Minmax regret 1-center problem on a network with a discrete set of scenarios

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Abstract

We consider the minmax regret 1-center problem on a general network with uncertainty on demands. Unlike other papers which considered the problem under interval uncertainty, we assume here that demand uncertainty is modelled by a finite number of discrete scenarios. We develop a solution approach based on the decomposition of the network into basic intervals and present a polynomial algorithm for the problem. Finally, we illustrate our approach with an example.

Key words : Robustness, Uncertainty modelling, 1-center location problem.

1 Introduction

In public service oriented siting problems, decision makers have to decide about the location of public services (offices, schools, hospitals, ambulance services, etc.), while emphasizing on the accessibility of people to services or services to people. In the case of most medical emergencies, the risk of loss of life increases with response time [17]. Consequently, one reasonable objective is to minimize response time given a limited budget.

The *absolute 1-center* problem is to locate a facility on a network so as to minimize the maximal response time to any customer, that is to find a point on the network that minimizes the maximum of the weighted distances between that point and the nodes (see, e.g., [11, 12, 14, 19, 20, 23]). This model is suitable, for example, for the location of a hospital on a network whose nodes represent a certain number of small cities or the location of an ambulance station in a campus with different buildings [8]. Distances

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are computed as shortest paths in the network and the weights of the nodes represent numbers of customers residing at the nodes (cities populations and students in buildings in the previous examples).

The absolute 1-center problem was first defined and solved by Hakimi [12] in 1964. Hakimi *et al.* [13] implemented Hakimi's method in $O(mn^2 \log n)$ time for the weighted case and $O(mn \log n)$ for the unweighted case, m being the number of edges of the graph and n the number of nodes. Further refinements of the procedure were obtained by Kariv and Hakimi [15], reducing the algorithms complexity to $O(mn \log n)$ for the weighted case and $O(mn)$ for the unweighted case.

In practice, model parameters are usually uncertain and several scenarios have to be considered since inputs are based on forecasts. In such a context, uncertain data render inappropriate the search of optimal solutions and require the use of robustness analysis ([21, 24]). Unlike deterministic or stochastic approaches which are aimed at determining the best solution for one instance of values (or scenario), robust approaches try to find a solution or a set of solutions that is acceptable for any considered scenario. In combinatorial optimization and particularly in location problems, the most used robustness criterion rely on *maximal regret* (see, e.g., [1, 5, 6, 9, 16]) : a robust solution is one that minimizes the maximal regret among all scenarios. We recall that the *regret* is the difference between the resulting output under a given scenario and the best possible output under the same scenario.

The *minmax regret 1-center* problem was considered by many authors in the context of interval data uncertainty, that is when uncertain weights and/or uncertain distances are represented by intervals. In the case of uncertain weights, Averbakh and Berman [3] developed an $O(mn^2 \log n)$ algorithm for the problem on a general network, the complexity is reduced to $O(n^2)$ for the problem on a tree [4]. Uncertainty on edge lengths makes the minmax regret 1-center problem on a network NP-hard [2]. On the other hand, for a tree, the problem is polynomially solvable in the case of uncertain node weights and uncertain edge lengths. Averbakh and Berman [4] presented an $O(n^6)$ algorithm for a weighted tree and an $O(n^2 \log n)$ algorithm for an unweighted tree. In [7], Burkard and Dollani reduced the complexity of these latter algorithms to $O(n^3 \log n)$ and $O(n \log n)$ respectively.

In all these papers, possible values of the parameters are represented by intervals, which is suitable when there is an imprecise knowledge about the possible outcomes. In many practical situations, however, uncertainty deals with the presence of several contrasting futures. In such a case, experts mainly use a discrete set of well-defined scenarios in order to represent future alternatives ([10, 22]). In the case of the 1-center problem on a network, nodes can represent cities and their associated weights the populations. To model future demands at different nodes, a decision-maker will represent possible trends of the demographic evolution of the different cities through discrete scenarios instead of intervals.

In this paper, we consider the minmax regret 1-center problem on a network under uncertain demands. We assume that nodes demands (or weights) are modelled by a finite set S of possible scenarios, q being the number of scenarios. The paper is organized as follows. In section 2, we introduce the minmax regret 1-center problem under scenario-based uncertainty. In section 3, we explain our solution approach and we present a polynomial algorithm in $O(mnq^2)$ time. We illustrate our approach with an example in section 4 and summarize our work in the final section.

2 Notations and problem formulation

Let $G = (V, E)$ be a graph composed of a set $V = \{v_i, i = 1, \dots, n\}$ of n nodes (or vertices) and a set E of m edges. We denote by $d(a, b)$ the minimum distance between two points a and b of G. A *point* of the graph corresponds either to a node or to any point along an edge. The length of each edge $e \in E$ is denoted by d_e. We assume that the matrix of shortest distances between nodes of G is given. We also assume that demands occur only at the nodes of the network and that they can be characterized by a weight vector $W = (w_1, w_2, \dots, w_n)$ where w_i is the weight associated with node v_i for $i = 1, \dots, n$.

For a given point $x \in G$, the maximal weighted distance between x and all the nodes of G, also called *cost* of x, is denoted by $D(x)$. We have:

$$D(x) = \max_{1 \leq i \leq n} w_i d(x, v_i) \tag{1}$$

The *absolute 1-center* problem defined above is then formulated as follows :

$$\min_{x \in G} D(x) \tag{2}$$

We remind that, on a general network, the distance $d(x, v_i)$ between a point x varying on an edge e and a given node v_i has three possible plots as shown in Figure 1. The third

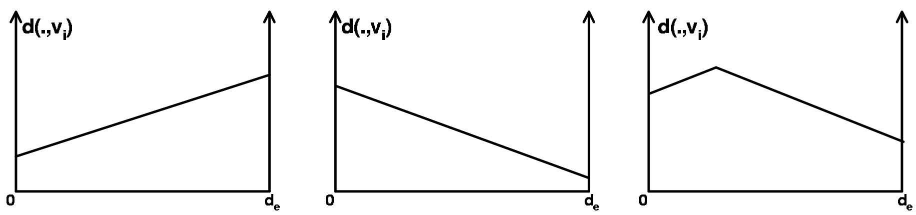

Figure 1: Plots of $d(., v_i)$ on a given edge e

case occurs when the shortest path from v_i to the first endpoint of e is different from the shortest path from v_i to the second endpoint of e. Therefore, the upper envelope $D(.)$, defined in equation (1), is piecewise linear and continuous on each edge of G.

Let us assume that the weights can take many different values and that there is a discrete and finite set S of possible scenarios (possible values of the parameters). For a given scenario s and a point x of G, the cost of solution x under scenario s is defined as follows:

$$D^s(x) = \max_{1 \leq i \leq n} w_i^s d(x, v_i) \tag{3}$$

where w_i^s denotes the weight of node v_i under scenario s. The regret of solution x (also called *opportunity loss* or *absolute deviation* [16]) is the difference between the cost of x under scenario s and the cost of the best solution under the same scenario:

$$R^s(x) = D^s(x) - D^s(x^{*s}) \tag{4}$$

where x^{*s} is the *absolute center* (the optimal solution of the 1-center problem) under scenario s. The *minmax regret 1-center* problem under scenario-based uncertainty is formulated as:

$$\min_{x \in G} \max_{s \in S} R^s(x) \tag{5}$$

In the following section, we present a solution method for solving (5).

3 Solution method

3.1 Principle

We describe here the different steps of our approach to solve the minmax regret 1-center problem under scenario-based uncertainty:

- We first use the Kariv and Hakimi's algorithm [15] to solve q classical 1-center problems corresponding to the q scenarios.
- Then, we decompose each edge of the network into intervals called basic intervals.
- For each basic interval, we determine a local solution of the problem, using a procedure developed by Kouvelis and Yu [16].
- Finally, the minmax regret 1-center is determined among all the local solutions given by the previous step.

The main idea of Kariv and Hakimi's algorithm is to determine, for a fixed scenario $s \in S$, the upper envelope D_e^s for each edge e of the network and find a best local minimum of D_e^s on e. The absolute center x^{*s} under scenario s is the best among the m local minima found. This method takes advantage of the piecewise linearity and continuity of function $D_e^s(.)$ on each edge $e \in E$ (see Figure 2). We recall that function $D_e^s(.)$ has the following expression:

$$D_e^s(x) = \max_{1 \le i \le n} w_i^s d(x, v_i) \quad \text{for all } x \in e \tag{6}$$

For a given edge $e \in E$ and a fixed scenario $s \in S$, function $D_e^s(.)$ has at most $3n$ breakpoints [15]. The list of the abscissas of the points where $D_e^s(.)$ breaks ($t_0^s, \ldots, t_6^s$ in Figure 2), as well as the values of the function at these points are determined by Kariv and Hakimi's algorithm.

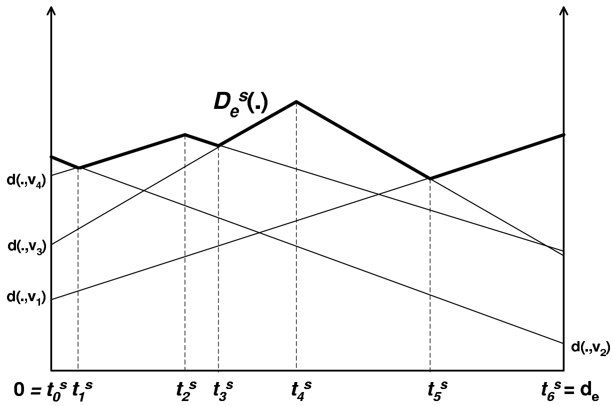

Figure 2: The upper envelope $D_e^s(.)$ for a given scenario s and a given edge e

We denote by $[a, b]$ a subinterval of an edge e such that a and b are the abscissas of two distinct breakpoints and there is no breakpoint between a and b. Points a and b are two breakpoints of either the same function $D_e^s(.)$ or two distinct functions $D_e^s(.)$ and $D_e^{s'}(.)$ (see Figure 3). We call such a subinterval a *basic interval* (term borrowed from Averbakh and Berman in [5]). Let us notice that, for every scenario s, the breakpoints of the regret function have the same abscissas than those of the cost function. Hence, basic intervals can be equally determined from cost function or from regret function. Given the number of breakpoints and the number of scenarios, we have at most $3nq$ basic intervals on each edge.

In order to find the robust center of G, we propose to determine, for every basic interval $[a, b]$, the point which minimizes the maximal regret function:

$$R_e(.) = \max_{s \in S} R_e^s(.) = \max_{s \in S} \{D_e^s(.) - D^s(x^{*s})\} \tag{7}$$

We call such a point a *local robust center* and denote it by t_{ab}^*. The robust center of the graph will be the one, among the $O(mnq)$ local robust centers, which minimizes the maximal regret. In the next subsection, we show how to find a local robust center on a given basic interval.

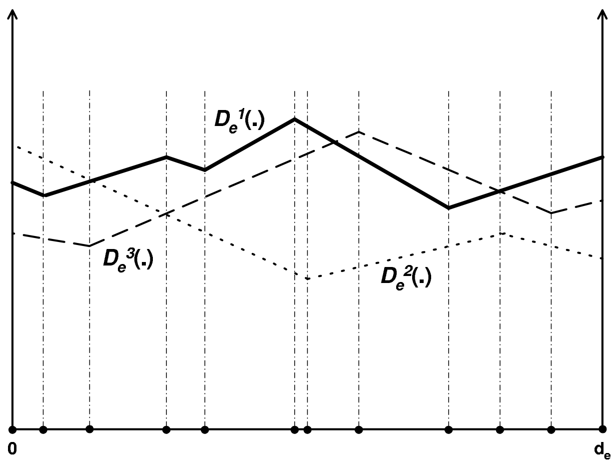

Figure 3: Representation of the basic intervals in the case of three scenarios

3.2 The local robust center

Let $[a, b]$ be a basic interval of an edge $e = (v_i, v_j)$ of G. For each point x on $[a, b]$, let t denote the distance from x to node v_i along edge e. For a fixed scenario s, function $D_e^s(.)$ is linear on $[a, b]$, by definition of a basic interval. Consequently, it can be written as follows:

$$D_e^s(t) = D_e^s(a) + \frac{D_e^s(b) - D_e^s(a)}{b - a}(t - a),\ \forall t \in [a, b] \tag{8}$$

If a and b are two breakpoints of the same function $D_e^s(.)$, then values $D_e^s(a)$ and $D_e^s(b)$ are given by Kariv and Hakimi's algorithm. Else, a and b are two breakpoints of either another function $D_e^{s'}(.)$, or two different functions $D_e^{s'}(.)$ and $D_e^{s''}(.)$, or one of $D_e^s(.)$ and the other of a different function $D_e^{s'}(.)$ (see Figure 3). Assume, for instance, that a is a breakpoint of $D_e^s(.)$ and b a breakpoint of another function $D_e^{s'}(.)$. Let c be the breakpoint of $D_e^s(.)$ adjacent to a and such that $b \in [a, c]$. Then, value $D_e^s(b)$ is :

$$D_e^s(b) = D_e^s(a) + \frac{D_e^s(c) - D_e^s(a)}{c - a}(b - a). \tag{9}$$

The other cases are processed similarly.

Hence, the regret $R_e^s(.)$ on $[a, b]$ under scenario s has the following form:

$$R_e^s(t) = (D_e^s(a) - D^s(x^{*s})) + \frac{D_e^s(b) - D_e^s(a)}{b - a}(t - a),\ \forall t \in [a, b] \tag{10}$$

where x^{*s} is the 1-center of G under scenario s. We see that $R_e^s(.)$ is a linear function in t on $[a, b]$ with slope:

$$\mu_e^s = \frac{D_e^s(b) - D_e^s(a)}{b - a}, \tag{11}$$

and y-intercept:

$$\lambda_e^s = D_e^s(a) - D^s(x^{*s}) - a\frac{D_e^s(b) - D_e^s(a)}{b - a}. \quad (12)$$

Consequently, the maximal regret function $R_e(.)$ on the basic interval $[a, b]$ is given by:

$$R_e(t) = \max_{s \in S}(\lambda_e^s + \mu_e^s t), \ \forall t \in [a, b] \quad (13)$$

Then, an equivalent way to state the minmax regret 1-center problem is :

$$\min_{\substack{[a,b] \subset e \\ e \in E}} \min_{a \leq t \leq b} R_e(t) \quad (14)$$

For a given basic interval $[a, b]$, we call *local robust center* and denote by t^*_{ab}, the solution of :

$$\min_{a \leq t \leq b} R_e(t) \quad (15)$$

In order to determine t^*_{ab}, we propose to use the procedure *FindMin(i,j)* developed by Kouvelis and Yu in the case of the robust 1-median problem on a tree [16]. We just remark that Kouvelis and Yu apply their procedure to an edge (v_i, v_j) while we apply it to a basic interval $[a, b]$. We outline hereafter a brief description of this procedure. Function $R_e(.)$ is piecewise linear and convex on a basic interval $[a, b]$ since it is the upper envelope of a set of linear functions. The point t^*_{ab} is then either on a, or on b or is the intersection of two linear functions. In the last case, only two linear functions are needed in specifying t^*_{ab} and the procedure proceeds by successively eliminating lines that are irrelevant for determining t^*_{ab}. Undeleted lines are stored in a list whose cardinality reduces as the procedure progresses. In the end, we get a list with two lines and the intersection of these two lines determines t^*_{ab}. For a thorough description as well as an algorithmic presentation of the procedure *FindMin(i,j)*, we invite the reader to consult chapter 4 of [16].

3.3 Algorithm

The following algorithm finds the minmax regret 1-center on a general network under scenario-based uncertainty on weights.

Algorithm

1. Preprocessing (*using Kariv and Hakimi's algorithm*)
  For all $s \in S$ **do**
    For all $e \in E$ **do**
      Compute the breakpoints of function $D_e^s(.)$;
    Determine x^{*s} and compute $D^s(x^{*s})$;

2. Local robust centers

For all $e \in E$ **do**

Determine all basic intervals $[a, b]$ of e;

For all basic intervals $[a, b]$ of e **do**

Compute μ_e^s and λ_e^s using (11) and (12);

Determine the point t_{ab}^* solution of (15) (*using Kouvelis and Yu's procedure*);

3. Robust center

For all $e \in E$ **do**

Determine the point t_e^* that minimizes $R_e(.)$ among all local robust centers on e;

The robust center is $x^* = \arg\min_{e\in E}(R_e(t_e^*))$.

Theorem 1 *Minmax regret 1-center problem on a general graph under scenario-based uncertainty on demand can be solved in* $O(mnq(\log n + q))$ *time.*

Proof:
Kariv and Hakimi's algorithm is in $O(mn \log n)$ time, so preprocessing phase can be performed in $O(qmn \log n)$ time.
Since there are at most $3nq$ basic intervals on each edge and m edges on the graph, there will be $O(mnq)$ local robust centers on G. As Kouvelis and Yu's procedure is in $O(q)$ time, phase 2 of the algorithm requires $O(mnq^2)$ elementary operations.
Phase 3 has a complexity of $O(mnq)$ since the robust center of G is chosen among the local robust centers found in the second phase. □

In the special case of a tree network, the minmax regret 1-center problem becomes easier to solve since the distance $d(x, v_i)$ between a point x varying on an edge e and a given node v_i is linear on e (corresponding to the first two cases of Figure 1). Thus, for a fixed scenario s and a given edge e, function $D_e^s(.)$ (and also the regret $R_e^s(.)$), which is the upper envelope of n linear functions, is a convex function. Hence, the maximal regret function $R_e(.)$ defined on edge e is also convex since it is the upper envelope of q convex functions. Therefore, $R_e(.)$ is the upper envelope of nq linear functions and a local robust center t_e^* can be determined on each edge e in $O(nq)$ time using Kouvelis and Yu's procedure. The minmax regret 1-center of the tree will correspond to the best among the $n-1$ local robust centers found. Consequently, using the $O(n \log n)$ algorithm proposed in [15] for the classical 1-center problem on a tree in the preprocessing phase, we have the following result:

Theorem 2 *Minmax regret 1-center problem on a tree under scenario-based uncertainty on demand can be solved in* $O(nq(n + \log n))$ *time.*

4 An illustrative example

Consider the graph G of Figure 4 where values on edges represent lengths. Uncertainty on node weights is modelled by three scenarios as shown in table 1.

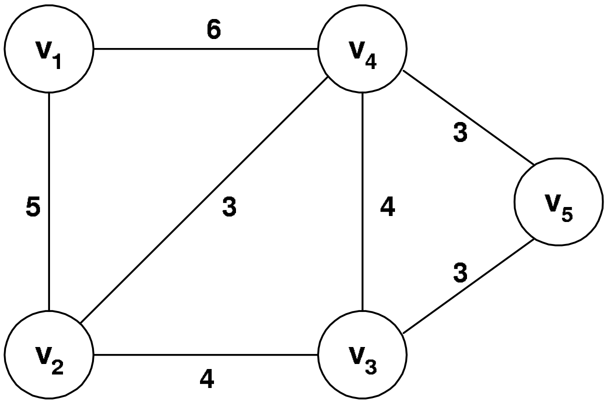

Figure 4: Example of a network

weights	v_1	v_2	v_3	v_4	v_5
s_1	10	20	10	15	10
s_2	10	15	20	10	10
s_3	10	10	15	20	10

Table 1: Weights under the three scenarios

The absolute center under scenario s_1 is the point x^{*1} of edge (v_2, v_4) at a distance 0.5 from node v_2. The absolute center under scenario s_2 is the point x^{*2} of edge (v_2, v_3) at a distance 1 from node v_2. Under scenario s_3, there are two absolute centers: the nodes v_2 and v_4 (see Figure 6). As an example, let us consider edge (v_2, v_3). Figure 5 shows the regret functions $R_e^i(.)$ on edge (v_2, v_3) under scenarios $s_i, i = 1, 2, 3$. There are seven basic intervals on this edge, which are $[0, 0.5]$, $[0.5, 1]$, $[1, 2.5]$, $[2.5, 2.8]$, $[2.8, 3]$, $[3, 3.67]$ and $[3.67, 4]$. Local robust centers on these basic intervals are $t_1^* = 0.5$, $t_2^* = 0.5$, $t_3^* = 1$, $t_4^* = 2.8$, $t_5^* = 3$, $t_6^* = 3.67$ and $t_7^* = 3.875$. The values of the maximal regret R_e at these points are $R_e(t_1^*) = R_e(t_2^*) = 10$, $R_e(t_3^*) = 20$, $R_e(t_4^*) = 44$, $R_e(t_5^*) = 40$, $R_e(t_6^*) = 26.6$ and $R_e(t_7^*) = 22.5$. Thus, the point which minimizes the maximal regret on edge (v_2, v_3) is t_1^*. Applying the same reasoning to the other edges gives that the robust center of G is the point x^* of (v_2, v_3) at a distance 0.5 from node v_2 (see Figure 6). The minmax regret is 10.

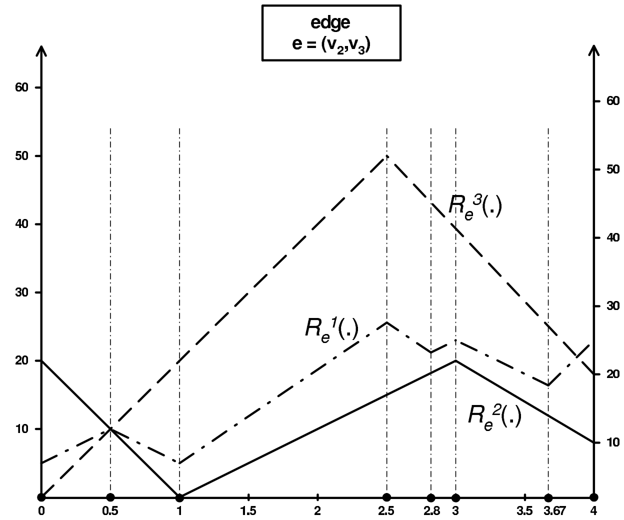

Figure 5: Regret functions and basic intervals on edge (v_2, v_3)

5 Conclusions

Robustness analysis is aimed at finding solutions or recommandations in a context where the imprecise, uncertain and generally badly known parameters of a problem make inappropriate the search of optimal solutions [21]. Imprecision is traditionally dealt with through the use of intervals, that is an imprecise parameter can take on any value between a lower and an upper bounds. On the other hand, uncertainty is modelled by a discrete and finite set of scenarios, since in real life, experts mainly use a discrete set of scenarios to represent future alternatives, rather than interval representations.

In this paper, we studied the minmax regret 1-center problem on a network with uncertain node weights. We developed a solution approach based on the decomposition of the network into basic intervals and built a polynomial algorithm using Kariv and Hakimi's algorithm [15], as well as a procedure developed by Kouvelis and Yu in [16]. We considered the case of a tree network and showed that the algorithm is simpler to implement in this case.

Another important robustness criterion relies on minimizing the maximal cost. The *minmax 1-center* problem on a network with a discrete set of scenarios on demands is easily solvable. Indeed, it can be restricted to a deterministic 1-center problem where each node weight is equal to its maximal value among all scenarios.

Finally, we would like to notice that, in the presence of a discrete set of scenarios, minmax regret and minmax cost criteria add some more complexity to the deterministic version of a problem, that is why it is reasonable to apply these robustness approaches only to problems which remain polynomially solvable in a deterministic context. One

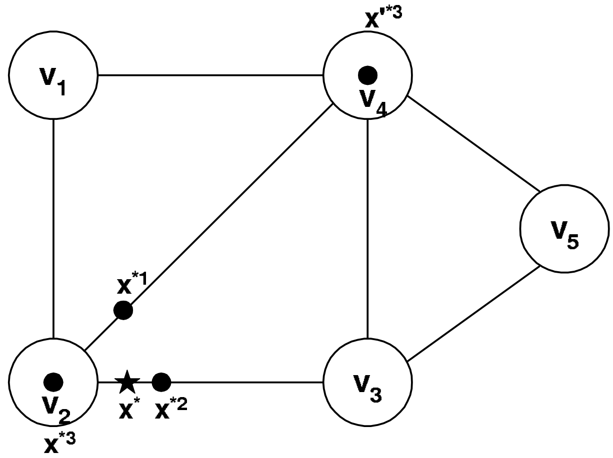

Figure 6: Absolute centers under the three scenarios and the robust center of G.

possible perspective is to study the p-center problem on a tree under scenario-based uncertainty. The deterministic version of this problem can be solved in polynomial time ([15, 18]).

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