Weighted coloring on planar, bipartite and split graphs: complexity and approximation (Preliminary version)

M. Demange[†], B. Escoffier[‡], J. Monnot[‡], V. Th. Paschos[‡], D. de Werra[*]

Abstract

We study complexity and approximation of MIN WEIGHTED NODE COLORING in planar, bipartite and split graphs. We show that this problem is **NP**-complete in planar graphs, even if they are triangle-free and their maximum degree is bounded above by 4. Then, we prove that MIN WEIGHTED NODE COLORING is **NP**-complete in P_8-free bipartite graphs, but polynomial for P_5-free bipartite graphs. We next focus ourselves on approximability in general bipartite graphs and improve earlier approximation results by giving approximation ratios matching inapproximability bounds. We next deal with MIN WEIGHTED EDGE COLORING in bipartite graphs. We show that this problem remains strongly **NP**-complete, even in the case where the input-graph is both cubic and planar. Furthermore, we provide an inapproximability bound of $7/6 - \varepsilon$, for any $\varepsilon > 0$ and we give an approximation algorithm with the same ratio. Finally, we show that MIN WEIGHTED NODE COLORING in split graphs can be solved by a polynomial time approximation scheme.

Key words : Graph coloring; ; weighted node coloring; weighted edge coloring; approximability; NP-completeness; planar graphs; bipartite graphs; split graphs.

1 Introduction

We give in this paper some complexity results as well as some improved approximation results for MIN WEIGHTED NODE COLORING, originally studied in Guan and Zhu [8]

[*]Ecole Polytechnique Fédérale de Lausanne, Switzerland, dewerra@ima.epfl.ch

[†]ESSEC, Dept. SID, France, demange@essec.fr

[‡]LAMSADE, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cédex 16, France, {escoffier,monnot,paschos}@lamsade.dauphine.fr

and more recently in [4]. A k-coloring of $G = (V, E)$ is a partition $\mathcal{S} = (S_1, \dots, S_k)$ of the node set V of G into stable sets S_i. In this case, the objective is to determine a node coloring minimizing k. A natural generalization of this problem is obtained by assigning a strictly positive integer weight $w(v)$ for any node $v \in V$, and defining the weight of stable set S of G as $w(S) = \max\{w(v) : v \in S\}$. Then, the objective is to determine $\mathcal{S} = (S_1, \dots, S_k)$ a node coloring of G minimizing the quantity $\sum_{i=1}^{k} w(S_i)$. This problem is easily shown **NP**-hard; it suffices to consider $w(v) = 1$, $\forall v \in V$ and MIN WEIGHTED NODE COLORING becomes the classical node coloring problem. Other versions of weighted colorings have been studied in Hassin and Monnot [9].

Consider an instance I of an **NP**-hard optimization problem Π and a polynomial time algorithm A computing feasible solutions for Π. Denote by $m_{\mathtt{A}}(I, S)$ the value of a Π-solution S computed by A on I and by $\mathrm{opt}(I)$, the value of an optimal Π-solution for I. The quality of A is expressed by the ratio (called approximation ratio in what follows) $\rho_{\mathtt{A}}(I) = m_{\mathtt{A}}(I, S)/\mathrm{opt}(I)$, and the quantity $\rho_{\mathtt{A}} = \inf\{r : \rho_{\mathtt{A}}(I) < r, I \text{ instance of } \Pi\}$. A very favorable situation for polynomial approximation occurs when an algorithm achieves ratios bounded above by $1 + \varepsilon$, for any $\varepsilon > 0$. We call such algorithms *polynomial time approximation schemes*. The complexity of such schemes may be polynomial or exponential in $1/\varepsilon$ (they are always polynomial in the sizes of the instances). A polynomial time approximation scheme with complexity polynomial also in $1/\varepsilon$ is called *fully polynomial time approximation scheme*.

This paper extends results on MIN WEIGHTED NODE COLORING, the study of which has started in [4]. We first deal with planar graphs and we show that, for this family, the problem studied is **NP**-complete, even if we restrict to triangle-free planar graphs with node-degree not exceeding 4.

We then deal with particular families of bipartite graphs. The **NP**-completeness of MIN WEIGHTED NODE COLORING has been established in [4] for general bipartite graphs. We show here that this remains true even if we restrict to planar bipartite graphs or to P_{21}-free bipartite graphs (for definitions of graph-theoretical notions used in this paper, the interested reader is referred to Berge [1]).

It is interesting to observe that these results are obtained as corollaries of a kind of generic reduction from the precoloring extension problem shown to be **NP**-complete in Bodlaender et al. [2], Hujter and Tuza [11, 12], Kratochvil [14]. Then, we slightly improve the last result to P_8-free bipartite graphs and show that the problem becomes polynomial in P_5-free bipartite graphs. Observe that in [4], we have proved that MIN WEIGHTED NODE COLORING is polynomial for P_4-free graphs and **NP**-complete for P_5-free graphs.

Then, we focus ourselves on approximability of MIN WEIGHTED NODE COLORING in (general) bipartite graphs. As proved in [4], this problem is approximable in such graphs within approximation ratio 4/3; in the same paper a lower bound of $8/7 - \varepsilon$, for any $\varepsilon > 0$, was also provided. Here we improve the approximation ratio of [4] by matching the $8/7$-lower bound of [4] with a same upper bound; in other words, we show here that MIN WEIGHTED NODE COLORING in bipartite graphs is approximable within approximation ratio bounded above by $8/7$.

We next deal with MIN WEIGHTED EDGE COLORING in bipartite graphs. In this problem we consider an edge-weighted graph G and try to determine a partition of the edges of G into matchings in such a way that the sum of the weights of these matchings is minimum (analogously to the node-model, the weight of a matching is the maximum of the weights of its edges). In [4], it is shown that MIN WEIGHTED EDGE COLORING is **NP**-complete for cubic bipartite graphs. Here, we slightly strengthen this result showing that this problem remains strongly **NP**-complete, even in cubic and planar bipartite graphs. Furthermore, we strengthen the inapproximability bound provided in [4], by reducing it from $8/7 - \varepsilon$ to $7/6 - \varepsilon$, for any $\varepsilon > 0$. Also, we match it with an upper bound of the same value, improving so the $5/3$-approximation ratio provided in [4].

Finally, we deal with approximation of MIN WEIGHTED NODE COLORING in split graphs. As proved in [4], MIN WEIGHTED NODE COLORING is strongly **NP**-complete in such graphs, even if the nodes of the input graph receive only one of two distinct weights. It followed that this problem cannot be solved by fully polynomial time approximation schemes, but no approximation study was addressed there. In this paper we show that MIN WEIGHTED NODE COLORING in split graphs can be solved by a polynomial time approximation scheme.

In the remainder of the paper, we shall assume that for any weighted node or edge coloring $\mathcal{S} = (S_1, \ldots, S_\ell)$ considered, we will have $w(S_1) \geqslant \ldots \geqslant w(S_\ell)$.

2 Weighted node coloring in triangle-free planar graphs

The node coloring problem in planar graphs has been shown **NP**-complete by Garey and Johnson [6], even if the maximum degree does not exceed 4. On the other hand, this problem becomes easy in triangle-free planar graphs (see Grotzsch [7]). Here, we show that the weighted node coloring problem is **NP**-complete in triangle-free planar graphs with maximum degree 4 by using a reduction from 3-SAT PLANAR, proved to be **NP**-complete in Lichtenstein [15]. This problem is defined as follows: Given a collection $\mathcal{C} = (C_1, \ldots, C_m)$ of clauses over the set $X = \{x_1, \ldots, x_n\}$ of boolean variables such that each clause C_j has at most three literals (and at least two), is there a truth assignment f satisfying $\mathcal{C}$? Moreover, the bipartite graph $BP = (L, R; E)$ is planar where $|L| = n$, $|R| = m$ and $[x_i, c_j] \in E$ iff the variable x_i (or $\overline{x_i}$) appears in the clause C_j.

Theorem 2.1 MIN WEIGHTED NODE COLORING *is **NP**-complete in triangle-free planar graphs with maximum degree 4.*

Proof : Let $BP = (L, R; E)$ be the bipartite graph representing an instance $(X, \mathcal{C})$ of 3-SAT PLANAR where $L = \{x_1, \ldots, x_n\}$, $R = \{c_1, \ldots, c_m\}$. We construct an instance $I = (G, w)$ of MIN WEIGHTED NODE COLORING by using two gadgets:

- The gadgets clause $F(C_j)$ are given in Figure 1 for clause C_j of size 3 and in Figure 2 for clause C_j of size 2. The nodes c_j^k are those that will be linked to the rest of the graph.

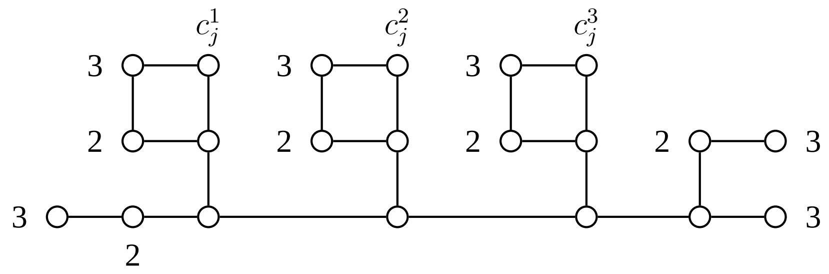

Figure 1: Graph $F(C_j)$ representing a clause C_j of size 3.

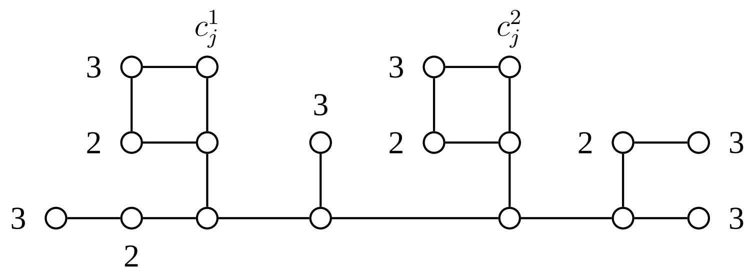

Figure 2: Graph $F(C_j)$ representing a clause C_j of size 2.

- The gadgets variable $H(x_i)$ is given in Figure 3 for variable x_i. Assume that x_i appears p_1 times positively and p_2 times negatively in $(X, \mathcal{C})$, then in $H(x_i)$ there are $2p = 2(p_1 + p_2)$ special nodes $x_i^k, \overline{x_i^k}$, $k = 1, \ldots, p$. These nodes form a path which meets node x_i^k, $\overline{x_i^k}$ alternatively.
- The weights of nodes which are not given in Figures 1, 2 and 3 are 1.

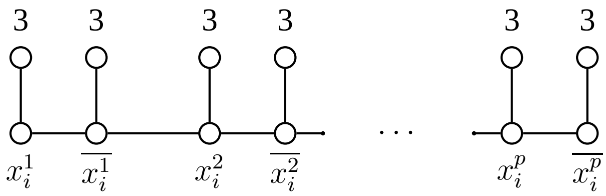

Figure 3: Graph $H(x_i)$ representing variable x_i

- These gadgets are linked together by the following process. If variable x_i appears positively (resp. negatively) in clause C_j, we link one of the variables $\overline{x_i^k}$ (resp. x_i^k), with a different k for each C_j, to one of the three nodes c_j^k of gadget $F(C_j)$. This can be done in a way which preserves the planarity of the graph.

  Indeed, for each node v of degree $\delta(v)$ in the planar graph BP, let's call $e_v^1, \cdots, e_v^{\delta(v)}$ the endpoints on v of the edges adjacent to v considered in a circular order. Then, for each edge in BP which joins node x_i in endpoint $e_{x_i}^k$ to node C_j in endpoint $e_{c_j}^l$, we put an edge from x_i^k (if x_i appears negatively in C_j, $\overline{x_i^k}$ otherwise) to c_j^l.

Observe that G is triangle-free and planar with maximum degree 4. Moreover, we assume that G is not bipartite (otherwise, we add a disjoint cycle Γ with $|\Gamma| = 7$ and $\forall v \in V(\Gamma), w(v) = 1$).

It is then not difficult to check that $(X, \mathcal{C})$ is satisfiable iff $opt(I) \leqslant 6$.

Let g be a truth assignment satisfying $(X, \mathcal{C})$. We set $S_1' = \{v : w(v) = 3\}$ and $S_2' = \{v : w(v) = 2\} \cup \{x_i^k : g(x_i) = 1\} \cup \{\overline{x_i^k} : g(x_i) = 0\}$. Since g satisfies the formula, we can color at least one node c_j^k with color 2 and then easily extend (S_1', S_2') to a coloring $\mathcal{S} = (S_1, S_2, S_3)$ of G with $S_i' \subseteq S_i$ for $i = 1, 2$. We have $w(S_1) = 3, w(S_2) = 2, w(S_3) = 1$ and then $val(\mathcal{S}) \leqslant 6$.

Conversely, let $\mathcal{S} = (S_1, \ldots, S_\ell)$ be a coloring of G with $val(\mathcal{S}) \leqslant 6$. Assume $w(S_1) \geqslant \ldots \geqslant w(S_\ell)$. We have $\ell \geqslant 3$ since G is not bipartite and $w(S_1) = 3$. We deduce $w(S_2) < 3$ (otherwise $val(\mathcal{S}) \geqslant 3 + 3 + 1$). Moreover, since each node of weight 2 is adjacent to a node of weight 3, we have $W(S_2) = 2$. For the same reasons as previously, we deduce $\ell = 3$ and $W(S_3) = 1$. We claim that for any $j = 1, \ldots, m$, $S_2 \cap \{c_j^1, c_j^2, c_j^3\} \neq \emptyset$ where c_j^1, c_j^2, c_j^3 are the nodes of $F(C_j)$ (with may be $c_j^3 = \emptyset$). Otherwise, we must have $\{c_j^1, c_j^2, c_j^3\} \subseteq S_3$ but in this case, we cannot colored $F(C_j)$ with 3 colors. Thus, setting $g(x_i) = 1$ iff $x_i^k \in S_2$, we deduce that g is a truth assignment satisfying $(X, \mathcal{C})$. □

3 Weighted node coloring in bipartite graphs

3.1 Complexity results

The **NP**-completeness of MIN WEIGHTED NODE COLORING in bipartite graphs has been proved in Demange et al. [4]. Here, we show that some more restrictive versions are also **NP**-complete, namely bipartite planar graphs and P_{21}-free bipartite graphs, i.e. bipartite graphs which do not contain induced chains of length 21 or more. We use a generic reduction from the precoloring extension node coloring problem (in short PREXT NODE COLORING). Then, using another reduction we improve this result to P_8-free bipartite graphs. This latter problem can be described as follows. Given a positive integer k, a graph $G = (V, E)$ and k pairwise disjoint subsets $V_1, \ldots, V_k$ of V, we want to decide if there exists a node coloring $\mathcal{S} = (S_1, \ldots, S_k)$ of G such that $V_i \subseteq S_i$, for all $i = 1, \ldots, k$. Moreover, we restrict us to some class of graphs $\mathcal{G}$: we assume that $\mathcal{G}$ is closed when we add a pending edge with a new node (i.e., if $G = (V, E) \in \mathcal{G}$ and $x \in V$, $y \notin V$, then $G + [x, y] \in \mathcal{G}$).

Theorem 3.1 *Let $\mathcal{G}$ be a class of graphs which is closed when we add a pending edge with a new node. If* PREXT NODE COLORING *is **NP**-complete for graphs in $\mathcal{G}$, then* MIN WEIGHTED NODE COLORING *is **NP**-complete for graphs in $\mathcal{G}$.*

Proof : Let $\mathcal{G}$ be such a class of graphs. We shall reduce PREXT NODE COLORING in $\mathcal{G}$ graphs to weighted node coloring in $\mathcal{G}$ graphs. Let $G = (V, E) \in \mathcal{G}$ and k pairwise disjoint subsets $V_1, \ldots, V_k$ of V. We build instance $I = (G', w)$ of weighted node coloring using several gadgets T_i, for $i = 1, \ldots, k$. The construction of T_i is given by induction as follows:

- T_1 is simply a root v_1 with weight $w(v_1) = 2^{k-1}$.
- Given $T_1, \ldots, T_{i-1}$, T_i is a tree with a root v_i of weight $w(v_i) = 2^{k-i}$ that we link to tree T_p via edge $[v_i, v_p]$ for each $p = 1, \ldots, i-1$.

Figure 4 illustrates the gadgets T_1, T_2, T_3. Now, $I = (G', w)$ where $G' = (V', E')$ is constructed in the following way:

- G' contains G.
- For all $i = 1, \ldots, k$, we replace each node $v \in V_i$ by a copy of the gadget T_i where we identify v with root v_i.
- For all $v \in V \setminus (\cup_{i=1}^{k} V_i)$ we set $w(v) = 1$.

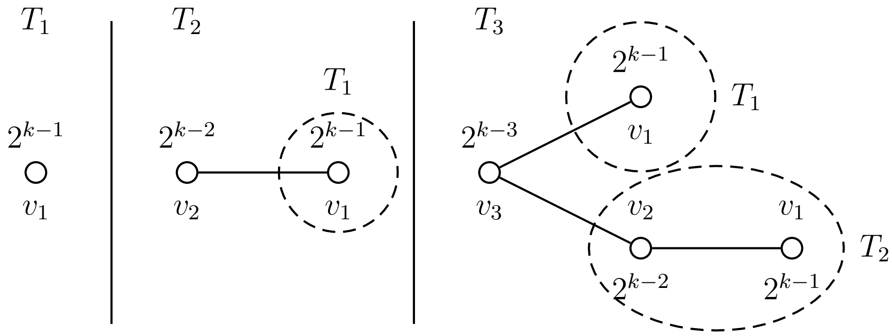

Figure 4: Gadgets for T_1, T_2 and T_3.

Note that, by hypothesis, $G' \in \mathcal{G}$. We prove that the precoloring of G (given by $V_1, \ldots, V_k$) can be extended to a proper node coloring of G using at most k colors iff $opt(I) \leqslant 2^k - 1$.

Let $\mathcal{S} = (S_1, \ldots, , S_k)$ with $V_i \subseteq S_i$ be a node coloring of G. We get $\mathcal{S}' = (S'_1, \ldots, , S'_k)$ where each stable S'_i is given by $S'_i = (S_i \setminus V_i) \cup \{v :\ \exists j \leqslant k,\ v \in T_j$ and $w(v) = 2^{k-i}\}$. It is easy to check that $\mathcal{S}'$ is a coloring of G' and $opt(I) \leqslant val(\mathcal{S}') = \sum_{i=1}^{k} 2^{k-i} = 2^k - 1$.

Conversely, let $\mathcal{S}' = (S'_1, \ldots, S'_\ell)$ with $w(S'_1) \geqslant \ldots \geqslant w(S'_\ell)$ be a weighted node coloring of G' with cost $val(\mathcal{S}') \leqslant 2^k - 1$. First, we prove by induction that $V'_i = \{v :\ \exists p \leqslant k,\ v \in T_p,\ w(v) = 2^{k-i}\}$ is a subset of S'_i, for all $i \leqslant k$. For $i = 1$, the result is true since otherwise we have $w(S'_1) = w(S'_2) = 2^{k-1}$ and then, $val(\mathcal{S}') \geqslant w(S'_1) + w(S'_2) = 2^k$. Now, assume that $V'_j \subseteq S'_j$ for $j < i$ and let us prove that $V'_i = \{v :\ \exists p \leqslant k,\ v \in T_p,\ w(v) = 2^{k-i}\} \subseteq S'_i$. By construction of gadget $T_j,\ j \geqslant i$, each node v of weight 2^{k-i} is adjacent to a node of weight 2^{k-p} for all $p < i$. Thus, $v \notin S'_p$. Now, if $V'_i \nsubseteq S'_i$, then $w(S'_i) = w(S'_{i+1}) = 2^{k-i}$ and we deduce $val(\mathcal{S}') \geqslant w(S'_1) + \ldots + w(S'_{i+1}) = \sum_{j=1}^{i} 2^{k-j} + 2^{k-i} = 2^k$, which is a contradiction. Since $V'_i \neq \emptyset$ for $i \leqslant k$, we deduce $\ell \geqslant k$. Consequently, $\ell = k$, since $\forall v \in V',\ w(v) \geqslant 1$. Now, getting $\mathcal{S} = (S_1, \ldots, S_k)$ where $S_i = (S'_i \setminus V'_i) \cup V_i$ for each $i = 1, \ldots, k$, we obtain a node coloring of G. □

Using the results of Kratochvil [14] on the **NP**-completeness of PREXT NODE COLORING in bipartite planar graphs and P_{13}-free bipartite graphs, we deduce:

Corollary 3.2 *In bipartite planar graphs,* MIN WEIGHTED NODE COLORING *is strongly* ***NP****-complete and it is not* $\frac{8}{7} - \varepsilon$*-approximable for all* $\varepsilon > 0$ *unless* ***P****=****NP****.*

Proof : PREXT NODE COLORING with $k = 3$ has been proved **NP**-complete in [14] for bipartite planar graphs. Since these graphs are closed when we add an pending edge with a new node, the result follows. Moreover, from the proof of Theorem 3.1 with $k = 3$, we deduce that it is **NP**-complete to distinguish whenever $opt(I) \leqslant 7$ and $opt(I) \geqslant 8$. □

Corollary 3.3 *In P_{21}-free bipartite graphs,* MIN WEIGHTED NODE COLORING *is strongly **NP**-complete and it is not $\frac{32}{31} - \varepsilon$-approximable for all $\varepsilon > 0$ unless **P**=**NP**.*

Proof : PREXT NODE COLORING with $k = 5$ has been proved **NP**-complete in [14] for P_{13}-free bipartite graphs. When, we add gadgets T_i with $i \leqslant 5$, G' becomes P_{21}-free bipartite graphs. Moreover, from the proof of Theorem 3.1 with $k = 5$, we deduce that it is **NP**-complete to distinguish whenever $opt(I) \leqslant 31$ and $opt(I) \geqslant 32$. □

In Hujter and Tuza [12], it is shown that PREXT NODE COLORING is **NP**-complete in P_6-free bipartite chordal graphs for unbounded k (a bipartite graph is chordal if the induced cycles of length at least 5 have a chord). Unfortunately, we cannot use this result in Theorem 3.1 since the resulting graph has an induced chain with arbitrarily large length. However, we can adapt their reduction to our problem.

Theorem 3.4 MIN WEIGHTED NODE COLORING *is **NP**-complete in P_8-free bipartite graphs.*

Proof : We shall reduce 3-SAT-3, proved to be **NP**-complete in Papadimitriou [17], to our problem. Given a collection $\mathcal{C} = (C_1, \ldots, C_m)$ of clauses over the set $X = \{x_1, \ldots, x_n\}$ of boolean variables such that each clause C_j has at most three literals and each variable appears 2 times positively and one time negatively, we construct an instance $I = (BP, w)$ in the following way:

- We start from $BP_1 = (L_1, R_1; E_1)$, a complete bipartite graph $K_{n,m}$ where $L_1 = \{x_1, \ldots, x_n\}$ and $R_1 = \{c_1, \ldots, c_m\}$. Moreover, each node of BP_1 has weight 1.
- There is also another bipartite graph BP_2 isomorphic to $K_{2n,2n}$ where a perfect matching has been deleted. More formally, $BP_2 = (L_2, R_2; E_2)$ where $L_2 = \{l_1, \ldots, l_{2n}\}$, $R_2 = \{r_1, \ldots, r_{2n}\}$ and $[l_i, r_j] \in E_2$ iff $i \neq j$. Finally, $w(l_i) = w(r_i) = 2^{2n-i}$ for $i = 1, \ldots, 2n$. Indeed, sets $\{l_{2i-1}, r_{2i-1}\}$ and $\{l_{2i}, r_{2i}\}$ will correspond to variable x_i and $\overline{x_i}$ respectively.
- Between BP_1 and BP_2, there is a set E_3 of edges. $[x_i, r_j] \notin E_3$ iff $j = 2i - 1$ or $j = 2i$ and $[l_i, c_j] \notin E_3$ iff $i = 2k - 1$ and x_k is in C_j or $i = 2k$ and $\overline{x_k}$ is in C_j.

Figure 5 illustrates the construction of the complement of BP with the clause $c_m = \overline{x_1} \vee x_2 \vee \overline{x_n}$.

Let us show that BP is P_8-free. We represent in Figure 6 the possible subgraphs on BP_1 (configuration A_1, A_2 and A_3) and on BP_2 (configuration B_1 to B_9) induced by a

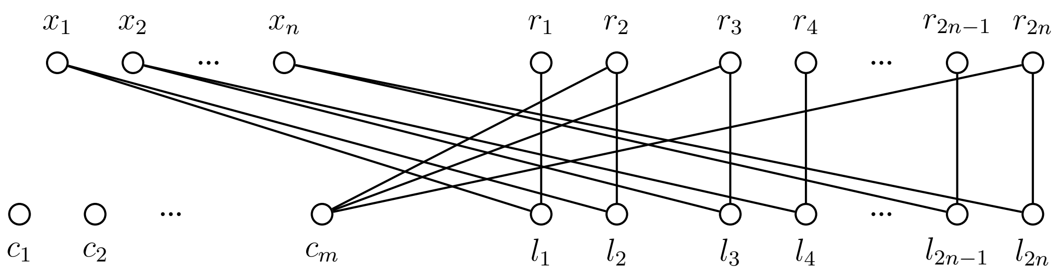

Figure 5: Complement of graph BP with the clause $c_m = \overline{x_1} \vee x_2 \vee \overline{x_n}$

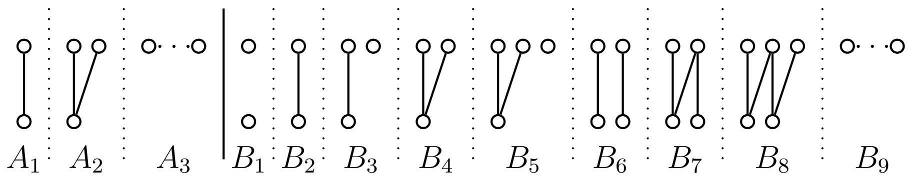

Figure 6: Subgraphs on BP_1 and BP_2 induced by a chain

chain on BP. In configurations A_3 and B_9, the number of nodes is arbitrary. Note that the upper line may correspond either to L_1 or R_1 for the left part (and L_2 or R_2 for the right part). Now we look at the possible ways to link a configuration A_i to a configuration B_j to obtain a chain of length (at least) 8.

- If we choose A_1, we easily see that it's impossible.
- If we choose A_2, the only way to have a chain of length at least 8 is to choose B_8 and link a node of A_2 to a node of B_8. In this case, we can see that the upper line corresponds to R_1 (left part) and L_2 (right part), and that there is a clause which contains a variable and its negation.
- If we choose A_3, the only possibility to have a chain of length at least 8 is to choose B_9. But in this case, the chain simply alternates a node of R_1 and a node of L_2. Then, at least one node of L_2 is not linked to at least 3 nodes of R_1, i.e. a literal appears in at least 3 clauses.

We claim that $(X, \mathcal{C})$ is satisfiable iff $opt(I) \leqslant 2^{2n} - 1$.

Let g be a truth assignment satisfying $(X, \mathcal{C})$. We build inductively the colors. $S_0 = \emptyset$ and for $i = 1, \ldots, n$, $S_{2i-1} = \{l_{2i-1}, r_{2i-1}\} \cup \{c_j : c_j \notin S_p, p < 2i-1, g(x_i) =$

1 and x_i is in $C_j\}$, $S_{2i} = \{l_{2i}, r_{2i}\} \cup \{c_j : c_j \notin S_p, p < 2i, g(x_i) = 0$ and $\overline{x_i}$ is in $C_j\}$. Finally, if $g(x_i) = 1$ then we add x_i to S_{2i}; otherwise, we add x_i to S_{2i-1}. We can easily see that $\mathcal{S} = (S_1, \dots, S_{2n})$ is a node coloring of BP with $val(\mathcal{S}) = 2^{2n} - 1$.

Conversely, let $\mathcal{S} = (S_1, \dots, S_\ell)$ be a node coloring of BP with $val(\mathcal{S}) = 2^{2n} - 1$. An inductive proof on i shows that $\{l_i, r_i\} \subseteq S_i$ (otherwise, we have $val(\mathcal{S}) \geqslant 2^{2n}$); consequently, $\ell = 2n$. Thus, setting $g(x_i) = 1$ if $x_i \in S_{2i}$ and $g(x_i) = 0$ if $x_i \in S_{2i-1}$, we obtain a truth assignment satisfying $(X, \mathcal{C})$. □

3.2 Polynomial result

We now prove that MIN WEIGHTED NODE COLORING is polynomial for P_5-free bipartite graphs, i.e., without induced chain on 5 nodes. Notice that in general P_5-free graphs, the weighted node coloring problem is **NP**-complete since on the one hand, the split graphs are P_5-free and on the other hand, we have proved in Demange et al. [4] that the weighted node coloring problem is **NP**-complete for split graphs. There are several characterizations of P_5-free bipartite graphs, see for example, Hammer et al. [10], Chung et al. [3] and Hujter and Tuza [11]. In particular, BP is a P_5-free bipartite graph iff BP is bipartite and each connected component of BP is $2K_2$-free, i.e., its complement is C_4-free.

Lemma 3.5 *In P_5-free bipartite graph, any optimal weighted node coloring uses at most 3 colors.*

Proof : Let $BP = (L, R; E)$ be a P_5-free bipartite graph with connected components $BP_1, \dots, BP_p$. Assume the reverse and let us consider an optimal solution $\mathcal{S}^* = (S_1^*, \dots, S_\ell^*)$ with $\ell \geqslant 4$ and $w(S_1^*) \geqslant \dots \geqslant w(S_\ell^*)$. Observe that, without loss of generality, we can assume that there exist a connected component BP_{k_0} colored with ℓ colors and any connected component BP_i using j colors is colored with colors $1, \dots, j$. Moreover, we also suppose that in any connected component BP_j, each node colored with color $i \geqslant 2$ is adjacent to nodes with colors $1, \dots, i-1$ (by applying greedy rule on $\mathcal{S}^*$).

We claim that there exist $1 \leqslant i < j \leqslant \ell$ such that $S_k^* \cap L \neq \emptyset$ and $S_k^* \cap R \neq \emptyset$ for $k = i, j$.

Otherwise, since $\ell \geqslant 4$, we must have $S_{i_0}^* \subseteq L$ (resp., $S_{i_0}^* \subseteq R$) and $S_{j_0}^* \subseteq L$ (resp., $S_{j_0}^* \subseteq R$) for some $i_0 < j_0$. In this case, by merging $S_{i_0}^*$ with $S_{j_0}^*$, we obtain a better node coloring that $\mathcal{S}^*$, which is a contradiction.

So, consider connected component BP_{k_0} and let $l_j \in S_j^* \cap L$ and $r_j \in S_j^* \cap L$ two nodes of BP_{k_0}. From this claim, we deduce there exist 2 other nodes l_i, r_i of BP_{k_0} such that $l_i \in S_i^* \cap L$, $r_i \in S_i^* \cap L$ and $[l_i, r_j] \in E$, $[l_j, r_i] \in E$. Since BP is bipartite, these 2 edges are independents which is a contradiction with characterization of P_5-free bipartite graphs. □

Let $BP_1, \ldots, BP_p$ be the connected components of BP where $BP_i = (L_i, R_i; E_i)$. Let $\mathcal{S}^* = (S_1^*, S_2^*, S_3^*)$ (with maybe some $S_i^* = \emptyset$) be an optimal solution with $w(S_1^*) \geqslant w(S_2^*) \geqslant w(S_3^*)$ and denote by $\mathcal{S}_i^* = (S_1^{*,i}, S_2^{*,i}, S_3^{*,i})$ the restriction of $\mathcal{S}^*$ to the subgraph BP_i. Remark that we may assume $w(S_1^{*,i}) \geqslant w(S_2^{*,i}) \geqslant w(S_3^{*,i})$ (otherwise, we can flip the color without increasing the weight). Moreover, we have:

Lemma 3.6 *We can always assume that one of these situations occurs, for any* $i = 1, \ldots, p$*:*

(i) $S_1^{*,i} = L_i$ *(resp.,* $S_1^{*,i} = R_i$*),* $S_2^{*,i} = R_i$ *(resp.,* $S_2^{*,i} = L_i$*) and* $S_3^{*,i} = \emptyset$*.*

(ii) $S_1^{*,i} \cap L_i \neq \emptyset$ *and* $S_1^{*,i} \cap R_i \neq \emptyset$*,* $S_2^{*,i} \subset R_i$ *(resp.,* $S_2^{*,i} \subset L_i$*) and* $S_3^{*,i} \subset L_i$ *(resp.,* $S_3^{*,i} \subset R_i$*).*

Proof : Let $BP = (L, R; E)$ be a P_5-free bipartite graph with connected components $BP_1, \ldots, BP_p$. Assume that $S_1^{*,i} \cap L_i = \emptyset$ or $S_1^{*,i} \cap R_i = \emptyset$. In this case, it is clear that we are in the first item (i) (since we have assumed $w(S_1^{*,i}) \geqslant w(S_2^{*,i}) \geqslant w(S_3^{*,i})$). Now, suppose $S_1^{*,i} \cap L_i \neq \emptyset$ and $S_1^{*,i} \cap R_i \neq \emptyset$; from the proof of Lemma 3.5, the result follows. $\square$

The algorithm computing an optimal solution is described by the following way:

P_5-FREEBIPARTITECOLOR

1 For all $k_1, k_2 \in \{w(v) : v \in V\}$, $k_1 \geqslant k_2$, do

1.1 For all connected component $BP_i = (L_i, R_i; E_i)$, $i = 1, \ldots, p$, do

1.1.1 If $L_i \cup R_i \setminus (L'_i \cup R'_i)$ is an independent set where $L'_i = \{v \in L_i : w(v) \leqslant k_1\}$ and $R'_i = \{v \in R_i : w(v) \leqslant k_2\}$ then set $S_{2,i}^{k_1,k_2} = L'_i$, $S_{3,i}^{k_1,k_2} = R'_i$ and $S_{1,i}^{k_1,k_2} = L_i \cup R_i \setminus (L'_i \cup R'_i)$;

1.1.2 Otherwise, if $L_i \cup R_i \setminus (L'_i \cup R'_i)$ is an independent set where $L'_i = \{v \in L_i : w(v) \leqslant k_2\}$ and $R'_i = \{v \in R_i : w(v) \leqslant k_1\}$ then set $S_{2,i}^{k_1,k_2} = R'_i$, $S_{3,i}^{k_1,k_2} = L'_i$ and $S_{1,i}^{k_1,k_2} = L_i \cup R_i \setminus (L'_i \cup R'_i)$;

1.1.3 Otherwise go to step 1;

1.1.4 Set $S_j^{k_1,k_2} = \cup_{i=1}^{p} S_{j,i}^{k_1,k_2}$ for $j = 1, 2, 3$ and $\mathcal{S}^{k_1,k_2} = (S_1^{k_1,k_2}, S_2^{k_1,k_2}, S_3^{k_1,k_2})$ (with maybe $S_1^{k_1,k_2} = \emptyset$);

2 Output $\mathcal{S} = argmin\{val(\mathcal{S}^{k_1,k_2}) : k_2 \leqslant k_1\}$;

This algorithm has a complexity $O(n|w|^2)$ where $|w| = |\{w(v) : v \in V\}|$. By applying a dichotomy technic on k_2, we can improve it to $O(n|w|log|w|)$. Note that this algorithm also computes the best node 2-coloring among the colorings using at most 2 colors (when $k_1 = w_{max}$).

Theorem 3.7 MIN WEIGHTED NODE COLORING *is polynomial in P_5-free bipartite graphs.*

Proof : Let $\mathcal{S}^* = (S_1^*, S_2^*, S_3^*)$ (with may be $S_1^* = \emptyset$) be an optimal solution satisfying Lemmas 3.5 and 3.6. We assume $w(S_2^*) \geqslant w(S_3^*)$ and if $\mathcal{S}^*$ is a node 3-coloring, then we have $w(S_1^*) = w_{max}$; otherwise $w(S_1^*) = 0$. Let $k_1 = w(S_2^*)$ and $k_2 = w(S_3^*)$; consider the step of algorithm corresponding to k_1, k_2. If $\mathcal{S}^*$ is a node 2-coloring, then the result is true. So, assume $S_1^* \neq \emptyset$; by construction, P_5-FREEBIPARTITECOLOR find an feasible solution $\mathcal{S}^{k_1,k_2}$ with $w(S_1^{k_1,k_2}) \leqslant w_{max}$, $w(S_2^{k_1,k_2}) \leqslant k_1$ and $w(S_3^{k_1,k_2}) \leqslant k_2$. Thus, we deduce the expected result. □

3.3 Approximation

In Demange et al. [4], a $\frac{4}{3}$-approximation is given for MIN WEIGHTED NODE COLORING and it is proved that a $(\frac{8}{7} - \varepsilon)$-approximation is not possible, for any $\varepsilon > 0$, unless **P**=**NP**, even if we consider arbitrarily large values of $opt(I)$. Using Corollary 3.2, we deduce that this lower bound also holds if we consider bipartite planar graphs. Here, we give a $\frac{8}{7}$-approximation in bipartite graphs.

BIPARTITECOLOR

1 Sort the nodes of BP in non-increasing weight order (i.e., $w(v_1) \geqslant \ldots \geqslant w(v_n)$);

2 For $i = 1$ to n do

- 2.1 Set $V_i = \{v_1, \ldots, v_i\}$;
- 2.2 Compute $\mathcal{S}_i^* = (S_1^i, S_2^i)$ (S_2^i may be empty) an optimal weighted node coloring in $BP[V_i]$ among the colorings using at most two colors;
- 2.3 Define node coloring $\mathcal{S}^i = (S_1^i, S_2^i, L \setminus V_i, R \setminus V_i)$ ($L \setminus V_i$ or/and $R \setminus V_i$ may be empty);

3 Output $\mathcal{S} = argmin\{val(\mathcal{S}^i) : i = 1, \ldots, n\}$;

The step 2.2 consists of computing the (unique) 2-coloration $(S^*_{1,j}, S^*_{2,j})$ (with $w(S^*_{1,j}) \geqslant w(S^*_{2,j})$) of each connected component $BP_j, j = 1 \ldots p$ of $BP[V_i]$ (with $S^*_{2,j} = \emptyset$ if BP_j is an isolated node). Then it merges the most expensive sets, i.e. it computes $S^i_1 = \cup^p_{j=1} S^*_{i,j}$ for $i = 1, 2$. It is easy to observe that $\mathcal{S}^*_i = (S^i_1, S^i_2)$ is the best weighted node coloring of $BP[V_i]$ among the colorings using at most 2 colors; such a coloring can be found in $O(m)$ time where $m = |E|$.

Theorem 3.8 *Algorithm* `BIPARTITECOLOR` *polynomially solves in time $O(nm)$ the weighted node coloring problem in bipartite-graphs within approximation ratio bounded above by $\frac{8}{7}$.*

Proof : Let $I = (BP, w)$ be a weighted bipartite-graph where $BP = (L, R; E)$ and $\mathcal{S}^* = (S^*_1, ..., S^*_l)$ be an optimal node coloring of I with $w(S^*_1) \geq ... \geq w(S^*_l)$. If $l < 3$, then `BIPARTITECOLOR` finds an optimal weighted node coloring which is $\mathcal{S}^n$ (corresponding to the step $i = n$). Now, assume $l \geqslant 3$ and let $i_j = \min\{k : \ v_k \in S^*_j\}$. We have $i_1 = 1$ and

$$opt(I) \geqslant w(v_{i_1}) + w(v_{i_2}) + w(v_{i_3}) \tag{3.1}$$

Let us examine several steps of this algorithm:

- when $i = i_2 - 1$, the algorithm produces a node 3-coloring $\mathcal{S}^{i_2-1} = (S^1_{i_2-1}, L \setminus S^1_{i_2-1}, R \setminus S^1_{i_2-1})$. Indeed, by construction $V_{i_2-1} \subseteq S^*_1$ is an independent set, and then, $\mathcal{S}^*_{i_2-1}$ is defined by $S^{i_2-1}_1 = V_{i_2-1}$ and $S^{i_2-1}_2 = \emptyset$. Moreover, $\forall v \notin V_{i_2-1}$, $w(v) \leqslant w(v_{i_2})$ and then

$$val(\mathcal{S}^{i_2-1}) \leqslant w(v_{i_1}) + 2w(v_{i_2}) \tag{3.2}$$

- when $i = i_3 - 1$, the algorithm produces on $BP[V_{i_3-1}]$ a node 2-coloring $\mathcal{S}^*_{i_3-1}$ with a cost $val(\mathcal{S}^*_{i_3-1}) \leqslant w(v_{i_1}) + w(v_{i_2})$ since the coloring $(S^*_1 \cap V_{i_3-1}, S^*_2 \cap V_{i_3-1})$ is a feasible node 2-coloring of $BP[V_{i_3-1}]$ with cost $w(v_{i_1}) + w(v_{i_2})$. Finally, since the weights are sorted in non-increasing order, we obtain:

$$val(\mathcal{S}^{i_3-1}) \leqslant w(v_{i_1}) + w(v_{i_2}) + 2w(v_{i_3}) \tag{3.3}$$

- when $i = n$ (the last step), the algorithm just produced a node 2-coloring satisfying:

$$val(\mathcal{S}^n) \leqslant 2w(v_{i_1}) \tag{3.4}$$

Using (3.2), (3.3) and (3.4), we deduce:

$$val(\mathcal{S}) \leqslant \min\{2w(v_{i_1}); w(v_{i_1}) + w(v_{i_2}) + 2w(v_{i_3}); w(v_{i_1}) + 2w(v_{i_2})\} \tag{3.5}$$

The convex combination of these 3 values with coefficients $\frac{1}{7}$, $\frac{4}{7}$ and $\frac{2}{7}$ respectively and the inequality (3.1) give the expected result, i.e.:

$$val(\mathcal{S}) \leqslant \frac{1}{7}\times 2w(v_{i_1}) + \frac{4}{7}\times(w(v_{i_1}) + w(v_{i_2}) + 2w(v_{i_3})) + \frac{2}{7}\times(w(v_{i_1}) + 2w(v_{i_2})) \leqslant \frac{8}{7}opt(I)$$

□

4 Weighted edge coloring in bipartite graphs

The weighted edge coloring problem on a graph G can be viewed as the weighted node coloring problem on $L(G)$ where $L(G)$ is the line graph of G. Here, for simplicity, we refer to the edge model.

4.1 Complexity results

Demange et al. [4] have proved that MIN WEIGHTED EDGE COLORING in bipartite cubic graphs is strongly **NP**-complete and a lower bound of $\frac{8}{7}$ is given for the approximation. Here, we slightly improve these complexity results. Indeed, we show that weighted edge coloring in bipartite cubic planar graphs is strongly **NP**-complete and we deduce that it is **NP**-complete to obtain an approximation within a ratio $\frac{7}{6} - \varepsilon$, for any $\varepsilon > 0$.

Theorem 4.1 MIN WEIGHTED EDGE COLORING *is strongly **NP**-complete in bipartite cubic planar graphs.*

Proof : We shall reduce PREXT EDGE COLORING in bipartite cubic planar graphs to our problem. Given a bipartite cubic planar graph $BP = (V, E)$ and 3 pairwise disjoint matchings $E_1, \ldots, E_3$ of E, the question of PREXT EDGE COLORING is to determine if it is possible to extend the edge precoloring $E_1, \ldots, E_3$ to a proper edge 3-coloring of G. Very recently, this problem has been shown **NP**-complete in Marx [16].

Let $BP = (V, E)$ and $E_1, \ldots, E_3$ be an instance of PREXT EDGE COLORING; we construct an instance $I = (BP', w)$ of weighted edge coloring such that the answer of PREXT EDGE COLORING instance is yes if and only if there exists an edge coloring $\mathcal{S}$ of I with cost $val(\mathcal{S}) \leqslant 6$.

The construction of instance I is the following :

- Each edge in E_1 receives weight 3.
- Each edge $[x, y] \in E_2$ is replaced by a gadget F_2 described in Figure 4.1, where we identify x and y to v_0 and v_9 respectively.
- Each edge in E_3 is replaced by a gadget F_3 which is the same as gadget F_2 except that we have exchanged weights 1 and 2.
- The other edges of G receive weight 1.

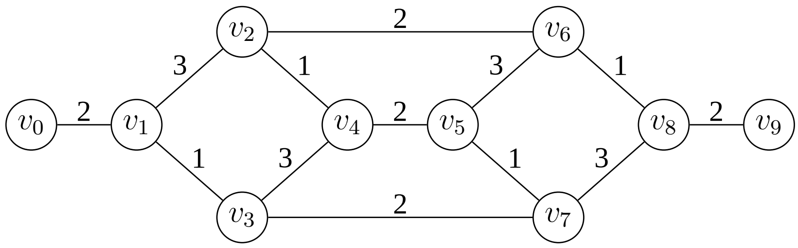

Figure 7: Gadget F_2 for $e \in E_2$.

Remark that BP' is still a bipartite cubic planar graph.

First of all, assume that BP admits an edge 3-coloring $\mathcal{S} = (M_1, M_2, M_3)$ where $E_i \subseteq M_i$ for any $i = 1, 2, 3$. We get a coloring $\mathcal{S}' = (M'_1, M'_2, M'_3)$ of BP' where $M'_1 = M_1 \cup \{e \in F_2 \cup F_3 : \ w(e) = 3\}$ and, for $i = 2, 3$, $M'_i = (M_i \setminus E_i) \cup \{e \in F_2 \cup F_3 : \ w(e) = 4 - i\}$. We can easily check that $opt(I) \leqslant val(\mathcal{S}') = 3 + 2 + 1 = 6$.

Conversely, consider an edge coloring $\mathcal{S}' = (M'_1, \dots, M'_\ell)$ of G' with $val(\mathcal{S}') \leqslant 6$ and assume $w(M'_1) \geqslant \dots \geqslant w(M'_\ell)$. We have $\ell \geqslant 3$ since $\Delta(BP') = 3$. Then, all the edges of weight 3 must be in the matching M'_1, and no edge of weight 2 is in M'_p with $p \geqslant 3$, since otherwise we have $val(\mathcal{S}') \geqslant 7$ ($3 + 3 + 1$ in the first case and $3 + 2 + 2$ in the second case). Moreover, each edge of weight 2 is adjacent to an edge of weight 3, and then, these edges are necessarily in M'_2. Finally, remark that the edges of the gadgets of weight 1 are adjacent to an edge of weight 2 and an edge of weight 3 and must be in M'_p with $p \geqslant 3$. Moreover, $p = 3$ and more generally $\ell = 3$ since $val(\mathcal{S}') \leqslant 6$. Now, consider the edge coloring (M_1, M_2, M_3) of BP where for any $i = 1, 2, 3$ we have $M_i = (M'_i \setminus \{e \in F_2 \cup F_3 : \ w(e) = 4 - i\}) \cup E_i$. We can easily see that (M_1, M_2, M_3) is a solution for the edge precoloring extension problem. $\square$

From the proof of Theorem 4.1, we deduce that computing an optimal weighted edge 3-coloring of a cubic bipartite graphs among edge 3-colorings is **NP**-complete. By the same technics, we can prove that more generally, finding an optimal weighted edge k-coloring of a cubic bipartite graphs among the edge colorings using at most k colors is **NP**-complete for any $k = 3, 4, 5$.

Corollary 4.2 *For all* $\varepsilon > 0$, MIN WEIGHTED EDGE COLORING *is not* $7/6 - \varepsilon$ *approximable in bipartite cubic planar graphs unless* ***P****=****NP****.*

4.2 Approximation result

In Demange et al. [4], a $\frac{5}{3}$-approximation is given for MIN WEIGHTED EDGE COLORING in bipartite graphs with maximum degree 3. Here, we give a $\frac{7}{6}$-approximation.

We need some notations: If $BP = [V, E]$ is a bipartite graph with node set $V = \{v_1, \ldots, v_n\}$, we always assume that its edges $E = \{e_1, \ldots, e_m\}$ are sorted in non-increasing weight order (i.e., $w(e_1) \geqslant \ldots \geqslant w(e_m)$). If V' is a subset of nodes and E' a subset of edges, $BP[V']$ and $BP[E']$ denote the subgraph of BP induced by V' and the partial graph of BP induced by E' respectively. For any $i \leqslant m$, we set $E_i = \{e_1, \ldots, e_i\}$ and $\overline{E_i} = E \setminus E_i$. Finally, V_i denotes the set of nodes of BP incident to an edge in E_i (so, it is the subset of non-isolated nodes of $BP[E_i]$).

Consider the following algorithm.

`BIPARTITEEDGECOLOR`

1 For $i = m$ downto 1 do

- 1.1 Apply algorithm `SOL1` on $BP[E_i]$;
- 1.2 If `SOL1`($BP[E_i]$)$\neq \emptyset$, complete in a greedy way all the colorings produced by `SOL1` on the edges of $\overline{E_i}$. Let $\mathcal{S}_{1,i}$ be a best one among these edge colorings of BP;
- 1.3 For $j = i$ downto 1 do
  - 1.3.1 Apply algorithm `SOL2` on $BP[E_j]$;
  - 1.3.2 If `SOL2`($BP[E_j]$)$\neq \emptyset$, complete in a greedy way all the colorings produced by `SOL2` on the edges of $\overline{E_j}$. Let $\mathcal{S}_{2,j,i}$ be a best one among these edge colorings of BP;
  - 1.3.3 Apply algorithm `SOL3` on $BP[E_j]$;
  - 1.3.4 If `SOL3`($BP[E_j]$)$\neq \emptyset$, complete in a greedy way all the colorings produced by `SOL3` on the edges of $\overline{E_j}$. Let $\mathcal{S}_{3,j,i}$ be a best one among these edge colorings of BP

2 Output $\mathcal{S} = argmin\{val(\mathcal{S}_{1,i}), val(\mathcal{S}_{k,j,i}) :\ k = 2, 3,\ j = 1, \ldots, i,\ i = 1, \ldots, m\}$.

The greedy steps 1.2, $1.2.2$ and $1.2.4$ can be described as follows: for each edge not yet colored, try to color it with an existing color, and otherwise take a new color. A simple argument shows that these edge colorings do not use more than 5 colors. Indeed, assume the reverse and let us consider an edge with color 6. Since the maximum degree of BP is 3, this edge is adjacent to at most 4 edges and then to at most 4 colors. Thus, we can recolor this edge with a missing color in $1, \ldots, 5$. Obviously, this result also holds for an optimal solution. More generally, in [4], we have proved that, in any graph G, there is an optimal weighted node coloring using at most $\Delta(G) + 1$ colors, where $\Delta(G)$ denotes the maximum degree of G. In our case, we have $G = L(H)$, the line graph of H, and we deduce $\Delta(L(H)) \leqslant 2(\Delta(H) - 1) + 1 = 2\Delta(H) - 1$.

The 3 algorithms `SOL1`, `SOL2` and `SOL3` are used on several partial graphs BP' of BP. In the following, V', E' and m' denote respectively the node set, the edge set and the number of edges of the current graph BP'. Moreover, we set $\overline{V_i'} = V' \setminus V_i'$ and $\overline{E_i'} = E' \setminus E_i'$. If $M = (M_1, \ldots, M_l)$ is an edge coloring of BP', we note $i_j = \min\{k : \; e_k \in M_j\}$ for $j = 1, \ldots, l$. We assume, for reason of readability, that some colors M_j may be empty (in this case $i_j = m' + 1$). The principle of these algorithms consist in finding a decomposition of BP' (a subgraph of BP) into two subgraph BP_1' and BP_2' having each a maximum degree 2. When there exists such a decomposition, we can color BP_1' and BP_2' with at most 2 colors respectively since BP is bipartite.

`SOL1`

1 For $j = m'$ downto 1 do

- 1.1 If the degree of $BP'[E_j']$ is at most 2 then
  - 1.1.1 Consider the graph BP'^j :
    - induced by the nodes of BP' incident to at least 2 edges of $\overline{E_j'}$;
    - restricted to the edges of $\overline{E_j'}$.
  - 1.1.2 Determine if there exists a matching M^j of BP'^j such that every node of $\overline{V_j'}$ is saturated;
  - 1.1.3 If such a matching is found, consider the decomposition $BP_{1,j}'$ and $BP_{2,j}'$ of BP' induced by $E_j' \cup M^j$ and $E' \setminus (E_j' \cup M^j)$ respectively;
  - 1.1.4 Find an optimal edge coloring (M_1^j, M_2^j) among the edge 2-colorings of $BP_{1,j}'$;
  - 1.1.5 Color greedily the edges of $BP_{2,j}'$ with two colors (M_3^j, M_4^j);
  - 1.1.6 Define $\mathcal{S}_1^j = (M_1^j, M_2^j, M_3^j, M_4^j)$ the edge coloring of BP';

2 **Output** $\{\mathcal{S}_1^j : j = 1, \ldots, m' - 1\}$**;**

Note that the step 1.1.2 is polynomial. Indeed, more generally, given a graph $G = [V, E]$ and a set $V' \subseteq V$, it is polynomial to determine if there exists a matching such that each node of V' is saturated. To see this, consider the graph G' where we add to G all missing edges between nodes of $V \setminus V'$. If $|V|$ is odd, then we add a node to the clique $V \setminus V'$. It is easy to see that G' has a perfect matching if and only if G has a matching such that each node of V' is saturated.

Lemma 4.3 *If $\mathcal{S} = (M_1, M_2, M_3, M_4)$ with $w(M_1) \geqslant \ldots \geqslant w(M_4)$ is an edge coloring of BP', then algorithm `SOL1` produces a solution $\mathcal{S}_1^j$ satisfying: $val(\mathcal{S}_1^j) \leqslant w(M_1) + w(M_2) + 2w(M_3)$*

Proof : Let $\mathcal{S} = (M_1, M_2, M_3, M_4)$ with $w(M_1) \geqslant \ldots \geqslant w(M_4)$ be an edge coloring of BP'. Let us examine the step of `SOL1` corresponding to $j = i_3 - 1$. By construction, $BP'[E'_{i_3-1}]$ is 2 edge colorable since we have $E'_{i_3-1} \subseteq M_1 \cup M_2$. Moreover, in the subgraph induced by $\overline{E'_{i_3-1}}$, each node of degree 3 has at least an edge of $M_1 \cup M_2$ incident to it. Thus, in BP'^j, there exists a matching where each node of $\overline{V'_{i_3-1}}$ is saturated. The subgraph BP'_{1,i_3-1} has a maximum degree 2 and contains by construction the subgraph $BP'[E'_{i_3-1}]$. Moreover, two any connected components of $BP'[E'_{i_3-1}]$ have not been merged in BP'_{1,i_3-1} since each edge $e = [x, y] \in M^{i_3-1}$ has at least one node (say x) satisfying $d_{BP'[E'_{i_3-1}]}(x) = 0$. Thus, any edge 2-coloring of $BP'[E_{i'_3-1}]$ can be extended to an edge 2-coloring of BP'_{1,i_3-1} . So, since $\forall e \in M^{i_3-1}, \forall e' \in E'_{i_3-1}\ w(e) \leqslant w(e')$, and $(M_1^{i_3-1}, M_2^{i_3-1})$ is an optimal weighted 2 edge coloring of BP'_{1,i_3-1}, we deduce:

$$w(M_1^{i_3-1}) + w(M_2^{i_3-1}) \leqslant w(M_1) + w(M_2) \tag{4.1}$$

By construction, BP'_{2,i_3-1} has no node with degree 3, and then BP'_{2,i_3-1} has a a maximum degree 2. Moreover, $\forall e \notin (M^{i_3-1} \cup E'_{i_3-1})$ we have $w(e) \leqslant w(e_{i_3}) = w(M_3)$. Thus, any edge coloring of BP'_{2,i_3-1} using at most 2 colors and in particular $(M_3^{i_3-1}, M_4^{i_3-1})$ satisfies:

$$w(M_3^{i_3-1}) + w(M_4^{i_3-1}) \leqslant 2w(M_3) \tag{4.2}$$

Combining (4.1) and (4.2), we obtain:

$$val(\mathcal{S}_1^{i_3-1}) \leqslant w(M_1) + w(M_2) + 2w(M_3)$$

□

`SOL2`

1 For $k = m'$ downto 1 do

  1.1 If E'_k is a matching :

   $1.1.1$ Determine if there exists a matching M_k of $BP'[\overline{V'_k}]$ such that each node of $BP'[\overline{V'_k}]$ having a degree 3 in BP' is saturated.

   $1.1.2$ If such a matching is found, consider the decomposition $BP'_{1,k}$ and $BP'_{2,k}$ of BP' induced by $E'_k \cup M_k$ and $E' \setminus (E'_j \cup M_k)$ respectively;

   $1.1.3$ Color $BP'_{1,k}$ with one color M_1^k;

   $1.1.4$ Color greedily $BP'_{2,k}$ with two colors M_2^k and M_3^k;

   $1.1.5$ Define $\mathcal{S}_2^k = (M_1^k, M_2^k, M_3^k)$ the edge coloring of BP';

2 Output $\{\mathcal{S}_2^k :\ k = 1, \ldots, m'\}$;

Lemma 4.4 *If $\mathcal{S} = (M_1, M_2, M_3)$ with $w(M_1) \geqslant w(M_2) \geqslant w(M_3)$ is an edge coloring of BP', then algorithm `SOL2` produces a solution $\mathcal{S}_2^k$ satisfying: $val(\mathcal{S}_2^k) \leqslant w(M_1) + 2w(M_2)$.*

Proof : Let $\mathcal{S} = (M_1, M_2, M_3)$ with $w(M_1) \geqslant w(M_2) \geqslant w(M_3)$ be an edge coloring of BP'. Let us examine the step of `SOL2` corresponding to $k = i_2 - 1$. By construction, $E'_{i_2-1} \subseteq M_1$ and among $M_1 \setminus E'_{i_2-1}$ there is a matching of $BP'[\overline{V'_{i_2-1}}]$ where each node of degree 3 is saturated (otherwise, $\mathcal{S} = (M_1, M_2, M_3)$ is not feasible). Thus, BP'_{1,i_2-1} can be considered and colored with one color $M_1^{i_2-1}$, and we have:

$$w(M_1^{i_2-1}) = w(M_1) \tag{4.3}$$

We also deduce that BP'_{2,i_2-1} has a maximum degree 2. Then, it can be edge colored with 2 colors $M_2^{i_2-1}$ and $M_3^{i_2-1}$. Moreover, since $\forall e \notin E'_{i_2-1}$, $w(e) \leqslant w(e_{i_2}) = w(M_2)$, we obtain:

$$w(M_2^{i_2-1}) + w(M_3^{i_2-1}) \leqslant 2w(M_2) \tag{4.4}$$

Using (4.3) and (4.4), we obtain:

$$val(\mathcal{S}_2^{i_2-1}) \leqslant w(M_1) + 2w(M_2)$$

□

`SOL3`

1 For $k = m'$ downto 1 do

- 1.1 Determine if there is a matching M_k in $BP'[\overline{E'_k}]$ such that each node of degree 3 in BP' is saturated.
- 1.2 If such a matching is found, consider the decomposition $BP'_{1,k}$ and $BP'_{2,k}$ of BP' induced by M_k and $E' \setminus M_k$ respectively;
- 1.3 Color $BP'_{1,k}$ with one color M_3^k;
- 1.4 Color greedily $BP'_{2,k}$ with two colors M_1^k and M_2^k;
- 1.5 Define $\mathcal{S}_3^k = (M_1^k, M_2^k, M_3^k)$ the edge coloring of BP';

2 Output $\{\mathcal{S}_3^k : k = 1, \ldots, m' - 1\}$;

Lemma 4.5 *If $\mathcal{S} = (M_1, M_2, M_3)$ with $w(M_1) \geqslant w(M_2) \geqslant w(M_3)$ is an edge coloring of BP', then algorithm `SOL3` produces a solution $\mathcal{S}_3^k$ satisfying: $val(\mathcal{S}_3^k) \leqslant 2w(M_1) + w(M_3)$*

Proof : Let $\mathcal{S} = (M_1, M_2, M_3)$ with $w(M_1) \geqslant w(M_2) \geqslant w(M_3)$ be an edge coloring of BP'. As previously, let us consider one particular iteration of `SOL3`. In this lemma, we study the case where $k = i_3 - 1$. By construction, we have $M_3 \subseteq \overline{E'_{i_3-1}}$ and M_3 contains a matching where each node of $BP'[\overline{E'_{i_3-1}}]$ having a degree 3 in BP' is saturated. Thus, BP'_{2,i_3-1} exists. Moreover, since $\forall e \in \overline{E'_{i_3-1}}$, $w(e) \leqslant w(e_{i_3}) = w(M_3)$, we obtain:

$$w(M_3^{i_3-1}) \leqslant w(M_3) \tag{4.5}$$

As previously, we deduce that BP'_{1,i_3-1} can be edge colored with 2 colors $M_1^{i_3-1}$ and $M_2^{i_3-1}$ and we have:

$$w(M_1^{i_3-1}) + w(M_2^{i_3-1}) \leqslant 2w(M_1) \tag{4.6}$$

Combining (4.5) and (4.6), we obtain:

$$val(\mathcal{S}_3^{i_3-1}) \leqslant 2w(M_1) + w(M_3)$$

□

Remark 4.6 *Observe that if a color $M_j^{i_3-1}$ is empty, then we can improve the bound : in this case, $val(\mathcal{S}_3^{i_3-1}) \leq 2w(M_1)$. This remark is also valid for algorithms* `SOL1` *and* `SOL2`*, and if several colors are empty. For* `SOL1` *for instance, if $M_2^{i_3-1}$ and $M_2^{i_3-1}$ are empty, then $val(\mathcal{S}_1^{i_3-1}) \leq w(M_1) + w(M_3)$.*

Theorem 4.7 `BIPARTITEEDGECOLOR` *produces a $\frac{7}{6}$ approximation for* MIN WEIGHTED EDGE COLORING *in bipartite graphs with maximum degree 3.*

Proof : Let $\mathcal{S}^* = (M_1^*, \ldots, M_5^*)$ with $w(M_1^*) \geqslant \ldots \geqslant w(M_5^*)$ be an optimal weighted edge coloring of BP. Denote by i_k^* the smallest index of an edge in M_k^* ($i_k^* = m + 1$ if the color is empty).

Consider the iteration of `BIPARTITEEDGECOLOR` corresponding to the cases $i = i_5^* - 1$ and $j = i_4^* - 1$. Then :

- applying lemma 4.3, we produce on $BP' = BP[E_i]$ an edge coloring of weight at most $w(M_1^*) + w(M_2^*) + 2w(M_3^*)$. Then the greedy coloring of the edges of $\overline{E_i}$ produces a coloring of weight at most

$$w(M_1^*) + w(M_2^*) + 2w(M_3^*) + w(M_5^*) \tag{4.7}$$

- Applying lemma 4.4, we produce on $BP' = BP[E_j]$ an edge coloring of weight at most $w(M_1^*) + 2w(M_2^*)$. Then the greedy coloring of the edges of $\overline{E_j}$ produces a coloring of weight at most

$$w(M_1^*) + 2w(M_2^*) + 2w(M_4^*) \tag{4.8}$$

- Applying lemma 4.5, we produce on $BP' = BP[E_j]$ an edge coloring of weight at most $2w(M_1^*) + w(M_3^*)$. Then the greedy coloring of the edges of $\overline{E_j}$ produces a coloring of weight at most

$$2w(M_1^*) + w(M_3^*) + 2w(M_4^*) \tag{4.9}$$

Note that if there is an empty color or several empty colors produced by one of the algorithms `SOL`i, then the bound are still valid. Indeed, for `SOL`3 for instance, according to Remark 4.6, the value of the coloring computed at step $j = i_3 - 1$ has a weight at most $2w(M_1^*)$, and the greedy step produces a coloring of value at most $2w(M_1^*) + 3w(M_4^*) \leqslant 2w(M_1^*) + w(M_3^*) + 2w(M_4^*)$.

Using (4.7), (4.8) and (4.9), we deduce that the coloring $\mathcal{S}$ computed by `BIPARTITE-EDGECOLOR` satisfies:

$$val(\mathcal{S}) \leqslant \min\{ \quad w(M_1^*) + w(M_2^*) + 2w(M_3^*) + w(M_5^*); \\ w(M_1^*) + 2w(M_2^*) + 2w(M_4^*); 2w(M_1^*) + w(M_3^*) + 2w(M_4^*)\} \quad (4.10)$$

The convex combination of these 3 values with coefficients $\frac{3}{6}$, $\frac{2}{6}$ and $\frac{1}{6}$ respectively and the inequality (4.10) give the expected result, that is:

$$w(\mathcal{S}) \leqslant \frac{7}{6}w(M_1^*) + \frac{7}{6}w(M_2^*) + \frac{7}{6}w(M_3^*) + w(M_4^*) + \frac{1}{2}w(M_5^*) \leqslant \frac{7}{6}opt(I)$$

□

5 Weighted node coloring in Split graphs

The split graphs are a class of graphs related to bipartite graphs. Formally, $G = (K_1, V_2; E)$ is a split graph if K_1 is a clique of G with size $|K_1| = n_1$ and V_2 is an independent set with size $|V_2| = n_2$. So, a split graph can be viewed as a bipartite graph where the left set is a clique. Since split graphs form a subclass of perfect graphs, the node coloring problem on split graphs is polynomial. On the other hand, in [4], it is proved that the weighted node coloring problem is strongly **NP**-complete in split graphs, even if the weights take only two values. Thus, we deduce that there is no fully polynomial time approximation scheme in such a class of graphs. Here, we propose a polynomial time approximation scheme using structural properties of optimal solutions. An immediate observation of split graphs is that any optimal node coloring $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$ satisfies $|K_1| \leqslant \ell \leqslant |K_1| + 1$ and any color S_i^* is a subset of V_2 with possibly one node of K_1. In particular, for any optimal node coloring $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$, there exists at most one index $i(\mathcal{S}^*)$ such that $S_{i(\mathcal{S}^*)}^* \cap K_1 = \emptyset$.

Lemma 5.1 *There is an optimal weighted node coloring $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$ with $w(S_1^*) \geqslant \ldots \geqslant w(S_\ell^*)$ and an index $i_0 \leqslant \ell + 1$ such that:*

- $\forall j < i_0\ S_j^* = \{v_j\} \cup \{v \in V_2 :\ v \notin \cup_{k=1}^{j-1} S_k^*$ *and* $[v, v_j] \notin E\}$ *for some* $v_j \in K_1$.
- $S_{i_0}^* = V_2 \setminus (S_1^* \cup \ldots \cup S_{i_0-1}^*)$.
- $\forall j > i_0\ S_j^* = \{v_j\}$ *for some* $v_j \in K_1$.

Proof : Let $G = (K_1, V_2; E)$ be a split graph and let $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$ with $w(S_1^*) \geqslant \ldots \geqslant w(S_\ell^*)$ be an optimal weighted node coloring of G. If $\ell = n_1$ (we recall that $n_1 = |K_1|$), then we set $i_0 = \ell + 1$ otherwise let i_0 be the unique i such that $S_i^* \cap K_1 = \emptyset$. We build set $S_i^{*'}$ by the following way:

- For $i = 1, \ldots, i_0 - 1$, $S_i^{*'} = \{v_i\} \cup \{v \in V_2 : \ v \notin \cup_{k=1}^{i-1} S_k^{*'}$ and $[v, v_i] \notin E\}$ where we assume that $S_i^* \cap K_1 = \{v_i\}$.
- $S_{i_0}^{*'} = V_2 \setminus (S_1^{*'} \cup \ldots \cup S_{i_0-1}^{*'})$.
- For $i = i_0 + 1, \ldots, \ell$, $S_i^{*'} = S_i^* \cap K_1$.

Thus, when $i_0 = \ell + 1$, the sets resulting from second and third items are empty. Let us prove that:

$$\forall i = 1, \ldots, \ell, \ w(S_i^{*'}) \leqslant w(S_i^*) \tag{5.1}$$

Since $w(S_1^*) \geqslant \ldots \geqslant w(S_\ell^*)$, we have $w(S_i^*) = \max\{w(v) : \ v \in K_1 \cup V_2 \setminus (S_1^* \cup \ldots \cup S_{i-1}^*)\}$. Moreover, by construction $\cup_{j=1}^{i-1} S_j^* \subseteq \cup_{j=1}^{i-1} S_j^{*'}$. Thus, the result follows.

Using inequality (5.1), we deduce that node coloring $\mathcal{S}^{*'} = (S_1^{*'}, \ldots, S_\ell^{*'})$ has a cost $val(\mathcal{S}^{*'}) \leqslant \sum_{i=1}^{\ell} w(S_i^*) = opt(I)$ and then, $\mathcal{S}^{*'}$ is an optimal weighted node coloring satisfying Lemma 5.1. □

SPLITNODECOLOR$_k$

1 For all subset $K_1' \subseteq K_1$ with $|K_1'| = p \leqslant k$ do

  1.1 For all bijection f: $\{1, \ldots, p\} \longmapsto K_1'$ do

    1.1.1 For $i = 1$ to p do

      1.1.1.1 Set $S_i^{K_1', f} = \{f(i)\} \cup \{v \in V_2 : \ v \notin \cup_{k=1}^{i-1} S_k^{K_1', f}$ and $[v, f(i)] \notin E\}$;

    1.1.2 Set $S_{p+1,f}^{K_1'} = V_2 \setminus (S_1^{K_1', f} \cup \ldots \cup S_p^{K_1', f})$;

    1.1.3 For $i = p + 2$ to $n_1 + 1$ (assume $K_1 \setminus K_1' = \{v_{p+2}, \ldots, v_{n_1+1}\}$) do

      1.1.3.1 Set $S_i^{K_1', f} = \{v_i\}$;

    1.1.4 Set $\mathcal{S}^{K_1', f} = (S_1^{K_1', f}, \ldots, S_{n_1+1}^{K_1', f})$;

2 Output $\mathcal{S} = argmin\{val(\mathcal{S}^{K_1', f})\}$;

This algorithm has a complexity-time $O(k! n^{k+1})$.

Theorem 5.2 *For all* $\varepsilon > 0$, *SPLITNODECOLOR*$_{\lceil \frac{1}{\varepsilon} \rceil}$ *produces a* $1+\varepsilon$ *approximation for* MIN WEIGHTED NODE COLORING *in split graphs.*

Proof : Let $G = (K_1, V_2; E)$ be a split graph and let $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$ with $w(S_1^*) \geqslant \ldots \geqslant w(S_\ell^*)$ be an optimal weighted node coloring of G satisfying Lemma 5.1. Let $k = \lceil \frac{1}{\varepsilon} \rceil$. If $i_0 \leqslant k$, then by construction the solution $\mathcal{S}$ returned by SPLITNODECOLOR$_k$ is optimal. So, assume $i_0 > k$ and let $K_1^{*'} = (\cup_{j=1}^{k} S_j^*) \setminus V_2$. Obviously, $|K_1^{*'}| = k$ and let $f^*(i) = S_i^* \cap K_1$ for $i = 1, \ldots, k$.

Let us examine the solution $\mathcal{S}^{K_1^{*'}, f^*}$ corresponding to the step $K_1' = K_1^{*'}$ and $f = f^*$ of SPLITNODECOLOR$_k$. By construction, we have

$$\forall i = 1, \ldots, k, \; S_i^{K_1^{*'}, f^*} = S_i^* \tag{5.2}$$

Moreover, since $K_1 \setminus K_1^{*'} \subseteq S_{k+1}^* \cup \ldots \cup S_\ell^*$ and $K_1 \setminus K_1^{*'}$ is a clique, we obtain:

$$\sum_{j=k+2}^{n_1+1} w(S_i^{K_1^{*'}, f^*}) \leqslant \sum_{j=k+1}^{\ell} w(S_i^*) \tag{5.3}$$

Thus, combining (5.2) and (5.2), we deduce:

$$val(\mathcal{S}^{K_1^{*'}, f^*}) - w(S_{k+1}^{K_1^{*'}, f^*}) \leqslant opt(I) \tag{5.4}$$

Moreover, by construction $w(S_{k+1}^{K_1^{*'}, f^*}) \leqslant w(S_k^*) \leqslant \ldots \leqslant w(S_1^*)$ and then

$$w(S_{k+1}^{K_1^{*'}, f^*}) \leqslant \frac{1}{k} \times opt(I) \tag{5.5}$$

Finally, using these two last inequalities with $\frac{1}{k} \leqslant \varepsilon$, we obtain the expected result. □

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