Boolean games revisited

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1. Introduction

2. Boolean games

3. Nash equilibria

4. Dominated strategies

5. Conclusion
1 Introduction

2 Boolean games

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5 Conclusion
- 2-players games with $p$ binary decision variables
- Each decision variable is controlled by only one player
- Zero-sum games
- Static games

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Example: Boolean $n$-players version of prisoners’ dilemma

- $n$ prisoners (denoted by 1, . . . , $n$).
- The same proposal is made to each of them:
  “Either you cover your accomplices ($C_i$, $i = 1, . . . , n$) or you denounce them ($\neg C_i$, $i = 1, . . . , n$).”
  - Denouncing makes you freed while your partners will be sent to prison (except those who denounced you as well; these ones will be freed as well),
  - But if none of you chooses to denounce, everyone will be freed.
Boolean \( n \)-players version of prisoners’ dilemma

- Normal form for \( n = 3 \):

\[
\begin{array}{ccc}
 & C_3 & \bar{C}_3 \\
1 & (1, 1, 1) & (0, 1, 0) \\
C_1 & (1, 0, 0) & (1, 1, 0) \\
C_2 & (0, 1, 1) & (0, 1, 1) \\
\bar{C}_1 & (1, 0, 1) & (1, 1, 1) \\
\bar{C}_2 & & \\
\end{array}
\]

- \( n \) prisoners: \( n \)-dimension matrix, therefore \( 2^n \) \( n \)-tuples must be specified.
**Boolean $n$-players version of prisoners’ dilemma**

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- Expressed much more compactly by Boolean game $G = (A, V, \pi, \Phi)$:

  - $A = \{1, \ldots, n\}$,
  - $V = \{C_1, \ldots, C_n\}$,
  - $\forall i \in \{1, \ldots, n\}$, $\pi_i = \{C_i\}$, and
  - $\phi = (\bigwedge_i C_i \vee \neg C_i)$.
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- $\forall i$, $i$ has 2 possible strategies: $s_{i_1} = \{C_i\}$ and $s_{i_2} = \{\overline{C_i}\}$
- the strategy $\overline{C_i}$ is a winning strategy for $i$.
- 8 strategy profiles for $G$
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- $s_-$ denotes the projection of $S$ on $A \setminus \{i\}$
- $S = \{C_1 C_2 C_3\}$. $s_1 = (C_2, C_3)$; $s_2 = (C_1, C_3)$; $s_3 = (C_1, C_2)$
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- A pure-strategy Nash equilibrium (PNE) is a strategy profile such as each player’s strategy is an optimal response to other players’ strategies. $S = \{s_1, \ldots, s_n\}$ is a PNE iff $\forall i \in \{1, \ldots, n\}, \forall s'_i \in 2^{\pi_i}, u_i(S) \geq u_i(s_{-i}, s'_i)$.

- 2 pure-strategy Nash equilibria: $C_1 C_2 C_3$ and $\overline{C}_1 \overline{C}_2 C_3$. 

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- 2 pure-strategy Nash equilibria: \( C_1 C_2 C_3 \) and \( \overline{C}_1 \overline{C}_2 \overline{C}_3 \)
Characterization

A strategy profile $S$ is a pure-strategy Nash equilibrium for a Boolean-game $G$ iff for all $i$, either

- $S \models \varphi_i$
- or $s_{-i} \models \neg \varphi_i$

Example: $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a \land \neg b$, $\varphi_2 = \neg a \land b$.

$G$ has 2 PNE:

- $S = ab$. We have $S \models \varphi_1$ and $(s_{-2} = a) \models \neg \varphi_2$.
- $S = \overline{a}b$. We have $(s_{-1} = b) \models \neg \varphi_1$ and $S \models \varphi_2$. 
Characterization

- $\exists i : \varphi_i =$ projection of $\varphi_i$ on variables of $\pi_{-i}$.
- $\exists i : \varphi_i$: obtained by forgetting in $\varphi_i$ all variables controlled by $i$.
- Example: $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a \land \neg b$, $\varphi_2 = \neg a \land b$.
  - $\exists 1 : \varphi_1 = (T \land \neg b) \lor (F \land \neg b) = \neg b$
  - $\exists 2 : \varphi_2 = (\neg a \land T) \lor (\neg a \land F) = \neg a$
Characterization

- \( \exists i : \varphi_i = \) projection of \( \varphi_i \) on variables of \( \pi_{-i} \).

- \( \exists i : \varphi_i : \) obtained by forgetting in \( \varphi_i \) all variables controlled by \( i \).

- Example: \( G = (A, V, \pi, \phi) \) with \( A = \{1, 2\} \), \( V = \{a, b\} \), \( \pi_1 = \{a\} \), \( \pi_2 = \{b\} \), \( \varphi_1 = a \land \neg b \), \( \varphi_2 = \neg a \land b \).
  - \( \exists 1 : \varphi_1 = (\top \land \neg b) \lor (\bot \land \neg b) = \neg b \)
  - \( \exists 2 : \varphi_2 = (\neg a \land \top) \lor (\neg a \land \bot) = \neg a \).
Characterization

- $\exists i : \varphi_i =$ projection of $\varphi_i$ on variables of $\pi_{-i}$.
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- Example: $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a \land \neg b$, $\varphi_2 = \neg a \land b$.
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Characterization

Simplification of the characterization

$S$ is a pure-strategy Nash equilibrium for $G$ if and only if

$$S \models \bigwedge_{i} (\varphi_i \lor \neg \exists i : \varphi_i)$$

Example: $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a \land \neg b$, $\varphi_2 = \neg a \land b$.

Recall that: $\exists 1 : \varphi_1 = \neg b$ and $\exists 2 : \varphi_2 = \neg a$.

$G$ has 2 PNE:

- $S = a\bar{b}$. We have $S \models \varphi_1 \land (\neg \exists 2 : \varphi_2)$.
- $S = \bar{a}b$. We have $S \models (\neg \exists 1 : \varphi_1) \land \varphi_2$. 
Complexity

**Nash equilibrium**

Deciding whether there is a pure-strategy Nash equilibrium in a Boolean game is $\Sigma^P_2$-complete. Completeness holds even under the restriction to two-players zero-sum games.

**Goals in DNF**

Let $G$ be a Boolean game. If every $\varphi_i$ is in DNF, then deciding whether there is a pure-strategy Nash equilibrium is NP-complete.

Completeness holds even if both we restrict the number of players to 2 and one player controls only one variable.
1 Introduction

2 Boolean games

3 Nash equilibria

4 Dominated strategies

5 Conclusion
**Boolean $n$-players version of prisoners’ dilemma**

- **Normal form for $n = 3$:**

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<tr>
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<th>$C_3$</th>
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- A strategy $s_i$ for player $i$ **strictly dominates** another strategy $s'_i$ if it does strictly better than it against all possible combinations of other players’ strategies: $\forall s_{-i} \in 2^{\pi-i}, u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i})$.

- $s_i$ **weakly dominates** $s'_i$ if it does at least as well against all possible combinations of other players’ strategies, and strictly better against at least one: $\forall s_{-i} \in 2^{\pi-i}, u_i(s'_i, s_{-i}) \leq u_i(s_i, s_{-i})$ and $\exists s_{-i} \in 2^{\pi-i}$ s.t. $u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i})$.

- Elimination of dominated strategies: $\overline{C}_1 \overline{C}_2 \overline{C}_3$
Boolean $n$-players version of prisoners’ dilemma

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- A strategy $s_i$ for player $i$ **strictly dominates** another strategy $s_i'$ if it does strictly better than it against all possible combinations of other players’ strategies: $\forall s_{-i} \in 2^{\pi-i}, u_i(s_i', s_{-i}) < u_i(s_i, s_{-i})$.

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- $s_i$ weakly dominates $s'_i$ if it does at least as well against all possible combinations of other players’ strategies, and strictly better against at least one: $\forall s_{-i} \in 2^{\pi-i}, u_i(s'_i, s_{-i}) \leq u_i(s_i, s_{-i})$ and $\exists s_{-i} \in 2^{\pi-i}$ s.t. $u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i})$.

- Elimination of dominated strategies: $\overline{C_1 \overline{C_2 \overline{C_3}}}$
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- A strategy \( s_i \) for player \( i \) **strictly dominates** another strategy \( s'_i \) if it does strictly better than it against all possible combinations of other players’ strategies: \( \forall s_{-i} \in 2^{\pi_{-i}}, \ u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i}). \)

- \( s_i \) **weakly dominates** \( s'_i \) if it does at least as well against all possible combinations of other players’ strategies, and strictly better against at least one: \( \forall s_{-i} \in 2^{\pi_{-i}}, \ u_i(s'_i, s_{-i}) \leq u_i(s_i, s_{-i}) \) and \( \exists s_{-i} \in 2^{\pi_{-i}} \) s.t. \( u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i}). \)

- Elimination of dominated strategies: \( \overline{C}_1 \overline{C}_2 \overline{C}_3 \)
Characterization

**Strict dominance**

Strategy $s_i$ **strictly dominates** strategy $s'_i$ if and only if:

- $s_i \models (\neg \exists -i : \neg \varphi_i)$ and
- $s'_i \models (\neg \exists -i : \varphi_i)$.

Example: $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a$, $\varphi_2 = \neg a \land b$. Strategy $s_1 = a$ strictly dominates $s'_1 = \overline{a}$. We compute: $\exists -1 : \varphi_1 = a$ and $\exists -1 : \neg \varphi_1 = \neg a$, and we have:

- $s_1 \models \neg (\neg a)$.
- $s'_1 \models \neg a$. 
Characterization

**Weak dominance**

Strategy $s_i$ weakly dominates strategy $s'_i$ if and only if:

- $(\varphi_i)_{s'_i} \models (\varphi_i)_{s_i}$ and
- $(\varphi_i)_{s_i} \not\models (\varphi_i)_{s'_i}$.

Example: $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a$, $\varphi_2 = \neg a \land b$.

Strategy $s_2 = b$ strictly dominates $s'_2 = \overline{b}$. We compute: $(\varphi_2)_{s'_2} = \bot$, $(\varphi_2)_{s_2} = \neg a$, and we have:

- $\bot \models \neg a$
- $\neg a \not\models \bot$
Complexity

Nash equilibrium

Deciding whether a given strategy $s'_i$ is weakly dominated is $\Sigma^p_2$-complete.

Hardness holds even if $\varphi_i$ is restricted to be in DNF.
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Extension of Harrenstein and al.'s Boolean games:

- Arbitrary number of players;
- Non zero-sum games;
- Characterization of Nash equilibria and dominated strategies;
- Computational complexity of the related problems;
The companion paper (PRICAI’06) considers extended Boolean games with ordinal preferences represented by prioritized goals and CP-nets with binary variables;

Computing *mixed strategy Nash equilibria* for Boolean games;

Defining and studying *dynamic* Boolean games (with complete or incomplete information).