Boolean games revisited

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Introduction

Boolean games as introduced in Harrenstein, Van der Hoek, Meyer, Witteveen (2001, 2004)

- 2-players games with p binary decision variables
- Each decision variable is controlled by only one player
- Zero-sum games
- Static games

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Example: Boolean n-players version of prisoners' dilemma

- n prisoners (denoted by $1, \ldots, n$).
- The same proposal is made to each of them:
  "Either you cover your accomplices (C_i, $i = 1, \ldots, n$) or you denounce them ($\neg C_i$, $i = 1, \ldots, n$)."
  - Denouncing makes you freed while your partners will be sent to prison (except those who denounced you as well; these ones will be freed as well),
  - But if none of you chooses to denounce, everyone will be freed.

Boolean n-players version of prisoners' dilemma

- Normal form for $n = 3$:

$3: C_3$		
1 \ 2	C_2	$\overline{C}_2$
C_1	(1, 1, 1)	(0, 1, 0)
$\overline{C}_1$	(1, 0, 0)	(1, 1, 0)

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- n prisoners : n-dimension matrix, therefore 2^n n-tuples must be specified.

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  - $A = \{1, \ldots, n\}$,
  - $V = \{C_1, \ldots, C_n\}$,
  - $\forall i \in \{1, \ldots, n\}, \pi_i = \{C_i\}$, and
  - $\varphi_i = (C_1 \wedge C_2 \wedge \ldots C_n) \vee \neg C_i$.

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- $\forall i$, i has 2 possible strategies: $s_{i_1} = \{C_i\}$ and $s_{i_2} = \{\overline{C_i}\}$
- the strategy $\overline{C}_i$ is a winning strategy for i.
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- $S = \{C_1 C_2 C_3\}$. $s_{-1} = (C_2, C_3)$; $s_{-2} = (C_1, C_3)$; $s_{-3} = (C_1, C_2)$

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- 2 pure-strategy Nash equilibria: $C_1 C_2 C_3$ and $\overline{C_1}\,\overline{C_2}\,\overline{C_3}$

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Characterization

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A strategy profile S is a pure-strategy Nash equilibrium for a Boolean-game G iff for all i, either

- $S \models \varphi_i$
- or $s_{-i} \models \neg\varphi_i$

Example : $G = (A, V, \pi, \phi)$ with $A = \{1,2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a \wedge \neg b$, $\varphi_2 = \neg a \wedge b$.
G has 2 PNE:

- $S = a\bar{b}$. We have $S \models \varphi_1$ and $(s_{-2} = a) \models \neg\varphi_2$.
- $S = \bar{a}b$. We have $(s_{-1} = b) \models \neg\varphi_1$ and $S \models \varphi_2$.

Characterization

- $\exists i : \varphi_i$ = projection of φ_i on variables of π_{-i}.
- $\exists i : \varphi_i$: obtained by forgetting in φ_i all variables controlled by i.
- Example : $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a \wedge \neg b$, $\varphi_2 = \neg a \wedge b$.
  - $\exists 1 : \varphi_1 = (\top \wedge \neg b) \vee (\bot \wedge \neg b) = \neg b$
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Characterization

Simplification of the characterization

S is a pure-strategy Nash equilibrium for G if and only if

$$S \models \bigwedge_i (\varphi_i \vee (\neg \exists i : \varphi_i))$$

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Recall that: $\exists 1 : \varphi_1 = \neg b$ and $\exists 2 : \varphi_2 = \neg a$.
G has 2 PNE:

- $S = a\bar{b}$. We have $S \models \varphi_1 \wedge (\neg \exists 2 : \varphi_2)$.
- $S = \bar{a}b$. We have $S \models (\neg \exists 1 : \varphi_1) \wedge \varphi_2$.

Complexity

Nash equilibrium

Deciding whether there is a pure-strategy Nash equilibrium in a Boolean game is Σ_2^p-**complete**. Completeness holds even under the restriction to two-players zero-sum games.

Goals in DNF

Let G be a Boolean game. If every φ_i is in DNF, then deciding whether there is a pure-strategy Nash equilibrium is **NP-complete**.

Completeness holds even if both we restrict the number of players to 2 and one player controls only one variable.

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- A strategy s_i for player i **strictly dominates** another strategy s'_i if it does strictly better than it against all possible combinations of other players' strategies: $\forall s_{-i} \in 2^{\pi_{-i}}$, $u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i})$.
- s_i **weakly dominates** s'_i if it does at least as well against all possible combinations of other players' strategies, and strictly better against at least one: $\forall s_{-i} \in 2^{\pi_{-i}}$, $u_i(s'_i, s_{-i}) \leq u_i(s_i, s_{-i})$ and $\exists s_{-i} \in 2^{\pi_{-i}}$ s.t. $u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i})$.
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$\overline{C}_1$	(1, 0, 1)	(1, 1, 1)

- A strategy s_i for player i **strictly dominates** another strategy s'_i if it does strictly better than it against all possible combinations of other players' strategies: $\forall s_{-i} \in 2^{\pi_{-i}}$, $u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i})$.
- s_i **weakly dominates** s'_i if it does at least as well against all possible combinations of other players' strategies, and strictly better against at least one: $\forall s_{-i} \in 2^{\pi_{-i}}$, $u_i(s'_i, s_{-i}) \leq u_i(s_i, s_{-i})$ and $\exists s_{-i} \in 2^{\pi_{-i}}$ s.t. $u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i})$.
- Elimination of dominated strategies: $\overline{C}_1 \overline{C}_2 \overline{C}_3$

Characterization

Strict dominance

Strategy s_i **strictly dominates** strategy s'_i if and only if:

- $s_i \models (\neg\exists - i : \neg\varphi_i)$ and
- $s'_i \models (\neg\exists - i : \varphi_i)$.

Example : $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a$, $\varphi_2 = \neg a \wedge b$.
Strategy $s_1 = a$ strictly dominates $s'_1 = \overline{a}$. We compute: $\exists - 1 : \varphi_1 = a$ and $\exists - 1 : \neg\varphi_1 = \neg a$, and we have:

- $s_1 \models \neg(\neg a)$.
- $s'_1 \models \neg a$.

Characterization

Weak dominance

Strategy s_i **weakly dominates** strategy s'_i if and only if:

- $(\varphi_i)_{s'_i} \models (\varphi_i)_{s_i}$ and
- $(\varphi_i)_{s_i} \not\models (\varphi_i)_{s'_i}$.

Example : $G = (A, V, \pi, \phi)$ with $A = \{1, 2\}$, $V = \{a, b\}$, $\pi_1 = \{a\}$, $\pi_2 = \{b\}$, $\varphi_1 = a$, $\varphi_2 = \neg a \wedge b$.
Strategy $s_2 = b$ strictly dominates $s'_2 = \overline{b}$. We compute : $(\varphi_2)_{s'_2} = \bot$, $(\varphi_2)_{s_2} = \neg a$, and we have:

- $\bot \models \neg a$
- $\neg a \not\models \bot$

Complexity

Nash equilibrium

Deciding whether a given strategy s'_i is weakly dominated is Σ_2^p-**complete**.

Hardness holds even if φ_i is restricted to be in DNF.

Conclusion

Extension of Harrenstein and al.'s Boolean games:

- Arbitrary number of players;
- Non zero-sum games;
- Characterization of Nash equilibria and dominated strategies;
- Computational complexity of the related problems;

Perspectives

- The companion paper (PRICAI'06) considers extended Boolean games with ordinal preferences represented by prioritized goals and CP-nets with binary variables;
- Computing *mixed strategy Nash equilibria* for Boolean games;
- Defining and studying *dynamic* Boolean games (with complete or incomplete information).