Central Limit Theorems

In this chapter we extend the ideas introduced in the previous one to time continuous Markov process with a stationary measure $\pi$ that is eventually non-reversible.

Consider a Markov process $X_t$ taking values in a complete separable metric space $E$ endowed with its Borel $\sigma$-algebra $\mathcal{E}$. Assume that there exists a stationary ergodic state $\pi$. Denote by $L^2(\pi)$ the Hilbert space of $\pi$-square integrable functions, by $\langle \cdot, \cdot \rangle_\pi$ the scalar product in $L^2(\pi)$ and by $\| \cdot \|_0$ the norm associated to this scalar product.

Denote by $L$ the generator of the Markov process $X_t$ in $L^2(\pi)$ and by $\mathcal{D}(L)$ its domain. Let $L^*$ be the adjoint of $L$ in $L^2(\pi)$. Since $\pi$ is stationary, $L^*$ is itself the generator of a Markov process. Assume that there exists a core $\mathcal{C} \subset \mathcal{D}(L) \cap \mathcal{D}(L^*)$ for both generators $L$ and $L^*$. We denote by $P_\pi$ the measure on the path space $D(\mathbb{R}_+, E)$ induced by the Markov process $X_t$ starting from $\pi$ and by $E_\pi$ expectation with respect to $P_\pi$.

Fix a function $V: E \to \mathbb{R}$ in $L^2(\pi)$ such that $\langle V \rangle_\pi = 0$, where $\langle \cdot \rangle_\pi$ stands for the expectation with respect to $\pi$. The object of this section is to find conditions on $V$ which guarantee a central limit theorem for

$$\frac{1}{\sqrt{t}} \int_0^t V(X_s) \, ds .$$

Assume first that there exists a solution $f$ in $\mathcal{D}(L)$ of the Poisson equation

$$-Lf = V . \quad (0.1)$$

In this case a central limit theorem follows from the central limit theorem for martingales stated in Lemma 2.1 below. Indeed, since $f$ belongs to the domain of the generator,

$$M_t = f(X_t) - f(X_0) - \int_0^t (Lf)(X_s) \, ds$$
is a martingale. With the additional assumption that also $f^2$ belongs to $\mathcal{D}(L)$, the quadratic variation of $M_t$ is given by
\[
\langle M, M \rangle_t = \int_0^t ds \{ (L f^2)(X_s) - 2f(X_s)(Lf)(X_s) \} .
\] (0.2)
so that $E_{\pi}(\langle M, M \rangle_t) = 2t(f, -Lf)_{\pi}$. If only $f \in \mathcal{D}(L)$, then (0.2) may not be valid, but still $\langle M, M \rangle_t$ will be an increasing additive functional and its expectation will be still given by $2t(f, -Lf)_{\pi}$. This can be proven by some standard approximation argument.

Since $f$ is the solution of the Poisson equation (0.1), we may write the additive functional in terms of the martingale $M_t$:
\[
\frac{1}{\sqrt{t}} \int_0^t V(X_s) ds = \frac{M_t}{\sqrt{t}} + \frac{f(X_0) - f(X_t)}{\sqrt{t}} .
\]
Since $f$ belongs to $L^2(\pi)$ and the measure is stationary, $[f(X_0) - f(X_t)]/\sqrt{t}$ vanishes in $L^2(\mathbb{P}_{\pi})$ as $t \uparrow \infty$. It remains to check that the martingale $M_t$ satisfies the assumptions of Lemma 2.1.

The increments of the martingale $M_t$ are stationary because $X_t$ under $\mathbb{P}_{\pi}$ is itself a stationary Markov process. On the other hand, since $\pi$ is ergodic, in view of the formula for the quadratic variation of the martingale in terms of $f$, $t^{-1}\langle M, M \rangle_t$ converges in $L^1(\mathbb{P}_{\pi})$, as $t \uparrow \infty$, to $2(f, (-L)f)_{\pi}$. Since the martingale $M_t$ vanishes at time 0, by Lemma 2.1, $t^{-1/2}M_t$, and therefore $t^{-1/2} \int_0^t V(X_s) ds$, converges in distribution to a Gaussian law with mean zero and variance $\sigma^2(V) = 2(f, (-L)f)_{\pi}$.

Of course the existence of a solution $f$ in $L^2(\pi)$ of the Poisson equation (0.1) is too strong and should be weakened.

The chapter is organized as follows. In section 1 we prove a central limit theorem for continuous time martingales. In section 2 we introduce some Hilbert spaces related to the space of finite time-variance functions and in section 3 we compute the time-variance of mean-zero $L^2$ functions. ♣

### 1 Central Limit Theorem for Martingales

On a probability space $(\Omega, P, \mathcal{F})$, consider a right-continuous, square-integrable martingale $\{M_t : t \geq 0\}$ with respect to a given increasing filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Assume $M_0 = 0$ and denote by $\langle M, M \rangle_t$ its quadratic variation.

**Lemma 2.1.** Assume that the increments of the martingale $M_t$ are stationary and that its quadratic variation converges in $L^1(P)$ to some positive constant $\sigma^2$: for every $t \geq 0$, $n \geq 1$ and $0 \leq s_0 < \cdots < s_n$,
\[
(M_{s_1} - M_{s_0}, \ldots, M_{s_n} - M_{s_{n-1}}) = (M_{t+s_1} - M_{t+s_0}, \ldots, M_{t+s_n} - M_{t+s_{n-1}})
\]
in distribution and
$$\lim_{t \to \infty} E\left[ \left( \frac{M_t M_t'}{t} - \sigma^2 \right) \right] = 0.$$  

Then, $M_t/\sqrt{t}$ converges in distribution to a mean zero Gaussian law with variance $\sigma^2$.

In this statement $E$ stands for the expectation with respect to $P$. The proof of this result is a simple consequence of Theorem 1.1.1.

**Proof:** Denote by $[a]$ the integer part of a real $a$. It is enough to prove that $M_N/\sqrt{N}$ converges in distribution, as $N \uparrow \infty$, to a mean zero Gaussian law with variance $\sigma^2$ because

$$E\left[ \left( \frac{M_t}{\sqrt{t}} - \frac{M_{[t]}}{\sqrt{[t]}} \right)^2 \right] \leq 2E\left[ \left( \frac{M_t - M_{[t]}}{\sqrt{t}} \right)^2 \right] + 2\left( \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{[t]}} \right)^2 E[M^2_{[t]}].$$

Since the increments of the martingale $M$ are stationary, $E[M^2_{[t]}] = [t]E[M^2_t]$. On the other hand, $E[(M_t - M_{[t]})^2] \leq E[(M_{[t]+1} - M_{[t]})^2] = E[M^2_t]$ so that the previous expression is bounded by

$$\frac{4}{t} E[M^2_t].$$

To prove that $M_N/\sqrt{N}$ converges in distribution, as $N \uparrow \infty$, to a mean zero Gaussian law with variance $\sigma^2$, we follow the proof of Theorem 1.1.1 until formula (1.1.5). The first and third terms of this formula are estimated in the same way. For each fixed $N$, the expectation which appears in the second term is equal to

$$E\left[ e^{i(\theta/\sqrt{N})M_j} \{ \sigma^2 - (M_{j+1} - M_j)^2 \} \right].$$

Denote by $\{j = t_0, t_1, \ldots, t_K = j + 1\}$ a partition of the interval $[j, j+1]$. Since $M_t$ is a martingale,

$$E\left[ e^{i(\theta/\sqrt{N})M_j} \{M_{j+1} - M_j\}^2 \right] = E\left[ e^{i(\theta/\sqrt{N})M_j} \sum_{k=0}^{K-1} (M_{t_{k+1}} - M_{t_{k}})^2 \right].$$

By Proposition 2.3.4 in [2], as the mesh of the partition decreases to 0, $\sum_{k=0}^{K-1} (M_{t_{k+1}} - M_{t_{k}})^2$ converges in probability (and in $L^1$) to $\langle M, M \rangle_{j+1} - \langle M, M \rangle_j$. In particular, since $e^{i(\theta/\sqrt{N})M_j}$ is absolutely bounded by 1,

$$E\left[ e^{i(\theta/\sqrt{N})M_j} \{ \sigma^2 - (M_{j+1} - M_j)^2 \} \right] = E\left[ e^{i(\theta/\sqrt{N})M_j} \{ \sigma^2 - \langle M, M \rangle_{j+1} - \langle M, M \rangle_j \} \right].$$

At this point, we may follow again the proof of Theorem 1.1.1 with $\langle M, M \rangle_{j+1} - \langle M, M \rangle_j$ in place of $Z^2_{j+1}$, keeping in mind that $\langle M, M \rangle_{j+1} - \langle M, M \rangle_j$ is integrable. In view of the proof of Theorem 1.1.1, to conclude the proof which just need that
\[ \lim_{K \to \infty} E \left[ \left( \frac{(M, M)_K}{K} - \sigma^2 \right) \right] = 0 \]
and this is part of our assumptions.

It follows from the stationarity assumption that \( \sigma^2 = E[M_1^2] \) because

\[
E[(M, M)_n] = \sum_{0 \leq j < n} E[(M, M)_{j+1} - (M, M)_j] = \sum_{0 \leq j < n} E[(M_{j+1} - M_j)^2] = nE[M_1^2].
\]

## 2 A Central Limit Theorem for Markov Processes

Consider the semi-norm \( \| \cdot \|_1 \) defined on the common core \( \mathcal{C} \) by

\[
\| f \|_1^2 = (f, (-L)f)_\pi.
\] (2.1)

Let \( \sim_1 \) be the equivalence relation in \( \mathcal{C} \) defined by \( f \sim_1 g \) if \( \| f - g \|_1 = 0 \) and denote by \( \mathcal{G}_1 \) the normed space \( (\mathcal{C}, \sim_1, \| \cdot \|_1) \). It is easy to see from definition (2.1) that the norm \( \| \cdot \|_1 \) satisfies the parallelogram identity so that \( \mathcal{H}_1 \), the completion of \( \mathcal{G}_1 \) with respect to the norm \( \| \cdot \|_1 \), is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle_1 \) given by polarization:

\[
\langle f, g \rangle_1 = \frac{1}{4} \left\{ \| f + g \|_1^2 - \| f - g \|_1^2 \right\}.
\]

Notice that in this definition only the symmetric part of the generator, \( S = (1/2)(L + L^*) \), plays a role because

\[
\| f \|_1^2 = \langle f, (-L)f \rangle_\pi = \langle f, (-S)f \rangle_\pi.
\]

It is also easy to check that for any \( f, g \) in \( \mathcal{C} \)

\[
\langle f, g \rangle_1 = \langle f, (-S)g \rangle_\pi
\]
and that \( \| c \|_1 = 0 \) for any constant \( c \).

Associated to the Hilbert space \( \mathcal{H}_1 \), is the dual space \( \mathcal{H}_{-1} \) defined as follows. For \( f \) in \( \mathcal{C} \), let

\[
\| f \|_{-1}^2 = \sup_{g \in \mathcal{C}} \left\{ 2\langle f, g \rangle_\pi - \| g \|_1^2 \right\}.
\] (2.2)

Denote by \( \mathcal{G}_{-1} \) the subspace of \( \mathcal{C} \) of all functions with finite \( \| \cdot \|_{-1} \) norm. Introduce in \( \mathcal{G}_{-1} \) the equivalence relation \( \sim_{-1} \) by stating that \( f \sim_{-1} g \) if \( \| f - g \|_{-1} = 0 \) and denote by \( \mathcal{G}_{-1} \) the normed space \( (\mathcal{G}_{-1}, \sim_{-1}, \| \cdot \|_{-1}) \). The completion of \( \mathcal{G}_{-1} \) with respect to the norm \( \| \cdot \|_{-1} \), denoted by \( \mathcal{H}_{-1} \), is again
a Hilbert space with inner product defined through polarization. We present in section 7 some properties of the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_{-1}$.

It is easy to check from the variational formula (2.2) for the $H_{-1}$ norm that for every function $f$ in $D(L)$ and every function $g$ in $L^2(\pi) \cap \mathcal{H}_{-1}$

$$\langle f, g \rangle_{\pi} \leq \|f\|_1 \|g\|_{-1}.$$  

The same variational formula permits to show that a function in $D(L)$ belongs to $\mathcal{H}_{-1}$ if and only if there exists a finite constant $C_0$ such that

$$\langle f, g \rangle_{\pi} \leq C_0 \|g\|_1$$  

for every $g$ in $D(L)$. In this case, $\|f\|_{-1} \leq C_0$. Finally, it is not difficult to show (cf. [?], Lemma 2.5) that a function $f$ belongs to $\mathcal{H}_{-1}$ if and only if there exists $h$ in the domain of $\sqrt{-S}$ such that $\sqrt{-S}h = f$. In this case $\|f\|_{-1} = \|h\|_0$, which means that

$$\|f\|_{-1}^2 = \langle h, h \rangle_{\pi} = \langle (-S)^{-1/2}f, (-S)^{-1/2}f \rangle_{\pi} = \langle (-S)^{-1}f, f \rangle_{\pi}$$

because $S$ is symmetric.

Fix a function $V$ in $L^2(\pi) \cap \mathcal{H}_{-1}$, $\lambda > 0$ and consider the resolvent equation

$$\lambda f_\lambda - L f_\lambda = V.$$  

Taking the scalar product with respect to $f_\lambda$ on both sides, by Schwarz inequality (2.3) we get that

$$\lambda \langle f_\lambda, f_\lambda \rangle_{\pi} + \|f_\lambda\|_1^2 \leq \|f_\lambda\|_1 \|V\|_{-1}.$$  

In particular,

$$\lambda \langle f_\lambda, f_\lambda \rangle_{\pi} \leq \|V\|_{-1}^2, \quad \|f_\lambda\|_1 \leq \|V\|_{-1}.$$  

Therefore, $\lambda f_\lambda$ vanishes in $L^2(\pi)$ as $\lambda \searrow 0$ and $f_\lambda$ is a bounded sequence in $\mathcal{H}_1$.

**Theorem 2.2.** Suppose that

$$\sup_{0 < \lambda \leq 1} \|Lf_\lambda\|_{-1} < \infty.$$  

Then the $P_\pi$-law of $t^{-1/2} \int_0^t V(X_s) \, ds$ converges to a mean zero Gaussian distribution with variance

$$\sigma^2(V) = \lim_{\lambda \to 0} 2 \|f_\lambda\|_1^2.$$  


Notice that
\[
\sup_{0<\lambda \leq 1} \|Lf_{\lambda}\|_{-1} < \infty \quad \text{if and only if} \quad \sup_{0<\lambda \leq 1} \|\lambda f_{\lambda}\|_{-1} < \infty \quad (2.8)
\]
because \( V \) belongs to \( \mathcal{H}_{-1} \).

The proof of this theorem is divided into two parts. We first show in the next section that the variance of \( t^{-1/2} \int_0^t V(X_s) \, ds \) is asymptotically bounded by a multiple of the \( \mathcal{H}_{-1} \) norm of \( V \). Moreover, in the particular case where the generator is symmetric, i.e., when \( \pi \) is a reversible measure, we prove that the variance converges to \( 2\|V\|_{L^2}^2 \). In section 4, we prove that a central limit theorem holds for \( t^{-1/2} \int_0^t V(X_s) \, ds \), \( V \) satisfying the assumptions of Theorem 2.2, provided the following two conditions are satisfied:

\[
\lim_{\lambda \to 0} \lambda \|f_{\lambda}\|_{0}^2 = 0 \quad \text{and} \quad \lim_{\lambda \to 0} \|f_{\lambda} - f\|_1 = 0
\]

for some \( f \) in \( \mathcal{H}_1 \). Finally, in section 5, we show that the bound (2.7) implies the previous two conditions.

3 The Time-Variance

We estimate in this section the limiting variance of the integral \( \int_0^t W(X_s) \, ds \) for a mean zero function \( W \) in \( L^2(\pi) \). Let

\[
\sigma^2(W) = \limsup_{t \to \infty} \mathbb{E}_{\pi} \left[ \left( \frac{1}{\sqrt{t}} \int_0^t W(X_s) \, ds \right)^2 \right].
\]

Denote by \( P_t \) the semi-group of the Markov process \( X_t \). Since \( \pi \) is invariant, a change of variables shows that for each fixed \( t \) the expectation on the right hand side of the previous formula is equal to

\[
\frac{1}{t} \int_0^t ds \int_0^s dr \mathbb{E}_{\pi}[W(X_{[s-r]} W(X_0)] = \frac{1}{t} \int_0^t ds \int_0^t dr \langle P_{s-r} W, W \rangle_{\pi} = 2 \int_0^t ds [1 - (s/t)] \langle P_s W, W \rangle_{\pi}.
\]

Denote by \( a^+ \) the positive part of \( a \): \( a^+ = \max\{0, a\} \), so that

\[
\sigma^2(W) = 2 \limsup_{t \to \infty} \int_0^t ds [1 - (s/t)]^+ \langle P_s W, W \rangle_{\pi}.
\]  

(3.1)

In the general case, it is not clear whether this limsup is in fact a limit or whether it is finite without some restrictions on \( W \). However, in the reversible case, the sequence is increasing in \( t \) because \( \langle P_s W, W \rangle_{\pi} = \langle P_{s/2} W, P_{s/2} W \rangle_{\pi} \) is a positive function. Hence, in the reversible case, by the monotone convergence theorem,
\[ \sigma^2(W) = 2 \lim_{t \to \infty} \int_0^\infty ds \left[ 1 - \left( \frac{s}{t} \right) \right] P_s W, W \]
\[ = 2 \int_0^\infty ds \langle P_s W, W \rangle_\pi . \]

In the general case, one can show that \( \sigma^2(W) \) defined in (3.1) is finite provided the function \( W \) belongs to the Sobolev space \( H_{-1} \): there exists a universal constant \( C_0 \) such that
\[ \sigma^2(W) \leq C_0 \|W\|_{L^2}^2 \tag{3.2} \]
for all functions \( W \) in \( H_{-1} \cap L^2(\pi) \). The main difference between \( \sigma^2(W) \) and \( \|W\|_{L^2}^2 \) is that while in the second term only the symmetric part of the generator is involved, in the first term the full generator appears. Formally,
\[ \sigma^2(W) = 2 \int_0^\infty \langle P_s W, W \rangle_\pi = 2 \langle W, (-L)^{-1} W \rangle_\pi \]
\[ = 2 \langle W, [(-L)^{-1}]^* W \rangle_\pi , \]
and \( \|W\|_{L^2}^2 = \langle W, (-S)^{-1} W \rangle_\pi . \)

In this formula and below, \( B^* \) represents the symmetric part of the operator \( B \) and \( A = (1/2)(L - L^*) \) is the asymmetric part of the generator \( L \). Since,
\[ \left\{ [(-L)^{-1}]^* \right\}^{-1} = -S + A^* (-S)^{-1} A \geq -S , \]
we have that \( [(-L)^{-1}]^* \leq (-S)^{-1} \), from what it follows that \( \sigma^2(W) \leq 2 \|W\|_{L^2}^2 \). We now present a rigorous proof of this informal argument.

**Lemma 2.3.** Fix \( T > 0 \) and a function \( V \) in \( L^2(\pi) \). There exists a finite universal constant \( C_0 \) such that,
\[ \mathbb{E}_\pi \left[ \sup_{0 \leq t \leq T} \left( \int_0^t V(X_s) \, ds \right)^2 \right] \leq C_0 T \|V\|_{L^2}^2 . \]

**Proof:** Recall that we denote by \( S \) the symmetric part of the generator in \( L^2(\pi) \). Assume that for every \( \varepsilon > 0 \), there exists a function \( h_\varepsilon \) in the core \( C \) such that
\[ \|Sh_\varepsilon - V\|_0 \leq \varepsilon , \quad \|h_\varepsilon\|_1 \leq \|V\|_{L^2} + \varepsilon . \tag{3.3} \]
We return to this approximation at the end of the proof.

Fix \( \varepsilon > 0 \) and let \( h = h_\varepsilon \). For each \( s \geq 0 \), denote by \( \mathcal{F}_s \) the \( \sigma \)-algebra generated by \( \{X_r, 0 \leq r \leq s\} \). Since \( h \) belongs to the core, define the \( (\mathbb{P}_\pi, \mathcal{F}_t) \) zero-mean martingale \( M_t \) by
\[ M_t = h(X_t) - h(X_0) - \int_0^t ds \langle Lh(X_s) \rangle . \]
In the same way, denote by \( \mathcal{H}_t^- \) the backward filtration generated by \( \{X_s, s \geq t\} \). Recall that \( L^* \) stands for the adjoint of the generator \( L \) with respect to the invariant measure \( \pi \). It is easy to check that the process \( \{M_t^-, 0 \leq t \leq T\} \) defined by

\[
M_t^- = h(X_{T-t}) - h(X_T) - \int_0^t ds (L^* h)(X_{T-s})
\]

is a \( (\mathcal{P}_\pi, \mathcal{F}_t^-) \) martingale, called the backward martingale.

A change of variables gives that

\[
M_T^- - M_{T-t}^- = h(X_0) - h(X_t) - \int_0^t ds (L^* h)(X_s).
\]

In particular,

\[
M_t + M_{T-t}^- = 2 \int_0^t ds (Sh)(X_s) = 2 \int_0^t ds \{V(X_s) + R(X_s)\},
\]

where \( R = Sh - V \). Therefore,

\[
\mathbb{E}_\pi \left[ \sup_{0 \leq t \leq T} \left( \int_0^t V(X_s) \, ds \right)^2 \right] = (1/4) \mathbb{E}_\pi \left[ \sup_{0 \leq t \leq T} \left( M_t + M_{T-t}^- + 2 \int_0^t R(X_s) \, ds \right)^2 \right].
\]

Since \( \|R\|_0 \leq \varepsilon \), by Schwarz’ and Doob’s inequality, the previous expression is bounded above by

\[
3\mathbb{E}_\pi[(M_T^-)^2] + 3\mathbb{E}_\pi[(M_T)^2] + 4T^2\varepsilon.
\]

The variances of these martingales are equal to \( 2T\|h\|^2 \leq 2T(\|V\|_{-1} + \varepsilon)^2 \).

The previous expression is thus bounded above by \( 12T(\|V\|_{-1} + \varepsilon)^2 + 4T^2\varepsilon \).

It remains to send \( \varepsilon \downarrow 0 \) to conclude the proof of the lemma.

We now turn to (3.3). Fix \( \varepsilon > 0 \) and for \( \lambda > 0 \) denote by \( f_\lambda \) the solution of the resolvent equation (2.5) with \( S \) in place of \( L \). By (2.6), there exists \( \lambda \) sufficiently small for which \( \|Sf_\lambda - V\|_0 \leq \varepsilon \) and \( \|f_\lambda\|_1 \leq \|V\|_{-1} \). Since \( f_\lambda \) belongs to the domain of \( S \), there exists \( h \) in the core \( C \) such that \( \|h - f_\lambda\|_0 \leq \varepsilon \) \( \|Sh - Sf_\lambda\|_0 \leq \varepsilon \). Thus \( \|Sh - V\|_0 \leq 2\varepsilon \) and \( \|h\|_1 \leq \|f_\lambda\|_1 + \|h - f_\lambda\|_1 \leq \|V\|_{-1} + \varepsilon \) because \( \|g\|_1^2 \leq \|g\|_0 \|Sg\|_0 \) for every \( g \) in the domain of \( S \). This concludes the proof of (3.3) and the one of the lemma. \( \square \)

Remark 2.4. We just proved the existence of a universal finite constant \( C_0 \) such that \( \sigma^2(W) \leq C_0\|W\|_{-1}^2 \). We will see in the next chapter some examples of functions \( W \) not in \( \mathcal{H}_{-1} \) and for which \( \sigma^2(W) < \infty \). We suspect that a central limit theorem holds for these functions with the usual scaling \( t^{-1/2} \).
4 The Resolvent Equation

We assume from now on that $V$ belongs to $\mathcal{H}_{-1} \cap L^2(\pi)$. Taking inner product with respect to $f_\lambda$ on both sides of the resolvent equation (2.5), we get that

$$\lambda \langle f_\lambda, f_\lambda \rangle_\pi + \|f_\lambda\|_1^2 = \langle V, f_\lambda \rangle_\pi.$$  \hspace{1cm} (4.1)

Since $f_\lambda$ belongs to $\mathcal{D}(L)$ and $V$ belongs to $L^2(\pi) \cap \mathcal{H}_{-1}$, by Schwarz inequality (2.3), the right hand side is bounded above by $\|V\|_{-1} \|f_\lambda\|_1$. In particular, $\|f_\lambda\|_1 \leq \|V\|_{-1}$ so that $\|V\|_{-1} \|f_\lambda\|_1 \leq \|V\|_1^2$ and

$$\lambda \langle f_\lambda, f_\lambda \rangle_\pi + \|f_\lambda\|_1^2 \leq \|V\|_1^2.$$ \hspace{1cm} (4.2)

As a simple consequence of (4.2) is that $(\lambda - L)^{-1}$ is bounded mapping from $\mathcal{H}_{-1}$ to $\mathcal{H}_1$.

**Lemma 2.5.** The operator $(\lambda - L)^{-1}$ extends from $\mathcal{C}$ to a bounded mapping from $\mathcal{H}_{-1}$ to $\mathcal{H}_1$. Moreover, for any $V \in \mathcal{H}_{-1}$ we have

$$\|(\lambda - L)^{-1}V\|_{-1} \leq \|V\|_{-1}.$$  

Moreover, it follows from (4.2) that

$$\lim_{\lambda \to 0} \lambda f_\lambda = 0 \text{ in } L^2(\pi) \text{ and } \sup_{0 < \lambda \leq 1} \|f_\lambda\|_1 < \infty.$$  \hspace{1cm} (4.3)

The purpose of this section is to show that a central limit theorem for $t^{-1/2} \int_0^t V(X_s) \, ds$ holds provided we can prove the following stronger statements:

$$\lim_{\lambda \to 0} \lambda \|f_\lambda\|_2^2 = 0 \text{ and } \lim_{\lambda \to 0} \|f_\lambda - f\|_1 = 0$$  \hspace{1cm} (4.4)

for some $f$ in $\mathcal{H}_1$.

**Proposition 2.6.** Fix a function $V$ in $L^2(\pi) \cap \mathcal{H}_{-1}$ and assume (4.4). Then, $t^{-1/2} \int_0^t V(X_s) \, ds$ converges in law to a mean zero Gaussian distribution with variance

$$\sigma^2(V) = 2 \lim_{\lambda \to 0} \|f_\lambda\|_1^2.$$  

It follows from (4.4) and (4.1) that

$$\sigma^2(V) = 2 \lim_{\lambda \to 0} \|f_\lambda\|_1^2 = 2 \lim_{\lambda \to 0} \langle V, f_\lambda \rangle_\pi.$$  \hspace{1cm} (4.5)

The idea of the proof of Proposition 2.6 is again to express $\int_0^t V(X_s) \, ds$ as the sum of a martingale and a term which vanishes in the limit. This is proved in (4.7) and Lemma 2.8 below. We start taking advantage of the resolvent equation (2.5) to build up a martingale closely related to $\int_0^t V(X_s) \, ds$.

For each fixed $\lambda > 0$, let $M^\lambda_t$ be the martingale defined by
\[ M_t^\lambda = f_\lambda(X_t) - f_\lambda(X_0) - \int_0^t (Lf_\lambda)(X_s) \, ds \]

so that
\[ \int_0^t V(X_s) \, ds = M_t^\lambda + f_\lambda(X_0) - f_\lambda(X_t) + \lambda \int_0^t f_\lambda(X_s) \, ds. \] *(4.6)*

**Lemma 2.7.** The martingale \( M_t^\lambda \) converges in \( L^2(\pi) \), as \( \lambda \downarrow 0 \), to a martingale \( M_t \).

**Proof:** It is enough to prove that \( M_t^\lambda \) is a Cauchy sequence in \( L^2(\pi) \). For \( \lambda, \lambda' > 0 \), since \( \pi \) is an invariant state, the expectation of the quadratic variation of the martingale \( M_t^\lambda - M_t^{\lambda'} \) is
\[ 2t \langle \{f_\lambda - f_{\lambda'}\}, (-L)\{f_\lambda - f_{\lambda'}\}\rangle_\pi = 2t \|f_\lambda - f_{\lambda'}\|_1^2. \]

By assumption (4.4), \( f_\lambda \) converges in \( \mathcal{H}_1 \). In particular, \( M_t^\lambda \) is a Cauchy sequence in \( L^2(\pi) \) and converges to a martingale \( M_t \). This proves the statement. \( \Box \)

It follows from this result and from identity (4.6) that \( f_\lambda(X_t) - f_\lambda(X_0) - \int_0^t \lambda f_\lambda(X_s) \, ds \) also converges in \( L^2(\pi) \) as \( \lambda \downarrow 0 \). Denote this limit by \( R_t \) so that
\[ \int_0^t V(X_s) \, ds = M_t + R_t. \] *(4.7)*

**Lemma 2.8.** \( t^{-1/2}R_t \) vanishes in \( L^2(\pi) \) as \( t \uparrow \infty \).

**Proof:** Putting together equation (4.6) with (4.7), we get that
\[ \frac{R_t}{\sqrt{t}} = \frac{1}{\sqrt{t}} \left\{ M_t^\lambda - M_t + f_\lambda(X_0) - f_\lambda(X_t) + \lambda \int_0^t f_\lambda(X_s) \, ds \right\}. \] *(4.8)*

We consider separately each term on the right hand side of this expression.

Since \( M_t^\lambda \) converges in \( L^2(\pi) \) to \( M_t \),
\[ \frac{1}{t} \mathbb{E}_\pi \left[ \left( M_t^\lambda - M_t \right)^2 \right] = \frac{1}{t} \lim_{\lambda' \to 0} \mathbb{E}_\pi \left[ \left( M_t^\lambda - M_t^{\lambda'} \right)^2 \right]. \]

In the previous Lemma, we computed the expectation of the quadratic variation of the martingale \( M_t^\lambda - M_t^{\lambda'} \). This calculation shows that the previous expression is equal to
\[ \lim_{\lambda' \to 0} \|f_\lambda - f_{\lambda'}\|_1^2 = \|f_\lambda - f\|_1^2. \]

In the last step we used assumption (4.4) which states that \( f_\lambda \) converges in \( \mathcal{H}_1 \) to some \( f \).
We now turn to the second term in (4.8). Since \( \pi \) is invariant, the expectation of its square is bounded by
\[
2t^{-1}E_\pi \left[ f_\lambda(X_t)^2 \right] + 2t^{-1}E_\pi \left[ f_\lambda(X_0)^2 \right] = 4t^{-1}\|f_\lambda\|_0^2.
\]
On the other hand, by Schwarz inequality, the expectation of the square of the third term in (4.8) is bounded by \( t\lambda^2\|f_\lambda\|_0^2 \).

Putting together all previous estimates, we obtain that
\[
\frac{1}{t}E_\pi[R_t^2] \leq 3\|f_\lambda - f\|_1^2 + 3(4t^{-1} + \lambda^2)\|f_\lambda\|_0^2
\]
for all \( \lambda > 0 \). Set \( \lambda = t^{-1} \) to conclude the proof of the lemma in view of hypotheses (4.4).

We may now prove Proposition 2.6. Recall equation (4.7). By the previous lemma the second term on the right hand side divided by \( \sqrt{t} \) vanishes in \( L^2(\pi) \) (and therefore in probability) as \( t \uparrow \infty \). On the other hand, by the martingale central limit theorem, \( M_t/\sqrt{t} \) converges in law to a mean zero Gaussian distribution with variance \( \sigma^2(V) \) given by
\[
E_\pi[M_t^2] = \lim_{\lambda \to 0} E_\pi[(M_\lambda^\lambda)^2] = \lim_{\lambda \to 0} E_\pi[\langle M^\lambda, M^\lambda \rangle_1] = 2\lim_{\lambda \to 0} \|f_\lambda\|_1^2.
\]
The last identity follows from the computation of the expectation of the quadratic variation of the martingale \( M^\lambda_t \) performed in the proof of Lemma 2.7.

5 An \( \mathcal{H}_{-1} \) estimate

In the previous section we showed that the central limit theorem for the additive functional \( t^{-1/2} \int_0^t V(X_s)ds \) follows from conditions (4.4) if \( V \) belongs to \( L^2(\pi) \cap \mathcal{H}_{-1} \). In the present section we prove that (4.4) follows from the bound (2.7) on the solution of the resolvent equation (2.5).

**Lemma 2.9.** Fix a function \( V \) in \( \mathcal{H}_{-1} \cap L^2(\pi) \) and denote by \( \{f_\lambda, \lambda > 0\} \) the solution of the resolvent equation (2.5). Assume that \( \sup_{\lambda > 0} \|L f_\lambda\|_{-1} \leq C_0 \) for some finite constant \( C_0 \). Then, there exists \( f \) in \( \mathcal{H}_1 \) such that
\[
\lim_{\lambda \to 0} \lambda(f_\lambda, f_\lambda)_\pi = 0 \quad \text{and} \quad \lim_{\lambda \to 0} f_\lambda = f \text{ strongly in } \mathcal{H}_1.
\]

**Proof:** We already proved in (4.2) that
\[
\sup_{0 < \lambda \leq 1} \|f_\lambda\|_1 \leq \|V\|_{-1} \quad \text{and} \quad \sup_{0 < \lambda \leq 1} \lambda(f_\lambda, f_\lambda)_\pi \leq \|V\|_{-1}^2.
\]
In particular, \( \lambda f_\lambda \) converges to 0 in \( L^2(\pi) \), as \( \lambda \downarrow 0 \). The proof is divided in several claims.
Claim 1. \( Lf_\lambda \) converges weakly in \( \mathcal{H}_1 \), as \( \lambda \downarrow 0 \), to \( -V \). Since \( \sup_{\lambda > 0} \|Lf_\lambda\|_{-1} \) is bounded, we only need to show that the set of weak limit points is reduced to \( -V \). Assume that \( Lf_\lambda \) converges weakly in \( \mathcal{H}_1 \) to some function \( U \) and fix \( g \) in \( L^2(\pi) \cap \mathcal{H}_1 \). Since \( g \) belongs to \( \mathcal{H}_1 \) and \( Lf_\lambda \) converges weakly to \( U \), \( \langle U, g \rangle_\pi = \lim_{\lambda \to 0} \langle Lf_\lambda, g \rangle_\pi \). On the other hand, since \( f_\lambda \) is the solution of the resolvent equation, \( \langle Lf_\lambda, g \rangle_\pi = -\langle V, g \rangle_\pi + \lim_{\lambda \to 0} \langle \lambda f_\lambda, g \rangle_\pi \). This latter expression is equal to \( -\langle V, g \rangle_\pi \) because \( g \) belongs to \( L^2(\pi) \) and \( \lambda f_\lambda \) converges strongly to 0 in \( L^2(\pi) \), as \( \lambda \downarrow 0 \). Thus, \( \langle U, g \rangle_\pi = -\langle V, g \rangle_\pi \) for all functions \( g \) in \( L^2(\pi) \cap \mathcal{H}_1 \). Since this set is dense in \( \mathcal{H}_1 \), \( U = -V \), proving the claim.

In the same way, since \( \sup_{\lambda > 0} \|f_\lambda\|_1 \) is bounded, each sequence \( \lambda_n \downarrow 0 \) has a subsequence still denoted by \( \lambda_n \), for which \( f_{\lambda_n} \) converges weakly in \( \mathcal{H}_1 \) to some function, denoted by \( W \).

Claim 2. \( W \) satisfies the relation \( \|W\|_1^2 = \langle W, V \rangle_\pi \). To check this identity, apply Mazur’s theorem ([21], Section V.1) to the sequences \( f_{\lambda_n}, Lf_{\lambda_n} \) to obtain sequences \( v_n, Lv_n \), which converge strongly to \( W, -V \), respectively. On the one hand, since \( v_n \) (resp. \( Lv_n \)) converges strongly in \( \mathcal{H}_1 \) (resp. \( \mathcal{H}_1 \)) to \( W \) (resp. \( -V \)), \( \langle v_n, Lv_n \rangle_\pi \) converges to \( -\langle W, V \rangle_\pi \). On the other hand, since \( -\langle v_n, Lv_n \rangle_\pi = \|v_n\|_1^2 \), it converges to \( \|W\|_1^2 \). Therefore, \( \|W\|_1^2 = \langle W, V \rangle_\pi \).

Claim 3. \( \lim_{\lambda \to 0} \lambda \langle f_\lambda, f_\lambda \rangle_\pi = 0 \). Suppose by contradiction that \( \lambda \langle f_\lambda, f_\lambda \rangle_\pi \) does not converge to 0 as \( \lambda \downarrow 0 \). In this case there exists \( \varepsilon > 0 \) and a subsequence \( \lambda_n \downarrow 0 \) such that \( \lambda_n \langle f_{\lambda_n}, f_{\lambda_n} \rangle_\pi \geq \varepsilon \) for all \( n \). We have just shown the existence of a sub-subsequence \( \lambda_{n'} \) for which \( f_{\lambda_{n'}} \) converges weakly in \( \mathcal{H}_1 \) to some \( W \) satisfying the relation \( \langle W, V \rangle_\pi = \|W\|_1^2 \). Since \( f_\lambda \) is solution of the resolvent equation,

\[
\lim_{n' \to \infty} \|f_{\lambda_{n'}}\|_1^2 \leq \lim_{n' \to \infty} \left\{ \lambda_{n'} \|f_{\lambda_{n'}}\|_1^2 + \|f_{\lambda_{n'}}\|_1^2 \right\} = \lim_{n' \to \infty} \langle f_{\lambda_{n'}}, V \rangle_\pi = \langle W, V \rangle_\pi = \|W\|_1^2 \leq \lim_{n' \to \infty} \|f_{\lambda_{n'}}\|_1^2.
\]

This contradicts the fact that \( \lambda_n \langle f_{\lambda_n}, f_{\lambda_n} \rangle_\pi \geq \varepsilon \) for all \( n \), so that \( \lim_{\lambda \to 0} \lambda \langle f_\lambda, f_\lambda \rangle_\pi = 0 \).

Claim 4. \( f_\lambda \) converges strongly in \( \mathcal{H}_1 \) as \( \lambda \downarrow 0 \). It follows also from the previous argument that \( f_{\lambda_n} \) converges to \( W \) strongly in \( \mathcal{H}_1 \). In particular, all sequences \( \lambda_n \) have subsequences \( \lambda_{n'} \) for which \( f_{\lambda_{n'}} \) converges strongly in \( \mathcal{H}_1 \). To show that \( f_\lambda \) converges strongly, it remains to check uniqueness of the limit.

Consider two decreasing sequences \( \lambda_n, \mu_n \), vanishing as \( n \to \infty \). Denote by \( W_1, W_2 \) the strong limit in \( \mathcal{H}_1 \) of \( f_{\lambda_n}, f_{\mu_n} \), respectively. Since \( f_\lambda \) is the solution of the resolvent equation,

\[
\langle \lambda_n f_{\lambda_n} - \mu_n f_{\mu_n}, f_{\lambda_n} - f_{\mu_n} \rangle_\pi + \|f_{\lambda_n} - f_{\mu_n}\|_1^2 = 0
\]

for all \( n \). Since \( f_{\lambda_n}, f_{\mu_n} \) converges strongly to \( W_1, W_2 \) in \( \mathcal{H}_1 \),
\[
\lim_{n \to \infty} \| f_{\lambda_n} - f_{\mu_n} \|^2_1 = \| W_1 - W_2 \|^2_1.
\] (5.3)

On the other hand, since \( \lambda \| f_\lambda \|^2 \) vanishes as \( \lambda \downarrow 0 \),
\[
\lim_{n \to \infty} (\lambda_n f_{\lambda_n} - \mu_n f_{\mu_n}, f_{\lambda_n} - f_{\mu_n})_\pi
= \lim_{n \to \infty} \left\{ (\lambda_n f_{\lambda_n}, f_{\mu_n})_\pi + (\mu_n f_{\mu_n}, f_{\lambda_n})_\pi \right\}.
\]

Each of these terms vanish as \( n \uparrow \infty \). Indeed,
\[
\lambda_n (f_{\lambda_n}, f_{\mu_n})_\pi = \lambda_n (f_{\lambda_n}, f_{\mu_n} - W_2)_\pi + \lambda_n (f_{\lambda_n}, W_2)_\pi
\]

By Schwarz inequality (2.3), the first term on the right hand side is bounded above by \( \| \lambda_n f_{\lambda_n} \|_1 \| f_{\mu_n} - W_2 \|_1 \), which vanishes because \( \lambda f_\lambda \) is bounded in \( \mathcal{H}_{-1} \) and \( f_{\mu_n} \) converges to \( W_2 \) in \( \mathcal{H}_1 \). The second term of the previous formula also vanishes in the limit because \( W_2 \) belongs to \( \mathcal{H}_1 \) and \( \lambda f_\lambda \) converges weakly to 0 in \( \mathcal{H}_{-1} \). This concludes the proof of the lemma. \(\square\)

Theorem 2.2 follows from this lemma and Proposition 2.6.

6 Some Examples

Fix a function \( V \) in \( L^2(\pi) \cap \mathcal{H}_{-1} \). In this section we present three conditions which guarantee that the solution \( f_\lambda \) of the resolvent equation (2.5) satisfies the bound (2.7).

6.1 Reversibility

Assume that the generator \( L \) is self-adjoint in \( L^2(\pi) \). In this case, by Schwarz inequality,
\[
\| (Lf, g)_\pi \| \leq \| f \|_1 \| g \|_1.
\]

Therefore, in view of the variational formula (2.4) for the \( \mathcal{H}_{-1} \) norm, for any \( f \) in \( L^2(\pi) \cap \mathcal{H}_1 \), \( Lf \) belongs to \( \mathcal{H}_{-1} \) and
\[
\| Lf \|_{-1} \leq \| f \|_1.
\]

In fact the equality holds. In particular, in the reversible case (2.7) follows from the elementary estimate (4.3). In this situation we have \( \sigma^2(V) = 2 \| V \|_{-1}^2 \).

6.2 Sector Condition

Assume now that the generator \( L \) satisfies the sector condition
\[
( f, Lg)_\pi^2 \leq C_0 (f, (-L)f)_\pi (g, (-L)g)_\pi
\] (6.1)
for some finite constant $C_0$ and every functions $f, g$ in the domain of the generator. In view of (2.4), for any function $g$ in $D(L)$,

$$\|Lg\|_1^2 \leq C_0\|g\|_1^2$$

and condition (2.7) follows from estimate (4.3).

Of course, $L$ satisfies a sector condition if and only if $A$, the asymmetric part of the generator, satisfies it:

$$\langle f, Ag \rangle_{\pi}^2 \leq C_0\langle f, (-L)f \rangle_{\pi} \langle g, (-L)g \rangle_{\pi}$$

for some finite constant $C_0$ and every functions $f, g$ in the core. In this case, $A$ can be extended as a bounded operator from $H_1$ to $H_{-1}$:

**Lemma 2.10.** Fix an operator $B$. Assume that $C$ is a core for $B$ and that $B$ satisfies the sector condition

$$\langle f, Bg \rangle_{\pi}^2 \leq C_0\langle f, (-L)f \rangle_{\pi} \langle g, (-L)g \rangle_{\pi}$$

for all $f, g$ in $C$ and a finite constant $C_0$. Then, the operator $B$ extends from $C$ to a bounded, linear mapping from $H_1$ to $H_{-1}$.

**Proof.** Suppose that $f \in C$. By definition of the $H_{-1}$ norm and the sector condition,

$$\|Bf\|_{-1}^2 = \sup_{g \in C} \{2\langle Bf, g \rangle_{\pi} - \|g\|_{1}^2 \} \leq C_0\|f\|_{1}^2.$$

We just obtained that

$$\langle g, A^*(-S)^{-1}Ag \rangle_{\pi} = \|Ag\|_1^2 \leq C_0\|g\|_1^2 = C_0\langle g, (-S)g \rangle_{\pi}$$

for all functions $g$ in $D(L)$. Hence, the sector condition requires that

$$A^*(-S)^{-1}A \leq C_0(-S)$$

for some finite constant $C_0$. This inequality states that the asymmetric part of the generator can be estimated by the symmetric part. Furthermore, in this case, in view of the computations performed just after (3.2),

$$(-S) \leq (-S) + A^*(-S)^{-1}A \leq (1 + C_0)(-S)$$

so that

$$C_1^{-1}\sigma^2(V) \leq \|V\|_{-1}^2 \leq C_1\sigma^2(V)$$

for some finite constant $C_1$. This means that under the sector condition, the limiting variance is finite if and only if the function belongs to $H_{-1}$.
6.3 Graded Sector Condition

Now, instead of assuming that the generator satisfies a sector condition on the all space, we decompose $L^2(\pi)$ as a direct sum of orthogonal spaces $A_n$ and assume that on each subspace $A_n$, the generator satisfies a sector condition with a constant which may be different on each $A_n$.

Assume that $L^2(\pi)$ can be decomposed as a direct sum $\oplus_{n \geq 0} A_n$ of orthogonal spaces. Functions in $A_n$ are said to have degree $n$. For $n \geq 0$, denote by $\pi_n$ the orthogonal projection on $A_n$ so that

$$f = \sum_{n \geq 0} \pi_n f \quad \text{and} \quad \pi_n f \in A_n$$

for all $n \geq 0$, $f$ in $L^2(\pi)$.

Suppose that the generator $L$ keeps the degree of a function or changes it by one: $L : D(L) \cap A_n \to A_{n-1} \oplus A_n \oplus A_{n+1}$. Denote by $L_-$ (resp. $L_+$ and $L_0$) the piece of the generator which decreases (resp. increases and keeps) the degree of a function. Assume that $L_0$ can be decomposed as $S_0 + B_0$, where $-S_0$ is a positive symmetric operator bounded by $-C_0 L$ for some positive constant $C_0$:

$$0 \leq \langle f, (-S_0)f \rangle_\pi \leq C_0 \langle f, (-L)f \rangle_\pi$$

for all functions $f$ in $D(L)$.

Since $-S_0$ is a positive operator, repeating the steps of Section 2 with $S_0$ in place of $L$, we define the Sobolev spaces $H_{0,1}, H_{0,-1}$ and the norms $\| \cdot \|_{0,1}, \| \cdot \|_{0,-1}$ associated to $S_0$. Since $S_0$ keeps the degree of a function,

$$\|f\|_{0,1}^2 = \langle f, (-S_0)f \rangle_\pi = \left( \sum_{n \geq 0} \pi_n f, (-S_0) \sum_{n \geq 0} \pi_n f \right)_\pi$$

$$= \sum_{n \geq 0} \langle \pi_n f, (-S_0) \pi_n f \rangle_\pi = \sum_{n \geq 0} \|\pi_n f\|_{0,1}^2 .$$

for all functions $f$ in the domain of the generator. By the same reasons, for a function $f$ in $L^2(\pi)$,

$$\|f\|_{0,-1}^2 = \sup_{g \in D(L)} \{ 2 \langle f, g \rangle_\pi - \|g\|_{0,1}^2 \} = \sum_{n \geq 0} \|\pi_n f\|_{0,-1}^2 .$$

In terms of the new norm $\| \cdot \|_{0,1}$, (6.2) translates to

$$\|f\|_{0,1} \leq \sqrt{C_0} \|f\|_1$$

for all function $f$ in the domain of the generator and some finite constant $C_0$. It follows from this inequality and from the variational formula for the $H_{0,-1}$, $H_{0,-1}$ norms that

$$\|f\|_{-1} \leq \sqrt{C_0} \|f\|_{a,-1}$$

(6.3)
for all functions $f$ in $L^2(\pi)$ and the same finite constant $C_0$.

Suppose now that a sector condition holds on each subspace $\mathcal{A}_n$ with a constant which depends on $n$: there exists $\beta < 1$ and a finite constant $C_0$ such that

\begin{align}
\langle f, (-L+)g \rangle_\pi^2 &\leq C_0 n^{2\beta} \langle f, (-S_0)\rangle_\pi \langle g, (-S_0)\rangle_\pi, \\
\langle g, (-L-)f \rangle_\pi^2 &\leq C_0 n^{2\beta} \langle f, (-S_0)\rangle_\pi \langle g, (-S_0)\rangle_\pi
\end{align}

for all $g$ in $\mathcal{D}(L) \cap \mathcal{A}_n$ and $f$ in $\mathcal{D}(L) \cap \mathcal{A}_{n+1}$. It follows from the previous assumptions and from the variational formula for the $\| \cdot \|_{-1,0}$ norm that

\begin{align}
\|L^+g\|_{0,-1} &\leq \sqrt{C_0} n^\beta \|g\|_{0,1}, \\
\|L^-f\|_{0,-1} &\leq \sqrt{C_0} n^\beta \|f\|_{0,1}
\end{align}

for all $g$ in $\mathcal{D}(L) \cap \mathcal{A}_n$ and $f$ in $\mathcal{D}(L) \cap \mathcal{A}_{n+1}$. The proof of Lemma 2.11 below shows that the restriction $\beta < 1$ is crucial.

Fix $k \geq 1$ and define the triple norms $\| \cdot \|_{k,0}$, $\| \cdot \|_{k,1}$ and $\| \cdot \|_{k,-1}$ by

\begin{align}
\|f\|_{k,0}^2 &= \sum_{n \geq 1} n^{2k} \|\pi_n f\|_0^2, \\
\|f\|_{k,1}^2 &= \sum_{n \geq 1} n^{2k} \|\pi_n f\|_{0,1}^2, \\
\|f\|_{k,-1}^2 &= \sum_{n \geq 1} n^{2k} \|\pi_n f\|_{0,-1}^2.
\end{align}

On several occasions we omit the dependence of the norms on $k$.

**Lemma 2.11.** Let $V$ be a function in $L^2(\pi)$ such that

$$\|V\|_{k,-1} < \infty$$

for some $k \geq 1$. Denote by $f_\lambda$ the solution of the resolvent equation (2.5). There exists a finite constant $C_1$ depending only on $\beta$, $k$ and $C_0$ such that

$$\lambda \|f_\lambda\|_{k,0}^2 + \|f_\lambda\|_{k,1}^2 \leq C_1 \|V\|_{k,-1}^2.$$

**Proof:** Consider an increasing sequence $\{t_n : n \geq 0\}$, to be fixed later, and denote by $T : L^2(\pi) \to L^2(\pi)$ the operator which is a multiple of the identity on each subspace $\mathcal{A}_n$:

$$Tf = \sum_{n \geq 0} t_n \pi_n f.$$

Apply $T$ to both sides of the resolvent equation and take the inner product with respect to $Tf_\lambda$ on both sides of the identity to obtain that

$$\lambda \langle Tf_\lambda, Tf_\lambda \rangle_\pi - \langle Tf_\lambda, LTf_\lambda \rangle_\pi = \langle Tf_\lambda, TV \rangle_\pi - \langle Tf_\lambda, [L,T]f_\lambda \rangle_\pi.$$

In this formula, $[L,T]$ stands for the commutator of $L$ and $T$ and is given by $LT - TL$. By assumption (6.2), the second term on the left hand side is bounded below by
\[ C_0^{-1}(T f_{\lambda}, (-S_0)T f_{\lambda})_\pi = C_0^{-1} \|T f_{\lambda}\|_{0,1}^2 = C_0^{-1} \sum_{n \geq 0} t_n^2 \|\pi_n f_{\lambda}\|_{0,1}^2. \]

Let \( \delta > 0 \). We now estimate the scalar product \( (T f_{\lambda}, [L, T] f_{\lambda})_\pi \) in terms of \( \|T f_{\lambda}\|_{0,1}^2 \). Since \( T \) commutes with any operator that keeps the degree, \([L, T] = [L_+ + L_- , T]\). To fix ideas, consider the operator \([L_+, T]\), the other expression being estimated in a similar way. Since \( L_+ \) increases the degree by one, by definition of the commutator,

\[ \pi_n [L_+, T] f = L_+ T \pi_{n-1} f - T L_+ \pi_{n-1} f = (t_{n-1} - t_n) L_+ \pi_{n-1} f \]

for all functions \( f \in D(L) \). Therefore,

\[
(T f_{\lambda}, [L_+, T] f_{\lambda})_\pi = \sum_{n \geq 0} (\pi_n T f_{\lambda}, \pi_n [L_+, T] f_{\lambda})_\pi = \sum_{n \geq 0} (t_{n-1} - t_n) t_n (\pi_n f_{\lambda}, L_+ \pi_{n-1} f_{\lambda})_\pi.
\]

By (6.4) and since the sequence \( t_n \) is increasing, the previous expression is bounded below by

\[
\sum_{n \geq 0} (t_{n} - t_{n-1}) t_n C_0 n^\beta \|\pi_n f_{\lambda}\|_{0,1} \|\pi_{n-1} f_{\lambda}\|_{0,1} \\
\leq \frac{1}{2} \sum_{n \geq 0} (t_{n} - t_{n-1}) t_n C_0 n^\beta \|\pi_n f_{\lambda}\|_{0,1}^2 \\
+ \frac{1}{2} \sum_{n \geq 0} (t_{n} - t_{n-1}) t_n C_0 n^\beta \|\pi_{n-1} f_{\lambda}\|_{0,1}^2.
\]

Since \( \beta < 1 \), there exists \( n_1 = n_1(C_0, \beta, \delta, k) \) such that

\[
C_0 n^\beta \left\{ \frac{1 - (n - 1)^{2k}}{n^{2k}} \right\} \leq \delta, \quad C_0 n^\beta \left\{ \frac{n^{2k}}{(n - 1)^{2k} - 1} \right\} \frac{n^{2k}}{(n - 1)^{2k}} \leq \delta
\]

for all \( n \geq n_1 \). Fix \( n_2 > n_1 \) and set \( t_n = n_1^k 1 \{ n < n_1 \} + n^k 1 \{ n_1 \leq n \leq n_2 \} + n_2^k 1 \{ n > n_2 \} \). With this definition, we obtain that the previous expression is bounded by

\[
\delta \sum_{n \geq 0} t_n^2 \|\pi_n f_{\lambda}\|_{0,1}^2 = \delta \|T f_{\lambda}\|_{0,1}^2.
\]

It remains to estimate \( (T f_{\lambda}, TV)_\pi \). By (2.3), and since \( 2ab \leq A^{-1} a^2 + Ab^2 \) for every \( A > 0 \),

\[
(T f_{\lambda}, TV)_\pi = \sum_{n \geq 0} t_n^2 \langle \pi_n f_{\lambda}, \pi_n V \rangle_\pi \leq \sum_{n \geq 0} t_n^2 \|\pi_n f_{\lambda}\|_{0,1} \|\pi_n V\|_{\pi,0,1} \\
\leq \delta \sum_{n \geq 0} t_n^2 \|\pi_n f_{\lambda}\|_{0,1}^2 + \delta^{-1} \sum_{n \geq 0} t_n^2 \|\pi_n V\|_{0,1}^2 \\
= \delta \|T f_{\lambda}\|_{0,1}^2 + \delta^{-1} \|TV\|_{0,1}^2.
\]
Putting together the previous three estimates, we obtain that
\[
\lambda \|Tf_\lambda\|^2_{0,0} + C_0^{-1} \|Tf_\lambda\|^2_{0,1} \leq 3\delta \|Tf_\lambda\|^2_{0,1} + \delta^{-1} \|TV\|^2_{0,-1}
\]
so that
\[
\lambda \|Tf_\lambda\|^2_{0,0} + \delta \|Tf_\lambda\|^2_{0,1} \leq \delta^{-1} \|TV\|^2_{0,-1}
\]
if we choose \(\delta = 1/4C_0\). Recall the definition of the sequence \(t_n\). This estimate holds uniformly in \(n_2\). Let \(n_2 \uparrow \infty\) and definite \(T'\) as the operator associated to the sequence \(t'_n\), where \(t'_n = n^k 1\{n \geq n_1\} + n_1 k^1 \{n < n_1\}\), to deduce that
\[
\|f_\lambda\|^2_{k,1} \leq \|T'f_\lambda\|^2_{0,1} \leq \delta^{-2} \|T'V\|^2_{0,-1} \leq \delta^{-2} n_1^{2k} \|V\|^2_{k,-1}.
\]
A similar argument shows that
\[
\lambda \|f_\lambda\|^2_{k,0} \leq \delta^{-1} \|V\|^2_{k,-1}.
\]
To conclude the proof of the lemma, it remains to recall that we fixed \(\delta = 1/4C_0\) and that \(n_1 = n_1(C_0,k,\beta,\delta)\).

Assume now that \(L_0 = S_0 + B_0\) satisfies a sector condition on each subset \(A_n\):
\[
\langle g, (-L_0)f \rangle_\pi \leq C_0 n^{2\gamma} \langle f, (-S_0)f \rangle_\pi \langle g, (-S_0)g \rangle_\pi
\]
for some \(\gamma > 0\) and all functions \(f, g\) in \(D(L) \cap A_n\). Notice that we do not impose any condition on \(\gamma\). By the variational formula for the norm \(\| \cdot \|_{0,-1}\),
\[
\|L_0f\|_{0,-1} \leq \sqrt{C_0} n^\gamma \|f\|_{0,1}
\]
for all functions \(f\) in \(D(L) \cap A_n\).

**Lemma 2.12.** Suppose that the generator \(L\) satisfies hypotheses (6.2), (6.4) and (6.7). Fix a function \(V\) such that
\[
\|V\|_{k,-1} < \infty
\]
for some \(k \geq (\beta \vee \gamma)\). Let \(f_\lambda\) be the solution of the resolvent equation (2.5). Then,
\[
\sup_{0 < \lambda \leq 1} \|L_\lambda f\|_{0,-1} < \infty .
\]

**Proof:** It follows from (6.3) that
\[
C_0^{-1} \|L_\lambda f\|_{0,-1}^2 \leq \|L_\lambda f\|_{0,-1}^2 = \sum_{n \geq 0} \|\pi_n L_\lambda f\|_{0,-1}^2 .
\]
Fix \(n \geq 0\). Since \(\pi_n L_\lambda f = L - \pi_{n+1} f_\lambda + L_0 \pi_n f_\lambda + L_+ \pi_{n-1} f_\lambda\), by (6.5), (6.8),
Some Examples

In particular, by Schwarz inequality, by Lemma 2.11 and since $k \geq (\beta \lor \gamma)$, the right hand side of (6.9) is bounded above by

$$C_1 \sum_{n \geq 0} n^{2k} \| \pi_n f \|_{0,1}^2 \leq C_1 \sum_{n \geq 0} n^{2k} \| \pi_n V \|_{0,1}^2$$

for some finite constant $C_2$ depending only on $C_0$, $\beta$ and $\gamma$. This proves the lemma.

Therefore, to prove a central limit theorem for an additive functional of a Markov process, it is enough to check whether its generator satisfies the graded sector conditions (6.2), (6.4) and (6.7).

### 6.4 Perturbations of Normal operators

In this section, instead of the sector condition, we suppose that $L$ can be decomposed on the core $\mathcal{C}$ as a normal operator plus a perturbation which satisfies a sector condition with respect to the normal operator. More specifically, assume that the operator $L$ can be written as $L = L_0 + B$ and that $\mathcal{C}$ is a common core for each of the operators $L_0$, $L_0^*$, $B$ and $B^*$. We impose three conditions on the operators $L_0$, $B$.

Suppose that the closure of $L_0 : \mathcal{C} \to L^2(\pi)$ is normal:

$$L_0^* L_0 = L_0 L_0^* . \tag{6.10}$$

This assumption requires the symmetric and the asymmetric part of the operator $L_0$ to commute. Indeed, if $S_0$, $A_0$ stand for the symmetric and the asymmetric part of the operator $L_0$, respectively, (6.10) is equivalent to the condition $S_0 A_0 = A_0 S_0$.

Suppose that the Dirichlet form associated to $L_0$ is equivalent to the Dirichlet form associated to $L$: there exist constants $0 < c_* < C_* < +\infty$ such that

$$c_* \langle f, (-L_0) f \rangle_\pi \leq \langle f, (-L_0^*) f \rangle_\pi \leq C_* \langle f, (-L_0) f \rangle_\pi , \tag{6.11}$$

for every function $f$ in the core $\mathcal{C}$. Denote by $\| \cdot \|_{0,\pm 1}$ the respective $\mathcal{H}_{\pm 1}$ norms for $L_0$. It follows from this condition that there exist constants $0 < c_1 < C_1 < +\infty$ such that

$$c_1 \| f \|_{0,\pm 1} \leq \| f \|_{\pm 1} \leq C_1 \| f \|_{0,\pm 1} \tag{6.12}$$

for all $f$ in $\mathcal{C}$.

Assume that $B$ satisfies a sector condition relative to $L_0$:...
\[ \langle f, Bg \rangle_\pi^2 \leq C_0 \langle f, (-L_0)f \rangle_\pi \langle g, (-L_0)g \rangle_\pi \]  \hspace{1cm} (6.13)

for some finite constant \( C_0 \) and every functions \( f, g \in \mathcal{C} \). In view of (6.12), a sector condition for \( B \) relative to \( L_0 \) holds if and only if it holds relative to \( L \).

It follows from this sector condition that \( B \) is a bounded mapping from \( \mathcal{H}_1 \) to \( \mathcal{H}_1 \): There exists a finite constant \( C_B > 0 \) such that

\[ \|Bf\|_{-1} \leq C_B \|f\|_1 \]  \hspace{1cm} (6.14)

for all \( f \in \mathcal{C} \). Indeed, by definition of the \( \mathcal{H}_1 \) norm, the sector condition (6.13) and the bound (6.12),

\[ \|Bf\|_{-1} = \sup_{\|\phi\|_1 = 1} \langle Bf, \phi \rangle_\pi \leq C_0 \|f\|_{0,1} \leq \frac{C_0}{C_1} \|f\|_1 . \]

**Proposition 2.13.** Assume hypotheses (6.10), (6.11), (6.13). Fix a function \( V \) in \( \mathcal{H}_{-1} \cap L^2(\pi) \) and denote by \( f_\lambda \) the solution of the resolvent equation (2.5). Then,

\[ \sup_{0 < \lambda \leq 1} \|Lf_\lambda\|_{-1} < +\infty . \]

**Proof.** Fix a function \( V \) in \( \mathcal{H}_{-1} \cap L^2(\pi) \). For \( \lambda > 0 \), denote by \( f^{(0)}_\lambda \) the solution of the resolvent equation

\[ \lambda f^{(0)}_\lambda - L_0 f^{(0)}_\lambda = V . \]

By the spectral decomposition of the normal operator \( L_0 \),

\[ f^{(0)}_\lambda = \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\lambda + \varphi + i\tau} E(d\varphi, d\tau) V , \]

where \( E(d\varphi, d\tau) \) is the spectral resolution of identity which corresponds to the normal operator \( L_0 \). Since \( V \) belongs to \( \mathcal{H}_{-1} \cap L^2(\pi) \),

\[ \|V\|_{0,-1}^2 = \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\varphi} \mu_V(d\varphi, d\tau) < \infty , \]

where \( \mu_V(d\varphi, d\tau) \) stands for the spectral measure of \( V \): \( \mu_V(d\varphi, d\tau) = \langle E(d\varphi, d\tau)V, V \rangle_\pi \).

Hence,

\[ \|L_0 f^{(0)}_\lambda\|_{0,-1}^2 = \int_0^{+\infty} \int_{-\infty}^{+\infty} \left| \frac{\varphi + i\tau}{\lambda + \varphi + i\tau} \right|^2 \frac{\mu_V(d\varphi, d\tau)}{\varphi} \leq \|V\|_{0,-1}^2 . \]

In particular, by (6.12),

\[ \|L_0 f^{(0)}_\lambda\|_{-1} \leq \frac{C_1}{C_1} \|V\|_{-1} . \]  \hspace{1cm} (6.15)
Denote by \(f_\lambda\) the solution of the full resolvent equation \((\lambda - L)f_\lambda = V\). Since \(L = L_0 + B\), we may rewrite the resolvent equation as
\[
(\lambda - L_0)f_\lambda = V + Bf_\lambda.
\]
Since \(L = L_0 + B\), by (6.15) with \(V + Bf_\lambda\) in place of \(V\),
\[
\sup_{0 < \lambda \leq 1} \|Lf_\lambda\|_{-1} \leq \sup_{0 < \lambda \leq 1} \|L_0f_\lambda\|_{-1} + \sup_{0 < \lambda \leq 1} \|Bf_\lambda\|_{-1}
\]
\[
\leq \frac{C_1}{c_1} \sup_{0 < \lambda \leq 1} \|V + Bf_\lambda\|_{-1} + \sup_{0 < \lambda \leq 1} \|Bf_\lambda\|_{-1}.
\]
By (6.14), the previous expression is bounded above by
\[
\frac{C_1}{c_1} \|V\|_{-1} + \left(1 + \frac{C_1}{c_1}\right) \sup_{0 < \lambda \leq 1} \|Bf_\lambda\|_{-1}
\]
\[
\leq \frac{C_1}{c_1} \|V\|_{-1} + C_B \left(1 + \frac{C_1}{c_1}\right) \sup_{0 < \lambda \leq 1} \|f_\lambda\|_1.
\]
By (4.2), this sum is less than or equal to \(C_2\|V\|_{-1}\) for some finite constant \(C_2\). Therefore, \(\sup_{0 < \lambda \leq 1} \|Lf_\lambda\|_{-1} < C_2\|V\|_{-1} < \infty\), which concludes the proof of the Proposition.

7 Variational principles

Fix a function \(V\) in \(H_{-1} \cap L^2(\pi)\) and denote by \(f_\lambda\) the solution of the resolvent equation (2.5). Assume that \(\sup_{\lambda \leq 1} \|Lf_\lambda\|_{-1} < \infty\). By Theorem 2.2, \(t^{-1/2} \int_0^t V(X_s)ds\) converges to a mean-zero Gaussian variable with variance \(\sigma^2(V) = 2 \lim_{\lambda \to 0} \|f_\lambda\|^2\). By Lemma 2.9, \(\lambda(f_\lambda, f_\lambda)\) vanishes as \(\lambda \downarrow 0\). Therefore,
\[
\sigma^2(V) = 2 \lim_{\lambda \to 0} \langle (\lambda - L)f_\lambda, f_\lambda\rangle = 2 \lim_{\lambda \to 0} \langle (\lambda - L)^{-1}V, V\rangle_\pi.
\]
We derive in this section variational formulas and upper and lower bounds for the variance \(\sigma^2(V)\). Throughout this section we assume \(\lambda \leq 1\).

7.1 Quadratic functional of the resolvent

Fix a function \(V\) in \(L^2(\pi)\). Note that we do not assume in this subsection that \(V\) belongs to \(H_{-1}\). Recall that \(S\) stand for the closure of the essential self-adjoint symmetric part of \(L\). Assume that
\[
(\lambda - S)^{-1}(C) \subset D(L^*)\).
\]
for each \(\lambda > 0\).
For $\lambda > 0$, let $\mathcal{H}_{1,\lambda}$ be the completion of $\mathcal{C}$ under the pre-Hilbert norm defined by $\|f\|_{1,\lambda}^2 := (\langle \lambda - L \rangle f, f)_\pi$. Since the operator $-L$ is non-negative definite, it is clear that $\| \cdot \|_{1,\lambda}$ is a norm. Also, since $\|f\|_{1,\lambda}^2 \geq \lambda \|f\|_0^2$, each Cauchy sequence in this norm is also Cauchy in $L^2(\pi)$. This fact allows for an obvious identification of $\mathcal{H}_{1,\lambda}$ with a dense subset of $L^2(\pi)$. For any $f \in L^2(\pi)$ define

$$\|f\|_{-1,\lambda}^2 := \sup_{g \in \mathcal{C}} \{2 \langle f, g \rangle_\pi - \|g\|_{1,\lambda}^2\}$$

being it finite, or not. Note that the expression inside braces on the right hand side of the previous formula is smaller than $2\|f\|_0\|g\|_0 - \lambda\|g\|_0^2$. In particular,

$$\|f\|_{-1,\lambda}^2 \leq \lambda^{-1}\|f\|_0^2 \quad (7.3)$$

for any $f \in L^2(\pi)$. Let $\mathcal{H}_{-1,\lambda}$ be the completion of $L^2(\pi)$ under the norm $\| \cdot \|_{-1,\lambda}$.

**Theorem 2.14.** Fix $V$ in $L^2(\pi)$ and assume that (7.2) holds. Then, for every $\lambda > 0$,

$$\langle V, (\lambda - L)^{-1}V \rangle_\pi = \sup_{g \in \mathcal{C}} \left\{2 \langle V, g \rangle_\pi - \|g\|_{1,\lambda}^2 - \|Ag\|_{-1,\lambda}^2\right\}$$

$$= \inf_{g \in \mathcal{C}} \left\{\|g\|_{1,\lambda}^2 + \|V + Ag\|_{-1,\lambda}^2\right\}.$$  

**Proof.** Fix $\lambda > 0$ and $V$ in $L^2(\pi)$. Denote by $G_\lambda : L^2(\pi) \to L^2(\pi)$ the resolvent operator $(\lambda - L)^{-1}$, by $G^*_\lambda$ the symmetric part of $G_\lambda$ and by $\mathcal{R}_\lambda$ the range of $G^*_\lambda$: $\mathcal{R}_\lambda = \{G^*_\lambda f : f \in L^2(\pi)\}$. By (??,??),

$$\langle V, (\lambda - L)^{-1}V \rangle_\pi = \langle V, G^*_\lambda V \rangle_\pi = \sup_{g \in \mathcal{R}_\lambda} \left\{2 \langle V, g \rangle_\pi - \|g\|_{[G^*_\lambda]^{-1}}\right\}.$$  

**A.** Denote by $\mathcal{R}^*_\lambda$ the range of $G^*_\lambda(\lambda - L^*)$ over the core $\mathcal{C}$: $\mathcal{R}^*_\lambda = \{G^*_\lambda(\lambda - L^*)g : g \in \mathcal{C}\}$. We claim that we can replace the set $\mathcal{R}_\lambda$ by $\mathcal{R}^*_\lambda$ in the previous variational formula.

Since $\mathcal{C}$ is a core for $L^*$, $\mathcal{R}^*_\lambda \subset \mathcal{R}_\lambda$. In particular,

$$\sup_{g \in \mathcal{R}^*_\lambda} \left\{2 \langle V, g \rangle_\pi - \|g\|_{[G^*_\lambda]^{-1}}\right\} \leq \sup_{g \in \mathcal{R}_\lambda} \left\{2 \langle V, g \rangle_\pi - \|g\|_{[G^*_\lambda]^{-1}}\right\}.$$  

To prove the reverse inequality, it is enough to show that for each $g$ in $\mathcal{R}_\lambda$, there exists a sequence $\{g_n : n \geq 1\}$ of functions in $\mathcal{R}^*_\lambda$ such that $\langle V, g_n \rangle_\pi$, $\langle g_n, [G^*_\lambda]^{-1}g_n \rangle_\pi$ converge, as $n \to \infty$, to $\langle V, g \rangle_\pi$, $\langle g, [G^*_\lambda]^{-1}g \rangle_\pi$, respectively.

Fix a function $g$ in $\mathcal{R}_\lambda$. By definition, $g = G^*_\lambda h$ for some $h$ in $L^2(\pi)$. $g$ may be rewritten as $G^*_\lambda(\lambda - L^*)(\lambda - L^*)^{-1}h$. If $h = (\lambda - L^*)^{-1}h$ belonged to the core $\mathcal{C}$, $g$ would be an element of $\mathcal{R}^*_\lambda$ and there would be nothing to prove. This may not be the case, but since $h$ belongs to the domain of $L^*$, $h$ may be approximated by functions in the core and one just need to check that the
approximating sequence has the required properties. The rigorous argument is as follows.

Since \((\lambda - L^*)^{-1}\) is a bounded operator in \(L^2(\pi)\), \(\tilde{h} = (\lambda - L^*)^{-1}h\) is well defined and belongs to \(\mathcal{D}(L^*)\), the domain of \(L^*\). Since \(\mathcal{C}\) is a core of \(L^*\), there exists a sequence \(\{h_n : n \geq 1\} \) in \(\mathcal{C}\) such that \(h_n, (\lambda - L^*)h_n\) converge in \(L^2(\pi)\), as \(n \uparrow \infty\), to \(\tilde{h}\), \((\lambda - L^*)h = h\), respectively. Since \(G_{\lambda}^*\) is a bounded operator, \(g_n = G_{\lambda}^*(\lambda - L^*)h_n\) converges in \(L^2(\pi)\) to \(G_{\lambda}^*h = g\). \(g_n\) belongs to \(\mathcal{R}_{\lambda}^*\) because each \(h_n\) is in the core \(\mathcal{C}\). Since \(g_n\) converges to \(g\) in \(L^2(\pi)\), \(\langle V, g_n \rangle_\pi\) converges to \(\langle V, g \rangle_\pi\). On the other hand, by definition of \(g_n, h_n, h\), and since \(g_n, (\lambda - L^*)h_n\) converge to \(g, h\), respectively,

\[
\lim_{n \to \infty} \langle g_n, [G_{\lambda}^*]^{-1}g_n \rangle_\pi = \lim_{n \to \infty} \langle g_n, (\lambda - L^*)h_n \rangle_\pi = \langle g, h \rangle_\pi = \langle g, [G_{\lambda}^*]^{-1}g \rangle_\pi.
\]

The claim is proved.

**B.** \(\mathcal{R}_{\lambda}^* = \{ (\lambda - L)^{-1}(\lambda - S)h : h \in \mathcal{C} \} \).

We start with an explicit formula for \(G_{\lambda}^*\) in terms of \(L, S\) and \(L^*\). For a function \(f\) such that \(f = (\lambda - L^*)h\) for some \(h \in \mathcal{C}\),

\[
G_{\lambda}^*f = [(\lambda - L)^{-1}]^*f = (\lambda - L)^{-1}(\lambda - S)(\lambda - L^*)^{-1}f.
\]

Fix a function \(g\) in \(\mathcal{R}_{\lambda}^*\) so that \(g = G_{\lambda}^*(\lambda - L^*)h\) for some function \(h \in \mathcal{C}\). Replace \(f\) by \((\lambda - L^*)h\) on both sides of the previous equation to obtain that

\[
g = (\lambda - L)^{-1}(\lambda - S)h. \tag{7.4}
\]

Hence, \(\mathcal{R}_{\lambda}^*\) is contained in the set \(\{ (\lambda - L)^{-1}(\lambda - S)h : h \in \mathcal{C} \}\).

The inverse of \(G_{\lambda}^*\) is formally given by

\[
[G_{\lambda}^*]^{-1} = \{[(\lambda - L)^{-1}]^*\}^{-1} = (\lambda - L^*)(\lambda - S)^{-1}(\lambda - L).
\]

In view of (7.4), \([G_{\lambda}^*]^{-1}g\) is well defined for all \(g\) in \(\mathcal{R}_{\lambda}^*\). Assume that \(g = (\lambda - L)^{-1}(\lambda - S)h\) for some function \(h \in \mathcal{C}\). By the previous formula, \([G_{\lambda}^*]^{-1}g = (\lambda - L^*)h\), so that \(g = G_{\lambda}^*(\lambda - L^*)h\). This proves that \(g\) belongs to \(\mathcal{R}_{\lambda}^*\) and the claim.

**C. For any \(V\) in \(L^2(\pi)\),**

\[
\langle V, (\lambda - L)^{-1}V \rangle_\pi = \sup_{g \in \mathcal{R}_{\lambda}^*} \{ 2\langle V, g \rangle_\pi - \| (\lambda - L)g \|^2_{1,\lambda} \}.
\]

Indeed, by the representation (7.4) of a function \(g\) in \(\mathcal{R}_{\lambda}^*\),

\[
\langle g, [G_{\lambda}^*]^{-1}g \rangle_\pi = \langle g, (\lambda - L^*)h \rangle_\pi = \langle (\lambda - L)g, (\lambda - S)^{-1}(\lambda - L)g \rangle_\pi
\]

because \(h = (\lambda - S)^{-1}(\lambda - L)g\). This last expression is equal to \(\| (\lambda - L)g \|^2_{1,\lambda}\).

Claim C follows from this identity and Claim A.

**D.** We may replace the set \(\mathcal{R}_{\lambda}^*\) in the variational formula of Claim C by \(\mathcal{C}\).
Fix a function $g$ in $\mathcal{R}_\lambda^*$. By (7.4), $g$ belongs to the domain of $L$. Since $C$ is a core for $L$, there exists a sequence $\{h_n : n \geq 1\}$ of functions in $C$ such that $h_n, (\lambda - L)h_n$ converge in $L^2(\pi)$ as $n \uparrow \infty$ to $g, (\lambda - L)g$, respectively. By (7.3), the convergence takes place also in $\mathcal{H}_{-1, \lambda}$. In particular
\[
\lim_{n \to \infty} \|h_n - g\|_0 = 0, \quad \lim_{n \to \infty} \|(\lambda - L)h_n\|_{-1, \lambda} = \|(\lambda - L)g\|_{-1, \lambda},
\]
so that
\[
\sup_{g \in \mathcal{R}_\lambda^*} \left\{2(V, g)_\pi - \|(\lambda - L)g\|_{-1, \lambda}^2\right\} \leq \sup_{g \in C} \left\{2(V, g)_\pi - \|(\lambda - L)g\|_{-1, \lambda}^2\right\}.
\]
To prove the reverse inequality, fix a function $g$ in the core $C$. $(\lambda - S)^{-1}(\lambda - L)g$ belongs to the domain of $S$. Since $C$ is a core for $S$, there exists a sequence $\{h_n : n \geq 1\}$ in $C$ such that $h_n, (\lambda - S)h_n$ converge in $L^2(\pi)$ as $n \uparrow \infty$ to $(\lambda - S)^{-1}(\lambda - L)g, (\lambda - L)g$, respectively.

Let $g_n = (\lambda - L)^{-1}(\lambda - S)h_n$. By Claim B, $g_n$ belongs to $\mathcal{R}_\lambda^*$. On the other hand, $(\lambda - L)g_n = (\lambda - S)h_n$ converges to $(\lambda - L)g$ in $L^2(\pi)$ and therefore in $\mathcal{H}_{-1, \lambda}$. Moreover, since $(\lambda - L)^{-1}$ is a bounded operator and since $(\lambda - S)h_n$ converges to $(\lambda - L)g, g_n = (\lambda - L)^{-1}(\lambda - S)h_n$ converges to $(\lambda - L)^{-1}(\lambda - L)g = g$ in $L^2(\pi)$.

Hence, for a function $g$ in $C$, we obtained a sequence $\{g_n : n \geq 1\}$ in $\mathcal{R}_\lambda^*$ such that $g_n$ converges to $g$ in $L^2(\pi)$ and $(\lambda - L)g_n$ converges to $(\lambda - L)g$ in $\mathcal{H}_{-1, \lambda}$. This shows that
\[
\sup_{g \in C} \left\{2(V, g)_\pi - \|(\lambda - L)g\|_{-1, \lambda}^2\right\} \leq \sup_{g \in \mathcal{R}_\lambda^*} \left\{2(V, g)_\pi - \|(\lambda - L)g\|_{-1, \lambda}^2\right\}
\]
and concludes the proof of the claim.

**E. Proof of the first statement of the Theorem.** Since $A$ is an anti-symmetric operator, for $g \in C$,
\[
\|(\lambda - L)g\|_{-1, \lambda}^2 = \langle (\lambda - S)g, g \rangle_\pi + \langle (\lambda - S)^{-1}Ag, Ag \rangle_\pi
\]
\[
= \|g\|_{1, \lambda}^2 + \|Ag\|_{-1, \lambda}^2.
\]
In view of Claim D, this concludes the proof the first variational formula.

**F. Lower bound for the second variational formula.**

To prove the second statement of the theorem, note that for each $g \in C$,
\[
\|Ag\|_{-1, \lambda}^2 = \sup_{h \in C} \left\{2\langle Ag, h \rangle_\pi - \|h\|_{1, \lambda}^2\right\}.
\]
In particular, by the first part of the theorem and since $A$ is anti-symmetric,
\[
\langle V, (\lambda - L)^{-1}V \rangle_\pi = \sup_{g \in C} \inf_{h \in C} \left\{2\langle V + Ah, g \rangle_\pi - \|g\|_{-1, \lambda}^2 + \|h\|_{1, \lambda}^2\right\}
\]
\[
\leq \inf_{h \in C} \sup_{g \in C} \left\{2\langle V + Ah, g \rangle_\pi - \|g\|_{-1, \lambda}^2 + \|h\|_{1, \lambda}^2\right\}
\]
\[
= \inf_{h \in C} \left\{\|V + Ah\|_{-2, \lambda}^2 + \|h\|_{1, \lambda}^2\right\}.
\]
G. Upper bound for the second variational formula.

To prove the reverse inequality, for any $\varepsilon > 0$, it is enough to exhibit a function $h$ in $\mathcal{C}$ such that

$$ (V, (\lambda - L)^{-1}V)_{\pi} \geq \|V + Ah\|_{-1,\lambda}^2 + \|h\|_{1,\lambda}^2 - \varepsilon. \tag{7.5} $$

Assume that there exist functions $h_0, g_0$ in $\mathcal{C}$ such that

$$ \begin{cases} 
(\lambda - L)h_0 = V, \\
(\lambda - L^*)g_0 = V.
\end{cases} \tag{7.6} $$

Of course this may not be possible. However, since $\mathcal{C}$ is a core for both $L$ and $L^*$, for any $\varepsilon > 0$, there exists $h, g$ in $\mathcal{C}$ such that $\|(\lambda - L)h - V\|_0 \leq \varepsilon$, $\|(\lambda - L^*)g - V\|_0 \leq \varepsilon$. We present below the modifications needed in the proof to handle the general case.

Let $g = (1/2)\{h_0 + g_0\}$, $h = (1/2)\{h_0 - g_0\}$.

It follows from the identities relating $h_0, g_0$ to $V$ that

$$ \begin{cases} 
(\lambda - S)g - Ah = V, \\
(\lambda - S)h - Ag = 0.
\end{cases} $$

Since $(\lambda - L)h_0 = V$, since $(\lambda - L)h_0, h_0)_{\pi} = (\lambda - S)h_0, h_0)_{\pi}$ and since $h_0 = g + h$,

$$ (V, (\lambda - L)^{-1}V)_{\pi} = (\lambda - S)h_0, h_0)_{\pi} = (\lambda - S)(g + h), (g + h))_{\pi} \\
= (\lambda - S)g, g)_{\pi} + (\lambda - S)h, h)_{\pi} + 2((\lambda - S)h, g)_{\pi}. $$

Since $(\lambda - S)h = Ag$, the last term vanishes. On the other hand, the first term can be rewritten as

$$ \|(\lambda - S)g\|_{-1,\lambda}^2 = \|Ah + V\|_{-1,\lambda}^2. $$

Hence, the penultimate formula becomes

$$ (V, (\lambda - L)^{-1}V)_{\pi} = \|h\|_{1,\lambda}^2 + \|V + Ah\|_{-1,\lambda}^2, $$

which proves the claim.

We now adapt the previous arguments to the case in which solutions of (7.6) do not exist in $\mathcal{C}$. Fix $\varepsilon > 0$. Since $\mathcal{C}$ is a common core of both $L$ and $L^*$, there exists $h, g$ in $\mathcal{C}$ such that

$$ \|(\lambda - L)h - V\|_0 \leq \varepsilon, \quad \|(\lambda - L^*)g - V\|_0 \leq \varepsilon. $$

Let $u = (\lambda - L)h - V, w = (\lambda - L^*)g - V$ and keep in mind that $\|u\|_0 \leq \varepsilon, \|w\|_0 \leq \varepsilon$. Taking scalar product with respect $h$ on both sides of
that the identity $u_\varepsilon = (\lambda - L)h_\varepsilon - V$ and applying Schwarz inequality we obtain that $\|h_\varepsilon\|_0 \leq \lambda^{-1}(\|V\|_0 + \varepsilon)$. A similar inequality holds for $g_\varepsilon$ so that

$$\|h_\varepsilon\|_0 \leq \lambda^{-1}(\|V\|_0 + \varepsilon), \quad \|g_\varepsilon\|_0 \leq \lambda^{-1}(\|V\|_0 + \varepsilon). \quad (7.7)$$

A similar argument gives that

$$\|(\lambda - L)^{-1}W\|_0 \leq \lambda^{-1}\|W\|_0, \quad \|(\lambda - S)^{-1}W\|_0 \leq \lambda^{-1}\|W\|_0 \quad (7.8)$$

for any function $W$ in $L^2(\pi)$.

Let $G_\varepsilon = (1/2)\{h_\varepsilon + g_\varepsilon\}$, $H_\varepsilon = (1/2)\{h_\varepsilon - g_\varepsilon\}$ and remark that the bounds (7.7) extend to $G_\varepsilon$, $H_\varepsilon$. From the equations for $h_\varepsilon$, $g_\varepsilon$, we obtain that

$$\{ (\lambda - S)G_\varepsilon - AH_\varepsilon = V + U_\varepsilon, \quad (\lambda - S)H_\varepsilon - AG_\varepsilon = W_\varepsilon, \}
$$

where $U_\varepsilon = (1/2)\{u_\varepsilon + w_\varepsilon\}$, $W_\varepsilon = (1/2)\{u_\varepsilon - w_\varepsilon\}$. Of course, $\|U_\varepsilon\|_0 \leq \varepsilon$, $\|W_\varepsilon\|_0 \leq \varepsilon$.

Since $V = (\lambda - L)h_\varepsilon - u_\varepsilon$ and $\langle h_\varepsilon, (\lambda - L)h_\varepsilon \rangle = \langle h_\varepsilon, (\lambda - S)h_\varepsilon \rangle$, we have that

$$\langle V, (\lambda - L)^{-1}V \rangle = \langle h_\varepsilon, (\lambda - S)h_\varepsilon \rangle - \langle h_\varepsilon, u_\varepsilon \rangle - \langle (\lambda - L)^{-1}u_\varepsilon, V \rangle.$$

By (7.7) and since $\|u_\varepsilon\|_0 \leq \varepsilon$, the second term on the right hand side is absolutely bounded by $\lambda^{-1}\varepsilon\{\|V\|_0 + \varepsilon\}$. On the other hand, by (7.8), the third term is less than or equal to $\lambda^{-1}\varepsilon\|V\|_0$. Therefore,

$$\langle V, (\lambda - L)^{-1}V \rangle \geq \langle h_\varepsilon, (\lambda - S)h_\varepsilon \rangle - 2\lambda^{-1}\varepsilon\{\|V\|_0 + \varepsilon\}.$$

Recall that $h_\varepsilon = G_\varepsilon + H_\varepsilon$ and that $(\lambda - S)H_\varepsilon = W_\varepsilon + AG_\varepsilon$. In particular, by the anti-symmetry of $A$,

$$\langle h_\varepsilon, (\lambda - S)h_\varepsilon \rangle = \langle H_\varepsilon, (\lambda - S)H_\varepsilon \rangle + 2\langle G_\varepsilon, W_\varepsilon \rangle + \langle G_\varepsilon, (\lambda - S)G_\varepsilon \rangle.$$

In view of (7.7), the second term on the right hand side is absolutely bounded by $2\|G_\varepsilon\|_0\|W_\varepsilon\|_0 \leq 2\varepsilon\lambda^{-1}\{\|V\|_0 + \varepsilon\}$. The third one can be rewritten as

$$\langle G_\varepsilon, V + AH_\varepsilon \rangle + \langle G_\varepsilon, U_\varepsilon \rangle$$

because $(\lambda - S)G_\varepsilon = V + AH_\varepsilon + U_\varepsilon$. By (7.7), the second term is less than or equal to $\varepsilon\lambda^{-1}\{\|V\|_0 + \varepsilon\}$, while the first one is equal to

$$\langle (\lambda - S)^{-1}(V + AH_\varepsilon), V + AH_\varepsilon \rangle + \langle (\lambda - S)^{-1}U_\varepsilon, V + AH_\varepsilon \rangle.$$

By (7.8) $\|\lambda - S\|^{-1}U_\varepsilon\|_0 \leq \varepsilon\lambda^{-1}$. On the other hand, since $V + AH_\varepsilon = (\lambda - S)G_\varepsilon - U_\varepsilon$, by (7.7), (7.8), $\|V + AH_\varepsilon\|_0 \leq \lambda^{-2}\{\|V\|_0 + \varepsilon\} + \varepsilon$. Since we assumed $\lambda < 1$, the previous displayed formula is bounded below by

$$\|V + AH_\varepsilon\|^2_{1,\lambda} - 2\varepsilon\lambda^{-3}\{\|V\|_0 + \varepsilon\}.$$

Putting all previous estimates together, we obtain that

$$\langle V, (\lambda - L)^{-1}V \rangle \geq \|H\|^2_{1,\lambda} + \|V + AH_\varepsilon\|^2_{1,\lambda} - 7\varepsilon\lambda^{-3}\{\|V\|_0 + \varepsilon\}.$$

This concludes the proof of (7.5) and therefore of the theorem. □
7.2 Bounds on the variance

In this subsection we present upper and lower bounds for the variance $\sigma^2(V)$. Recall that $f_\lambda$ stands for the solution of the resolvent equation (2.5).

**Theorem 2.15.** Assume that $V$ belongs to $L^2(\pi) \cap H_1$. Then,
\[
\sup_{g \in \mathcal{C}} \left\{ 2 \langle V, g \rangle_\pi - \| g \|_1^2 - \| Ag \|_1^2 \right\} \leq \left( \frac{1}{2} \right) \sigma^2(V) \leq \inf_{h \in \mathcal{C}} \left\{ \| h \|_1^2 + \| V + Ah \|_1^2 \right\}.
\]

This result is a straightforward consequence of (7.1), Theorem 2.14 and next lemma.

**Lemma 2.16.** For any $g$ in $\mathcal{C}$ and $h$ in $L^2(\pi)$,
\[
\lim_{\lambda \to 0} \| g \|_{1,\lambda} = \| g \|_1, \quad \lim_{\lambda \to 0} \| h \|_{-1,\lambda} = \| h \|_{-1}.
\]

**Proof.** The proof of the first identity is obvious and follows directly from the definition of the norm $\| \cdot \|_{1,\lambda}$. To show the second one, fix $h$ in $L^2(\pi)$. For each $f$ in $\mathcal{C}$,
\[
\liminf_{\lambda \to 0} \| h \|_{-1,\lambda}^2 \geq \liminf_{\lambda \to 0} \left\{ 2 \langle h, f \rangle_\pi - \| f \|_{1,\lambda}^2 \right\} = 2 \langle h, f \rangle_\pi - \| f \|_1^2.
\]

Taking the supremum over all $f$ in $\mathcal{C}$, we conclude that $\liminf_{\lambda \to 0} \| h \|_{-1,\lambda} \geq \| h \|_{-1}$.

On the other hand, we claim that
\[
\| h \|_{-1,\lambda} \leq \| h \|_{-1}
\]
for all $h$ in $L^2(\pi)$ and all $\lambda > 0$. Indeed, since $\| h \|_{2,\lambda}^2 \leq \lambda^{-1} \| h \|_0^2 < \infty$, for any $\varepsilon > 0$ there exists $f_\varepsilon$ in $\mathcal{C}$ such that
\[
\| h \|_{2,\lambda}^2 \leq 2 \langle h, f_\varepsilon \rangle_\pi - \langle (\lambda - S) f_\varepsilon, f_\varepsilon \rangle_\pi + \varepsilon \\
\leq 2 \langle h, f_\varepsilon \rangle_\pi - \langle (-S) f_\varepsilon, f_\varepsilon \rangle_\pi + \varepsilon \leq \| h \|_{-1}^2 + \varepsilon.
\]

Letting $\varepsilon \downarrow 0$, we conclude the proof of the claim and of the one of the lemma.

7.3 Variational formulas for the variance

To strengthen the conclusion of Theorem 2.15, we need further assumptions on $V$. Denote by $f_\lambda^*$ the solution of the resolvent equation (2.5) with $L^*$ in place of $L$. Denote by $f_0$ the limit in $H_1$ of the sequence $f_\lambda$ given by Lemma 2.9. Assume that $f_\lambda^*$ converges to some $f_0^*$ in $H_1$ and that for any $\varepsilon > 0$ there exists $h_\varepsilon, h_\varepsilon^*$ in $\mathcal{C}$ such that
\[
\| L h_\varepsilon - V \|_{-1} < \varepsilon \quad \text{and} \quad \| h_\varepsilon - f_0 \|_1 < \varepsilon,
\]
\[
\| L^* h_\varepsilon^* - V \|_{-1} < \varepsilon \quad \text{and} \quad \| h_\varepsilon^* - f_0^* \|_1 < \varepsilon.
\]
Theorem 2.17. Assume that $V$ belongs to $L^2(\pi) \cap \mathcal{H}_{-1}$ and that (7.9) holds. Then,

$$
\sigma^2(V) = 2 \inf_{h \in \mathcal{C}} \left\{ \| h \|_1^2 + \| V + Ah \|_{-1}^2 \right\} \\
= 2 \sup_{g \in \mathcal{C}} \left\{ (V, g)_{\pi} - \| g \|_1^2 - \| Ag \|_{-1}^2 \right\}.
$$

Proof. We start with the first identity, whose proof is similar to the one of Claim G in Theorem 2.14. In view of Theorem 2.15, we just need to prove the lower bound for $\sigma^2(V)$.

Fix $\varepsilon > 0$. According to assumption (7.9), there exists $h_\varepsilon, g_\varepsilon$ in $\mathcal{C}$ such that $\| Lh_\varepsilon + V \|_{-1} \leq \varepsilon$, $\| L^* g_\varepsilon + V \|_{-1} \leq \varepsilon$, $\| h_\varepsilon - f_0 \|_1 \leq \varepsilon$, $\| g_\varepsilon - f_0^* \|_1 \leq \varepsilon$.

Let $u_\varepsilon = -Lh_\varepsilon - V, w_\varepsilon = -L^* g_\varepsilon - V$ and keep in mind that

$$
\| u_\varepsilon \|_{-1} \leq \varepsilon, \quad \| w_\varepsilon \|_{-1} \leq \varepsilon, \quad \| h_\varepsilon \|_1 \leq \| f_0 \|_1 + \varepsilon, \quad \| g_\varepsilon \|_1 \leq \| f_0^* \|_1 + \varepsilon.
$$

Let $G_\varepsilon = (1/2)\{ h_\varepsilon + g_\varepsilon \}$, $H_\varepsilon = (1/2)\{ h_\varepsilon - g_\varepsilon \}$. From the equations for $h_\varepsilon, g_\varepsilon$, we obtain that

$$
\begin{align*}
& \left\{ -SG_\varepsilon - AH_\varepsilon = V + U_\varepsilon, \\
& -SH_\varepsilon - AG_\varepsilon = W_\varepsilon,
\right.
\end{align*}
$$

where $U_\varepsilon = (1/2)\{ u_\varepsilon + w_\varepsilon \}$, $W_\varepsilon = (1/2)\{ u_\varepsilon - w_\varepsilon \}$. Of course, $\| U_\varepsilon \|_{-1}, \| W_\varepsilon \|_{-1}$ are bounded by $\varepsilon$, and $\| G_\varepsilon \|_1, \| H_\varepsilon \|_1$ are less than or equal to $(1/2)(\| f_0 \|_1 + \| f_0^* \|_1 + 2\varepsilon)$.

Since $f_\lambda$ converges in $\mathcal{H}_1$ to $f_0$, by the estimates on $\| u_\varepsilon \|_{-1}, \| h_\varepsilon \|_1$ and since $-Lh_\varepsilon = V + u_\varepsilon$,

$$
\begin{align*}
(1/2)\sigma^2(V) &= \lim_{\lambda \to 0} (V, f_\lambda) + \varepsilon \| V \|_{-1} \\
& \geq (V + u_\varepsilon, h_\varepsilon) - \varepsilon \{ \| V \|_{-1} + \| f_0 \|_1 + \varepsilon \} \\
& = \| h_\varepsilon \|_1^2 - \varepsilon \{ \| V \|_{-1} + \| f_0 \|_1 + \varepsilon \}.
\end{align*}
$$

Recall that $h_\varepsilon = G_\varepsilon + H_\varepsilon$ and that $-SH_\varepsilon = W_\varepsilon + AG_\varepsilon$. In particular, by the anti-symmetry of $A$,

$$
\| h_\varepsilon \|_1^2 = (h_\varepsilon, -Sh_\varepsilon)_{\pi} = (H_\varepsilon, -SH_\varepsilon)_{\pi} + 2(G_\varepsilon, W_\varepsilon)_{\pi} + (G_\varepsilon, -SG_\varepsilon)_{\pi}.
$$

The second term on the right hand side of this identity is absolutely bounded by $2\| G_\varepsilon \|_1 \| W_\varepsilon \|_{-1} \leq \varepsilon \{ \| f_0 \|_1 + \| f_0^* \|_1 + 2\varepsilon \}$. The third one can be rewritten as

$$
\| G_\varepsilon \|_1^2 = \| SG_\varepsilon \|_{-1}^2 = \| V + AH_\varepsilon + U_\varepsilon \|_{-1}^2.
$$

Since $\| U_\varepsilon - SG_\varepsilon \|_{-1} \leq \varepsilon + \| G_\varepsilon \|_1 \leq \varepsilon + (1/2)\{ \| f_0 \|_1 + \| f_0^* \|_1 + 2\varepsilon \}$, and since $\| u + v \|^2 \geq \| u \|^2 - 2 \| u \| \| v \|$, the previous expression is bounded below by
\[ \|V + AH_\varepsilon\|^2_{-1} - 2\varepsilon\{\|f_0\|_1 + \|f_0^*\|_1 + 2\varepsilon\} . \]

Assembling all previous estimates, we obtain that for every \( \varepsilon \), there exists a function \( H_\varepsilon \) in \( \mathcal{C} \) such that
\[
(1/2)\sigma^2(V) \geq \|H_\varepsilon\|^2_1 + \|V + AH_\varepsilon\|^2_{-1} - \varepsilon C_0 ,
\]
where \( C_0 = 4\{\|V\|_{-1} + \|f_0\|_1 + \|f_0^*\|_1 + 2\varepsilon\} \). This concludes the proof of the first statement of the theorem.

We now turn to the second identity. In view of Theorem 2.15, we just need to prove an upper bound for \( \sigma^2(V) \). By definition of \( h_\varepsilon \),
\[
(1/2)\sigma^2(V) = \lim_{\lambda \to 0} \|f_k\|^2_1 \leq \|h_\varepsilon\|^2_1 + \varepsilon\{3\varepsilon + 2\|f_0\|_1\} .
\]

Since \( h_\varepsilon = H_\varepsilon + G_\varepsilon \), \( \|h_\varepsilon\|^2_1 \) can be rewritten as
\[
\|H_\varepsilon\|^2_1 + \|G_\varepsilon\|^2_1 + 2\langle -S \rangle H_\varepsilon, G_\varepsilon \rangle = \|H_\varepsilon\|^2_1 + \|G_\varepsilon\|^2_1 + 2\langle G_\varepsilon, W_\varepsilon \rangle_\pi
\]
because \( -SH_\varepsilon = AG_\varepsilon + W_\varepsilon \) and \( A \) is anti-symmetric. The last term is absolutely bounded by \( \varepsilon\{\|f_0\|_1 + \|f_0^*\|_1 + 2\varepsilon\} \).

On the one hand, in view of the equation relating \( H_\varepsilon \) and \( G_\varepsilon \) to \( V \), \( \|G_\varepsilon\|^2_1 + \|H_\varepsilon\|^2_1 \) can be written as
\[
\langle AH_\varepsilon + V + U_\varepsilon, G_\varepsilon \rangle_\pi + \langle AG_\varepsilon + W_\varepsilon, H_\varepsilon \rangle_\pi = \langle V, G_\varepsilon \rangle_\pi + \langle U_\varepsilon, G_\varepsilon \rangle_\pi + \langle W_\varepsilon, H_\varepsilon \rangle_\pi
\]
because \( A \) is anti-symmetric. By Schwarz inequality, the last two terms are absolutely bounded by \( \varepsilon\{\|f_0\|_1 + \|f_0^*\|_1 + 2\varepsilon\} \). On the other hand, by the same relations, \( \|G_\varepsilon\|^2_1 + \|H_\varepsilon\|^2_1 \) can be written as
\[
\|G_\varepsilon\|^2_1 + \|SH_\varepsilon\|^2_{-1} = \|G_\varepsilon\|^2_1 + \|AG_\varepsilon + W_\varepsilon\|^2_{-1}
\]
Since \( AG_\varepsilon = -SH_\varepsilon - W_\varepsilon \), \( \|AG_\varepsilon\|_{-1} \leq \|H_\varepsilon\|_1 + \|W_\varepsilon\|_{-1} \leq \{\|f_0\|_1 + \|f_0^*\|_1 + 2\varepsilon\} \). In particular, the last term of the previous identity is bounded below by \( \|AG_\varepsilon\|^2_{-1} - 2\varepsilon\{\|f_0\|_1 + \|f_0^*\|_1 + 2\varepsilon\} \).

Therefore, writing \( A = \|H_\varepsilon\|^2_{-1} + \|G_\varepsilon\|^2_1 \) as \( 2A - A \), we obtain that \( \|H_\varepsilon\|^2_{-1} + \|G_\varepsilon\|^2_1 \) is bounded above by
\[
2\langle V, G_\varepsilon \rangle_\pi - \|G_\varepsilon\|^2_1 - \|AG_\varepsilon\|^2_{-1} + 4\varepsilon\{\|f_0\|_1 + \|f_0^*\|_1 + 2\varepsilon\} .
\]

We have thus proved that for any \( \varepsilon > 0 \) there exists a function \( G_\varepsilon \) in the core \( \mathcal{C} \) such that
\[
(1/2)\sigma^2(V) \leq 2\langle V, G_\varepsilon \rangle_\pi - \|G_\varepsilon\|^2_1 - \|AG_\varepsilon\|^2_{-1} + C_1\varepsilon ,
\]
where \( C_1 = 7\varepsilon\{\|f_0\|_1 + \|f_0^*\|_1 + 2\varepsilon\} \). This concludes the proof of the second identity of the theorem. \( \square \)
7.4 Variational principles in the graded sector context.

Recall the formalism and the notation introduced in Subsection 6.3. Assume that the hypotheses (6.2), (6.5), (6.8) hold. Denote by $A_{\pm 1}^n$ for the subspaces of $\mathcal{H}_{\pm 1}$ obtained by the closures of $A_n \cap \mathcal{C}$ in the respective norms. Let also $P_N := \bigoplus_{n=0}^N A_n$ and $P_{\pm 1}^N := \bigoplus_{n=0}^N A_{\pm 1}^n$. For a given $N \geq 0$ define $\Pi_N := \sum_{n=0}^N \pi_n$, where $\pi_n$ is the orthogonal projection of $L^2(\pi)$ onto $A_n$. We can also extend the graded structure on $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ spaces in an obvious fashion. Then, $\mathcal{H}_{\pm 1} = \bigoplus_{n \geq 1} A_{\pm 1}^n$. Using the results of Section 2.7 we conclude the following.

**Proposition 2.18.** $S$ extends to a unitary isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_{-1}$. Furthermore, for each $N \geq 1$ the respective restriction of the operator is a unitary isomorphism between $P_1^N$ and $P_{-1}^N$.

We let

$$Jf := AS^{-1}f \quad (7.10)$$

for any $f \in S(\mathcal{C}) \cap P_{-1}^N$. We have the following.

**Proposition 2.19.** For each $N \geq 1$ the operator $J$ extends to a bounded mapping from $P_{-1}^N$ into $P_{N+1}^N$.

**Proof.** Suppose that $f = Sg$, $g \in P_1^N$ and $\phi \in \mathcal{C}$ is such that $\|\phi\|_1 = 1$. We have

$$\langle Jf, \phi \rangle_P = \langle Ag, \Pi_{N+1}\phi \rangle_P. \quad (7.11)$$

According to (2.32) (and analogous estimates for $L_-, L_0$) we can estimate the right hand side of (7.11) by

$$C(N+1)^{1/2}\|g\|_1\|\phi\|_1 = C(N+1)^{1/2}\|f\|_{-1}. \quad \text{(7.12)}$$

The conclusion now follows from the fact that $\mathcal{C} \cap P_1^N$ is dense in $P_{N}^1$. \hfill \Box

We say that $V \in L^2(\pi)$ is *local* if $V \in P_N$ for a certain $N \geq 0$. Recall that from the results of Section 6.6.3 it follows that if $V$ is local then $f_\lambda = (\lambda-L)^{-1}V \rightarrow f_0$ and $f_\lambda^* = (\lambda-L^*)^{-1}V \rightarrow f_0^*$ in $\mathcal{H}_1$, as $\lambda \rightarrow 0+$. In particular, (7.1) holds.

**Theorem 2.20.** Suppose that $V \in \mathcal{H}_{-1} \cap L^2$ is local. Then, both (??) and (??) hold.

**Proof.** It suffices only to prove the hypothesis A) is satisfied. The argument contained in Section 2.6.3 proves that one can find $N \geq 1$ sufficiently large and $\lambda > 0$ sufficiently close to 0 such that $h_0 := \Pi_N f_\lambda$ satisfies

$$\|Lh_0 - V\|_{-1} < \frac{\varepsilon}{2} \quad \text{and} \quad \|h_0 - f_0\|_1 < \frac{\varepsilon}{2}. \quad (7.12)$$

We show that $h_0$ can be replaced by an element $h$ from $\mathcal{C} \cap P_1^N$. This follows from density of $\mathcal{C} \cap P_1^N$ in $P_N^1$ and the fact that $L = (I + J)S$ on $P_N^1$. \hfill \Box
7.5 Estimates of the variance

In this section we give some estimates of the variance of an additive functional of the Markov process \( \{X_t : t \geq 0\} \). These bounds shall be useful in proving the superdiffusive behavior of a tracer particle in turbulent diffusion models of Section 10.3.

Fix a function \( V \) in \( L^2(\pi) \) and let

\[
Y_t := \int_0^t ds V(X_s).
\]

**Proposition 2.21.** Let \( f_\lambda \) be the solution of the resolvent equation (2.5). For every \( t > 0 \),

\[
E_\pi [Y_t^2] \leq 12 t \langle f_{1/t}, V \rangle_\pi.
\]

**Proof.** Fix \( \lambda > 0 \) and recall the definition of the martingale \( M^\lambda_t \) introduced just before (4.6). By this latter formula, \( Y_t \) is equal to

\[
\lambda \int_0^t ds f_\lambda(X_s) + M^\lambda_t + f_\lambda(X_0) - f_\lambda(X_t).
\]

By the proof of Lemma 2.7, the expectation of the quadratic variation of the martingale \( M^\lambda_t \) is equal to \( 2t \|f_\lambda\|^2 \). Therefore,

\[
E_\pi [Y_t^2] \leq 4 \{ (\lambda t)^2 \|f_\lambda\|^2_0 + 2t \|f_\lambda\|^2 + 2 \|f_\lambda\|^2_0 \}.
\]

By (4.1), \( \lambda \|f_\lambda\|^2_0 + \|f_\lambda\|^2 \) is less than or equal to \( \langle f_\lambda, V \rangle_\pi \). Choosing \( \lambda = 1/t \) we can therefore estimate the right hand side of the previous equation by \( 12t \langle f_\lambda, V \rangle_\pi \) and the proposition follows. \( \square \)

**Proposition 2.22.** Fix a function \( V \) in \( L^2(\pi) \) and denote by \( f_\lambda \) the solution of the resolvent equation (2.5). Suppose that there exist finite constants \( \kappa < 1 \), \( C_0 > 0 \) such that \( \langle f_\lambda, V \rangle_\pi \geq C_0 \lambda^{-\kappa} \) for all \( \lambda \) in \( (0,1] \). Then for any \( \kappa' \in (0, \kappa) \) there exists \( C_1 > 0 \) such that

\[
\int_0^t E_\pi [Y_s^2] ds \geq C_1 t^{2 + \kappa'}
\]

for all \( t \) sufficiently large.

**Proof.** Let \( \varphi(t) = \int_0^t E_\pi [Y_s^2] ds \). Of course, \( \varphi(t) \) is a non-decreasing positive function. Moreover, by Schwarz inequality,

\[
\varphi(t) = \int_0^t ds E_\pi [Y_s^2] \leq (1/3) t^3 \|V\|^2_0.
\]

On the other hand, a long but straightforward computation gives that
\[ \int_0^{+\infty} e^{-\lambda t} \varphi(t) \, dt = \frac{2}{\lambda^3} \int_0^{+\infty} e^{-\lambda t} (P_{tV} V)_\pi \, dt = \frac{2}{\lambda^3} (f_\lambda, V)_\pi , \]

where \( f_\lambda \) is the solution of the resolvent equation (2.5). Hence, by assumption,

\[ \int_0^{+\infty} e^{-\lambda t} \varphi(t) \, dt \geq C_0 \lambda^{-3-\kappa} \]

for all \( 0 < \lambda \leq 1 \).

The function \( \varphi(t) \) satisfies therefore all assumptions of Lemma 2.23 below with \( \varrho = 1 \). In particular, for all \( \kappa' < \kappa \), there exists a constant \( C_1 \) such that \( \varphi(t) \geq C_1 t^{2+\kappa'} \) for all \( t \) sufficiently large. This is the content of this proposition. \( \square \)

We conclude this section with a Tauberian type estimate.

**Lemma 2.23.** Suppose that \( \varphi : [0, +\infty) \to \mathbb{R} \) is a non-decreasing, positive function, for which there exist \( \varrho, C, \xi > 0 \) and \( \kappa \in (0, \varrho) \) such that for all \( t > 0 \), \( \varphi(t) \leq Ct^{2+\varrho} \); and for all \( \lambda \in (0, 1) \),

\[ \int_0^{+\infty} dt \, e^{-\lambda t} \varphi(t) \geq C \lambda^{-3-\kappa} . \]

Then, for any \( \kappa' \in (0, \kappa) \) there exists \( \hat{c} > 0 \) such that \( \varphi(t) \geq \hat{c} t^{2+\kappa'} \) for all \( t \) sufficiently large.

**Proof.** Fix \( \kappa' < \kappa \) and choose \( \gamma > 0 \) such that \( (\kappa - 2\gamma)/(1 + \gamma) > \kappa' \). By assumption,

\[ C \lambda^{-3-\kappa} \leq \int_0^{+\infty} e^{-\lambda t} \varphi(t) \, dt = \frac{1}{\lambda} \int_0^{+\infty} e^{-t} \varphi(t/\lambda) \, dt . \]

Last expression can be rewritten as

\[ \frac{1}{\lambda} \int_0^{\lambda^{-1}} e^{-t} \varphi(t/\lambda) \, dt + \frac{1}{\lambda} \int_{\lambda^{-1}}^{+\infty} e^{-t} \varphi(t/\lambda) \, dt . \]

Since \( \varphi \) is monotone and since \( \varphi(t) \leq Ct^{2+\varrho} \) the previous sum can be estimated by

\[ \frac{1}{\lambda} \varphi(\lambda^{-1} - \gamma) + \frac{C}{\lambda} \int_{\lambda^{-1}-\gamma}^{+\infty} e^{-t} (t/\lambda)^{2+\varrho} \, dt \leq \frac{1}{\lambda} \varphi(\lambda^{-1} - \gamma) + \frac{c_1}{\lambda^{3+\varrho}} e^{-\gamma/2} \]

for some constant \( c_1 > 0 \) because the integral \( \int_0^{+\infty} \exp(-t/2) t^{2+\varrho} \, dt \) is finite. Up to this point, we proved that

\[ C \lambda^{-3-\kappa} \leq \frac{1}{\lambda} \varphi(\lambda^{-1} - \gamma) + \frac{c_1}{\lambda^{3+\varrho}} e^{-\gamma/2} . \]
Moving the second expression on the right hand side to the left hand side, and using the fact that
\[
\frac{c_1}{\lambda^{1+\theta}} e^{-\lambda^{-1}/2} \leq \frac{1}{2} \lambda^{-3-\kappa}
\]
for sufficiently small \( \lambda > 0 \), we obtain that \( \varphi(\lambda^{-1-\gamma}) \geq C_0 \lambda^{-2-\kappa} \) for some finite constant \( C_0 \) and all \( \lambda \) sufficiently small. Replacing \( \lambda \) by \( t^{-1/(1+\gamma)} \), we deduce that \( \varphi(t) \geq C_0 t^{2+\kappa_0} \) for all \( t \) sufficiently large, where \( \kappa_0 = (\kappa - 2\gamma)/(1 + \gamma) > \kappa' \). This concludes the proof of the lemma. \( \square \)

8 Transient Markov processes

We prove in this section some estimates involving the Dirichlet form and the Green function of transient Markov processes which have their own interest and which will be needed in later chapters.

Consider a transient, irreducible Markov process \( \{X_t, t \geq 0\} \) on a countable state space \( \mathcal{E} \), reversible with respect to the counting measure. Denote by \( r: \mathcal{E} \times \mathcal{E} \to \mathbb{R}_+ \) the rate at which the process jumps from \( x \) to \( y \) and assume that \( \sum_{y \in \mathcal{E}} r(x, y) < \infty \) for all \( x \) in \( \mathcal{E} \). Since the process is reversible with respect to the counting measure, \( r(x, y) = r(y, x) \) for all \( x, y \) in \( \mathcal{E} \).

The generator of the Markov process reads
\[
(Lf)(x) = \sum_{y \in \mathcal{E}} r(x, y)[f(y) - f(x)].
\]

Denote by \( G(x, y) \) the Green function:
\[
G(x, y) = \int_0^\infty dt \, p_t(x, y),
\]
which is finite because we assumed the process to be transient. Here \( p_t(x, y) \) stands for the transition probability function of the Markov chain. For \( x \) in \( \mathcal{E} \), denote by \( \mathbb{P}_x \) the probability measure on the path space \( D(\mathbb{R}_+, \mathcal{E}) \) induced by the Markov chain \( X_t \) starting from \( x \).

**Proposition 2.24.** Let \( V: \mathcal{E} \to \mathbb{R}_+ \) be a positive function with compact support. Then, for all finitely supported \( u: \mathcal{E} \to \mathbb{R} \),
\[
\sum_{y \in \mathcal{E}} u(y)^2 V(y) \leq (C_0/2) \sum_{x, y} r(x, y)\{u(y) - u(x)\}^2,
\]
where
\[
C_0 = C_0(V) = \sup_{x \in \mathcal{E}} \sum_{y \in \mathcal{E}} G(x, y)V(y).
\]
Central Limit Theorems

Proof. Notice that $C_0(V)$ is finite because $V$ has compact support, the process is transient and $G(x, y) \leq G(y, y)$ for all $x, y$ in $\mathcal{E}$. Let $W(x) = \sum_{y \in \mathcal{E}} G(x, y)V(y)$ so that $LW = -V$ and

$$
\sum_{y \in \mathcal{E}} u(y)^2 V(y) \leq C_0 \sum_{y \in \mathcal{E}} \frac{u(y)^2}{W(y)} V(y) = -C_0 \sum_{y \in \mathcal{E}} \frac{u(y)^2}{W(y)} (LW)(y).
$$

After a change of variables, since the process is reversible with respect to the counting measure, the previous expression becomes

$$
(C_0/2) \sum_{x,y \in \mathcal{E}} r(x, y) \left\{ u(y)^2 \frac{W(y)}{W(x)} - u(x)^2 \frac{W(x)}{W(y)} \right\} (W(y) - W(x))
$$

$$
= (C_0/2) \sum_{x,y \in \mathcal{E}} r(x, y) \left\{ u(y)^2 + u(x)^2 - \frac{u(y)^2}{W(y)} W(x) - \frac{u(x)^2}{W(x)} W(y) \right\}.
$$

Since $2ab \leq a^2 + b^2$, last expression is bounded above by

$$
(C_0/2) \sum_{x,y \in \mathcal{E}} r(x, y) (u(y) - u(x))^2,
$$

which concludes the proof of the proposition. \qed

Consider a subset $\mathcal{E}_0$ of $\mathcal{E}$ and exclude all jumps to or from $\mathcal{R} = \mathcal{E} \setminus \mathcal{E}_0$. Consider on $\mathcal{E}_0$ the induced Markov chain $\{X_t^0 : t \geq 0\}$ with generator given by

$$(L_0,f)(x) = \sum_{y \in \mathcal{E}_0} r(x, y) [f(y) - f(x)].$$

We claim that the induced Markov chain is transient modulo degenerated cases. More precisely, for each subset $A$ of $\mathcal{E}$, denote by $H_A$ the hitting time of $A$:

$$H_A = \inf \{t \geq 0 : X_t \in A\}.$$

Let

$$\theta(x) = \mathbb{P}_x [H_\mathcal{R} < \infty].$$

Lemma 2.25. Assume that $\theta(x) < 1$ for all $x$ in $\mathcal{E}_0$. Then the process is transient.

Proof. To prove that the process is transient, we will show that the function $\theta$ is a bounded, nonconstant, superharmonic function.

Fix $x$ in $\mathcal{E}_0$. Conditioning on the first jump of the process, we obtain that $\theta$ solves the equation

$$
\begin{cases}
(L\theta)(x) = 0 & \text{for } x \in \mathcal{E}_0, \\
\theta(x) = 1 & \text{for } x \in \mathcal{R}.
\end{cases}
$$
Therefore, for \( x \) in \( \mathcal{E}_0 \),
\[
(L_0 \theta)(x) = \sum_{y \in \mathcal{E}_0} r(x, y) \{\theta(y) - \theta(x)\} = -\sum_{y \in \mathcal{R}} r(x, y) \{\theta(y) - \theta(x)\}
\]
\[
= -\{1 - \theta(x)\} R(x) ,
\]
where \( R(x) = \sum_{y \in \mathcal{R}} r(x, y) \). This identity shows that \( \theta \) is a bounded superharmonic function for the Markov chain \( \{X^0_t : t \geq 0\} \). By irreducibility, \( R \) can not vanish uniformly on \( \mathcal{E}_0 \) so that \( \theta \) is not constant. This proves the lemma.

Denote by \( G_0 \) the Green function of the induced Markov chain. We now obtain estimates on \( G_0 \) assuming that
\[
b = \sup_{x \in \mathcal{E}_0} \theta(x) < 1 .
\]

We proved in Lemma 2.25 that \(- (L_0 \theta)(x) = (1 - \theta(x)) R(x) \). By definition of \( b \), this latter quantity is bounded below by \((1 - b) R(x) \) so that \(- (L_0 \theta)(x) \geq (1 - b) R(x) \). Applying the Green’s function on both sides of this inequality, we obtain that
\[
\sum_{y \in \mathcal{E}_0} G_0(x, y) R(y) \leq \frac{\theta(x)}{1 - b}
\]
for all \( x \) in \( \mathcal{E}_0 \).

**Proposition 2.26.** For all \( x, y \) in \( \mathcal{E}_0 \),
\[
G_0(x, y) \leq G(x, y) + \frac{\theta(x)}{1 - b} G(y, y) , \quad G_0(x, x) \leq \frac{1}{1 - b} G(x, x) .
\]

**Proof.** Fix a positive function \( U : \mathcal{E}_0 \to \mathbb{R}_+ \) supported on \( \mathcal{E}_0 \). Let
\[
W(x) = \sum_{y \in \mathcal{E}} G(x, y) U(y) .
\]
The function \( W \) is nonnegative and solves \( L W = -U \). A computation shows that for \( x \) in \( \mathcal{E}_0 \),
\[
(L_0 W)(x) = (LW)(x) - \sum_{y \in \mathcal{R}} r(x, y) \{W(y) - W(x)\} \leq -U(x) + A(x) W(x) .
\]
The latter term is bounded above by \(- U(x) + C_1 A(x) \), where \( C_1 = \sup_{x \in \mathcal{E}_0} W(x) \). Thus, \( U(x) \leq (L_0 W)(x) + C_1 A(x) \). Applying \( G_0 \) on both sides, in view of (8.1) we get that
\[
(G_0 U)(x) \leq W(x) + C_1 \sum_{y \in \mathcal{E}_0} G_0(x, y) A(y) \leq W(x) + C_1 \frac{\theta(x)}{1 - b} , \quad (8.2)
\]
Taking the supremum over $x$ and recalling the definition of $C_1$ we obtain that
\[ \sup_{x \in E_0} \sum_{y \in E_0} G_0(x, y)U(y) \leq \frac{1}{1 - \beta} \sup_{x \in E} \sum_{y \in E} G(x, y)U(y). \]

Since $U$ vanishes on the set $R$, we may replace on the right hand side $E$ by $E_0$ in both occurrences.

Fix $z$ in $E_0$ and take $U(y) = 1\{y = z\}$ in (8.2). Since $C_1 = \sup_{x \in E_0} W(x) = \sup_{x \in E_0} G(x, z) = G(z, z)$, we obtain that
\[ G_0(x, z) = (G_0U)(x) \leq G(x, z) + \frac{\theta(x)}{1 - b} G(z, z), \]
which is the first statement of the lemma. Taking $z = x$ in this inequality and recalling that $\theta(x) \leq b$, we prove the second statement of the lemma.

We conclude this section applying the previous results to the case of symmetric random walks on $\mathbb{Z}^d$, $d \geq 3$, for later applications in Chapter 4. Fix a symmetric finite range probability $p$ on $\mathbb{Z}^d$ and denote by $\{X_t : t \geq 0\}$ the random walk which jumps from $x$ to $x + y$ at rate $p(y)$. The process is reversible with respect to the counting measure. Hereafter we adopt all the notation introduced in this section.

It is well known that the random walk is transient in dimension $d \geq 3$. Exclude all jumps from and to the origin and consider the induced random walk. In this case,
\[ \theta(x) = \mathbb{P}_x[H_{\{0\}} < \infty] < 1 \]
for all $x \neq 0$. Moreover, since the probability has finite range,
\[ b = \sup_{x \neq 0} \theta(x) = \max_{|x| \leq A} \theta(x) < 1. \]

Hence, the induced process is transient and the estimates derived in Proposition 2.26 are in force.

9 Notes and References

We proved in this chapter a stronger result: for any continuous bounded function $H$ on $\mathbb{R}$
\[ \lim_{t \to \infty} \left( \mathbb{E}_x \left[ H \left( t^{-1/2} \int_0^t V(X_s) \, ds \right) \right] - \int_\mathbb{R} H(x)G_\sigma(y) \, dy \right)^2 \right)_\pi = 0, \]
where $G_\sigma$ is the Gaussian density with variance $\sigma^2(V)$.
[?], [?], [?], [2], [?]