A note on the hardness results for the labeled perfect matching problems in bipartite graphs

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Abstract

In this note, we strengthen the inapproximation bound of $O(\log n)$ for the labeled perfect matching problem established in J. Monnot, The Labeled perfect matching in bipartite graphs, *Information Processing Letters* **96** (2005) 81-88, using a self improving operation in some hard instances. It is interesting to note that this self improving operation does not work for all instances. Moreover, based on this approach we deduce that the problem does not admit constant approximation algorithms for connected planar cubic bipartite graphs.

Keywords: labeled matching; bipartite graphs; approximation and complexity; inapproximation bounds.

1 Introduction

A matching M on a graph $G = (V, E)$ is a subset of edges that are pairwise non adjacent; M is said perfect if it covers the vertex set V of G . In the labeled perfect matching problem (LBELED *Min PM* in short), we are given a simple graph $G = (V, E)$ on $|V| = 2n$ vertices which contains a perfect matching together with a color (or label) function $\mathcal{L} : E \rightarrow \{c_0, \dots, c_q\}$ on the edge set of G . For $i = 0, \dots, q$, we denote by $\mathcal{L}_i \subseteq E$ the set of edges of color c_i . The goal of LBELED *Min PM* is to find a perfect matching on G that uses a minimum number of colors. Alternatively, if $G[\mathcal{L}'] = (V, E')$ where $E' = \{e \in E : \mathcal{L}(e) \in \mathcal{L}'\}$ denotes the subgraph induced by the edges of colors $\mathcal{L}' \subseteq \{c_0, \dots, c_q\}$, then LBELED *Min PM* aims at finding a subset $\mathcal{L}'$ of minimum size such that $G[\mathcal{L}']$ contains a perfect matching. In [9] a generalization, called perfect matching under categorization, has been studied. In this framework, each edge e has also a non-negative weight $w(e)$, and the colors are called categories (thus, $q + 1$ indicates the number of categories). The goal is to find a perfect matching M of E minimizing $\sum_{i=0}^q \max_{e \in \mathcal{L}_i \cap M} w(e)$. In [9], it is shown that, on the one hand, the problem is polynomial when the number of categories (i.e., colors) is fixed, and on the other hand, the problem is **NP**-hard when the weights take values 0 or 1 and the graph is a collection of disjoint 4-cycles. Note that the case $w(e) = 1, \forall e \in E$ corresponds to LBELED *Min PM*. Very recently, some approximation results are obtained for LBELED *Min PM* when the graphs are bipartite 2-regular or complete bipartite $K_{n,n}$, [7]. In particular, it is shown that the 2-regular bipartite case is

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equivalent to the minimum satisfiability problem, and that a greedy algorithm picking at each iteration a monocolored matching of maximum size provides a $\frac{r+H_r}{2}$ -approximation in complete bipartite graphs where r is the maximum of times that a color appears in the graph and H_r is the r -th harmonic number. Moreover, it is proved that **LABELED Min PM** is not $O(\log n)$ -approximable in bipartite complete graphs. In [6], this problem is motivated by some applications in timetable problems. Several related works concerning some matching problems on colored graphs can be found in [3, 4, 5]

In this note, we prove first that **LABELED Min PM** is not in **APX** whenever the bipartite graphs have a maximum degree of 3. Hence, there is a gap of approximability between graphs of maximum degree 2 and 3 since we can easily deduce from [7] that **LABELED Min PM** is 2-approximable in bipartite graphs of maximum degree 2. Using a weaker complexity hypothesis, we can even obtain that **LABELED Min PM** is not $2^{O(\log^{1-\varepsilon} n)}$ -approximable in bipartite graphs of maximum degree 3 on n vertices, unless **NP** $\subseteq$ **DTIME** $(2^{O(\log^{1/\varepsilon} n)})$. Dealing with the unbounded degree case, this yields to the fact that **LABELED Min PM** is not in **polyLog-APX**, unless **P** = **NP**.

In the following, given an instance I , we denote by $opt(I)$ and $apx(I)$ the value of an optimal and an approximate solution, respectively for **LABELED Min PM**. We say that an algorithm $\mathcal{A}$ is a ρ -approximation (with $\rho \geq 1$) if $apx(I) \leq \rho \times opt(I)$ for any instance I .

Finally, in order to simplify the proofs exposed in the rest of the paper, the results concern a variation of **LABELED Min PM**, where the value of each perfect matching M is given by $val_1(M) = val(M) - 1$. This problem is denoted **LABELED Min PM₁** and we have for any instance I , $apx_1(I) = apx(I) - 1$ and $opt_1(I) = opt(I) - 1$. It is important to note that a $\rho(n)$ -approximation of **LABELED Min PM** becomes a $2\rho(n)$ -approximation of **LABELED Min PM₁**, and conversely a $\rho(n)$ -approximation of **LABELED Min PM₁** remains a $\rho(n)$ -approximation of **LABELED Min PM**. Actually, since **LABELED Min PM** is simple, [8] (i.e., the restriction to $opt(I) \leq k$ is polynomial), we can see that **LABELED Min PM** and **LABELED Min PM₁** are asymptotically equivalent to approximate. Hence, the proposed results for **LABELED Min PM₁** also hold for **LABELED Min PM**.

2 A self improving operation on some classes of graphs

We now propose a self improving operation for some classes of instances $\mathcal{P}_k$ described as follows. $I = (H, \mathcal{L}) \in \mathcal{P}_k$ where $H = (V, E)$ if and only if the following properties are satisfied:

- (i) H is planar of maximum degree k and connected.
- (ii) $\exists u, v \in V$ such that $[u, u_1]$ and $[v, v_1]$ for some $u_1, v_1 \in V$ are the only edges incident to u and v (the vertices u and v will be called the extreme vertices of H). Moreover, these two edges have color c_0 , i.e., $\mathcal{L}([u, u_1]) = \mathcal{L}([v, v_1]) = c_0$.
- (iii) H is bipartite and admits a perfect matching (in particular, H has an even number of vertices).
- (iv) $H[\{c_0\}] = (V, \mathcal{L}_0)$, the subgraph induced by edges of color c_0 does not have any perfect matching and the subgraph $H[\mathcal{L}(E) \setminus \{c_0\}]$ induced by edges of colors different from c_0 is acyclic.

- (v) if $H' = H \setminus \{u, v\}$ denotes the subgraph induced by $V \setminus \{u, v\}$, then $H'[\{c_0\}] = (V \setminus \{u, v\}, \mathcal{L}_0)$ has a perfect matching denoted by M_{c_0} .

We have $\mathcal{P}_1 = \emptyset$ and $\mathcal{P}_2$ is the set of odd paths from u to v alternating matchings M and M_{c_0} where M_{c_0} is only colored by color c_0 . Finally, we define the class $\mathcal{P}_*$ by $\mathcal{P}_* = \cup_k \mathcal{P}_k$.

Restricted label squaring operation. Given an instance $I = (H, \mathcal{L}) \in \mathcal{P}_k$ of Labeled Min PM, its *label squaring* instance is $I^2 = (H^2, \mathcal{L}^2)$ with $H^2 = (V^2, E^2)$, where

1. The graph H^2 is created by removing each edge $e = [x, y]$ of H with color different from c_0 and placing instead of it a copy $H(e)$ of H , such that x and y are now identified with u and v of $H(e)$, respectively.
2. For each copy $H(e)$ of H and for an edge e' in $H(e)$ with color different from c_0 , the new color of e' is $\mathcal{L}^2(e') = (\mathcal{L}(e), \mathcal{L}(e'))$. The remaining edges of copy $H(e)$ keep their color c_0 , that is if $\mathcal{L}(e') = c_0$, then $\mathcal{L}^2(e') = c_0$.

Let us prove that classes $\mathcal{P}_k$ are closed under restricted label squaring operation.

Lemma 2.1 *If $I \in \mathcal{P}_k$, then $I^2 \in \mathcal{P}_k$.*

Proof: Let $I \in \mathcal{P}_k$. The proofs of (i) and (ii) are obvious since u and v have degree 1.

For (iii), since H and $H \setminus \{u, v\}$ admit a perfect matching, we deduce that $u \in L$ and $v \in R$ where (L, R) is the bipartition of H . Thus, we can extend the bipartition to H^2 by taking for each $H(e)$ a copy of the bipartition. Finally, it is easy to verify that H^2 admits a perfect matching if H does. Actually, given a perfect matching M of H , a perfect matching M^2 of H^2 can be constructed as follows: for any edge e of H with a color different from c_0 , if $e \in M$, then take for $H(e)$ a copy of M ; if $e \notin M$, then take for $H(e) \setminus \{u, v\}$ a copy of a perfect matching of $H \setminus \{u, v\}$ of color c_0 . Moreover, given any edge e in H with color c_0 , then $e \in M^2$ iff $e \in M$.

For (iv) assume on the contrary that is $H^2[\{c_0\}]$ admits a perfect matching M and $H[\{c_0\}]$ does not. By hypothesis, in each copy $H([x, y])$, the vertices x and y are not saturated by M because some vertices (in fact, an even positive number of vertices) of $H([x, y])$ are not saturated by M (otherwise, the restriction of M to $H([x, y])$ is a perfect matching for H), and x and y are the only vertices of $H([x, y])$ that do not need to be saturated by M (if other vertices are not saturated by M in $H([x, y])$, then M is not a perfect matching for H^2). Hence, the edges of M which do not belong to $H(e)$ form a perfect matching of $H[\{c_0\}]$, contradiction. Moreover, using Property (ii), it is easy to verify that the subgraph $H^2[\mathcal{L}^2(E^2) \setminus \{c_0\}]$ is acyclic whenever $H[\mathcal{L}(E) \setminus \{c_0\}]$ is acyclic.

For (v) let M_{c_0} be a perfect matching of $H' = H \setminus \{u, v\}$ only using color c_0 . We complete M_{c_0} by taking for each copy $H(e)$ a copy of M_{c_0} . In this way, we obtain a perfect matching of $H^2 \setminus \{u, v\}$ that uses color c_0 only. ■

We now propose an approximation preserving reduction using the label squaring operation on $\mathcal{P}_k$.

Theorem 2.2 *Let $I = (H, \mathcal{L}) \in \mathcal{P}_k$. Any solution of I^2 with value $\text{apx}(I^2) \leq \rho \text{opt}(I^2)$ for Labeled Min PM₁, can be polynomially converted into a solution of I for Labeled Min PM₁ with a value $\text{apx}(I) \leq \sqrt{\rho} \text{opt}(I)$.*

Proof: Let M^* be an optimal perfect matching of $I \in \mathcal{P}_k$ and let $\mathcal{L}^*$ be the set of colors used by M^* (we have $\text{opt}(I) = |\mathcal{L}^*|$). We construct a perfect matching M^2 for H^2 as follows. For each edge e of H , we do the following. If $\mathcal{L}(e) \neq c_0$ and $e \in M^*$, then we take for $H(e)$ a copy of M^* using colors $(\mathcal{L}(e), l)$ for $l \in \mathcal{L}^*$ and color c_0 . If $\mathcal{L}(e) \neq c_0$ and $e \notin M^*$, then we take for $H(e) \setminus \{u, v\}$ a perfect matching for $H \setminus \{u, v\}$ of color c_0 . If $\mathcal{L}(e) = c_0$, then $e \in M^2$ iff $e \in M$. This matching uses $(\text{opt}(I) - 1)^2 + 1$ colors and thus

$$\text{opt}_1(I^2) \leq \text{opt}_1^2(I) \quad (1)$$

Now, consider an approximate perfect matching M^2 of H^2 with value $\text{apx}(I^2)$ and let $H(e_1), \dots, H(e_p)$ be the copies of H such that the restriction of M^2 to $H(e_i)$ is a perfect matching. Hence, we may always assume that $M^2 \setminus (\cup_{i=1}^p H(e_i))$ only uses color c_0 . Because of Property (v) and the fact that, in each copy $H(e)$ with $e \notin \{e_1, \dots, e_p\}$, there is an even number of vertices (from Property (iii)) and so neither u nor v is saturated. Therefore, if we denote $\mathcal{L}' = \{\mathcal{L}(e_i) : i = 1, \dots, p\}$ the set of colors of these edges in H , then for any $c_j \in \mathcal{L}'$ there exists a perfect matching $M_{c_j, k} \subseteq M^2$ in copy $H(e_k)$ with $\mathcal{L}(e_k) = c_j$. Let M_{c_j} be one of these perfect matchings of H , one that minimizes $|\mathcal{L}(M_{c_j, k})|$ for any $c_j \in \mathcal{L}'$, where $|\mathcal{L}(M_{c_j, k})|$ is the number of colors used by $M_{c_j, k}$ and let M_0 be a perfect matching of H containing edges $\{e_1, \dots, e_p\}$ and all the other edges having color c_0 .

The approximate perfect matching M of I will be given by one of the matchings M_{c_j} or M_0 with value $\text{apx}(I) = \min\{|\mathcal{L}(M_0)|, |\mathcal{L}(M_{c_j})| : c_j \in \mathcal{L}'\}$. Thus, we deduce that $\text{apx}_1(I) = \text{apx}(I) - 1 = \min\{|\mathcal{L}(M_0)| - 1, |\mathcal{L}(M_{c_j})| - 1 : c_j \in \mathcal{L}'\}$ and hence:

$$\text{apx}_1^2(I) \leq (|\mathcal{L}(M_0)| - 1) \min\{|\mathcal{L}(M_{c_j})| - 1 : c_j \in \mathcal{L}'\} \leq \sum_{c_j \in \mathcal{L}'} (|\mathcal{L}(M_{c_j})| - 1) \leq \text{apx}_1(I^2) \quad (2)$$

The last inequality is valid because for $c_{j_1} \neq c_{j_2}$, the colors used in $M_{c_{j_1}}$ are different (except for c_0) from the ones used in $M_{c_{j_2}}$. Applying inequality (2) with an optimal perfect matching M^2 of H^2 , we obtain $\text{opt}_1^2(I) \leq \text{opt}_1(I^2)$. Using inequality (1), we deduce $\text{opt}_1^2(I) = \text{opt}_1(I^2)$ and the expected result follows. ■

3 Inapproximability results

In [7], an inapproximability bound of $O(\log n)$ is obtained for LABELED *Min PM* in complete bipartite graphs via a reduction from Set Cover. A slight modification of this reduction allows us to obtain the same result for instances in $\mathcal{P}_*$.

Theorem 3.1 LABELED *Min PM*₁ is not $c \log n$ approximable for some constant $c > 0$ for instances in $\mathcal{P}_*$ having $2n$ vertices, unless $\mathbf{P} = \mathbf{NP}$.

Proof: Given a family $\mathcal{S} = \{S_1, \dots, S_{n_0}\}$ of subsets of a ground set $X_0 = \{x_1, \dots, x_{m_0}\}$ (we assume that $\cup_{i=1}^{n_0} S_i = X_0$), a set cover of X_0 is a sub-family $\mathcal{S}' = \{S_{f(1)}, \dots, S_{f(p)}\} \subseteq \mathcal{S}$ such that $\cup_{i=1}^p S_{f(i)} = X_0$; MINSC is the problem of determining a minimum-size set cover $\mathcal{S}^* = \{S_{f^*(1)}, \dots, S_{f^*(q)}\}$ of X_0 . Given an instance $I_0 = (\mathcal{S}, X_0)$ of MINSC, its characteristic graph $G_{I_0} = (L_0, R_0; E_{I_0})$ is a bipartite graph with a left set $L_0 = \{l_1, \dots, l_{n_0}\}$ that represents the members of the family $\mathcal{S}$ and a right set $R_0 = \{r_1, \dots, r_{m_0}\}$ that

Figure 1: The gadget $H(x_j)$.

represents the elements of the ground set X ; the edge-set E_{I_0} of the characteristic graph is defined by $E_{I_0} = \{[l_i, r_j] : x_j \in S_i\}$.

From I_0 , we construct the instance $I = (H, \mathcal{L})$ of Labeled *Min PM*₁ containing (n_0+1) colors $\{c_0, c_1, \dots, c_{n_0}\}$, described as follows:

- For each element $x_j \in X_0$, we build a gadget $H(x_j)$ that consists of a bipartite graph of $2(d_{G_{I_0}}(r_j) + 3)$ vertices and $3d_{G_{I_0}}(r_j) + 4$ edges, where $d_{G_{I_0}}(r_j)$ denotes the degree of vertex $r_j \in R$ in G_{I_0} . The graph $H(x_j)$ is illustrated in Figure 1 where the vertices $\{l_{f(1)}, \dots, l_{f(p)}\}$ are the neighbors of r_j in G_{I_0} and $p = d_{G_{I_0}}(r_j)$.
- A distinct color is associated to each subset of $\mathcal{S}$. More precisely, assume that vertices $\{l_{f(1)}, \dots, l_{f(p)}\}$ with $p = d_{G_{I_0}}(r_j)$ are the neighbors of r_j in G_{I_0} , then color $H(x_j)$ as follows: for any $k = 1, \dots, p$, $\mathcal{L}(v_{3,j}, l_{j,f(k)}) = c_{f(k)}$ and the other edges receive color c_0 .
- We complete $H = \cup_{x_j \in X} H(x_j)$ by adding edges $[v_{2,j}, v_{1,j+1}]$ with color c_0 for $j = 1, \dots, m_0 - 1$.
- Finally, we set $u = v_{1,1}$ and $v = v_{2,m_0}$.

Clearly, $I \in \mathcal{P}_*$ and has $2n = 2 \sum_{r_j \in R} (d_{G_{I_0}}(r_j) + 3) = 2|E_{I_0}| + 6m_0$ vertices.

Let $\mathcal{S}^*$ be an optimal set cover on I_0 . We can associate to each element x_j a subset S_k of $\mathcal{S}^*$ that "covers" x_j . Then, we construct a perfect matching M^* as follows. The restriction of M^* to gadget $H(x_j)$ is a perfect matching M_j : $[l_{j,f(k)}, v_{3,j}] \in M_j$ and $[r_{j,f(k)}, v_{4,j}] \in M_j$ (S_k be the subset covering x_j); all the others edges of M_j are of color c_0 . Thus, the perfect matching M^* of $I = (H, \mathcal{L})$ uses exactly $(|\mathcal{S}^*| + 1)$ colors. Conversely, let M be a perfect matching on I . We have the following property: in each gadget $H(x_j)$, the edge of this gadget which is adjacent to $v_{2,j}$ is in M (this is true for $H(x_{m_0})$, and this can easily be

proved by induction for all the other gadgets). This is also the case for the edge of this gadget which is adjacent to $v_{1,j}$. This implies that, in each gadget $H(x_j)$, there is exactly one edge of color different from c_0 that belongs to M . We include the subset S_k associated with this edge in the set cover. In conclusion, the subset $\mathcal{S}' = \{S_k : c_k \in \mathcal{L}(M)\}$ of $\mathcal{S}$ is a set cover of X using $(|\mathcal{L}(M)| - 1)$ sets.

Now, it is well known that the set cover problem is **NP**-hard to approximate within factor $c \log n_0$ for some constant $c > 0$, [2]. This result also applies to instances $(X, \mathcal{S})$ when $|X|$ and $|\mathcal{S}|$ are polynomially related (i.e., $|X|^q \leq |\mathcal{S}| \leq |X|^p$ for some constants p, q).

Hence, given such an instance $I_0 = (X, \mathcal{S})$, from any algorithm A solving **LABELED Min PM₁** within a performance ratio $\rho_A(I) \leq \frac{c}{q+1} \times \log(n)$ for a bipartite graph on $2n$ vertices, we can deduce an algorithm for **MINSC** that guarantees the performance ratio $c \frac{1}{q+1} \log(n) \leq c \frac{1}{q+1} \log(n_0^{q+1}) = c \log(n_0)$, a contradiction. ■

Starting from the **APX**-completeness result for the vertex cover problem in cubic graphs [1], we are able to obtain the following result.

Corollary 3.2 *LABELED Min PM₁ for instances in $\mathcal{P}_3$ is not in **PTAS**, unless $P=NP$.*

Proof: Starting from the restriction of set cover where each element x_i is covered by exactly two sets (this case is usually called vertex cover where the sets and elements can respectively be viewed as vertices and edges), we apply the same proof as in Theorem 3.1. The instance I becomes an element of $\mathcal{P}_3$, and using for instance the **APX**-completeness result of [1], the expected result follows. ■

By applying the well known method of self improving, we obtain the two following results:

Theorem 3.3 *LABELED Min PM₁ for instances in $\mathcal{P}_3$ is not in **APX**, unless $P = NP$.*

Proof: Assume the reverse and let A be a polynomial algorithm solving **LABELED Min PM₁** within a constant performance ratio ρ . Let $\varepsilon > 0$ (with $\varepsilon < \rho - 1$) and choose the smallest integer q such that:

$$q \geq -\log_2 \log_\rho(1 + \varepsilon) \quad (3)$$

Consider now an instance $I = (H, \mathcal{L}) \in \mathcal{P}_3$ and use the restricted label squaring operation on I . We produce the instance $I^2 = (H^2, \mathcal{L}^2)$ and by repeating q times this operation on I^2 , we obtain thanks to Lemma 2.1 the instance $I^{2^q} = (H^{2^q}, \mathcal{L}^{2^q}) \in \mathcal{P}_3$, in time $P(|I|)$ for some polynomial P since on the one hand, I^2 is obtained from I in time $O(|I|^2)$ (we have $|V(H^2)| = O(|V(H)|^2)$ and $|\mathcal{L}^2(E(H^2))| = O(|\mathcal{L}(E(H))|^2)$) and on the other hand, we repeat this operation a constant number of times. Using Theorem 2.2, from the ρ -approximation on I^{2^q} given by A , we obtain a $\rho^{2^{-q}}$ -approximation on I . Thanks to inequality (3), we deduce $\rho^{2^{-q}} \leq 1 + \varepsilon$. Hence, we obtain a polynomial time approximation scheme for instances in $\mathcal{P}_3$, a contradiction with Corollary 3.2. ■

Theorem 3.4 *For any $\varepsilon > 0$ LABELED Min PM₁ is not $2^{O(\log^{1-\varepsilon} n)}$ -approximable for instances in $\mathcal{P}_3$ on n vertices, unless $NP \subseteq DTIME\left(2^{O(\log^{1/\varepsilon} n)}\right)$.*

Proof: Let $\varepsilon > 0$ and $I = (H, \mathcal{L}) \in \mathcal{P}_3$ where H has n vertices. Choose the smallest integer p such that $n^{2^p} \geq 2^{\log^{1/\varepsilon} n}$. Thus, $2^{2^p \times \log n} \geq 2^{\log^{1/\varepsilon} n}$ and then,

$$2^{p \times \varepsilon} \geq \log^{1-\varepsilon} n \quad (4)$$

Using the restricted label squaring operation on I , we produce the instance $I^2 = (H^2, \mathcal{L}^2)$. By repeating p times this operation on I^2 , we obtain the instance $I^{2^p} = (H^{2^p}, \mathcal{L}^{2^p}) \in \mathcal{P}_3$. Since H has n vertices, we derive that the number n' of vertices of H^{2^p} and the number $|\mathcal{L}^{2^p}(E(H^{2^p}))|$ of colors of H^{2^p} satisfy:

$$n' \leq n^{2^p} \text{ and } |\mathcal{L}^{2^p}(E(H^{2^p}))| \leq |\mathcal{L}(E(H))|^{2^p} \quad (5)$$

Now, assume that we have a $f(n')$ -approximation on I^{2^p} where $f(n') \leq 2^{c \times \log^{1-\varepsilon} n'}$ for some $c > 0$. Using Theorem 2.2, we obtain a $f(n')^{2^{-p}}$ -approximation on I . Using inequalities (4) and (5), we deduce:

$$\begin{aligned} \text{apx}_1(I) &\leq f(n')^{2^{-p}} \text{opt}_1(I) \\ &\leq 2^{c \times \frac{\log^{1-\varepsilon} n'}{2^p}} \text{opt}_1(I) \\ &\leq 2^{c \times \frac{\log^{1-\varepsilon} n}{2^{\varepsilon \times p}}} \text{opt}_1(I) \\ &\leq 2^c \text{opt}_1(I) \end{aligned}$$

By definition of p , $n^{2^{p-1}} < 2^{\log^{1/\varepsilon} n}$, and thus $n' \leq n^{2^p} < 2^{2 \log^{1/\varepsilon} n}$. In conclusion, using inequality (5), we obtain a constant approximation for I in time $\text{poly}(n') = 2^{O(\log^{1/\varepsilon} n)}$, and thus, a contradiction to Theorem 3.3. $\blacksquare$

It is natural to ask the question whether the problem is easier in cubic bipartite graphs. Here, we prove that the answer is negative.

Theorem 3.5 *Labeled Min PM₁ is not in APX in connected planar cubic bipartite graphs, unless P = NP.*

Proof: The proof consists of two steps. First, using a reduction quite similar to the one of Corollary 3.2, we prove that Theorem 3.4 also holds for the sub-family $\mathcal{P}'_3$ of $\mathcal{P}_3$ where each vertex has a degree 3, except u and v . Then, we transform any instance of $\mathcal{P}'_3$ into a connected planar cubic bipartite graph.

Let us prove the first point. Let $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_n\}$ be an instance of vertex cover. We associate to any edge $e_j = [x, y]$ a gadget $H(e_j)$ described in Figure 2. All edges of $H(e_j)$, except $[v_{3,j}, l_{j,x}]$ and $[v_{3,j}, l_{j,y}]$ have color c_0 . We have $\mathcal{L}([v_{3,j}, l_{j,x}]) = c_x$ and $\mathcal{L}([v_{3,j}, l_{j,y}]) = c_y$. Finally, $H(e_j)$ is linked to $H(e_{j+1})$ using the graph depicted in Figure 3 where each edge is colored with c_0 . We conclude this part of the proof by using arguments similar to ones given in the proof of Corollary 3.2 (the gadgets considered in both reductions having the same basic properties). Clearly, Labeled Min PM₁ is APX-hard in class $\mathcal{P}'_3$. Since the restricted label squaring operation also preserves the membership in $\mathcal{P}'_3$, we deduce that Labeled Min PM₁ is not in APX when the instances are restricted to $\mathcal{P}'_3$.

Now, let us prove the second point. Given $I \in \mathcal{P}'_3$ with $I = (G, \mathcal{L})$, we consider the instance I' where G is duplicated 3 times into G_1, G_2, G_3 . If u_i, v_i denote the extreme

Figure 2: The gadget $H(e_j)$ for $e_j = [x, y]$.

Figure 3: The gadget linking $H(e_j)$ to $H(e_{j+1})$.

vertices of G_i , we shrink vertices u_1, u_2, u_3 into u and v_1, v_2, v_3 into v . Clearly, this new graph G' is connected bipartite, planar and cubic. Given any perfect matching M' of G' , the two edges of color c_0 adjacent to u and v in M' necessarily lie in the same copy of G , because there is an even number of vertices in each copy. Finally, since we can restrict ourselves to perfect matchings M' of G' that use only color c_0 for exactly two copies of G , the result follows. $\blacksquare$

Obviously, a proof similar to the one given in Theorem 3.4 can be applied to the Theorem 3.5 leading to the following conclusion.

Corollary 3.6 *For any $\varepsilon > 0$ LABELED Min PM_1 is not $2^{O(\log^{1-\varepsilon} n)}$ -approximable in connected planar cubic bipartite graphs on n vertices, unless $\mathbf{NP} \subseteq \mathbf{DTIME}\left(2^{O(\log^{1/\varepsilon} n)}\right)$.*

Dealing with the unbounded degree case (that is instances of $\mathcal{P}_*$), we can deduce the following stronger result:

Theorem 3.7 *LABELED Min PM_1 for instances in $\mathcal{P}_*$ is not in **polyLog-APX**, unless $P = \mathbf{NP}$.*

Proof: Assume on the contrary that LABELED Min PM_1 is $f(n)$ -approximable with $f(n) \leq c \log^k n$ for some constants $c > 0$ and $k \geq 1$. Let $I = (H, \mathcal{L}) \in \mathcal{P}_*$ where H has $2n$ vertices.

Let $p = \lceil \log k \rceil + 1$. Using as previously 2^p times the restricted label squaring operation on I , we produce in polynomial time the instance $I^{2^p} = (H^{2^p}, \mathcal{L}^{2^p}) \in \mathcal{P}_*$. The same arguments as in Theorem 3.4 allow us to obtain a contradiction with Theorem 3.1 since from Property (iv) of Lemma 2.1, the inequality (5) also holds. ■

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