replaced $g(T_{j+1} \circ \cdots \circ T_n \eta) - g(\eta)$ by $\sum_{j+1 \leq k \leq n} g(T_{k+1} \circ \cdots \circ T_n \eta) - g(T_{k+1} \circ \cdots \circ T_n \eta)$ and applied Schwarz inequality.

By a change of variables, we obtain that the previous sum is equal to

$$\frac{1}{2An} \sum_{j=1}^{n} \int \{ f(T_j \eta) - f(\eta) \}^2 \mu(d\eta) + \frac{An}{2} \sum_{k=1}^{n} \int \{ g(T_k \eta) - g(\eta) \}^2 \mu(d\eta).$$

In view of (3.2), this expression is bounded above by

$$\frac{2}{A} \langle -L f, f \rangle + 2An^2 \langle -L g, g \rangle.$$

To conclude the proof of the lemma, it remains to minimize over $A$. 

\[ \square \]
Fix a cycle $C = \{0, x_1, \ldots, x_n = 0\}$ of length $n$ and recall from (3.1) the expression of the generator $L_C$ associated to the cycle $C$. It can be written as

$$L_C = \sum_{z \in \mathbb{Z}^d} L_{C+z}^0,$$

where $L_{C+z}^0$ is the generator of an exclusion process in which all particles not in the set $C + z$ are frozen while the ones in $C + z$ may jump from $z + x_j$ to $z + x_{j+1}$. The formal generator of this process is given by

$$(L_{C+z}^0 f)(\eta) = \frac{1}{n} \sum_{j=0}^{n-1} \eta(z + x_j)[1 - \eta(z + x_{j+1})][f(\sigma^{z+x_j, z+x_{j+1}}) - f(\eta)].$$

An elementary computation shows that the Dirichlet form associated to the generator $L_{C+z}^0$ is given by

$$h_{L_{C+z}^0 f; f} = \frac{1}{4n} \sum_{j=0}^{n-1} E_{\nu_0}[\{f(\sigma^{z+x_j, z+x_{j+1}}) - f(\eta)\}^2].$$

Notice that the indicator function $\eta(z + x_j)[1 - \eta(z + x_{j+1})]$ gave place to a factor $1/4$. Summing over $z$ we obtain the full Dirichlet form:

$$\langle -L_{C+z}^0 f, f \rangle = \sum_{z \in \mathbb{Z}^d} \langle -L_{C+z}^0 f, f \rangle = \frac{1}{4n} \sum_{z \in \mathbb{Z}^d} \sum_{j=0}^{n-1} D_{z+x_j, z+x_{j+1}}(f), \quad (3.3)$$

where $D_{x,y}(f)$ stands for the piece of the Dirichlet form associated to jumps over the bond $\{x, y\}$:

$$D_{x,y}(f) = E_{\nu_0}[\{f(\sigma^{x,y}) - f(\eta)\}^2]. \quad (3.4)$$

**Lemma 3.8.** For every local functions $f, g$,

$$|\langle L_C f, g \rangle| \leq 4n^2 \langle -L_C f, f \rangle^{1/2} \langle -L_C g, g \rangle^{1/2}.$$  

**Proof.** Fix two local functions $f, g$. We have just seen that $\langle L_C f, g \rangle$ can be written as $\sum_{z \in \mathbb{Z}^d} \langle L_{C+z}^0 f, g \rangle$. By Lemma 3.9 below, this expression is less than or equal to

$$4n^2 \sum_{z \in \mathbb{Z}^d} \langle -L_{C+z}^0 f, f \rangle^{1/2} \langle -L_{C+z}^0 g, g \rangle^{1/2}.$$  

By Schwarz inequality, this expression is bounded above by

$$4n^2 \left\{ \sum_{z \in \mathbb{Z}^d} \langle -L_{C+z}^0 f, f \rangle \sum_{z \in \mathbb{Z}^d} \langle -L_{C+z}^0 g, g \rangle \right\}^{1/2} = 4n^2 \langle -L_C f, f \rangle^{1/2} \langle -L_C g, g \rangle^{1/2}.$$  

This concludes the proof of the lemma. \qed
Lemma 3.9. For every local functions $f, g$, 
\[
|\langle L_0^f, g \rangle| \leq 4n^2(-L_0^f, f)^{1/2}(-L_0^g, g)^{1/2}.
\]

Proof. For $0 \leq j \leq n - 1$, denote by $T_j$ the operator $\sigma_{x_j}^+$. Fix a configuration $\xi$ in $\{0, 1\}^C$ with a particle at the origin and observe that $T_{n-1} \cdots T_0 \xi$ is a configuration with a particle at the origin and where each other particle moved one step backward (two steps if the particle is originally at $x_1$). More precisely, if $\{0, x_{j_1}, \ldots, x_{j_m}\}$, $j_1 < \cdots < j_m$, stands for the occupied sites of the configuration $\xi$, the occupied site of $T_{n-1} \cdots T_0 \xi$ are
\[
\begin{cases}
\{0, x_{j_1-1}, \ldots, x_{j_m-1}\} & \text{if } x_{j_1} \neq x_1, \\
\{0, x_{j_2-1}, \ldots, x_{j_m-1}, x_{j_m}\} & \text{if } x_{j_1} = x_1.
\end{cases}
\]
In particular, for configurations $\xi$ with a particle at the origin, $T^{n-1} \xi = \xi$ if $T = T_{n-1} \cdots T_0$. This identity fails, however, if there is no particle at the origin.

By definition of the operators $T_j$, we may write the scalar product $\langle L_0^f, g \rangle$ as
\[
\frac{1}{n} \sum_{j=0}^{n-1} \int \eta(x_j)[1 - \eta(x_{j+1})] \{f(T_j \eta) - f(\eta)\} g(\eta) \nu_\alpha(d\eta).
\]

The presence of the indicators $\eta(x_j)[1 - \eta(x_{j+1})]$ in the previous formula prevents us in applying Lemma 3.7 directly. The proof, however, goes through.

Notice that we may take out the indicator $[1 - \eta(x_{j+1})]$ in the previous formula since the difference $f(T_j \eta) - f(\eta)$ vanishes if $\eta(x_{j+1}) = 1$. For $0 \leq k \leq n - 2$, perform the change of variable $\eta = T_{j-1} \cdots T_0 \circ T^k \xi$ to rewrite the previous sum as
\[
\frac{1}{n(n-1)} \sum_{k=0}^{n-2} \sum_{j=0}^{n-1} \int \xi(0) \{f(T_j \circ \cdots \circ T_0 \circ T^k \xi) - f(T_{j-1} \circ \cdots \circ T_0 \circ T^k \xi)\}
\times g(T_{j-1} \circ \cdots \circ T_0 \circ T^k \xi) \nu_\alpha(d\xi).
\]
Observe that the indicator $\eta(x_j)$ is transformed in $\xi(0)$ by the change of variables. Since the configuration $\xi$ has a particle at the origin, $\sum_{k=0}^{n-2} \sum_{j=0}^{n-1} \{f(T_j \circ \cdots \circ T_0 \circ T^k \xi) - f(T_{j-1} \circ \cdots \circ T_0 \circ T^k \xi)\} = f(T^{n-1} \xi) - f(\xi)$ vanishes. In particular, we may add $g(\xi)$ in the previous formula and rewrite it as
\[
\frac{1}{n(n-1)} \sum_{k=0}^{n-2} \sum_{j=0}^{n-1} \int \xi(0) \{f(T_j \circ \cdots \circ T_0 \circ T^k \xi) - f(T_{j-1} \circ \cdots \circ T_0 \circ T^k \xi)\}
\times \{g(T_{j-1} \circ \cdots \circ T_0 \circ T^k \xi) - g(\xi)\} \nu_\alpha(d\xi).
\]
It remains to repeat the arguments presented at the end of the proof of Lemma 3.7 to estimate the previous expression by
\[ \frac{1}{2\, A n} \sum_{j=0}^{n-1} \int \eta(x_j) \{f(T_j \eta) - f(\eta)\}^2 \nu_\alpha(d\eta) \]
\[ + \frac{A n(n - 1)}{2} \sum_{j=0}^{n-1} \int \eta(x_j) \{g(T_j \eta) - g(\eta)\}^2 \nu_\alpha(d\eta) \]

for every \( A > 0 \). To conclude the proof of the lemma it remains to optimize over \( A \) and to recall the expression (3.3) of the Dirichlet form \( \langle -L^0_{C,f} f, f \rangle \). \( \square \)

We conclude this section with an alternative proof of the sector condition for generators associated to cycle measures. It gives a factor \( 3/2 \) instead of 2.

**Lemma 3.10.** Fix a cycle \( C \) of length \( n \). There exists a finite universal constant \( C_0 \) such that
\[
|\langle -L_{C} f, g \rangle| \leq C_0 n^{3/2} \langle -L_{C} f, f \rangle^{1/2} \langle -L_{C} g, g \rangle^{1/2}
\]
for every local functions \( f, g \).

**Proof.** Denote the cycle \( C \) by \( \{y_0, \ldots, y_{n-1}\} \) and let \( y_n = y_0 \). Fix two local functions \( f, g \) and recall the definition of the generators \( L^0_{C+z} \) as well as the computations presented just before the statement of the lemma. By definition of \( L^0_{C+z} \),
\[
\langle L_C f, g \rangle = \sum_{z \in \mathbb{Z}^d} \langle L^0_{C+z} f, g \rangle.
\]

Fix a site \( z \). Denote by \( \mathcal{F}_z \) the \( \sigma \)-algebra generated by all occupation variables outside \( C + z \) and the total number of particles in \( C + z \):
\[
\mathcal{F}_z = \sigma \left( \sum_{y \in C+z} \eta(y), \{\eta(y), y \notin C+z\} \right).
\]

Denote by \( g_z \) the conditional expectation of \( g \) given \( \mathcal{F}_z \):
\[
g_z = \mathbb{E}\left[g \mid \mathcal{F}_z\right].
\]

Since \( g_z \) depends on the variables \( \{\eta(y_j), 1 \leq j \leq n\} \) only through their sum, \( \langle L^0_{C+z} f, g \rangle = 0 \) and
\[
\langle L^0_{C+z} f, g \rangle = \langle L^0_{C+z} f, g - g_z \rangle = E \left[ \mathbb{E}\left[\langle L^0_{C+z} f, g - g_z \mid \mathcal{F}_z\rangle\right] \right].
\]

Observe that the expectation \( E[\cdot \mid \mathcal{F}_z] \) corresponds to an expectation with respect to the canonical measure on \( C + z \) since all particles outside \( C + z \) are frozen and since \( \sum_{y \in C+z} \eta(y) \) is fixed. Hence, \( g_z \) is the expectation of \( g \) with respect to the canonical measure on \( C + z \). It follows from the explicit formula for the generator \( L^0_{C+z} \) that the previous conditional expectation is equal to
\[ \frac{1}{n} \sum_{j=0}^{n-1} E \left[ \eta(z + y_j)[1 - \eta(z + y_{j+1})] (T^{z+y_j,z+y_{j+1}} f)(\eta)[g(\eta) - g_z(\eta)] \mid \mathcal{F}_z \right]. \]

By Schwarz inequality, this expression is bounded above by
\[ \frac{A}{2n} \sum_{j=0}^{n-1} E \left[ (T^{z+y_j,z+y_{j+1}} f)(\eta)^2 \mid \mathcal{F}_z \right] + \frac{1}{2A} E \left[ [g(\eta) - g_z(\eta)]^2 \mid \mathcal{F}_z \right] \]
for every \( A > 0 \). Taking expectation with respect to \( \nu_\alpha \), in view of (3.4), the first term becomes
\[ \frac{2A}{n} \sum_{j=0}^{n-1} D_{z+y_j,z+y_{j+1}}(f). \]

On the other hand, the second one is the variance of \( g \) with respect to the canonical measure on \( C^+z \). Thus, by the spectral gap for the symmetric simple exclusion process on a finite torus, there exists a finite and universal constant \( B_0 \) which makes the second term in the penultimate expression bounded by
\[ B_0 n^2 \sum_{j=0}^{n-1} E \left[ [g(\sigma^{z+y_j,z+y_{j+1}}) - g(\eta)]^2 \mid \mathcal{F}_z \right]. \]

Taking expectation with respect to \( \nu_\alpha \) and recollecting all previous estimates, we obtain that
\[ |\langle L^0_{C^+z} f, g \rangle| \leq \frac{2A}{n} \sum_{j=0}^{n-1} D_{z+y_j,z+y_{j+1}}(f) + \frac{B_0 n^2}{2A} \sum_{j=0}^{n-1} D_{z+y_j,z+y_{j+1}}(g). \]

Summing over \( z \) and minimising over \( A \) we conclude that
\[ |\langle L_C f, g \rangle|^2 \leq B_1 \left\{ \sum_{z \in \mathbb{Z}^d} \left\{ \sum_{j=0}^{n-1} D_{z+y_j,z+y_{j+1}}(f) \right\} \left\{ \sum_{z \in \mathbb{Z}^d} \left\{ \sum_{j=0}^{n-1} D_{z+y_j,z+y_{j+1}}(g) \right\} \right\} \right\} \]
for some finite universal constant \( B_1 \). This concludes the proof of the lemma in view of the formula (3.3) for the Dirichlet form \( \langle -L_C f, f \rangle \).

**4 Duality**

The proof of the central limit theorem for additive functionals in the context of asymmetric exclusion processes relies on the concept of degree of a local function and on the decomposition of the space \( L^2(\nu_\alpha) \) in a direct sum of functions of fixed degree. This is the content of this section.
For each $n \geq 0$, denote by $E_n$ the subsets of $\mathbb{Z}^d$ with $n$ points and let $E = \bigcup_{n \geq 0} E_n$ be the class of finite subsets of $\mathbb{Z}^d$. For each $A$ in $E$, let $\Psi_A$ be the local function

$$\Psi_A = \prod_{x \in A} \frac{\eta(x) - \alpha}{\sqrt{\chi(\alpha)}},$$

where $\chi(\alpha) = \alpha(1 - \alpha)$. By convention, $\Psi_\emptyset = 1$. It is easy to check that $\{\Psi_A, A \in E\}$ is an orthonormal basis of $L^2(\nu_\alpha)$. For each $n \geq 1$, denote by $G_n$ the subspace of $L^2(\nu_\alpha)$ generated by $\{\Psi_A, A \in E_n\}$, so that $L^2(\nu_\alpha) = \bigoplus_{n \geq 0} G_n$. Functions in $G_n$ are said to have degree $n$.

Consider a local function $f$. Since $\{\Psi_A : A \in E\}$ is a basis of $L^2(\nu_\alpha)$, there exists a finite supported function $f : E \to \mathbb{R}$ such that

$$f = \sum_{A \in E} f(A) \Psi_A = \sum_{n \geq 0} \sum_{A \in E_n} f(A) \Psi_A.$$

The function $f$ represents the Fourier coefficients of the local function $f$ and is denoted by $\mathcal{F}f$ when we want to stress its dependence on $f$. $\mathcal{F} : E \to \mathbb{R}$ is function of finite support because $f$ is a local function. Moreover, the Fourier coefficients $f(A)$ depend not only on $f$ but also on the density $\alpha$; $f(A) = f(\alpha, A)$.

Denote by $C$ the space of finite supported functions $f : E \to \mathbb{R}$, by $\mu$ the counting measure on $E$ and by $\langle \cdot, \cdot \rangle_\mu$ the inner product in $L^2(\mu)$. For any two cylinder functions $f = \sum_{A \in E} f(A) \Psi_A$, $g = \sum_{A \in E} g(A) \Psi_A$,

$$\langle f, g \rangle_{\nu_\alpha} = \sum_{A, B \in E} f(A) g(B) \langle \Psi_A, \Psi_B \rangle_{\nu_\alpha} = \sum_{A \in E} f(A) g(A) = \langle f, g \rangle_\mu .$$

In particular, the map $\mathcal{F} : L^2(\nu_\alpha) \to L^2(\mu)$ is an isomorphism.

To examine how the generator acts on the Fourier coefficients we need to introduce some notation. For a subset $A$ of $\mathbb{Z}^d$ and $x, y$ in $\mathbb{Z}^d$, denote by $A_{x,y}$, the set defined by

$$A_{x,y} = \begin{cases} (A \setminus \{x\}) \cup \{y\} & \text{if } x \in A, y \notin A, \\ (A \setminus \{y\}) \cup \{x\} & \text{if } x \notin A, y \in A, \\ A & \text{otherwise}. \end{cases} \quad (4.1)$$

Fix a local function $f = \sum_{A \in E} (\mathcal{F}f)(A) \Psi_A$. We claim that

$$Lf = \sum_{A \in E} (\mathcal{L}_\alpha \mathcal{F}f)(A) \Psi_A,$$

where

$$\mathcal{L}_\alpha = \mathcal{G} + (1 - 2\alpha)\mathcal{K} + \sqrt{\chi(\alpha)} \mathcal{J}_+ + \sqrt{\chi(\alpha)} \mathcal{J}_- \quad (4.2)$$
and $\mathfrak{S}$, $\mathfrak{A}$, $\mathfrak{J}_+$, $\mathfrak{J}_-$ are the operators defined by

$$
(\mathfrak{S}f)(A) = (1/2) \sum_{x, y \in \mathbb{Z}^d} s(y - x) [f(A_{x, y}) - f(A)],
$$

$$
(\mathfrak{A}f)(A) = \sum_{x \in A \atop y \notin A} a(y - x) [f(A_{x, y}) - f(A)],
$$

$$
(\mathfrak{J}_+ f)(A) = 2 \sum_{x, y \in A} a(y - x) f(A \setminus \{y\}),
$$

$$
(\mathfrak{J}_- f)(A) = 2 \sum_{x, y \notin A} a(y - x) f(A \cup \{y\}).
$$

In other words, $T L = L \tau$.

In general $L_\alpha$ is not the generator of a Markov process. However, in the particular case where the transition probability $p(\cdot)$ is symmetric, the operators $\mathfrak{A}$, $\mathfrak{J}_+$, $\mathfrak{J}_-$ vanish and $L_\alpha$ becomes $\mathfrak{S}$, which restricted to $E_\alpha$ corresponds to the generator of the Markov process in which $n$ particles evolve on $\mathbb{Z}^d$ according to symmetric exclusion random walks. In fact, $\mathfrak{S}$ is the operator on the Fourier coefficients associated to the symmetric part of the generator since, by (4.2),

$$
S f = \sum_{A \in E} (\mathfrak{S} f)(A) \Psi_A. \quad (4.3)
$$

To prove (4.2), we compute $L \Psi_A$ for finite subsets $A$ of $\mathbb{Z}^d$. By definition of the generator $L$,

$$
(L \Psi_A)(\eta) = \sum_{x, y \in \mathbb{Z}^d} \eta(x) [1 - \eta(y)] p(y - x) [\Psi_A(\sigma^{x,y} \eta) - \Psi_A(\eta)]
$$

$$
= \sum_{x, y \in \mathbb{Z}^d} \eta(x) p(y - x) [\Psi_A(\sigma^{x,y} \eta) - \Psi_A(\eta)]
$$

because on the set $\eta(x)\eta(y) = 1$, $\sigma^{x,y} \eta = \eta$. Moreover, $\Psi_A(\sigma^{x,y} \eta) - \Psi_A(\eta)$ vanishes unless $x \in A$, $y \notin A$ or $x \notin A$, $y \in A$. In these two cases, $\Psi_A(\sigma^{x,y} \eta) = \Psi_{A_+,y}(\eta)$. These remarks together with the elementary identity

$$
\eta(z) \Psi_B = \begin{cases} 
(1 - \alpha) \Psi_B + \sqrt{\chi(\alpha)} \Psi_{B \setminus \{z\}} & \text{if } z \in B, \\
\alpha \Psi_B + \sqrt{\chi(\alpha)} \Psi_{B \cup \{z\}} & \text{if } z \notin B
\end{cases}
$$

give that

$$
L \Psi_A = \sum_{x \in A \atop y \notin A} s(y - x) [\Psi_{A_{x,y}} - \Psi_A] + (2\alpha - 1) \sum_{x \in A \atop y \notin A} a(y - x) [\Psi_{A_{x,y}} + \Psi_A]
$$

$$
+ 2 \sqrt{\chi(\alpha)} \sum_{x \in A \atop y \notin A} a(y - x) \Psi_{A \cup \{y\}} - 2 \sqrt{\chi(\alpha)} \sum_{x \in A \atop y \notin A} a(y - x) \Psi_{A \setminus \{x\}},
$$
from which (4.2) follows after a change of variables.

We claim that the operators $\mathcal{A}$ and $\mathcal{J} + \mathcal{J}$ are anti-symmetric with respect to the counting measure. More precisely, for any finitely supported functions $f, g : \mathcal{E} \to \mathbb{R}$,

$$
\langle \mathcal{A} f, g \rangle_\mu = -\langle f, \mathcal{A} g \rangle_\mu , \quad \langle \mathcal{J} + f, g \rangle_\mu = -\langle f, \mathcal{J} - g \rangle_\mu .
$$

(4.4)

Moreover, for any finite supported functions $f : \mathcal{E}_1 \to \mathbb{R}$, $g : \mathcal{E}_0 \to \mathbb{R}$,

$$
\langle \mathcal{J} f, g \rangle(\phi) = 0 , \quad \langle \mathcal{J} + g \rangle(\{z\}) = 0
$$

(4.5)

for all $z$ in $\mathbb{Z}$.

The proof of identities (4.4) is based on the following observation. Since $a$ is anti-symmetric, $\sum_{z \in \mathbb{Z}} a(z) = 0$. In particular, for any finite set $A$ and any site $y$, $\sum_{x \in A} a(y - x) = -\sum_{y \not\in A} a(y - x)$.

Fix two functions $f, g$ in $\mathcal{E}$. By the observation just mentioned and since $a(\cdot)$ is anti-symmetric, $\sum_{x \in A, y \not\in A} a(y - x) = -\sum_{y \not\in A} a(y - x) = 0$, so that $\sum_{A \in \mathcal{E}} \sum_{x \in A, y \not\in A} a(y - x) f(A) g(A) = 0$. Therefore, by definition of $\mathcal{A}$,

$$
\langle \mathcal{A} f, g \rangle_\mu = \sum_{A \in \mathcal{E}} \sum_{x \in A, y \not\in A} a(y - x) f(A) g(A).
$$

Changing variables $B = A_{x,y}$, $x' = y$, $y' = x$, since $a(\cdot)$ is anti-symmetric, we obtain that the previous sum is equal to

$$
-\sum_{A \in \mathcal{E}} \sum_{x \in A, y \not\in A} a(y - x) f(A) g(A_{x,y}) = -\langle f, \mathcal{A} g \rangle_\mu.
$$

This proves the first identity in (4.4). The second one is analogous. On the other hand, (4.5) follows immediately from the definition of the operators $\mathcal{J}$, $\mathcal{J}$ and the fact that $\sum_{y \not\in A} a(y - x) = 0$ for all $x$.

The basis $\{\Psi_A, A \in \mathcal{E}\}$ permitted to decompose the generator $L$ of the simple exclusion process in four pieces: $\mathcal{E}$, $\mathcal{A}$, $\mathcal{J}$ and $\mathcal{J}$. We may now define the operators $\mathcal{A}$, $\mathcal{J}$, $\mathcal{J}$ on $L^2(\nu_\alpha)$ corresponding to $\mathcal{A}$, $\mathcal{J}$ and $\mathcal{J}$: For a local function $f$, let

$$
(A f)(\eta) = \sum_{B \in \mathcal{E}} (\mathcal{A} \mathcal{T} f)(B) \Psi_B(\eta) , \quad (\mathcal{J} f)(\eta) = \sum_{B \in \mathcal{E}} (\mathcal{J} \mathcal{T} f)(B) \Psi_B(\eta) , \quad (\mathcal{J} f)(\eta) = \sum_{B \in \mathcal{E}} (\mathcal{J} \mathcal{T} f)(B) \Psi_B(\eta) .
$$

Of course, with these definitions, $\mathcal{T} A = \mathcal{A} \mathcal{T}$, $\mathcal{T} \mathcal{J} = \mathcal{J} \mathcal{T}$.

The description of a local function and of the generator in terms of their Fourier coefficients have several applications. One of the first is the possibility to obtain simple criteria to describe local functions with finite $\mathcal{H}$ norm.

Consider the semi-norm $\| \cdot \|_1 = \| \cdot \|_{\nu_\alpha, 1}$ defined on the space of cylinder functions $\mathcal{C}$ by
\[ \| f \|_1^2 = \langle f, (-L)f \rangle_{\nu_\alpha} = \langle f, (-S)f \rangle_{\nu_\alpha}. \]

It is easy to see that this norm satisfies the parallelogram identity because it comes from a scalar product. Notice, moreover, that only the symmetric part of the operator appears in the definition. Since the parallelogram identity holds, the space of local functions endowed with the norm \( \| \cdot \|_1 \) induces a Hilbert space \( \mathcal{H}_1(S) = \mathcal{H}_1(S, \mathcal{C}, \nu_\alpha) \) with inner product \( \langle \cdot, \cdot \rangle_{\nu_\alpha,1} \) defined by

\[ \langle f, g \rangle_{\nu_\alpha,1} = \langle f, (-S)g \rangle_{\nu_\alpha} \]

for local functions \( f, g \).

An elementary computation, based on the change of variables \( \xi = \sigma^{x,y} \eta \), shows that

\[ \langle f, (-S)g \rangle_{\nu_\alpha} = \frac{1}{4} \sum_{x, y \in \mathbb{Z}^d} s(y - x) \int (T^{x,y} f)(T^{x,y} g) \, d\nu_\alpha, \]

where, for a local function \( f \),

\[ (T^{x,y} f)(\eta) = f(\sigma^{x,y} \eta) - f(\eta). \quad (4.6) \]

We may also define the Hilbert space \( \mathcal{H}_1 \) at the level of the Fourier coefficients. Since \( \mathcal{A} \) and \( \mathcal{J}_+ + \mathcal{J}_- \) are anti-symmetric, the symmetric part of the operator \( \mathcal{L}_\alpha \) with respect to the counting measure \( \mu \) is \( \mathcal{S} \). In particular, the Sobolev space \( \mathcal{H}_1 \) in the present context is the Hilbert space induced by the finite supported functions \( \mathcal{C} \) endowed with the scalar product \( \langle \cdot, \cdot \rangle_{\mu,1} \) defined by

\[ \langle f, g \rangle_{\mu,1} = \langle f, (-\mathcal{S})g \rangle_{\mu}. \]

This Hilbert space is denoted by \( \mathcal{H}_1(\mathcal{S}) = \mathcal{H}_1(\mathcal{S}, \mathcal{C}, \mu) \).

An elementary computation shows that for any two finite supported functions \( f, g : \mathcal{E} \to \mathbb{R} \),

\[ \langle f, g \rangle_{\mu,1} = \langle f, (-\mathcal{S})g \rangle_{\mu} \]

\[ = -\frac{1}{2} \sum_{x, y \in \mathbb{Z}^d} s(y - x) \sum_{A \in \mathcal{E}} \{ f(A) \{ g(A_{x,y}) - g(A) \} \} \]

\[ = \frac{1}{4} \sum_{x, y \in \mathbb{Z}^d} s(y - x) \sum_{A \in \mathcal{E}} \{ f(A_{x,y}) - f(A) \} \{ g(A_{x,y}) - g(A) \} . \]

The last identity is obtained through a change of variables \( B = A_{x,y} \).

Of course the Hilbert spaces \( \mathcal{H}_1(S) \), \( \mathcal{H}_1(\mathcal{S}) \) are isomorphic since for any local functions \( f, g \), by (4.2),

\[ \langle f, g \rangle_{\nu_\alpha,1} = \langle f, (-S)g \rangle_{\nu_\alpha} = \langle \mathcal{T} f, -\mathcal{T}(Sg) \rangle_{\mu} \]

\[ = \langle \mathcal{T} f, (-\mathcal{S}) \mathcal{T} g \rangle_{\mu} = \langle \mathcal{T} f, \mathcal{T} g \rangle_{\mu,1} . \]
Denote by $\pi_n$ the orthogonal projection on $\mathcal{G}_n$, the space of functions of degree $n$, or the restriction to $\mathcal{E}_n$ of a finite supported function $f : \mathcal{E} \to \mathbb{R}$: 
\((\pi_n f)(A) = f(A)1\{A \in \mathcal{E}_n\}\). With this definition, $\mathfrak{T}(\pi_n f) = \pi_n \mathfrak{T} f$.

An important simplification appears clearly when working on the level of the Fourier coefficients in computations involving the $H_1$ norm. For example, in view of the previous formulae, we have that
\[
\|f\|_1^2 := \langle f, f \rangle_{\mu_1} = \left(\frac{1}{4}\right) \sum_{n \geq 0} \sum_{x,y \in \mathbb{Z}^d} s(y-x) \sum_{A \in \mathcal{E}_n} (f(A_{x,y}) - f(A))^2.
\]

In particular, the square of the $H_1$ norm of a finite supported function $f$ is just the sum of the squares of the $H_1$ norm of $\pi_n f$:
\[
\|f\|_1^2 = \sum_{n \geq 0} \|\pi_n f\|_1^2. \quad (4.7)
\]

The following alternative formula for the $H_1$ norm of a function of degree $n$ is sometimes useful. Fix a function $f : \mathcal{E}_n \to \mathbb{R}$. By definition of $A_{x,y}$, in the penultimate formula we may restrict the sum to sets $A$ which either contain $x$ and do not contain $y$ or the opposite. In particular, by symmetry,
\[
\|f\|_1^2 = \left(\frac{1}{2}\right) \sum_{x,y \in \mathbb{Z}^d} s(y-x) \sum_{A \in \mathcal{E}_n, A \ni x, A \not\ni y} (f(A_{x,y}) - f(A))^2.
\]

Performing the change of variables $A' = A \cup \{y\}$, we get that
\[
\|f\|_1^2 = \left(\frac{1}{2}\right) \sum_{x,y \in \mathbb{Z}^d} s(y-x) \sum_{A \in \mathcal{E}_{n-1}, A \ni x, A \not\ni y} (f(A \setminus \{x\}) - f(A \setminus \{y\}))^2. \quad (4.8)
\]

Once the spaces $\mathcal{H}_1(S)$, $\mathcal{H}_1(\mathcal{G})$ have been introduced, we may examine the dual Hilbert spaces $\mathcal{H}_{-1}(S)$, $\mathcal{H}_{-1}(\mathcal{G})$. Recall from Section 2.2 that $\mathcal{H}_{-1}(S) = \mathcal{H}_{-1}(S, \mathcal{C}, \nu_o)$ is the Hilbert space induced by local functions and the norm $\| \cdot \|_{-1}$ defined by
\[
\| f \|_{-1}^2 = \sup_{g \in \mathcal{C}} \left\{ 2 \langle f, g \rangle_{\nu_o} - \langle g, g \rangle_{\nu_o,1} \right\},
\]
while $\mathcal{H}_{-1}(\mathcal{G}) = \mathcal{H}_{-1}(\mathcal{G}, \mathcal{C}, \mu)$ is the Hilbert space induced by finitely supported functions $f : \mathcal{E} \to \mathbb{R}$ and the norm $\| \cdot \|_{-1}$ defined by
\[
\| f \|_{-1}^2 = \sup_{g \in \mathcal{C}} \left\{ 2 \langle f, g \rangle_{\mu} - \langle g, g \rangle_{\mu,1} \right\}.
\]

Since for local functions $f$, $g$, $\langle f, g \rangle_{\nu_o} = \langle \mathfrak{T} f, \mathfrak{T} g \rangle_{\mu}$ and $\langle g, g \rangle_{\nu_o,1} = \langle \mathfrak{T} g, \mathfrak{T} g \rangle_{\mu,1}$, the previous supermom coincide and $\|f\|_{-1}^2 = \|\mathfrak{T} f\|_{-1}^2$ so that the spaces $\mathcal{H}_{-1}(S)$ and $\mathcal{H}_{-1}(\mathcal{G})$ are isomorph.
We claim that for finite supported function \( f \) defined on \( E \),
\[
\| f \|_{-1}^2 = \sum_{n \geq 0} \| \pi_n f \|_{-1}^2 .
\]
Indeed, by (4.7),
\[
\| f \|_{-1}^2 = \sup_{g \in \mathcal{E}} \left\{ 2 \langle f, g \rangle_{\mu} - \langle g, g \rangle_{\mu,1} \right\} = \sup_{g \in \mathcal{E}} \sum_{n \geq 0} \left\{ 2 \langle \pi_n f, \pi_n g \rangle_{\mu} - \langle \pi_n g, \pi_n g \rangle_{\mu,1} \right\} = \sum_{n \geq 0} \sup_{g \in \mathcal{E}} \left\{ 2 \langle \pi_n f, g \rangle_{\mu} - \langle g, g \rangle_{\mu,1} \right\} = \sum_{n \geq 0} \| \pi_n f \|_{-1}^2 .
\]

The penultimate identity follows from the fact that \( \langle \pi_n f, g \rangle_{\mu} \) is equal to \( \langle \pi_n f, \pi_n g \rangle_{\mu} \) for all finitely supported function \( g \).

Fix a finitely supported function \( f : E_n \to \mathbb{R} \). Since \( \mathcal{S} \) restricted to \( E_n \) corresponds to the generator of \( n \) symmetric random walks evolving on \( \mathbb{Z}^d \) with an exclusion rule,
\[
\| f \|_{-1}^2 = \langle f, (-\mathcal{S})^{-1} f \rangle_{\mu} = \sum_{A,B \in E_n} f(A) \mathfrak{G}_n(A,B) f(B) ,
\]
where \( \mathfrak{G}_n(A,B) \) is the Green function associated to \( \mathcal{S} \) restricted to \( E_n \).

Since \( n \) random walks with exclusion evolving on \( \mathbb{Z}^d \) corresponds essentially to a random walk in dimension \( nd \), which is transient if \( nd \geq 3 \), \( \mathfrak{G}_n(A,B) < \infty \) if \( nd \geq 3 \). We have in particular the following result.

**Lemma 3.11.** Fix \( 0 \leq \alpha \leq 1 \). A local zero-mean function \( f \) belongs to \( \mathcal{H}_{-1}(S, \nu_\alpha) \) if
\[
\sum_{A \in E_n} (\mathcal{I} f)(\alpha, A) = 0
\]
for all \( n \) such that \( nd \leq 2 \).

**Proof.** Fix \( 0 \leq \alpha \leq 1 \) and a local zero-mean function \( f \). We have just seen that
\[
\| f \|_{-1}^2 = \| \mathcal{I} f \|_{-1}^2 = \sum_{n \geq 1} \| \pi_n \mathcal{I} f \|_{-1}^2 .
\]
The sum starts from \( n = 1 \) because by assumption \( (\pi_0 \mathcal{I} f)(\phi) \), which is the expectation of \( f \) with respect to \( \nu_\alpha \), vanishes.

Let \( \mathfrak{G}_n \) be the Green function associated to the Markov generator \( \mathcal{S} \) restricted to \( E_n \). By [?], \( G_n(A,B) \) is uniformly bounded for \( nd \geq 3 \). In particular, in this range and since \( \mathcal{I} f \) is a finite supported function,
\[ \| \pi_n \mathcal{F} f \|_1^2 = \sum_{A, B \in \mathcal{E}_n} f(A) \Theta_n(A, B) f(B) < \infty, \]

where \( f = \mathcal{F} f \).

Fix now \( n \) such that \( nd \leq 2 \), a finite supported function \( \mathfrak{h} : \mathcal{E} \to \mathbb{R} \) and a set \( A_0 \) in \( \mathcal{E}_n \). By assumption,

\[ \sum_{A \in \mathcal{E}_n} f(A) \mathfrak{h}(A) = \sum_{A \in \mathcal{E}_n} f(A) \{ \mathfrak{h}(A) - \mathfrak{h}(A_0) \}. \]

Since \( f \) has finite support, by Schwarz inequality, the previous expression is bounded above by \( C(f) \| \mathfrak{h} \|_1 \) for some finite constant \( C(f) \) depending only on \( f \). This shows that \( \| \pi_n f \|_{-1} \) is finite for \( nd \leq 2 \) and concludes the proof of the lemma.

Fix a local function \( f = \sum_{A \in \mathcal{E}} f(A) \Psi_A \) and two densities \( \alpha, \beta \). Here the Fourier decomposition is performed with respect to the measure \( \nu_\alpha \) so that \( \Psi_A(\eta) = \Psi_A(\alpha, \eta) \), \( f(A) = f(\alpha, A) \). Denote by \( |A| \) the total number of sites of a finite subset \( A \) of \( \mathbb{Z}^d \). Since \( \nu_\beta \) is a product measure, \( E_{\nu_\beta}[\Psi_A(\alpha)] = \chi(\alpha)^{-|A|/2} (\beta - \alpha)^{|A|} \),

\[ E_{\nu_\beta}[f] = \sum_{n \geq 0} \frac{(\beta - \alpha)^n}{\chi(\alpha)^{n/2}} \sum_{A \in \mathcal{E}_n} f(\alpha, A). \]

In particular, \( E_{\nu_\alpha}[f] = f(\alpha, \phi) \) and more generally,

\[ \frac{d^n}{d\beta^n} E_{\nu_\beta}[f] \bigg|_{\beta = \alpha} = \frac{n!}{\chi(\alpha)^{n/2}} \sum_{A \in \mathcal{E}_n} f(\alpha, A). \]

The assumptions of the previous lemma can thus be restated as \( E_{\nu_\alpha}[f] = 0 \) in dimension \( d \geq 3 \) and

\[ \frac{d^n}{d\beta^n} E_{\nu_\beta}[f] \bigg|_{\beta = \alpha} = 0 \quad \text{for } n = 0, 1, 2 \text{ in dimension } 1, \]

\[ \frac{d^n}{d\beta^n} E_{\nu_\beta}[f] \bigg|_{\beta = \alpha} = 0 \quad \text{for } n = 0, 1 \text{ in dimension } 2. \]

We conclude this section with an example to illustrate that there might be a central limit theorem for an additive functional of a Markov process without having a solution of the Poisson equation in \( L^2 \).

**Example 3.12** Assume that the transition probability \( p(\cdot) \) is symmetric, fix a density \( 0 < \alpha < 1 \) and consider the local zero-mean function \( V(\eta) = \eta(0) - \alpha \) which can be written as \( \sqrt{\chi(\alpha)} \Psi_{\{0\}} \). Taking Fourier coefficients, the Poisson equation \( -L f = V \) may be transformed into the elliptic equation \( -\mathcal{G} f = \mathfrak{V} \),

where \( \mathfrak{V}(A) = \sqrt{\chi(\alpha)} 1\{A = \{0\} \} \). This last transformation illustrates the
utility of the duality introduced in this section. While the equation \(-L f = V\) is an equation on an infinite-dimensional space, \(-\mathcal{G} f = \mathcal{V}\) is an equation on the finite-dimensional space \(\mathbb{Z}^d\) because \(V\) has degree one and \(\mathcal{G}\) maps \(\mathcal{E}_1\) into \(\mathcal{E}_1\). (More generally, \(\mathcal{G}\) preserves the degrees, i.e., maps \(\mathcal{E}_n\) into \(\mathcal{E}_n\).) Recall that \(\mathcal{G}\) restricted to \(\mathcal{E}_1\) corresponds to the generator of a symmetric random walk on \(\mathbb{Z}^d\) and that we denoted by \(\mathcal{G}_1(\cdot, \cdot)\) the Green kernel associated to \(\mathcal{G}\) restricted to \(\mathcal{E}_1\). Since the symmetric random walk is transient in dimension \(d \geq 3\), \(\mathcal{S} f = V\) has a solution in \(\mathbb{Z}^d\) given by \(f(x) = \mathcal{G}_1(0, x)\). Here, to keep notation simple, we denoted the set \(\{x\}\) simply by \(x\) so that \(\mathcal{G}_1(0, x)\) and \(f(x)\) stand for \(\mathcal{G}_1(\{x\}, \{y\})\) and \(f(\{x\})\). It is well known that \(\mathcal{G}_1(0, x)\) decays as \(|x|^{2-d}\) so that \(f\) belongs to \(L^2(\mu)\) only in dimension \(d \geq 5\). Thus, the Poisson equation has a solution in \(L^2(\mu)\) only in dimension \(d \geq 5\). Nevertheless, a central limit theorem holds for the time integral of \(V\) whenever \(V\) belongs to \(H_{-1}(L, \nu_0)\).

5 The asymmetric case, \(\alpha = 1/2\)

In this section and the next we extend the central limit theorem to asymmetric exclusion processes in dimension \(d \geq 3\):

**Theorem 3.13.** Assume that \(d \geq 3\) and consider a zero-mean local function \(V\) in \(H_{-1}(L, \nu_0)\). Then,

\[
\frac{1}{\sqrt{t}} \int_0^t V(\eta_s) \, ds
\]

converges in distribution, as \(t \uparrow \infty\), to a zero-mean Gaussian variable with variance \(\sigma^2(V) = 2 \lim_{\lambda \to 0} \|f_\lambda\|^2_1\), where \(f_\lambda\) is the solution of the resolvent equation (2.1).

Recall from (4.2) the representation of the operator \(\mathcal{L}_\alpha\) as a sum of four pieces and from Section 2.6 the general approach to prove central limit theorems for additive functionals under a graded sector condition. In the present context and with the notation introduced in Section 4, \(S_0 = S\), \(B_0 = \mathcal{A}\), \(L_+ = \mathcal{J}_+\), \(L_- = \mathcal{J}_-\).

By Theorem 2.2.2 and Lemma 2.2.12, a central limit theorem holds if the generator satisfies the following three assumptions:

(A1) There exists a finite constant \(C_0\) such that

\[
0 \leq \langle f, (-\mathcal{G}) f \rangle_\mu \leq C_0 \langle f, (-\mathcal{L}_\alpha) f \rangle_\mu \quad (5.1)
\]

for all finite supported functions \(f: \mathcal{E} \to \mathbb{R}\).
There exists \( \beta < 1 \) and a finite constant \( C_0 \) such that
\[
\langle f, (-\mathcal{J}_\pm) g \rangle_\mu^2 \leq C_0 n^{2\beta} \langle f, (-\mathcal{G}) f \rangle_\mu \langle g, (-\mathcal{G}) g \rangle_\mu
\]  
for all finite supported functions \( f : \mathcal{E}_n \to \mathbb{R} \), \( g : \mathcal{E}_{n+1} \to \mathbb{R} \).

(A2) There exists a finite constant \( C_0 \) such that
\[
\langle f, (-\mathcal{A}) g \rangle_\mu^2 \leq C_0 n^{2\gamma} \langle f, (-\mathcal{G}) f \rangle_\mu \langle g, (-\mathcal{G}) g \rangle_\mu
\]  
for all finite supported functions \( f, g : \mathcal{E}_n \to \mathbb{R} \).

(A3) There exist finite constants \( \gamma \) and \( C_0 \) such that
\[
\langle f, (-\mathcal{A}) g \rangle_\mu^2 \leq C_0 n^{2\gamma} \langle f, (-\mathcal{G}) f \rangle_\mu \langle g, (-\mathcal{G}) g \rangle_\mu
\]  
for all finite supported functions \( f, g : \mathcal{E}_n \to \mathbb{R} \).

Since \( \mathcal{A}, \mathcal{J}_- + \mathcal{J}_+ \) are anti-symmetric operators, the first condition holds trivially with \( C_0 = 1 \) and an equality in place of the inequality. In particular, the norms \( \| \cdot \|_{0,1}, \| \cdot \|_{0,-1} \) introduced in Section 2.6 coincide with the norms \( \| \cdot \|_1, \| \cdot \|_{-1} \).

The second assumption is proved in Lemma 3.14 below in dimensions \( d \geq 3 \) with \( \beta = 1/2 \) and constitutes the main result of this section. To avoid proving the third estimate, we assume in this section that the density \( \gamma = 1 \), in which case the asymmetric part \( (1 - 2\alpha)\mathcal{A} \) vanishes.

As we said before, Theorem 3.13 in the case \( \gamma = 1 \) follows from Theorem 2.2.2, Lemma 2.2.12 and Lemma 3.14.

In the remaining of this section we examine the graded sector condition for the asymmetric exclusion process in dimension \( d \geq 3 \).

Lemma 3.14. Assume that \( d \geq 3 \). There exists a finite constant \( C_0 \), depending only on the probability \( p(\cdot) \), such that for every \( n \geq 0 \),
\[
\langle \mathcal{J}_- f, g \rangle_\mu^2 \leq C_0 n (\langle \mathcal{G} f \rangle_\mu \langle -\mathcal{G} g \rangle_\mu
\]  
for any finitely supported functions \( f : \mathcal{E}_n \to \mathbb{R} \), \( g : \mathcal{E}_{n+1} \to \mathbb{R} \).

Since \( -\mathcal{J}_- \) is the adjoint of \( \mathcal{J}_+ \), a similar bound holds for \( \mathcal{J}_- \).

Proof. Fix \( n \geq 0 \) and two finite supported functions \( f : \mathcal{E}_n \to \mathbb{R} \), \( g : \mathcal{E}_{n+1} \to \mathbb{R} \).

By the explicit formula for \( \mathcal{J}_+ \),
\[
\langle \mathcal{J}_+ f, g \rangle_\mu = 2 \sum_{A \in \mathcal{E}_n} \sum_{x, y \in A} a(y - x) \{ f(A \setminus \{ y \}) \ g(A) \}.
\]

Notice that the first sum is in fact carried over sets \( A \) in \( \mathcal{E}_{n+1} \). Since \( a(\cdot) \) is anti-symmetric, a change of variables \( x' = y, y' = x \) permits to rewrite the previous scalar product as
\[
\sum_{A \in \mathcal{E}_{n+1}} \sum_{x, y \in A} a(y - x) \{ f(A \setminus \{ y \}) - f(A \setminus \{ x \}) \} \ g(A).
\]

Since \( a(\cdot) \) is absolutely bounded by \( s(\cdot) \), the elementary inequality \( 2ab \leq \gamma a^2 + \gamma^{-1} b^2 \) gives that the previous expression is less than or equal to
\[
\frac{\gamma}{2} \sum_{A \in E_{n+1}} \sum_{x,y \in A} s(y - x) \left\{ (f(A \setminus \{y\}) - f(A \setminus \{x\}) \right\}^2 + \frac{1}{2\gamma} \sum_{A \in E_{n+1}} g(A)^2 W(A)
\]

for all \(\gamma > 0\), where \(W : E \to \mathbb{R}\) is the function defined by

\[
W(A) = \sum_{x,y \in A} s(y - x).
\]

By (4.8), the first expression in (5.4) is equal to \(\gamma(-\mathcal{S}f, f)_\mu\). By Lemma 3.16, the second term is bounded by \(C_1 \gamma^{-1} n(-\mathcal{S}g, g)_\mu\). Therefore, (5.4) is less than or equal to

\[
\gamma(-\mathcal{S}f, f)_\mu + C_1 n\gamma^{-1} (-\mathcal{S}g, g)_\mu
\]

for every \(\gamma > 0\). It remains to minimise over \(\gamma\) to conclude the proof of the lemma.

Next result is a straightforward consequence of the lemma.

**Corollary 3.15.** There exists a finite constant \(C_0\), depending only on the probability \(p(\cdot)\), such that

\[
\|3_\pm f\|_1^2 \leq C_0 n \|f\|_1^2
\]

for any finitely supported functions \(f : E_n \to \mathbb{R}\). In particular,

\[
\|3_\pm f\|_{k,-1} \leq C_0 \|f\|_{k+1,1}
\]

for any finitely supported functions \(f : E_n \to \mathbb{R}\) and every every \(k \geq 0\).

We conclude this section with an estimate on two point correlation functions.

**Lemma 3.16.** Let \(W : E_n \to \mathbb{R}\) be defined by \(W(A) = \sum_{x,y \in A} s(y - x)\). There exists a finite constant \(C_0\), depending only on \(p(\cdot)\) such that

\[
\sum_{A \in E_n} g(A)^2 W(A) \leq C_0 n (-\mathcal{S}g, g)_\mu
\]

for all \(n \geq 1\) and all finitely supported function \(g : E_n \to \mathbb{R}\).

**Proof.** We need some sort of spectral gap since we have to estimate the square of \(g\) multiplied by the function \(W\) by the \(L^2(\mu)\) norm of the gradient of \(g\). The idea is to reduce the problem to an estimate on symmetric random walks on \(\mathbb{Z}^d\).

Inverting the order of summation, we may write the expression we want to estimate by
The asymmetric case, $\alpha = 1/2$

\[ \sum_{x,y \in \mathbb{Z}^d} s(y - x) \rho(\{x, y\})^2, \]

where, for each set $\{x, y\}$ in $E_2$,

\[ \rho(\{x, y\})^2 = \sum_{A \in E_n \atop A \ni y, x} g(A)^2. \]

Note that $\rho$ inherits a finite support from $g$.

Consider the irreducible, continuous-time random walk on $\mathbb{Z}^d$ with transition probability $p(x, y) = s(y - x)$. Since the process is transient in $d \geq 3$ and since $s(\cdot)$ has finite support, by Proposition 2.24, there exists a finite constant $C_1$, depending only on $s(\cdot)$ such that

\[ \sum_{x \in \mathbb{Z}^d} s(x) F(x)^2 \leq C_1 \sum_{x, y \in \mathbb{Z}^d} s(y - x)[F(y) - F(x)]^2 \]

for all finitely supported functions $F : \mathbb{Z}^d \rightarrow \mathbb{R}$.

Since $\rho$ has a finite support, replacing the origin by a site $x_0$ and taking $F(x) = \rho(\{x_0, x\})$, we obtain that

\[ \sum_{x \in \mathbb{Z}^d} s(x - x_0) \rho(\{x_0, x\})^2 \leq C_1 \sum_{x, y \in \mathbb{Z}^d} s(y - x) \{\rho(\{x_0, y\}) - \rho(\{x_0, x\})\}^2. \]

Summing over $x_0$, the left hand side of the previous inequality becomes the expression we want to estimate. On the other hand, by definition of $\rho$ and by Schwarz inequality, we have that

\[ \{\rho(\{x_0, y\}) - \rho(\{x_0, x\})\}^2 \leq \sum_{A \in E_{n+1} \atop A \ni x_0, x, y} \{g(A \setminus \{y\}) - g(A \setminus \{x\})\}^2. \]

Summing over $x_0$ in the penultimate formula, we conclude that there exists a finite constant $C_1$, depending only on $s(\cdot)$, for which

\[ \sum_{A \in E_n} g(A)^2 \sum_{x, x_0 \in A} s(x - x_0) \leq C_1 \sum_{x, x_0, y \in \mathbb{Z}^d} s(y - x) \sum_{A \in E_{n+1} \atop A \ni x_0, x, y} \{g(A \setminus \{y\}) - g(A \setminus \{x\})\}^2. \]

Since $A$ has $n + 1$ elements, summing over $x_0$, the right hand side becomes

\[ C_1 n \sum_{x, y \in \mathbb{Z}^d} s(y - x) \sum_{A \in E_{n+1} \atop A \ni x} \{g(A \setminus \{y\}) - g(A \setminus \{x\})\}^2. \]

It follows from (4.8) that this expression is equal to $2C_1 n (-\mathcal{G} g, g)_n$. \hfill \square
6 The asymmetric case, $\alpha \neq 1/2$

We prove in this section Theorem 3.13 in the case $\alpha \neq 1/2$ by checking the assumptions of Lemma 2.2.12. Besides conditions (5.1), (5.2), which have been proved in the previous section, in this general context we also need to verify the validity of (5.3).

This assumption is not expected to hold because in the context of asymmetric simple exclusion processes, the operator $\mathfrak{A}$ is essentially a gradient, while $\mathfrak{S}$ is a Laplacian. This condition requires therefore that

$$(f, \nabla g)^2 \leq C_0 \langle \nabla f, \nabla f \rangle \langle \nabla g, \nabla g \rangle$$

which can not hold because it would imply a Poincaré inequality in infinite volume, that is, a bound on the $L^2$ norm of a zero-mean function in terms of the $L^2$ norm of its gradient: $\|f\| \leq C_0 \|
abla f\|$.

Nevertheless, if we return to the proof of Lemma 2.2.12 and recall that $f_\lambda$ stands for the solution of the resolvent equation, we see that the bound

$$\|\pi_n \mathfrak{A} f_\lambda\|_{-1}^2 \leq C_0 n \|\pi_n \mathfrak{S} V\|_{-1}^2,$$

or the one presented in Lemma 3.17, is sufficient to conclude the proof of that lemma. Since $f_\lambda$ is essentially $(\nabla - \Delta)^{-1} V$, $\Delta$ being the Laplacian, the previous inequality states that

$$\langle \nabla (\nabla - \Delta)^{-1} V, (-\Delta)^{-1} \nabla (\nabla - \Delta)^{-1} V \rangle \leq C_0 \langle V, (-\Delta)^{-1} V \rangle.$$

Formulating this estimate in Fourier variables, it is not difficult to show that such a bound should hold.

Recall the definition of the tripe norm defined in the previous chapter. In terms of Fourier coefficients, these norms can be written as

$$\|f\|_{k}^2 = \sum_{n \geq 0} n^{2k} \|\pi_n f\|^2, \quad \|f\|_{k,1}^2 = \sum_{n \geq 0} n^{2k} \|\pi_n f\|_1^2,$$

$$\|f\|_{k,-1}^2 = \sum_{n \geq 0} n^{2k} \|\pi_n f\|_{-1}^2.$$

**Lemma 3.17.** Let $f_\lambda$ be the solution of the resolvent equation (2.1). There exists a finite constant $C_0$, depending only on $\alpha$ and $p(\cdot)$, such that

$$\|\pi_n \mathfrak{A} f_\lambda\|_{-1}^2 \leq C_0 n \|\pi_n \mathfrak{S} V\|_{-1}^2 + C_0 n^3 \sum_{j=n-1}^{n+1} \|\pi_j f_\lambda\|_1^2$$

for all $n \geq 1$. In particular,

$$\|\mathfrak{A} f_\lambda\|_{k,-1}^2 \leq C_0 \|\mathfrak{S} V\|_{k+1,-1}^2 + C_0 \|f_\lambda\|_{k+2,1}^2.$$
Proof. Let \( w = TV, \ w_1 = w + \sqrt{\lambda(\alpha)} \{ J_+ + J_- \} f_\lambda \) so that
\[
\lambda f_\lambda - (\mathcal{S} + (1 - 2\alpha)\mathcal{A}) f_\lambda = w_1. \tag{6.1}
\]
By Lemma 3.14, there exists a finite constant \( C_0 = C_0(\alpha) \) such that
\[
\| \pi_n w_1 \|_1^2 \leq 2 \| \pi_n w \|_1^2 + C_0 n \sum_{j=n-1}^{n+1} \| \pi_j f_\lambda \|_1^2 \tag{6.2}
\]
for all \( n \geq 1 \).

The operator \( \mathcal{S} + (1 - 2\alpha)\mathcal{A} \) does not change the degree of a function. We may therefore examine equation (6.1) on each set \( E_n \):
\[
\lambda f_{\lambda,n} - (\mathcal{S} + (1 - 2\alpha)\mathcal{A}) f_{\lambda,n} = \pi_n w_1,
\]
where \( f_{\lambda,n} = \pi_n f_\lambda \). Since \( n \) is fixed until estimate (6.5), we omit the operator \( \pi_n \) in the next formulae.

The main idea of this proof is to approximate the operator \( \mathcal{S} + (1 - 2\alpha)\mathcal{A} \) by a convolution operator that can be analysed through Fourier transforms. Fix \( n \geq 1 \) and let \( X_n = (\mathbb{Z}^d)^n \). We consider a set \( A \) in \( E_n \) as an equivalent class of \( n! \) sets of distinct points of \( \mathbb{Z}^d \). A function \( f : E_n \to \mathbb{R} \) can be lifted into a symmetric function \( Bf \) on \( X_n \), that vanishes on \( X_n \setminus E_n \):
\[
Bf(x_1, \ldots, x_n) = \begin{cases} f(\{x_1, \ldots, x_n\}) & \text{if } x_i \neq x_j \text{ for } i \neq j, \\ 0 & \text{otherwise.} \end{cases}
\]
The operators \( \mathcal{S}, \mathcal{A} \) can also be extended in a natural way to \( X_n \). Denote by \( \{e_j, 1 \leq j \leq n\} \) the canonical basis of \( \mathbb{R}^n \) and consider on \( X_n \) the operators \( \mathcal{S}^o, \mathcal{A}^o \) defined by
\[
(\mathcal{S}^o f)(x) = \sum_{1 \leq j \leq n} s(z) \{ f(x + ze_j) - f(x) \},
\]
\[
(\mathcal{A}^o f)(x) = \sum_{1 \leq j \leq n} a(z) \{ f(x + ze_j) - f(x) \}.
\]
In this formula and below, \( x = (x_1, \ldots, x_n) \) is an element of \( X_n \), so that each \( x_j \) belongs to \( \mathbb{Z}^d \) and \( x + ze_j = (x_1, \ldots, x_{j-1}, x_j + z, x_{j+1}, \ldots, x_n) \).

Denote by \( \| \cdot \|_{X_n,1} \) the \( H_1 \) norm associated to the generator \( \mathcal{S}^o \); for each function \( f : X_n \to \mathbb{R} \),
\[
\| f \|_{X_n,1}^2 = \frac{-1}{n!} \sum_{x \in X_n} f(x) (\mathcal{S}^o f)(x)
\]
and denote by \( \| \cdot \|_{X_n,-1} \) its dual norm defined by
\[ \|f\|_{\mathcal{X}_{n-1}}^2 = \frac{-1}{n!} \sum_{x \in \mathcal{X}_n} f(x) \left( (\mathcal{G}^o)^{-1} f(x) \right). \]

Lifting the resolvent equation (6.1) to \( \mathcal{X}_n \) and adding and subtracting \( \mathcal{G}^o \mathcal{B}_\lambda + (1 - 2\alpha) \mathcal{A}^o \mathcal{B}_\lambda \), we obtain that

\[ \lambda \mathcal{B}_\lambda - \left\{ \mathcal{G}^o + (1 - 2\alpha) \mathcal{A}^o \right\} \mathcal{B}_\lambda = w_2, \quad (6.3) \]

where

\[ w_2 = \mathcal{B} w_1 + \left\{ \mathcal{B} \mathcal{S} - \mathcal{G}^o \mathcal{B} \right\} f_\lambda + (1 - 2\alpha) \left\{ \mathcal{B} \mathcal{A}^o - \mathcal{A}^o \mathcal{B} \right\} f_\lambda. \]

We claim that \( w_2 \) has finite \( H_{-1}(\mathcal{X}_n) \) norm. Indeed, for each \( n \geq 1 \), by (6.7) and Lemma 3.19 below, there exists a finite constant \( C_0 \) such that

\[ \|\pi_n \mathcal{B} w_1\|_{\mathcal{X}_{n-1}}^2 \leq \|\pi_n w_1\|_{\mathcal{X}_{n-1}}^2, \]

\[ \|\mathcal{B} \mathcal{S} \pi_n f_\lambda - \mathcal{G}^o \mathcal{B} \pi_n f_\lambda\|_{\mathcal{X}_{n-1}}^2 \leq C_0 n^2 \|\pi_n f_\lambda\|_{\mathcal{X}_{n-1}}^2, \]

\[ \|\mathcal{B} \mathcal{A} \pi_n f_\lambda - \mathcal{A}^o \mathcal{B} \pi_n f_\lambda\|_{\mathcal{X}_{n-1}}^2 \leq C_0 n^2 \|\pi_n f_\lambda\|_{\mathcal{X}_{n-1}}^2 \]

so that

\[ \|\pi_n w_2\|_{\mathcal{X}_{n-1}}^2 \leq 2\|\pi_n w_1\|_{\mathcal{X}_{n-1}}^2 + C_0 n^2 \|\pi_n f_\lambda\|_{\mathcal{X}_{n-1}}^2 \quad (6.4) \]

for some finite constant \( C_0 \).

It remains to examine the resolvent equation (6.3) through Fourier analysis. Let \( T_{n,d} = [-\pi, \pi]^d \) and denote by \( \hat{f}_\lambda : (T^d)^n \rightarrow \mathbb{C} \) the Fourier transform of \( \mathcal{B} f_\lambda \):

\[ \hat{f}_\lambda(k) = \sum_{x \in \mathcal{X}_n} e^{ix \cdot k} (\mathcal{B} f_\lambda)(x). \]

In this formula, \( x \cdot k = \sum_{1 \leq j \leq n} x_j \cdot k_j \). It follows from the resolvent equation (6.3) that \( \hat{f}_\lambda \) is the solution of

\[ \hat{\lambda f}_\lambda(k) - \left\{ \hat{\mathcal{G}^o}(k) + (1 - 2\alpha) \hat{\mathcal{A}^o}(k) \right\} \hat{f}_\lambda(k) = \hat{w}_2(k), \]

where \( \hat{\mathcal{G}^o}, \hat{\mathcal{A}^o} \) are the functions associated to the operators \( \mathcal{G}^o, \mathcal{A}^o \):

\[ -\hat{\mathcal{G}^o}(k) = 2 \sum_{1 \leq j \leq n} s(z) \{1 - \cos(k_j \cdot z)\}, \]

\[ -\hat{\mathcal{A}^o}(k) = 2i \sum_{1 \leq j \leq n} a(z) \sin(k_j \cdot z). \]

The \( H_{-1}(\mathcal{X}_n, \mathcal{G}^o) \) norm of a function \( u : \mathcal{X}_n \rightarrow \mathbb{R} \) has a simple and explicit expression in terms of the Fourier transform:

\[ \|u\|_{\mathcal{X}_{n-1}}^2 = \frac{1}{n!(2\pi)^{nd}} \int_{T_{n,d}} dk |\hat{u}(k)|^2 \frac{1}{\hat{\mathcal{G}^o}(k)}. \]
Since \( \mathcal{B}f_\lambda \) is the solution of the resolvent equation (6.3), for every \( \lambda > 0 \),
\[
\| \mathfrak{A}^\circ \mathcal{B}f_\lambda \|_{X_{n-1}}^2 = \frac{-1}{n!(2\pi)^d} \int_{\mathbb{T}_{n,d}} \left| \frac{\mathcal{A}_\circ(k)}{\lambda - \mathfrak{S}_\circ(k) - (1 - 2\alpha)\mathfrak{A}_\circ(k)} \right|^2 \frac{|\mathfrak{w}_2(k)|^2}{\mathfrak{S}_\circ(k)} \, dk.
\]

It follows from the explicit formulae for the functions \( \mathfrak{S}_\circ, \mathfrak{A}_\circ \) and a Taylor expansion for \(|k|\) small that the previous expression is bounded by
\[
\frac{-C_0}{n!(2\pi)^d} \int_{\mathbb{T}_{n,d}} \frac{|\mathfrak{w}_2(k)|^2}{\mathfrak{S}_\circ(k)} \, dk = C_0 \|w_2\|_{X_{n-1}}^2
\]
for some finite constant \( C_0 \). We have thus proved that
\[
\| \mathfrak{A}^\circ \mathcal{B}f_\lambda \|_{X_{n-1}}^2 \leq C_0 \|w_2\|_{X_{n-1}}^2.
\]

We may now conclude the proof of Lemma 3.17. Fix \( n \geq 1 \). By (6.7), by the estimate presented just before (6.4) and by the inequality just derived, there exists a finite constant \( C_0 \), which may change from line to line, such that
\[
\|\pi_n \mathfrak{A}f_\lambda\|_{X_{n-1}}^2 = \|\mathfrak{A}\pi_n f_\lambda\|_{X_{n-1}}^2 \leq C_0 n \|\mathfrak{B}\mathfrak{A}\pi_n f_\lambda\|_{X_{n-1}}^2
\]
\[
\leq C_0 \left\{ n^2 \|\pi_n f_\lambda\|_{X_{n-1}}^2 + \|\mathfrak{A}^\circ \mathfrak{B}\pi_n f_\lambda\|_{X_{n-1}}^2 \right\}
\]
\[
\leq C_0 \left\{ n^3 \|\pi_n f_\lambda\|_{X_{n-1}}^2 + n \|\pi_n w_2\|_{X_{n-1}}^2 \right\}.
\]

In particular, by (6.4) and (8.10),
\[
\|\pi_n \mathfrak{A}f_\lambda\|_{X_{n-1}}^2 \leq C_0 \left\{ n^3 \|\pi_n f_\lambda\|_{X_{n-1}}^2 + n \|\pi_n w_1\|_{X_{n-1}}^2 \right\}
\]
\[
\leq C_0 \left\{ n^3 \sum_{j=n+1}^{n+1} \|\pi_j f_\lambda\|_{X_{n-1}}^2 + n \|\pi_n w_1\|_{X_{n-1}}^2 \right\}.
\]

This concludes the proof of the lemma. \( \square \)

**Lemma 3.18.** There exists a finite constant \( C_0 \) such that for any function \( f : \mathcal{E}_n \to \mathbb{R} \) in \( \mathcal{F}_1 \),
\[
\|f\|_1^2 \leq \|\mathfrak{B}f\|_{X_{n-1}}^2 \leq C_0 n \|f\|_1^2.
\]

**Proof.** The first inequality is elementary and follows from the explicit formulae for the respective \( H_1 \) norms. To prove the second inequality, recall from (5.5) the definition of the function \( W \). We also denote by \( W \) the lifted function \( \mathfrak{B}W \). A simple computation shows that there exists a finite constant \( C_0 \) such that
\[
\left| \mathfrak{B}\mathfrak{S}(x) - \mathfrak{S}^\circ \mathfrak{B}f(x) \right| \leq C_0 W(x) |\mathfrak{B}f(x)|
\]
for every \( x \) in \( \mathcal{E}_n \) and \( f : \mathcal{E}_n \to \mathbb{R} \).
We are now in a position to prove the second bound. By definition,
\[ k \mathcal{B} f k_{X_n,1}^2 = \frac{1}{n!} \sum_{x \in \mathcal{E}_n} (\mathcal{B} f)(x)(\mathcal{S}^\circ \mathcal{B} f)(x). \]

Since \( \mathcal{B} f \) vanishes outside \( \mathcal{E}_n \), we may restrict the sum to \( \mathcal{E}_n \). Now, adding and subtracting \( (\mathcal{B} \mathcal{S} f)(x) \) in this expression and recalling (8.14), we obtain that
\[ k \mathcal{B} f k_{X_n,1}^2 \leq \|f\|_1^2 + \frac{C_0}{n!} \sum_{x \in \mathcal{E}_n} W(x) \{\mathcal{B} f(x)\}^2 \]
\[ = \|f\|_1^2 + C_0 \sum_{A \in \mathcal{E}_n} W(A) f(A)^2. \]

By Lemma 3.16, the second term of the previous formula is bounded by \( C_0 n \|f\|_1^2 \), which concludes the proof of the lemma.

It follows from this result that
\[ \frac{1}{C_0 n} \|f\|_1^2 - 1 \leq k \mathcal{B} f k_{X_n,1}^2 \leq \|f\|_1^2 - 1. \]

(6.7)

Lemma 3.19. There exists a finite constant \( C_0 \) such that
\[ k \mathcal{B} f - \mathcal{S}^\circ \mathcal{B} f k_{X_n,1}^2 \leq C_0 n^2 \|f\|_1^2, \quad k \mathcal{B} f - \mathcal{S}^\circ \mathcal{B} f k_{X_n,1}^2 \leq C_0 n^2 \|f\|_1^2 \]
for all \( n \geq 1 \) and all functions \( f : \mathcal{E}_n \to \mathbb{R} \).

Proof. We prove the first estimate and leave to the reader the details of the second. Fix \( n \geq 1 \) and a function \( h : \mathcal{X}_n \to \mathbb{R} \). We need to estimate the scalar product
\[ \frac{1}{n!} \sum_{x \in \mathcal{X}_n} h(x) \{\mathcal{B} \mathcal{S} f(x) - \mathcal{S}^\circ \mathcal{B} f(x)\} \]
(6.8)
in terms of the \( H_1(\mathcal{X}_n) \) norm of \( h \) and the \( H_1(\mathcal{E}_n) \) norm of \( f \). There are two possible cases. Either \( x \) belongs to \( \mathcal{E}_n \) or \( x \) does not belong to \( \mathcal{E}_n \).

In the first case, by (8.14), the expression inside braces in previous formula is absolutely bounded by \( C_0 W(x) \|\mathcal{B} f(x)\| \) for some finite constant \( C_0 \). Therefore, the corresponding piece in the previous formula is bounded above by
\[ \frac{1}{n!} \sum_{x \in \mathcal{E}_n} W(x) h(x) \|\mathcal{B} f(x)\| \leq \frac{1}{2\ell n!} \sum_{x \in \mathcal{E}_n} W(x) h(x)^2 + \frac{\ell}{2} \sum_{A \in \mathcal{E}_n} W(A) f(A)^2 \]
for every \( \ell > 0 \).

If \( x \) does not belong to \( \mathcal{E}_n \), the corresponding piece of the scalar product writes...
because in this case $\mathcal{B}\tilde{f}(x) = \mathcal{B}f(x) = 0$. Since $\mathcal{B}f$ vanishes outside $\mathcal{E}_n$, it is implicit in the previous formula that the sum is restricted to all $x$ such that $x + ze_j$ belongs to $\mathcal{E}_n$. Since $x + ze_j \in \mathcal{E}_n$ and $x \notin \mathcal{E}_n$, $x_j = x_k$ for some $k$. In particular, since $2ab \leq \ell a^2 + \ell^{-1}b^2$ for every $\ell > 0$, a change of variables gives that the first term of the previous formula is bounded above by

$$\frac{1}{n!\ell} \sum_{x \in X_n} h(x)^2 \tilde{W}(x) + \frac{\ell}{n!} \sum_{x \in X_n} \mathcal{B}f(x)^2 W(x),$$

where $\tilde{W}(x) = \sum_{j \neq k} 1\{x_j = x_k\}$. We may of course replace the sum over $X_n$ by a sum over $\mathcal{E}_n$ in the second term, loosing the factor $n!$.

Adding together all previous estimates, we obtain that the scalar product (6.8) is bounded above by

$$\frac{C_0}{n!\ell} \sum_{x \in X_n} h(x)^2 \tilde{W}(x) + W(x) + C_0 \ell \sum_{A \in \mathcal{E}_n} f(A)^2 W(A).$$

By Lemma 3.16, the second term is less than or equal to $C_1 n\ell\|f\|_2^2$ for some finite constant $C_1$. On the other hand, a simple adaptation of the proof of the same lemma gives that the first term is bounded by $C_1 n\ell^{-1}\|b\|_{X_n,1}^2$. To conclude the proof, it remains to minimise over $\ell$ and to recall the variational formula for the $H_{-1}$ norm of a function.

7 References and Remarks

Lemma is due to Sethuraman and Xu [7]

Lemma 3.7 is due to Varadhan [8]

Lemma 3.5 is due to Von Weiszacker and Winkler [9], [10] (cf. also [11]).