Core of convex distortions of a probability on a non atomic space

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Abstract: This paper characterizes the core of a differentiable convex distortion of a probability measure on a non atomic space by identifying it with the set of densities which dominate the derivative of the distortion, for second order stochastic dominance. Furthermore the densities that have the same distribution as the derivative of the distortion are the extreme points of the core. These results are applied to the differentiability of a Choquet integral with respect to a distortion of a probability measure (respectively the differentiability of a Yaari's or Rank Dependent Expected utility function). A Choquet integral is differentiable at x if and only if x has a strictly increasing quantile function. The superdifferential of a Choquet integral at any point is then fully characterized. Examples of uses of these results in simple models where some agent is a Rank Dependent Expected Utility (RDEU) maximizer are then given. In particular, efficient risk sharing among an expected utility maximizer and a RDEU maximizer and among two RDEU maximizers is characterized.

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1 Introduction

Capacities that are convex distortions of a probability measure were introduced in Game Theory mainly as examples (see Delbaen [1974]). They have been, more recently, the subject of wider interest because of their use in the Rank Dependent Expected Utility (RDEU) model. RDEU theory was first presented by Quiggin [1982] and later rediscovered independently by Yaari [1987] in a special case. It does not rely on the independence axiom and accounts for a number of violations of Expected Utility as Allais' paradox. The axiomatic aspects of the RDEU model have extensively been analysed. As RDEU only depends on the distribution of a random variable, following Machina's papers [1982,1989], other authors have discussed the differentiability of the Yaari Utility or of the Rank Dependent Expected Utility as functionals over lotteries (or probabilities) (see for example Chew et al [1987] or Wang [1993] and the bibliographies listed therein). However when applying the theory to most economic problems, it is more natural to use the differentiability of those utility functions as defined on random variables (or *acts*) because of the existing constraints.

Let Q be a given probability. Let f be a convex increasing differentiable function from $[0,1]$ onto itself, satisfying $f(0) = 0,\ f(1) = 1$ and let $f(Q)$ be the capacity which is the distortion of Q by f. It may easily be checked that any element in the core of $f(Q)$ is a probability, absolutely continuous with respect to Q which may be identified with its density with respect to Q. As the Choquet integral $E_{f(Q)}(x)$ is a concave homogeneous function and furthermore

$$E_{f(Q)}[x] = \min_{P \in \text{core}(f(Q))} E_P[x],$$

it follows from standard convex analysis results that $\partial E_f(x)$ the superdifferential of E_f at $x \in L^\infty$ equals the set of densities of the minimizing probabilities. Furthermore E_f is Gateaux-differentiable at x if and only if $\partial E_f(x)$ is essentially a singleton.

This of course, raises the question of characterizing core$(f(Q))$ and its extremal points, since if E_f is Gateaux-differentiable at x, then $\partial E_f(x)$ is an extremal point of the core. Our first result is that if the probability space is non atomic, then core$(f(Q))$ can be identified with the set of densities which dominate the derivative of the distortion, f', for second order stochastic dominance and the extreme points of core$(f(Q))$ are the densities that have same distribution as f'. Applying those results to the Choquet integral with respect to a distortion of a probability measure, we rederive the formula

$$E_{f(Q)}(x) = \min_{\{h \succeq_2 f'\}} E_Q(hx) = \int_0^1 f'(t) F_x^{-1}(1-t)dt$$

(with F_x^{-1} the quantile function of x) and show that the minimum in $E_{f(Q)}(x)$ is uniquely attained if and only if F_x^{-1} is strictly increasing. We then give an explicit formula for the derivative. We further characterize the superdifferential of $E_{f(Q)}$ at any x using again convex analysis tools.

We now turn to the examples which motivate this paper. The first example is a portfolio selection problem. A RDEU investor can invest part of her money in a risky asset with price p and the rest in a savings account at zero interest rate. Short selling of the risky asset is allowed as well as borrowing. We show that there exists a range of prices for which the investor does not buy nor sell the risky asset. In other words, we show that there is a no-trade equilibrium supported by a whole interval of prices. If the investor had been an expected utility maximizer with a differentiable utility index, the only case where she wouldn't have traded the security, would have been if the price of the asset was equal to its expected payoff.

The second example deals with the characterization of Pareto-Optima in a model where an Expected Utility maximizer and a Rank Dependent Expected Utility maximizer with convex distortion f exchange risk. We show that if aggregate risk has a strictly increasing quantile function and if $f'(0) = 0$ (gains that occur with a low probability are not taken into account), then the consumption of the RDEU agent is a function of aggregate risk that reaches an upper limit: it is constant on a range of high values of the aggregate wealth. Thus the RDEU agent who believes that high values of aggregate endowment have a low probability and decide to disregard them will reach satiation. Symmetrically if the Rank Dependent Expected Utility maximizer has a distortion fulfilling $f'(1) = \infty$, then his consumption is constant and strictly positive, for low values of aggregate endowment. In other words a RDEU agent, extremely sensitive to the certainty effect, insures himself a constant minimal consumption. Similar result may also be proven for risk sharing between two RDEU maximizers, the first with a convex distortion fulfilling $f_1'(0) > 0$ and the second $f_2'(0) = 0$ (or $f_1'(1) < \infty$ and $f_2'(1) = \infty$). In an insurance setting, this means that if the insurer is extremely pessimistic about very likely events (or very sensitive to the certainty effect) while the insured is moderately pessimistic (an expected utility maximizer is *moderately* pessimistic), then the insurer offers a contract with an upper limit. On the contrary, if the insured is extremely pessimistic about events occuring with probability one while the insurer is moderately pessimistic, then the optimal contract includes a deductible for high values of the loss (in other words, the insured's wealth is constant for high values of risk).

The paper is organized as follows: in section two, we characterize cores of convex distortions of a probability measure on a non atomic space. Section three is devoted to differentiability of Choquet integrals. The last section is devoted to the two examples.

2 Core of convex distortions on non-atomic probability spaces

2.1 Non-atomic probability spaces

Definition 1 *Let $(\Omega, \mathcal{A}, Q)$ be a probability space. A measurable set $A \in \mathcal{A}$ is an atom of Q if $Q(A) > 0$ and $B \subset A$ implies $Q(B) = 0$ or $Q(B) = Q(A)$. A probability space $(\Omega, \mathcal{A}, Q)$ is non-atomic if it has no atoms.*

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $[0,1]$ and λ (also denoted dt through the sequel) the Lebesgue measure on $[0,1]$.

Definition 2 *A random variable $s : \Omega \mapsto [0,1]$ is a* measure-preserving map (m.p. for short) *(or has uniform distribution) if and only if:*

$$\lambda(B) = Q(s^{-1}(B)), \ \textit{for all } B \in \mathcal{B}.$$

It is easy to check that s is m.p. if and only if

$$\int_\Omega \psi(s(\omega))dQ(\omega) = \int_0^1 \psi(t)dt, \text{ for all measurable function } \psi$$

for which these integrals are well defined.

Let y be a nonnegative random variable and $F_y : \mathbf{R}_+ \to [0,1]$ denote its distribution function $F_y(t) = Q(\{y \le t\})$. Let F_y^{-1} denote the generalized inverse of F_y (sometimes also called the *nondecreasing rearrangement* of y):

$$F_y^{-1}(t) = \inf\{x \in \mathbf{R}_+ : F_y(x) \ge t\}, \text{ for } t \in]0,1[$$

$$F_y^{-1}(0) = \operatorname{essinf} y$$

where we recall that

$$\operatorname{essinf} y = \sup\{t \in \mathbf{R}_+ : Q(\{y \ge t\}) = 1\}$$

For further use, let us remark that F_y^{-1} is left continuous and continuous at 0.

We shall make extensive use of two important results under the assumption that $(\Omega, \mathcal{A}, Q)$ is non-atomic. The first one is due to Ryff [1970] and the second one is a generalization of Hardy-Littlewood's inequality. For a proof of these results, we refer to Chong and Rice [1970] and Schmeidler [1979].

Property 1 *Let y be a random variable on $(\Omega, \mathcal{A}, Q)$. Then there exists a measure-preserving map $s : \Omega \mapsto [0,1]$ such that $y = F_y^{-1} \circ s$ a.e..*

Property 2 *Let y_1 and y_2 be a pair of random variables on $(\Omega, \mathcal{A}, Q)$. Then the following inequalities hold (with the integrals written below possibly taking infinite values)*

$$\int_0^1 F_{y_1}^{-1}(t) F_{y_2}^{-1}(t) dt \geq \int_\Omega y_1 y_2 dQ \geq \int_0^1 F_{y_1}^{-1}(1-t) F_{y_2}^{-1}(t) dt \tag{1}$$

2.2 Core of convex distortions

We recall that a capacity on a measurable space $(\Omega, \mathcal{A})$ is a set function $\nu : \mathcal{A} \to [0,1]$ such that $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$ and for all $(A, B) \in \mathcal{A} \times \mathcal{A}$, $A \subset B$ implies $\nu(A) \leq \nu(B)$. The core of the capacity ν, denoted core(ν), is the set of additive set functions that weakly exceed ν everywhere. A capacity ν is convex if for all $(A, B) \in \mathcal{A} \times \mathcal{A}$, $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$.

Let $(\Omega, \mathcal{A}, Q)$ be a non atomic probability space. The core of a convex distortion of Q is defined as follows: let $f : [0,1] \to [0,1]$ be a strictly convex increasing differentiable function such that $f(0) = 0$, $f(1) = 1$. It may easily be verified that $\nu := f(Q)$ is a convex capacity. The core of $f(Q)$ is then defined by:

$$\text{core}(f(Q)) = \{P \text{ finitely additive s.t. } P(A) \geq f(Q(A)),\ \forall\, A \in \mathcal{A}\}.$$

The core of $f(\lambda)$, core$(f(\lambda))$ is defined similarly.

General properties of the core of an exact capacity may be found in Schmeidler [1972]. For the sake of completeness, let us review some basic properties of core$(f(Q))$. Let us first remark that any element of core$(f(Q))$ is in fact a probability measure. Moreover, any probability P in Core $f(Q)$ is absolutely continuous with respect to Q. Indeed if $Q(A) = 0$, then $Q(A^c) = 1$, hence $f(Q(A^c)) = 1$ and $P(A^c) = 1$. Hence core$(f(Q))$ may be identified with a subset of $L^1_+(Q)$. Clearly core$(f(Q))$ is convex and $\sigma(L^1, L^\infty)$ closed. Furthermore it may easily be checked that core$(f(Q))$ fulfills Dunford-Pettis criterion (see appendix for details). Hence, it will from now on be identified with a $\sigma(L^1, L^\infty)$ compact convex subset of $L^1_+(Q)$.

We first prove that if $h \in$ core$(f(Q))$, then $f'(1) \geq h \geq f'(0)$ Q-a.e. In particular, if $f'(1) < +\infty$ (an assumption that we do not need in the sequel), then core$(f(Q))$ is a bounded subset of $L^\infty_+(Q)$. It is interesting to note that in our second example (see section 4.2), in order to describe phenomena which do not appear in an expected utility model with differentiable utility index, we will assume $f'(0) = 0$ or $f'(1) = +\infty$.

Proposition 1 *Let h be in* $\mathrm{core}(f(Q))$ *then:*

$$f'(1) \geq h \geq f'(0) \ Q\text{-}a.e.$$

The proof is given in the appendix.

We next notice that *nondecreasing* elements of $\mathrm{core}(f(\lambda))$ may easily be characterized :

Lemma 1 *Let $h : [0,1] \to \mathbf{R}^+$ be a non decreasing density function. Then*

$$h \in \mathrm{core}(f(\lambda)) \text{ if and only if } \int_0^x h(t)dt \geq f(x), \ \forall \, x \in [0,1].$$

The proof may be found in the appendix.

On the way to characterize $\mathrm{core}(f(Q))$, our first step consists of the following result:

Lemma 2 *Let h be a density function on Ω. Then $h \in \mathrm{core}(f(Q))$ if and only if $F_h^{-1} \in \mathrm{core}(f(\lambda))$.*

Proof. Let us first prove that if $F_h^{-1} \in \mathrm{core}(f(\lambda))$, then $h \in \mathrm{core}(f(Q))$. Indeed let $A \in \mathcal{A}$. As $1_{[0,Q(A)]}$ is the nonincreasing rearrangement of 1_A, from (1) , we have:

$$\int_A hdQ = \int_\Omega 1_A hdQ \geq \int_0^1 F_h^{-1} 1_{[0,Q(A)]} dt = \int_0^{Q(A)} F_h^{-1} dt$$

Since $F_h^{-1} \in$ Core $f(\lambda)$, we get:

$$\int_A hdQ \geq \int_0^{Q(A)} F_h^{-1} dt \geq f(Q(A))$$

hence the desired result.

To show the converse, assume $h \in \mathrm{core}(f(Q))$. From property 1, there exists a measure-preserving map $s : \Omega \to [0,1]$ such that $h = F_h^{-1} \circ s$. Let $B \in \mathcal{B}$. We then have:

$$\begin{aligned} \int_B F_h^{-1}(t)dt = \int_{s^{-1}(B)} F_h^{-1}(s(\omega))dQ(\omega) &= \int_{s^{-1}(B)} hdQ \\ &\geq f(Q(s^{-1}(B)) = f(\lambda(B)) \end{aligned}$$

Hence $F_h^{-1} \in \mathrm{core}(f(\lambda))$.

$\square$

We now recall the definition and characterizations of second order stochastic dominance, restricted to nonnegative random variables.

Property 3 *Let x be a nonnegative random variable on $(\Omega_1, \mathcal{A}_1, Q_1)$ and y be a nonnegative random variable on $(\Omega_2, \mathcal{A}_2, Q_2)$. Then x dominates y in the sense of second order stochastic dominance (denoted $x \succeq_2 y$) if any of the following equivalent conditions is fulfilled:*

1. $\int_0^t F_y(s)ds \geq \int_0^t F_x(s)ds, \ \forall \ t \in \mathbf{R}_+$,
2. $\int_0^t F_x^{-1}(s)ds \geq \int_0^t F_y^{-1}(s)ds, \ \forall \ t \in [0,1]$,
3. $E_{Q_1}[U(x)] \geq E_{Q_2}[U(y)]$ *for every concave* $U : \mathbf{R}^+ \to \mathbf{R}$ *for which these expectations exist.*

Similarly, $x \sim_2 y$ if all the previous statements hold with equality. Hence $x \sim_2 y$ if and only if x and y are identically distributed ($F_x^{-1} = F_y^{-1}$ or $F_x = F_y$). We shall from now on simply write $x \sim y$.

Let us now state our first main characterization result.

Theorem 1 *Let h be a density function on Ω. Then the following statements are equivalent;*

1. $h \in \mathrm{core}(f(Q))$,
2. $F_h^{-1} \in \mathrm{core}(f(\lambda))$,
3. $\int_0^x F_h^{-1}(t)dt \geq f(x), \forall x \in [0,1]$,
4. $h \succeq_2 f'$.

Proof. The equivalence between 1 and 2 follows from lemma 2, that between 2 and 3 from lemma 1 and the last equivalence follows from the characterization of second order stochastic dominance. □

We now describe $\mathrm{core}(f(Q))$ in a finer way. This representation result, whose proof may be found in the appendix also follows from the theorem by Ryff that we cite in section 3.1.

Theorem 2

$$\mathrm{core}(f(Q)) = \overline{\mathrm{co}}\{f' \circ s, \ s : \Omega \to [0,1] \ \textit{measure preserving}\}$$

where the closure is taken for the $L^1(Q)$ topology.

3 Differentiability of a Choquet integral

3.1 Extreme points of $\text{core}(f(Q))$

We recall that an extreme point x of a convex compact subset K of a topological vector space is *exposed* if there exists a supporting hyperplane of K at x which intersects with K only at x.

We first state an important result of J. Ryff [1967] for non atomic spaces:

Theorem 3 (J. V. Ryff) *Every extreme point of the set* $\{h : h \succeq_2 x\}$ *is exposed and the set of extreme points of* $\{h : h \succeq_2 x\}$ *is* $\{h : h \sim x\}$.

We therefore get the following result:

Corollary 1 *Every extreme point of* $\text{core}(f(Q))$ *is exposed and the set of extreme points of* $\text{core}(f(Q))$ *is the set of densities:*

$$\{f' \circ s,\ s : \Omega \to [0,1] \text{ is measure preserving } \}.$$

One may therefore deduce from Corollary 1 and Krein-Milman's theorem, an alternate proof of Theorem 2.

3.2 Choquet Integral with respect to a convex distortion

From now on, we assume that f is *strictly* convex on $[0,1]$.

Let y be in $L^\infty(Q)$. The Choquet integral of y with respect to the capacity $f(Q)$, denoted $E_f(y)$ is defined by

$$E_f(y) = \int_{-\infty}^{0} (f(Q(\{y \geq t\})) - 1)dt + \int_0^\infty f(Q(\{y \geq t\}))dt$$

As the capacity $f(Q)$ is convex, it is well known (see Schmeidler [1986]) that the Choquet integral is given by:

$$E_f(y) := \min_{h \in \text{core}(f(Q))} \int_\Omega hydQ \tag{2}$$

Clearly E_f is concave positively homogeneous and upper semi continuous for both $\sigma(L^\infty, L^1)$ and $\sigma(L^\infty, (L^\infty)')$ topologies. Since the infimum in (2) is attained at at least one extreme point, it follows from Theorem 3 that we also have

$$E_f(y) := \min_{h \sim f'} \int_\Omega hydQ = \min_{s(.) \text{ m.p.}} \int_\Omega f'(s(\omega))y(\omega)dQ \tag{3}$$

For further use, let us define:

$$M(y) := \left\{ h \in \text{core}(f(Q)) \text{ such that } \int_\Omega hydQ = E_f(y) \right\} \tag{4}$$

We shall prove that $M(y)$ is the superdifferential of E_f at y. The nonemptiness of $M(y)$ follows from the compactness of core$(f(Q))$ and $M(y)$ clearly is a closed convex subset of core$(f(Q))$. Hence, $M(y)$ is the closed convex hull of its extreme points. It may easily be checked that the set of extreme points of $M(y)$, $EM(y)$, is the set of extreme points of core$(f(Q))$ which belong to $M(y)$:

$$EM(y) := \left\{ f' \circ s \text{ , } s \text{ m.p. such that } \int_\Omega f'(s(\omega))y(\omega)dQ(\omega) = E_f(y) \right\} \tag{5}$$

3.3 Minimizers of the Choquet Integral

The aim of this section is to study differentiability properties of the Choquet integral. We shall give a necessary and sufficient condition on y (more precisely on the distribution of y) for E_f to be Gateaux-differentiable at y. This problem reduces to characterizing the y's at which $M(y)$ reduces to a single point. Since $M(y)$ is the closed convex hull of $EM(y)$, $M(y)$ reduces to a single point if and only if $EM(y)$ does.

Let $y \in L^\infty$ and let s_0 be a measure preserving map (whose existence is given by property 1) such that

$$y = F_y^{-1} \circ s_0$$

We then have:

Lemma 3 *The density $h_0 := f'(1 - s_0)$ belongs to $EM(y)$ and*

$$E_f(y) = \int_0^1 f'(1-t)F_y^{-1}(t)dt$$

Proof. Let $h \sim f'$. By (1), we have:

$$\int_\Omega hydQ \geq \int_0^1 f'(1-t)F_y^{-1}(t)dt$$

Since s_0 is measure preserving,

$$\int_0^1 f'(1-t)F_y^{-1}(t)dt = \int_\Omega f'(1-s_0)ydQ$$

Hence

$$E_f(y) := \min_{h \sim f'} \int_\Omega hydQ = \int_\Omega f'(1-s_0)ydQ = \int_0^1 f'(1-t)F_y^{-1}(t)dt.$$

□

Remark. The formula $E_f(y) = \int_0^1 f'(1-t)F_y^{-1}(t)dt$ may be derived directly without convexity assumption on f nor reference to the structure of the Core.

It follows from the previous Lemma that $h_0 := f'(1-s_0)$ belongs to $EM(y)$ for any measure preserving s_0 such that $y = F_y^{-1} \circ s_0$.

Theorem 4 *Let $y \in L^\infty$, then $M(y)$ contains only one element if and only if F_y^{-1} is increasing. In that case, $M(y) = \{f'(1-F_y(y))\}$.*

Proof. Let us first prove that the requirement of the Theorem is necessary. Assume that F_y^{-1} is not increasing. Then there exists y_0 such that $Q(\{y = y_0\}) > 0$. Let $h_0 = f'(1-s_0)$ be as in Lemma 3 and let

$$h_1 := \frac{\int_{\{y=y_0\}} h_0 dQ}{Q(\{y=y_0\})} \cdot 1_{\{y=y_0\}} + h_0 \cdot 1_{\{y \neq y_0\}}$$

Then $\int_\Omega h_1 y dQ = \int_\Omega h_0 y dQ$ and $h_0 \neq h_1$ on $\{y = y_0\}$ for otherwise s_0 would have an atom, contradicting the fact that s_0 is measure preserving. By Jensen's inequality $h_1 \succeq_2 h_0$ and $h_1 \in M(y)$ contradicting the hypothesis that the solution is unique.

Assume now that F_y^{-1} is increasing and that $M(y) \neq \{h_0\}$. Then there exists $h_1 \in EM(y)$ with $h_1 \neq h_0$. Define the strictly concave function V on $[f'(0), f'(1)]$ by $V'(x) = F_y^{-1}(1 - {f'}^{-1}(x))$ for all $x \in [f'(0), f'(1)]$. Then $V'(h_0) = F_y^{-1}(s_0) = y$. Since $h_0 \sim h_1$, we have

$$0 = \int_\Omega (V(h_0) - V(h_1))dQ.$$

Since V is strictly concave on $[f'(0), f'(1)]$ and $h_0 \neq h_1$, we then have:

$$0 > \int_\Omega V'(h_0)(h_0 - h_1)dQ = \int_\Omega y(h_0 - h_1)dQ = 0$$

hence a contradiction.
Since F_y^{-1} is injective, there exists a unique measure preserving map s_0 such that $y = F_y^{-1} \circ s_0$, a.e. and this map is given by:

$$s_0 = F_y(y) \text{ a.e.}.$$

□

Remark. Note that the minimizer $h_0 = f'(1-s_0) = f'(1-F_y(y))$ is a nonincreasing function of y, in particular, it is anticomonotone with y (see definition in section 4).

In the next Theorem, we characterize $M(y)$ in the case where y may have atoms.

Theorem 5 *Let $y \in L^\infty$, let $h_0 = f'(1-s_0)$ be as in Lemma 3 and let*

$$A_y := \{\omega : Q(\{y = y(\omega)\}) > 0\}$$

If $h_1 = f'(1-s_1) \in EM(y)$ (with s_1 measure preserving), we then have:

1. *$s_1 = s_0 = F_y(y)$ Q-a.e. on $\Omega \setminus A_y$,*
2. *$F_y^{-1} \circ s_1 = y$ Q-a.e. on Ω.*

Hence $M(y)$ is the closed convex hull (for the L^1 topology) of all $f'(1-s)$ with s measure preserving and such that $y = F_y^{-1} \circ s$.

The proof may be found in the appendix. Note that Theorem 5 implies that any $h \in M(y)$ can be written in the form $h = f'(1-T)$ where $T(\omega) \in [F_y(y(\omega)_-), F_y(y(\omega))]$ Q-a.e..

3.4 Differentiability of a Choquet integral

We shall now apply our previous results to characterize the points at which E_f is differentiable and to describe the supergradient of E_f at any point $y \in L^\infty$.

Let us first recall that $g : L^\infty \to \mathbf{R}$ is Gateaux-differentiable at y if for all $x \in L^\infty$, the limit

$$Dg(y)(x) = \lim_{t \to 0^+} \frac{1}{t} [g(y+tx) - g(y)]$$

exists and $x \mapsto Dg(y)(x)$ is a continuous linear form in the norm topology.

As E_f is concave, we recall that the superdifferential $\partial E_f(y)$ of $E_f(.)$ at $y \in L^\infty$ is defined by

$$\partial E_f(y) := \left\{ h \in (L^\infty)' \text{ s.t. } E_f(y) - E_f(y') \geq \int_\Omega h(y-y')dQ, \text{ for all } y' \in L^\infty \right\}$$

Since

$$E_f(y) := \min_{h \in \text{core}(f(Q))} \int_\Omega hydQ$$

and since core$(f(Q))$ is $\sigma(L^1, L^\infty)$ compact, it follows from standard results in convex analysis that $\partial E_f(y) = M(y)$ so that in particular $\partial E_f(y) \subset L^1$ (for the sake of completeness, a proof is given in the appendix). Furthermore E_f is Gateaux-differentiable at y if and only if $\partial E_f(y)$ contains only one point h (see Aubin [1998] Corollary 4.2). We therefore have:

Corollary 2 *Let $y \in L^\infty$, then*

1. *E_f is Gateaux-differentiable at y if and only if F_y^{-1} is increasing. In this case:*
$$DE_f(y)(z) = \int_\Omega f'(1 - F_y(y)) \cdot z dQ, \ \forall\ z \in L^\infty$$
2. *More generally the supergradient of $E_f(.)$ at y is the closed convex hull (for the L^1 topology) of all $f'(1-s)$ with s measure preserving satisfying $y = F_y^{-1} \circ s$.*
3. *On the set of ω's such that $Q(\{y = y(\omega)\}) = 0$, every element of $\partial E_f(y)$ is equal to $f'(1 - F_y(y(\omega)))$ Q-a.e..*

Note that this corollary implies that any $h \in \partial E_f(y)$ can be written in the form $h = f'(1 - T)$ where $T(\omega) \in [F_y(y(\omega)_-), F_y(y(\omega))]$ Q-a.e..

Let $U : \mathbf{R}_+ \to \mathbf{R}_+$ be a strictly concave increasing, C^1 function. Let $E_f(U(y))$ denote the Choquet integral of $U(y)$ with respect to the capacity $f(Q)$. We also have $E_f(U(y)) := \min_{h \in \text{core}(f(Q))} \int_\Omega hU(y)dQ$. We may now state:

Corollary 3 *Let $y \in \text{int}(L_+^\infty)$, then*

1. *$E_f(U)$ is Gateaux-differentiable at y if and only if F_y^{-1} is increasing. In this case:*
$$DE_f(U(y))(z) = \int_\Omega f'(1 - F_y(y)) \cdot U'(y) \cdot z dQ, \ \forall\ z \in L_+^\infty$$
2. *More generally, the supergradient of $E_f(U(.))$ at y is the closed convex hull (for the L^1 topology) of all $f'(1-s) \cdot U'(y)$ with s measure preserving satisfying $y = F_y^{-1} \circ s$.*
3. *On the set of ω's such that $Q(\{y = y(\omega)\}) = 0$, every element of $\partial E_f(U(y))$ is equal to $f'(1 - F_y(y(\omega))) \cdot U'(y(\omega))$ Q-a.e..*

Proof. The first assertion follows by composition from the previous corollary and from the fact that the map $y \mapsto U(y)$ is differentiable (even in the sense of Fréchet) at any $y \in \text{int}L_+^\infty$ with derivative $h \in L^\infty \mapsto U'(y)h$. The proof of the second assertion follows from a representation of supergradients of infima of concave functions (see Aubin [1998] and Valadier [1969]). $\square$

4 Examples

4.1 A portfolio selection Problem

Dow and Werlang [1992] studied an optimal portfolio choice problem with one risky asset and one non-risky asset in a non expected utility model. They proved, that there exists a range of prices at which the investor has no position in the risky asset. Let us give a simple proof of this result in the RDEU model.

Taken as primitive is a non atomic probability space $(\Omega, \mathcal{A}, Q)$. A RDEU investor is characterized by a strictly concave, increasing, C^1 utility index U and an increasing convex, C^1 probability perception function $f : [0,1] \to [0,1]$ that satisfies $f(0) = 0,\ f(1) = 1$. She has w dollars as initial wealth. She may invest α dollars on a risky asset with random payoff $X : \Omega \to \mathbf{R}_+$ and which costs p or put the money on a savings account at zero interest rate. If she invests α dollars on the risky asset, her random future wealth is:

$$W = w - \alpha + \frac{\alpha X}{p}$$

and as $F^{-1}_{U(W)} = U(F^{-1}_W)$, her utility is

$$E_f(U(W)) = \int_0^1 U(F^{-1}_W(t)) f'(1-t) dt$$

For $\alpha \geq 0,\ F^{-1}_W(t) = w - \alpha + \frac{\alpha}{p} F^{-1}_X(t)$, while if $\alpha \leq 0$:

$$F^{-1}_W(t) = w - \alpha + \frac{\alpha}{p} F^{-1}_X(1-t).$$

Therefore the investor's indirect utility over α is

$$V(\alpha) = \int_0^1 U(w - \alpha + \frac{\alpha}{p} F^{-1}_X(t)) f'(1-t) dt \quad \text{if} \quad \alpha \geq 0$$

$$V(\alpha) = \int_0^1 U(w - \alpha + \frac{\alpha}{p} F^{-1}_X(1-t)) f'(1-t) dt \quad \text{if} \quad \alpha \leq 0$$

Clearly V is a concave function, that is differentiable at any $\alpha \neq 0$. It is optimal for the agent not to buy nor to sell the asset (in other words 0 is optimal) if and only if $V'(0_+) \leq 0 \leq V'(0_-)$.

As $V'(0_-) = U'(w)(-1 - \frac{E_f(-X)}{p})$ and $V'(0_+) = U'(w)(-1 + \frac{E_f(X)}{p})$, 0 is optimal if and only if:

$$E_f(X) \leq p \leq -E_f(-X).$$

We recall that in the EU model, 0 is optimal if and only if $p = E(X)$.

4.2 Risk sharing rules

We may now deduce from the results of section 3 properties of interior Pareto Optima when one agent is EU and the other RDEU in a non atomic probability space $(\Omega, \mathcal{A}, Q)$.

We consider a pure exchange economy under uncertainty. In each state of the world, a single good is available for consumption. Let $L^\infty_+ := L^\infty_+(Q)$ be the set of contingent consumptions. There are 2 agents. Agent i is characterized by her utility, $u_i : L^\infty_+ \to \mathbf{R}$. We only specify aggregate endowment $w \in L^\infty_+$ in the economy.

Definition 3 *A utility function $v : L^\infty_+ \to \mathbf{R}$ is strongly risk averse (respectively strictly strongly risk averse) if $x \succeq_2 y$ (respectively $x \succ_2 y$) implies $v(x) \geq v(y)$ (resp $v(x) > v(y)$).*

Examples and characterizations of strongly risk averse utilities may be found in Chateauneuf, Cohen and Meilijson [1997] and in Chew et al [1995]. Strong risk aversion does not imply concavity or quasi-concavity of v.

Let us first recall a definition:

Definition 4 *A pair of maps $(x, y) \in (L^\infty)^2$ is comonotone if there exists a subset $B \subset \Omega \times \Omega$ of measure one for $P \bigotimes P$ such that*

$$[x(s) - x(s')]\,[y(s) - y(s')] \geq 0,\ \forall (s, s') \in B \times B.$$

Similarly, a pair $(x, y) \in (L^\infty)^2$ is anti-comonotone if the pair $(x, -y)$ is comonotone.

A useful characterization of comonotonicity of a pair of random variables x and y is as follows:

Let $w = x + y$ and a :=essinf w and b :=esssup w, then (x, y) is comonotone if and only if there exists a pair of non decreasing functions (h_1, h_2), $h_i : [a, b] \to \mathbf{R}$ such that $h_1 + h_2 =$Id, $x = h_1(w)$, and $y = h_2(w)$ a.e..

For a proof of this characterization, see Denneberg [1994]. Note that the previous statement implies that both h_1 and h_2 are 1-Lipschitz continuous functions.

Let us introduce the following assumptions:

U1 u_i is $\sigma(L^\infty, L^1)$ upper semi-continuous, $i = 1, 2$,

U2 u_i is strictly strongly risk averse, $i = 1, 2$.

The following result is an infinite dimensional extension of Dana [2000] (see also Chateauneuf, Cohen and Kast [1997] and Landsberger and Meilijson [1994]).

Theorem 6 *Assume that utilities* u_1*,* u_2 *fulfill* **U1** *and that* F_w^{-1} *is increasing. Then:*

1. *there exist Pareto optimal allocations,*
2. *if furthermore utilities fulfill* **U2***, then any pair of Pareto Optimal allocations is comonotone.*

The proof may be found in the appendix.

Conditions **U1** and **U2** are fairly stringent since most strictly strongly risk averse utility function are not quasi-concave. Since closed convex sets for topologies $\sigma(L^\infty, L^1)$ and $\tau(L^\infty, L^1)$ coincide , a quasi-concave utility function is $\sigma(L^\infty, L^1)$ upper semi-continuous if and only if it is upper semi-continuous in the Mackey-topology $\tau(L^\infty, L^1)$.

Proposition 2 *Let the Rank dependent expected utility* u *be defined by* $u(x) = E_f(U(x))$ *for all* $x \in L^\infty_+$*. If* U *is strictly concave increasing and* $f : [0,1] \to [0,1]$ *is convex increasing with* $f(0) = 0$*,* $f(1) = 1$*, then* u *fulfills* **U1** *and* **U2**.

Proof. It follows from Chew et al. [1987] that u is strictly strongly averse and strictly concave. As shown by Bewley [1972], for any fixed $h \in \text{core}(f(Q))$, the map $x \mapsto E_Q(hU(x))$ is $\tau(L^\infty, L^1)$ continuous as a separable state dependent utility function, hence $\sigma(L^\infty, L^1)$ upper semi-continuous. Therefore

$$x \mapsto \min_{h \in \text{core}(f(Q))} E_Q(hU(x))$$

is $\sigma(L^\infty, L^1)$ upper semi-continuous. □

Let us now consider a pure exchange economy where agent 1 is an EU maximizer characterized by an increasing strictly concave, C^1 utility index U_1 and agent 2 is a RDEU maximizer characterized by an increasing strictly concave, C^1 utility index U_2 with $U_2(0) = 0$ and an increasing strictly convex, C^1 probability perception function $f : [0,1] \to [0,1]$ that satisfies $f(0) = 0,\ f(1) = 1$.

In order to ensure existence of interior solutions, assume further:

U3 $\lim_{x \to 0} U_i'(x) = \infty$, $i = 1, 2$.

and make the following regularity assumption on w:
U4 The measure dF_w is absolutely continuous with respect to Lebesgue measure, F_w^{-1} is strictly increasing, $a := \text{essinf} w > 0$ and $b := \text{essinf} w < +\infty$.

Our next result is a characterization of efficient Exchange of risk between an EU agent and an RDEU agent.

Proposition 3 *Assume* **U1-U4** *and* $f'(0) = 0$. *If* (x_1^*, x_2^*) *is a Pareto Optimal allocation, then there exist continuous non decreasing functions* $x_i : [a, b] \to \mathbf{R}^+,\ i = 1, 2$ *such that* $x_i^* = x_i(w)$. *Furthermore, either* $(x_1^*, x_2^*) \in \{(0, w), (w, 0)\}$ *or* $x_i^* > 0$ *Q-a.e. for* $i = 1, 2$. *In that case* x_2 *cannot be strictly increasing. In fact, it is constant in a neighbourhood of* $F_w^{-1}(1)$.

The proof may be found in the appendix.

The next proposition extends the previous property to efficient exchange of risk between two RDEU agents, with strictly convex increasing differentiable distortions f_1 and f_2.

Proposition 4 *Assume* **U1-U4**, $f_1'(0) > 0$ *and* $f_2'(0) = 0$. *If* (x_1^*, x_2^*) *is a Pareto Optimal allocation, then there exist continuous non decreasing functions* $x_i : [a, b] \to \mathbf{R}^+,\ i = 1, 2$ *such that* $x_i^* = x_i(w)$. *Furthermore, either* $(x_1^*, x_2^*) \in \{(0, w), (w, 0)\}$ *or* $x_i^* > 0$ *Q-a.e. for* $i = 1, 2$. *In that case,* x_2 *cannot be strictly increasing. In fact, it is constant in a neighbourhood of* $F_w^{-1}(1)$.

Remark. An anologous result holds in the case $f_1'(1) < +\infty$, $f_2'(1) = +\infty$. Indeed, if (x_1^*, x_2^*) is a Pareto Optimal allocation, then there exist continuous non decreasing functions $x_i : \mathbf{R}^+ \to \mathbf{R}^+,\ i = 1, 2$ such that $x_i^* = x_i(w)$. Furthermore, if $x_i^* > 0,\ i = 1, 2$, then x_2 cannot be strictly increasing as it is constant in a neighbourhood of $F_w^{-1}(0)$. The proof of this result is very similar to the one given in the appendix and is therefore omitted.

In an insurance setting (see Carlier and Dana [2001]), if the insurer is extremely pessimistic about very likely losses (in other words if the slope of his distortion is infinite at 1) while the insured is moderately pessimistic (the slope of his distortion is finite at 1), then the insurer offers a contract with an upper limit. On the contrary, if the insured is very sensitive to the certainty effect (he is extremely pessimistic about events of high probability), while the insurer is moderately pessimistic, then the optimal contract is such that the insured's wealth is constant for high values of the loss. Similarly if the insured (respectively the insurer) pays no attention to events of low probability (the slope of his distortion is 0 at 0) , then an optimal contract gives no insurance (respectively full insurance) for low values of the damage.

5 Appendix

5.1 Weak compactness of $\text{core}(f(Q))$.

Property 4 *A subset $H \in L^1(\Omega, \mathcal{A}, Q)$ is uniformly integrable if and only if*

1. $\sup_{h \in H} E_Q(|h|) < \infty$
2. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $A \in \mathcal{A}$ and $Q(A) \leq \delta$ implies*

$$\sup_{A \in H} \int_A |h| dQ \leq \varepsilon.$$

Let us recall that a subset $H \in L^1(\Omega, \mathcal{A}, Q)$ is relatively weakly-compact if and only if it is uniformly integrable. The proof of this classical result may be found for instance in Meyer [1966].

Proposition 5 *If f is continuous, $\text{core}(f(Q))$ is uniformly integrable.*

Proof. We first have $E_Q(h) = 1$, $\forall h \in \text{core}(f(Q))$. Furthermore, for any $A \in \mathcal{A}$, one has

$$\int_A h(\omega) dQ = 1 - \int_{A^c} h(\omega) dQ \leq 1 - f(1 - Q(A))$$

Hence the second requirement in property 4 follows from the continuity of f at 1. $\square$

5.2 Proof of Proposition 1

Proof. For any $A \in \mathcal{A}$, one has

$$\int_A h dQ = 1 - \int_{A^c} h dQ \leq 1 - f(1 - Q(A))$$

In particular for $A = \{h \geq \alpha\}$ with $\alpha \in \mathbf{R}_+$, we get:

$$\alpha Q(\{h \geq \alpha\}) \leq 1 - f(1 - Q(\{h \geq \alpha\}))$$

Further, if α is such that $Q(\{h \geq \alpha\}) > 0$, divide the previous inequality by $Q(\{h \geq \alpha\})$, to obtain from the convexity of f:

$$\alpha \leq \frac{1 - f(1 - Q(\{h \geq \alpha\}))}{Q(\{h \geq \alpha\})} \leq f'(1)$$

since f is convex. Hence, if $\alpha > f'(1)$, then $Q(\{h \geq \alpha\}) = 0$.

Assume that $f'(0) > 0$ and let $\alpha \in \mathbf{R}_+$ be such that $Q(\{h \leq \alpha\}) > 0$. We have:

$$\alpha Q(\{h \leq \alpha\}) \geq \int_{\{h \leq \alpha\}} h(\omega) dQ \geq f(Q(\{h \leq \alpha\}))$$

Hence

$$\alpha \geq \frac{f(Q(\{h \leq \alpha\}))}{Q(\{h \leq \alpha\})} \geq f'(0).$$

□

5.3 Proof of Lemma 1

Proof. Clearly the condition is necessary (without the assumption that h is non decreasing). To prove that the condition is sufficient, define

$$\mathcal{C} = \{A \in \mathcal{B} : \int_A h(t) dt \geq f(\lambda(A))\}.$$

By definition, $\mathcal{C}$ contains all intervals $]0, x[$. Let us prove that $\mathcal{C}$ contains all intervals $]y, y + x[\subset [0, 1]$. Let $F(x) = \int_0^x h(t) dt$. Then $F(x)$ is convex increasing and verifies $F(0) = 0$ and $F(1) = 1$. Moreover since F is convex, the map $y \rightarrow \frac{1}{x}(F(y + x) - F(y))$ is non decreasing, hence

$$F(y' + x) - F(y') \geq F(y + x) - F(y) \geq F(x) \geq f(x), \ \forall y' > y \qquad (6)$$

Hence $\mathcal{C}$ contains all intervals $]y, y + x[\subset [0, 1]$. Moreover if $I_1 =]a, b[$ and $I_2 =]c, d[$, with $a < b < c < d$, using (6) with $y = a$ and $x = b - a$, we get:

$$F(b) - F(a) \geq F(b - a)$$

and $y' = c, \ x = d - c, \ y = b - a$

$$F(d) - F(c) \geq F(b - a + d - c) - F(b - a)$$

hence

$$F(b) - F(a) + F(d) - F(c) \geq F(b - a + d - c) \geq f(b - a + d - c) = f(\lambda(I_1 \cup I_2))$$

hence $I_1 \cup I_2 \in \mathcal{C}$. Finally, let $O = \cup_n]a_n, b_n[$ with $a_0 \geq 0, \ a_n < b_n < a_{n+1} < b_{n+1} \leq 1, \forall \ n$. Since F is continuous, increasing and convex we have

$$\begin{aligned} \sum_n F(b_n) - F(a_n) &= \lim_{N \to \infty} \sum_{n=0}^{N} F(b_n) - F(a_n) \\ &\geq \lim_{N \to \infty} F(\sum_{n=0}^{N} (b_n - a_n)) \\ &= F(\sum_{n=0}^{\infty} (b_n - a_n)) \geq f(\sum_{n=0}^{\infty} (b_n - a_n)) \end{aligned}$$

As any open set is a countable union of disjoint open intervals, $\mathcal{C}$ contains all open sets. Since Lebesgue measure and any absolutely continuous measure with respect to it, is regular, for any Borel set B, we have:

$$\int_B h = \inf_{O \text{ open } \supset B} \int_O h$$

hence:

$$\int_B h = \inf_{O \supset B} \int_O h \geq \inf_{OB} f(\lambda(O)) = f(\inf_{O \supset B} \lambda(O)) = f(\lambda(B)$$

which finally proves that $\mathcal{C} = \mathcal{B}$. □

5.4 Proof of Theorem 2

Proof. Define:

$$B := \overline{\text{co}}\{f' \circ s,\ s : \Omega \to [0,1] \text{ measure preserving}\}$$

Obviously $\{f' \circ s, s : \Omega \to [0,1] \text{ m.p. }\} \subset \text{core}(f(Q))$. As $\text{core}(f(Q))$ is convex and closed in $L^1_+(Q)$, then $B \subset \text{core}(f(Q))$.

Assume that there exists $h_0 \in \text{core}(f(Q)) \setminus B$. By the Hahn-Banach theorem, there exist $\psi \in L^\infty(Q)$ and $\varepsilon > 0$ such that

$$\int \psi h_0 dQ \leq -\varepsilon + \inf_{h \in B} \int \psi h dQ \tag{7}$$

Since for all $h \in B$

$$\int h_0 dQ = \int h dQ = 1 \tag{8}$$

inequality (7) also holds true for $\psi + \|\psi\|_\infty + 1$. Hence we may without loss of generality assume that $\psi > 0$. Hence $F_\psi^{-1}(1-t)$ is nonincreasing and positive. From (1), we have

$$\int \psi h dQ \geq \int_0^1 F_\psi^{-1}(1-t) f'(t) dt,\ \forall h \in B \tag{9}$$

and

$$\int \psi h_0 dQ \geq \int_0^1 F_\psi^{-1}(1-t) F_{h_0}^{-1}(t) dt \tag{10}$$

Using (7), (9), (10), we thus have

$$\int_0^1 F_\psi^{-1}(1-t) F_{h_0}^{-1}(t) dt \leq -\varepsilon + \int_0^1 F_\psi^{-1}(1-t) f'(t) dt \tag{11}$$

Since $h_0 \succeq_2 f'$, for any non increasing step function g, we have

$$\int_0^1 g(t) F_{h_0}^{-1}(t) dt \geq \int_0^1 g(t) f'(t) dt \tag{12}$$

hence inequality (12) holds true for $g \in L^\infty_+$ nonincreasing, in particular, it holds true for $F_\psi^{-1}(1-t)$ contradicting (11). □

5.5 Proof of Theorem 5

Proof. As in the proof of Theorem 4, consider the concave function V defined by by $V'(x) = F_y^{-1}(1 - f'^{-1}(x))$ for all $x \in [f'(0), f'(1)]$. Then $V'(h_0) = y$. Let $h_1 = f'(1 - s_1) \in EM(y)$ (with s_1 measure preserving). Since $h_0 \sim h_1$, $\int_\Omega yh_0dQ = \int_\Omega yh_1dQ$ and $V'(h_0) = y$, we get:

$$0 = \int_\Omega (V(h_0) - V(h_1))dQ = \int_\Omega V'(h_0)(h_0 - h_1)dQ$$

Since V is concave

$$V(h_0) - V(h_1) \geq V'(h_0)(h_0 - h_1)$$

therefore

$$V(h_0) - V(h_1) = V'(h_0)(h_0 - h_1) \; Q\text{-a.e.} \tag{13}$$

Hence V is affine between $h_0(\omega)$ and $h_1(\omega)$.

By definition of V, note that V is affine only on segments of the form $[f'(1 - \beta_i), f'(1 - \alpha_i)]$ with $[\alpha_i, \beta_i) = \{F_y^{-1} = y_i\}$ and y_i atom of y i.e. $Q(\{y = y_i\}) > 0$.

With (13) we then get that for Q-a.e. ω, one of the following conditions must hold:

- either $s_1(\omega) = s_0(\omega)$,
- or there exists y_i, an atom of y, such that:
  $(s_0(\omega), s_1(\omega)) \in [\alpha_i, \beta_i]^2$

In particular if $\omega \notin A_y$, since $y(\omega) = F_y^{-1}(s_0(\omega))$ then s_0 does not belong to any of the segments $[\alpha_i, \beta_i]$ where F_y^{-1} is constant. Hence the second case does not hold and $s_0 = s_1$ a.e. in $\Omega \setminus A_y$.

In the case where there exists an atom of y, y_i such that $(s_0(\omega), s_1(\omega)) \in [\alpha_i, \beta_i]^2$, then $y_i = F_y^{-1}(s_1(\omega)) = F_y^{-1}(s_0(\omega)) = y(\omega)$. In both cases, we have $F_y^{-1} \circ s_1 = F_y^{-1} \circ s_0 = y$.

We have thus proved:

$$EM(y) \subset \{f'(1 - s) : s \text{ m.p. and } F_y^{-1} \circ s = y\}$$

and the converse inclusion holds by Lemma 3, so the last assertion of the Theorem follows.

□

5.6 Differentiability of a Choquet integral

Let us prove that $\partial E_f(y) = M(y)$ (hence $\partial E_f(y) \subset L^1$) .

First, it is obvious that $M(y) \subset \partial E_f(y)$.

Assume that there exists $\mu_0 \in \partial E_f(y)$ and $\mu_0 \notin \text{core} f(Q)$. As Core $f(Q)$ is a $\sigma((L^\infty)', L^\infty)$ compact convex subset of $(L^\infty)'$, there exist $\psi \in L^\infty(Q)$ and $\varepsilon > 0$ such that

$$< \psi, \mu_0 > \leq -\varepsilon + \inf_{h \in \text{core}(f(Q))} \int \psi(\omega) h(\omega) dQ(\omega)$$

Hence

$$< \psi, \mu_0 > \leq -\varepsilon + E_f(\psi) \tag{14}$$

As $E_f(y)$ is positively homogeneous and $E_f(y) - E_f(y') \geq < \mu_0, y - y' >$, by taking $y' = \frac{1}{2}y$ and $y' = 2y$, we obtain

$$E_f(y) = < \mu_0, y > \tag{15}$$

Since: $E_f(y) - E_f(\psi) \geq < \mu_0, y - \psi >$, we have

$$E_f(\psi) \leq < \mu_0, \psi > \leq -\varepsilon + E_f(\psi) \tag{16}$$

and we get the contradiction. Therefore $\mu_0 \in \text{core} f(Q) \subset L^1$ and $E_f(y) = < \mu_0, y >$ therefore $\partial E_f(y) = M(y)$.

5.7 Proof of Theorem 6

Proof.

Let $\alpha \in (0, 1)$. Assume first **U1**. Then, any solution to the problem

$$\max\{\alpha u_1(x_1) + (1 - \alpha) u_2(x_2) : x_1 \geq 0, \; x_2 \geq 0, \, x_1 + x_2 = w\}$$

is Pareto Optimal.

By Alaoglu's Theorem, the convex set:

$$\{(x_1, x_2) \in (L^\infty_+)^2 \text{ such that } x_1 + x_2 = w\}$$

is weak-star closed, is $\sigma(L^\infty, L^1)$ compact and the function $\alpha u_1(x_1) + (1 - \alpha) u_2(x_2)$ is $\sigma(L^\infty, L^1)$ upper semi-continuous, hence a solution exists.

Assume further **U2**. Let $(x_i^*)_{i=1,2}$ be a Pareto Optimal allocation. Since the allocation $(E(x_1^* \mid w), E(x_2^* \mid w))$ is feasible and $E(x_i^* \mid w) \succeq_2 x_i^*, \; i = 1, 2$, $u_i(E(x_i^* \mid w)) \geq u_i(x_i), \; i = 1, 2$ with a strict inequality if $E(x_i^* \mid w) \neq x_i$, we may, without loss of generality, assume that the x_i^* are functions of w. Hence let us assume that $x_i^* = x_i(w),\ i = 1,2$. Assume that $w \in [a,b]$ a.e. and that the $x_i,\ i = 1,2$ are not comonotone. Let us prove that $(x_i^*)_{i=1,2}$ is dominated for second order stochastic dominance by some allocation.

There exist two compact subsets of $[a,b]$, K_1 and K_2 that we may assume of same measure with respect to dF_w, such that

$$\begin{array}{l} x_1(s) > x_1(s') \\ x_2(s) < x_2(s') \end{array} \quad \text{for all } (s,s') \in K_1 \times K_2$$

Without loss of generality, we may assume by Lusin's theorem that $(x_i)_{i=1,2}$ are continuous on $K_i,\ i = 1,2$. There exists then $\varepsilon > 0$ such that:

$$\begin{array}{l} x_1(s) - \varepsilon > x_1(s') + \varepsilon \\ x_2(s) + \varepsilon < x_2(s') - \varepsilon \end{array} \quad \text{for all } (s,s') \in K_1 \times K_2$$

Define $\widetilde{x_i} : [a,b] \to \mathbf{R}_+$ by

$$\widetilde{x_1} = (x_1 - \varepsilon)1_{K_1} + (x_1 + \varepsilon)1_{K_2} + x_1 1_{(K_1 \cup K_2)^c}$$

$$\widetilde{x_2} = (x_2 + \varepsilon)1_{K_1} + (x_2 - \varepsilon)1_{K_2} + x_2 1_{(K_1 \cup K_2)^c}$$

Then $(\widetilde{x}_i)_{i=1,2}$ fulfills $0 \leq \widetilde{x}_i,\ \widetilde{x}_1 + \widetilde{x}_2 = Id$. Furthermore, for any strictly concave utility function U, we have:

$$E(U(\widetilde{x_1}(w)) - U(x_1(w))) > -\varepsilon \int_{K_1} U'(x_1 - \varepsilon) dF_w + \varepsilon \int_{K_2} U'(x_1 + \varepsilon) dF_w > 0$$

since $x_1(s) - \varepsilon > x_1(s') + \varepsilon,\ \forall s \in K_1,\ s' \in K_2$ and U' is decreasing and

$$E(U(\widetilde{x_2}(w)) - U(x_2(w))) > \varepsilon \int_{K_1} U'(x_2 + \varepsilon) dF_w - \varepsilon \int_{K_2} U'(x_2 - \varepsilon) dF_w > 0$$

which implies that $u_i(\widetilde{x_i}(w)) > u_i(x_i^*),\ i = 1,2$ contradicting the assumption that $(x_i^*)_{i=1,2}$ is Pareto Optimal. Hence $(x_i^*)_{i=1,2}$ must be comonotone. □

5.8 Proof of Proposition 3

Proof. Let (x_1^*, x_2^*) be Pareto Optimal allocation, as x_1^* and x_2^* are comonotone by Theorem 6, there exist non decreasing functions $x_i : [a,b] \to \mathbf{R}_+,\ i = 1,2$ such that $x_i^* = x_i(w)$. Since $x_1(w) + x_2(w) = w$, x_1, x_2 are 1-Lipschitz continuous.

There exists $\alpha \in [0,1]$ such that x_2^* solves

$$\begin{cases} \max \alpha E_f(U_2(X_2)) + (1-\alpha)E(U_1(w-X_2)) \\ \text{s.t.} \\ 0 \le X_2 \le w \end{cases}$$

If $\alpha = 0$ (respectively $\alpha = 1$) then $(x_1^*, x_2^*) = (w, 0)$ (respectively $(x_1^*, x_2^*) = (0, w)$) and there is nothing to prove. We therefore assume $\alpha \in (0,1)$. Let us prove then that for $i = 1, 2$, $x_i^* > 0$ Q-a.e. (equivalently $x_i > 0$ everywhere). Assume for instance that $x_2(c) = 0$ for some $c \in (a, b]$. Since x_2 is non decreasing this implies that $x_2 = 0$ on $[a, c]$. Let $\varepsilon > 0$ be small enough for the function $x_2^\varepsilon := x_2 + \varepsilon 1_{[a,c]}$ to satisfy $x_2^\varepsilon(w) \le w$ on $[a, b]$ and define $x_1^\varepsilon := w - x_2^\varepsilon$. By Corollary 3, $f'(1 - F_{x_2^\varepsilon(w)}(x_2^\varepsilon(w)))U'(x_2^\varepsilon(w))$ belongs to the superdifferential of $E_f(U_2)$ at $x_2^\varepsilon(w)$ so that:

$$E_f(U_2(x_2^\varepsilon(w))) - E_f(U_2(x_2^*)) \ge \int_{\{w \le c\}} f'(1 - F_{x_2^\varepsilon(w)}(\varepsilon))U_2'(\varepsilon)\varepsilon dQ.$$

Hence there exists a constant $c_0 > 0$ such that

$$E_f(U_2(x_2^\varepsilon(w))) - E_f(U_2(x_2^*)) \ge c_0 U_2'(\varepsilon)\varepsilon \tag{17}$$

Similarly, there exists a constant c_1 such that

$$E(U_1(x_1^\varepsilon(w))) - E(U_1(x_1^*)) \ge -c_1\varepsilon \tag{18}$$

Using (17), (18) and **U3**, we get that for $\varepsilon > 0$ small enough

$$\alpha E_f(U_2(x_2^\varepsilon(w))) + (1-\alpha)E(U_1(x_1^\varepsilon(w))) > \alpha E_f(U_2(x_2^*)) + (1-\alpha)E(U_1(x_1^*))$$

which yields the desired contradiction. One can prove similarly that $x_1^* > 0$ Q-a.e.

Let us prove now that (x_1, x_2) cannot be both strictly increasing. If it was so, since the constraints $0 \le x_2 \le w$ are not binding, then for any $y \in L^\infty$, one would have:

$$0 = \int_\Omega U_2'(x_2^*(\omega))f'(1 - F_{x_2^*}(x_2^*(\omega)))y(\omega)dQ - \alpha \int_\Omega U_1'(x_1^*(\omega))y(\omega)dQ$$

As $F_{x_2^*}(x_2^*) = F_w(w)$, we get

$$\alpha U_1'(x_1(w(\omega))) = f'(1 - F_w(w(\omega)))U_2'(x_2(w(\omega))),\ Q\text{-a.e.} \tag{19}$$

hence using **U4**, we have

$$\alpha U_1'(x_1(w)) = f'(1 - F_w(w))U_2'(x_2(w)) \text{ for all } w \in [F_w^{-1}(0), F_w^{-1}(1)] \tag{20}$$

Let $w \to F_w^{-1}(1)$, as $U_2'(x_2(F_w^{-1}(1)))$ is finite, $U_1'(x_1(w)) \to 0$, hence $x_1(w) \to \infty$, a contradiction. Hence x_2 cannot be increasing in a neighborhood of $F_w^{-1}(1)$.

Now note that in the general case where x_2 is only nondecreasing, then by Theorem 5 and Corollary 3 (20) is satisfied for every w which does not belong to a segment on which x_2 is constant. It remains to show that x_2 is constant in a neighbourhood of $F_w^{-1}(1)$; if not there would exist an increasing sequence w_n with limit $F_w^{-1}(1)$ and such that (20) holds for evey w_n. Passing to the limit as previously, we would have $x_1(w_n) \to +\infty$, hence a contradiction.

□

5.9 Proof of Proposition 4

Proof.

The first assertions of the proposition can be proved exactly as for Proposition 3. Assume (x_1^*, x_2^*) is a Pareto Optimal allocation with $x_i^* > 0,\ i = 1, 2$, by Theorem 6, they are comonotone, hence non decreasing functions of w, $x_1^* = x_1(w),\ x_2^* = x_2(w)$. Let us first show that (x_1, x_2) cannot be both strictly increasing. If they were, we would get as first order conditions,

$$\alpha f_1'(1 - F_w(w(\omega)))U_1'(x_1(w(\omega))) = f_2'(1 - F_w(w(\omega)))U_2'(x_2(w(\omega))) \quad (21)$$

hence, for all $w \in [F_w^{-1}(0), F_w^{-1}(1)]$:

$$\alpha f_1'(1 - F_w(w))U_1'(x_1(w)) = f_2'(1 - F_w(w))U_2'(x_2(w)) \quad (22)$$

As $w \to F_w^{-1}(1),\ f_2'(1 - F_w(w)) \to 0$. As $U_2'(x_2(F_w^{-1}(1)))$ is finite and $f_1'(1 - F_w(w)) \geq f_1'(0),\ U_1'(x_1(w)) \to 0$, hence $x_1(w) \to \infty$, a contradiction.

Let us now show that x_2 is constant in a neighbourhood of $F_w^{-1}(1)$. By Theorem 5 and Corollary 3, the first order conditions are:

$$\alpha f_1'(1 - s_1(\omega))U_1'(x_1^*(\omega)) = f_2'(1 - s_2(\omega))U_2'(x_2^*(\omega)) \quad (23)$$

with $s_i(\omega) = F_w(w(\omega))$ for all ω which does not belong to a level set of x_i^* with positive measure. If x_2 is not constant in a neighbourhood of $F_w^{-1}(1)$, then there exists an increasing sequence w_n with limit $F_w^{-1}(1)$ and such that for almost every $\omega \in \{w = w_n\}$, $s_2(\omega) = F_w(w_n)$. Passing to the limit, we would have $x_1(w_n) \to +\infty$, hence a contradiction. □

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