

# Overlapping sets of priors and the existence of efficient allocations and equilibria for risk measures

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November 15, 2007

## Abstract

The overlapping expectations and the collective absence of arbitrage conditions introduced in the economic literature to insure existence of Pareto optima and equilibria with short-selling when investors have a single belief about future returns, is reconsidered. Investors use measures of risk. The overlapping sets of priors and the Pareto equilibrium conditions introduced by Heath and Ku for coherent risk measures are respectively reinterpreted as a weak no-arbitrage and a weak collective absence of arbitrage conditions and shown to imply existence of Pareto optima and Arrow Debreu equilibria.

**Keywords:** Overlapping sets of priors, collective absence of arbitrage, equilibria with short-selling, measures of risk.

**JEL Classification:** C62, D50.

We thank M.H Nguyen for comments and an anonymous referee for many helpful suggestions.

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# 1 Introduction

The problem of the existence and characterization of Pareto optima and equilibria in markets with short-selling, has recently been addressed by Barrieu and El Karoui [?], Jouini et al [?], Filipovic and Kupper [?] and Burgert and Rüschemdorf [?] for convex measures of risk in infinite markets. Existence of an equilibrium for finite markets with short-selling is an old problem in the economic literature. It has first been considered in the early seventies by Grandmont [?], Hart [?] and Green [?] since Debreu's standard theorems on existence of equilibrium could not be applied, investors' sets of portfolios being unbounded below. In these early papers, investors were assumed to hold a single homogeneous or heterogeneous probabilistic belief and be von Neumann-Morgenstern risk averse utility maximizers. Two sufficient conditions for existence of an equilibrium were given:

- the *overlapping expectations* condition which expresses that investors are sufficiently similar in their beliefs and risk tolerances so that there exists a non empty set of prices (*the no-arbitrage prices*) for which no agent can make costless unbounded utility nondecreasing purchases
- *the no unbounded utility arbitrage* condition, a collective absence of arbitrage condition, which requires that investors do not engage in mutually compatible, utility nondecreasing trades.

These conditions have later been weakened and shown to be equivalent under adequate conditions and under further assumptions, necessary for existence of equilibrium (see e.g. Page [?], Page and Wooders [?]). They have been generalized to abstract economies (see Werner [?] and Nielsen [?]). Other sufficient conditions were given. For a review of the subject in finite dimension, see Al-louch et al [?], Dana et al [?], Page [?],[?]. The theory has also been developed for infinite markets but the conditions given above do not generalize to the infinite dimension (see for example Brown and Werner [?]) and it is in general difficult to provide sufficient conditions on the primitives of an economy to have an equilibrium.

This paper provides sufficient conditions for existence of Pareto optima and equilibria when agents use convex measures of risk in *finite markets with short-selling*. In contrast, with the papers of Barrieu and El Karoui [?] who deal with families of  $\rho$ -dilated risk measures and Jouini et al [?] who consider law invariant convex monetary utilities, it makes no specific assumptions on the risk measures. However it assumes that there is a finite number of states of the world and uses finite dimensional convex analysis techniques. It builds on one hand, on the economic literature on equilibrium with short-selling and on the other hand on a paper by Heath and Ku [?]. Heath and Ku [?] introduced a condition now denoted *HKPE* that they called the *Pareto equilibrium* condition which requires

that if investors do engage in mutually compatible, utility nondecreasing trades, then those trades do not increase their utilities. They showed, for a subclass of measures of risk, the equivalence between *HKPE* and an *overlapping sets of priors* condition (see their proposition 4.2). They however did not address the question of existence of Pareto optima and equilibria. This paper makes two main contributions. The first is to relate Heath and Ku's *HKPE* and overlapping sets of priors condition to a weakening of the no unbounded utility arbitrage condition and to a weakening of the no-arbitrage price condition mentioned above. The second is to show that these conditions are sufficient conditions for existence of Pareto optima and Arrow Debreu equilibria. Two types of proof are provided, one uses the sup-convolution, the other applies standard results in the theory of equilibrium with short-selling. Following Heath and Ku [?], the case of constraints is also considered.

The paper is organized as follows. Section 2 presents the model and recall concepts in equilibrium theory. Two concepts of Pareto optima are introduced, one for complete preferences represented by measures of risk, the other for incomplete preferences associated to agents' priors. Section 3 contains the main results of the paper, the equivalence between *HKPE* and an overlapping sets of priors condition. These conditions are then shown to be sufficient for existence of an efficient allocation. A first proof of existence of efficient allocations is given by using the sup-convolution. Finally necessary conditions for existence of an efficient allocation are given. Section 4 relates the overlapping sets of priors condition and *HKPE* to the theory of arbitrage and equilibrium. Another proof of existence of efficient allocations and equilibria based on general equilibrium techniques is provided. Section 5 deals with the case of constraints on trades.

## 2 The model

We consider a standard Arrow-Debreu model of complete contingent security markets. There are two dates, 0 and 1. At date 0, there is uncertainty about which state  $s$  from a state space  $\Omega = \{1, \dots, k\}$  will occur at date 1. At date 0, agents trade contingent claims for date 1. The space of contingent claims is the set of random variables from  $\Omega \rightarrow \mathbb{R}$ . The random variable  $X$  which equals  $x_1$  in state 1,  $x_2$  in state 2 and  $x_k$  in state  $k$ , is identified with the vector  $X = (x_1, \dots, x_k)$ . Let  $\Delta = \{\pi \in \mathbb{R}_+^k : \sum_{s=1}^k \pi_s = 1\}$  be the probability simplex in  $\mathbb{R}^k$  and  $\pi \in \Delta$ . We note  $E_\pi(X) := \sum_{l=1}^k \pi_l x_l$  and for  $p \in \mathbb{R}^k$ ,  $p \cdot X := \sum_{l=1}^k p_l x_l$ .

There are  $m$  agents indexed by  $i = 1, \dots, m$ . Agent  $i$  has an endowment  $E^i \in \mathbb{R}^k$  of contingent claims. Let  $E = \sum_{i=1}^m E^i$  denote aggregate endowment.

We assume that each agent has a preference order  $\succeq_i$  over  $\mathbb{R}^k$  represented by a monetary utility function  $V^i$  where we recall that

**Definition 1** *A function  $V : \mathbb{R}^k \rightarrow \mathbb{R}$  is a monetary utility function if it is concave monotone and has the cash invariance property*

$$V(X + C) = V(X) + C, \text{ for any } X \in \mathbb{R}^k, C \text{ constant}$$

*A positively homogeneous monetary utility function is a monetary utility function that is positively homogeneous.*

Monetary utility functions can be identified with *convex measures of risk* (see Föllmer and Schied [?]) and positively homogeneous monetary utility functions with *coherent risk measures* (see Artzner et al [?]) by defining  $\rho = -V$ . We recall that monetary utility functions have the following representation

$$V(X) = \min_{\pi \in \Delta} E_{\pi}(X) + c(\pi) \quad (1)$$

where

$$c(\pi) = \sup_{X \in \mathbb{R}^k} V(X) - E_{\pi}(X) \in \mathbb{R} \cup \{+\infty\} \quad (2)$$

which is convex, lower semi-continuous, is the conjugate function of  $V$ . Let

$$P = \text{dom } c = \{\pi \in \Delta \mid c(\pi) < \infty\} \quad (3)$$

be the set of effective priors associated with  $V$ . Clearly, we also have:

$$V(X) = \min_{\pi \in P} E_{\pi}(X) + c(\pi) \quad (4)$$

Positively homogeneous monetary utility functions are obtained when  $c$  is an indicator function  $\delta_P$  (in other words,  $c(\pi) = 0$  if  $\pi \in P$  and  $c(\pi) = \infty$  otherwise). In that case,  $P = \{\pi \in \Delta : c(\pi) = 0\}$  is a convex compact subset of  $\Delta$  and we have  $V(X) = \min_{\pi \in P} E_{\pi}(X)$ .

We next recall standard concepts in equilibrium theory. An allocation  $(X^i)_{i=1}^m \in (\mathbb{R}^k)^m$  is attainable if  $\sum_{i=1}^m X^i = E$ . A trade  $(W^i)_{i=1}^m \in (\mathbb{R}^k)^m$  is feasible if  $\sum_{i=1}^m W^i = 0$ . The set of individually rational attainable allocations  $A$  is defined by

$$A = \left\{ (X^i)_{i=1}^m \in (\mathbb{R}^k)^m \mid \sum_{i=1}^m X^i = E \text{ and } V^i(X^i) \geq V^i(E^i) \text{ for all } i \right\}.$$

**Definition 2** *An attainable allocation  $(X^i)_{i=1}^m$  is Pareto optimal if there exists no feasible trade  $(W^i)_{i=1}^m$  such that  $V^i(X^i + W^i) \geq V^i(X^i)$  for all  $i$  with a strict inequality for some  $i$ . It is individually rational Pareto optimal if it is Pareto optimal and  $V^i(X^i) \geq V^i(E^i)$  for all  $i$ .*

**Definition 3** A pair  $(X^*, p^*) \in A \times \mathbb{R}^k \setminus \{0\}$  is a contingent Arrow-Debreu equilibrium if

1. for each agent  $i$  and  $X^i \in \mathbb{R}^k$ ,  $V^i(X^i) > V^i(X^{i*})$  implies  $p^* \cdot X^i > p^* \cdot X^{i*}$ ,
2. for each agent  $i$ ,  $p^* \cdot X^{i*} = p^* \cdot E^i$ .

Assertions 1 and 2 express that  $X^{i*}$  solves investor's  $i$  maximization problem at price  $p^*$ . Markets clear since  $X^*$  is attainable.

We also define a weaker concept of Pareto optimality under incomplete preferences. Let  $P$  be a set of priors. Consider the following incomplete preferences on pairs  $(X, Y) \in \mathbb{R}^k \times \mathbb{R}^k$  defined by

$$X \succeq_P Y \quad \text{iff} \quad E_\pi(X) \geq E_\pi(Y) \text{ for all } \pi \in P \quad (5)$$

Given  $\mathcal{P} = (P^i)_{i=1}^m$  a family of set of priors, an attainable allocation  $(X^i)_{i=1}^m$  is  $\mathcal{P}$ -Pareto optimal if there exists no feasible trade  $(W^i)_{i=1}^m$  such that  $E_\pi(X^i + W^i) \geq E_\pi(X^i)$  for all  $i$  and all  $\pi \in P^i$  with a strict inequality for some  $i$  and some  $\pi \in P^i$ . Equivalently:

*HKPE: there exists no feasible trade  $(W^i)_{i=1}^m$  such that  $E_\pi(W^i) \geq 0$  for all  $i$  and all  $\pi \in P^i$  with a strict inequality for some  $i$  and some  $\pi \in P^i$ .*

Hence, for  $m$  incomplete preferences defined by (??), either HKPE is fulfilled and any attainable allocation is  $\mathcal{P}$ -Pareto optimal or HKPE is not fulfilled and there exists no  $\mathcal{P}$ -Pareto optimal allocation.

### 3 Existence of efficient allocations and equilibria

Given a convex subset  $A \subseteq \mathbb{R}^p$ , the relative interior of  $A$ ,  $\text{ri } A$ , is the interior which results when  $A$  is regarded as a subset of its affine hull  $\text{aff } A$ .

#### 3.1 Heath and Ku's Pareto equilibrium condition

Heath and Ku [?] introduced the HKPE condition for a subclass of risk measures and called it the *Pareto equilibrium* condition. They showed its equivalence with non-emptiness of the intersection of the relative interiors of agents' sets of priors (see their proposition 4.2). They however did not address the question of existence of Pareto optima and equilibria in the sense of definitions 2 and 3. The next theorem which contains the main result of the paper, may be viewed as an elaboration of Heath and Ku's [?] proposition 4.2. It establishes that HKPE is a sufficient condition for existence of a Pareto allocation or equivalently of an equilibrium for monetary utilities.

**Theorem 1** *Let  $V^i$  fulfill (??) for each  $i$ . Then the following are equivalent:*

1.  $\bigcap_i \text{ri } P^i \neq \emptyset$ ,
2. there exists no feasible trade  $W^1, \dots, W^n$ , with  $E_\pi(W^i) \geq 0$  for all  $\pi \in P^i$  and all  $i$  with a strict inequality for some  $i$  and  $\pi \in P^i$ ,
3. Any attainable allocation is  $P$ -Pareto optimal.  
Any of the previous assertions imply any of the following assertions:
4. there exists an individually rational Pareto optimal allocation,
5. there exists an equilibrium.

The equivalence between 2 and 3 follows from the definition of  $\mathcal{P}$ -Pareto optima, that between 1 and 2 is proven in lemma ?? below. 5 implies 4 follows from the first welfare theorem. In the remainder of the paper, we shall provide two types of proofs for 1 implies 4 or 5. The first uses the sup-convolution, the second builds on general equilibrium techniques.

**Lemma 1** *Let  $(P^i)_{i=1}^m$  be a family of convex sets of probabilities. Then the following are equivalent:*

1.  $\bigcap_i \text{ri } P^i \neq \emptyset$ ,
2. there exists no feasible trade  $W^1, \dots, W^n$ , with  $E_\pi(W^i) \geq 0$  for all  $\pi \in P^i$  and all  $i$  with a strict inequality for some  $i$  and  $\pi \in P^i$ .

**Proof:** We first have that  $\bigcap_i \text{ri } P^i \neq \emptyset$  iff  $\text{cone } \bigcap_i \text{ri } P^i \neq \emptyset$ , equivalently iff

$$\text{cone } \bigcap_i \text{ri } P^i = \bigcap_i \text{cone ri}(P^i) = \bigcap_i \text{ri cone}(P^i) \neq \emptyset,$$

the last equality following from Rockafellar's [?] corollary 6.6.1. Let  $(f^i)_{i=1}^m$  be a family of convex functions with domain  $\text{ri cone } P^i$  for  $i = 1, \dots, m$ . From Rockafellar's [?] corollary 16.2.2, the condition  $\bigcap_i \text{ri cone } P^i \neq \emptyset$  is equivalent to the inexistence of a feasible trade  $(Z^i)_{i=1}^m$  such that

$$\sum_i \sup_{\pi \in P^i, \lambda_i \geq 0} \lambda_i E_{\pi^i}(Z^i) \leq 0 \tag{6}$$

$$\sum_i \sup_{\pi \in P^i, \lambda_i \geq 0} \lambda_i E_{\pi^i}(-Z^i) > 0 \tag{7}$$

Since (??) is equivalent to  $E_{\pi^i}(Z^i) \leq 0$ , for all  $i$  and  $\pi \in P^i$ ,  $\bigcap_i \text{ri } P^i \neq \emptyset$  is thus equivalent to the inexistence of a feasible trade  $(Z^i)_{i=1}^m$  such that  $E_{\pi^i}(Z^i) \leq 0$ , for all  $i$  and  $\pi \in P^i$  with a strict inequality for some  $i$  and  $\pi \in P^i$  which is assertion 2. ■

**Corollary 1** *Let  $V^i$  fulfill (??) for all  $i$  and  $P^i$  be independent of  $i$ . Then there exists an individually rational Pareto optimal allocation and an equilibrium.*

**Proof:** Let  $P$  denote the common set of priors. Since  $P$  is convex,  $\text{ri } P \neq \emptyset$ . ■

### 3.2 Sup-convolution

We now provide a proof based on the sup-convolution. This approach to efficient sharing has been used by Barrieu and El Karoui [?], Filipovic and Kupper [?], Jouini et al [?], Burgert and Rüschemdorf [?] in an infinite dimensional framework.

As is well known, from the monetary invariance, an attainable allocation is Pareto optimal for aggregate endowment  $E$  if and only if it solves the following problem:

$$\sup \sum_{i=1}^m V^i(X^i) \text{ subject to } \sum_{i=1}^m X^i = E.$$

For  $X \in \mathbb{R}^k$ , let  $\square_i V^i(X) = \sup \left\{ \sum_{i=1}^m V^i(X^i), \sum_{i=1}^m X^i = X \right\}$  be the sup-convolution of the  $V^i$ . Since  $V^i$  is finite for every  $i$ ,  $\square_i V^i(X) > -\infty$  and  $\text{dom } \square_i V^i = \mathbb{R}^k$  if and only if  $\bigcap_i \text{dom } c^i = \bigcap_i P^i \neq \emptyset$ . In that case,  $\square_i V^i$  is a monetary utility (the representative agent's utility when aggregate endowment is  $X$ ) and  $\square_i V^i$  and  $\sum_{i=1}^m c^i$  are conjugate. Furthermore, from Rockafellar's theorem 16.4 [?], a sufficient condition for existence of a Pareto optimum  $(X^1, \dots, X^m)$  is that

$$\bigcap_i \text{ri dom } c^i = \bigcap_i \text{ri } P^i \neq \emptyset. \quad (8)$$

We have thus proven that assertion 1 in theorem ?? implies assertion 4. Let us now show, that assertion 4 implies existence of an equilibrium. The proof provided does not use a fixed point theorem, contrary to the standard proofs of existence.

Let us first remark that  $\pi \in \partial \square_i V^i(X)$  iff  $\pi \in \bigcap_i \partial V^i(X^i)$  for any Pareto optimum  $(X^1, \dots, X^m)$  associated with  $X$ . Indeed,

$$\pi \in \partial \square_i V^i(X) \text{ iff } \square_i V^i(X) = \sum_{i=1}^m c^i(\pi) + E_\pi(X).$$

Since  $\square_i V^i(X) = \sum_{i=1}^m V^i(X^i)$  for any Pareto optimum  $(X^1, \dots, X^m)$  associated with  $X$  and  $c^i(\pi) + E_\pi(X^i) - V^i(X^i) \geq 0$ , for all  $\pi \in \Delta$ , we obtain that  $V^i(X^i) = c^i(\pi) + E_\pi(X^i)$  for all  $i$ , equivalently,  $\pi \in \bigcap_i \partial V^i(X^i)$ .

Therefore, a pair  $((X^*)_{i=1}^m, p^*) \in A \times \mathbb{R}^k \setminus \{0\}$  is a contingent Arrow-Debreu equilibrium, when aggregate endowment is  $E$  iff

1.  $(X^{*i})_{i=1}^m$  is Pareto optimal,
2.  $p^* \in \lambda \partial \square_i V^i(E)$  for some  $\lambda > 0$ ,
3.  $p^* \cdot X^{*i} = p^* \cdot E^i$  for all  $i$ .

As remarked by Filipovic and Kupper [?], given a Pareto optimum  $(X^1, \dots, X^m)$  and any  $p \in \lambda \partial \square_i V^i(E)$ , we have that  $(X^1 + p \cdot (E^1 - X^1), \dots, X^m + p \cdot (E^m - X^m), p)$  is an equilibrium. Indeed,  $(X^1 + p \cdot (E^1 - X^1), \dots, X^m + p \cdot (E^m - X^m))$  is Pareto optimal since

$$\sum_{i=1}^m V^i(X^i + p \cdot (E^i - X^i)) = \sum_{i=1}^m V^i(X^i) + \sum_{i=1}^m p \cdot (E^i - X^i) = \sum_{i=1}^m V^i(X^i)$$

since  $(X^1, \dots, X^m)$  is attainable. By construction  $p$  fulfills assertion 2 and  $p \cdot (X^i + p \cdot (E^i - X^i)) = p \cdot E^i$  for all  $i$ .

### 3.3 Necessary conditions for existence

Theorem ?? provides sufficient conditions for existence of efficient allocation. We next give necessary conditions for existence of an efficient allocation .

**Proposition 1** *Let  $V^i$  fulfill (?? ) for each  $i$ . If there exists an efficient allocation, then*

1.  $\bigcap_i P^i \neq \emptyset$ ,
2. *there exists no feasible trade  $W^1, \dots, W^n$  fulfilling  $E_\pi(W^i) > 0$  for all  $\pi \in P^i$  and for all  $i$ .*

**Proof:** To prove assertion one, if  $X^*$  is efficient, then for every  $i$ , there exists  $\lambda^i > 0$ , such that  $\bigcap_i \lambda^i \delta V^i(X^{i*}) \neq \emptyset$ . As  $\delta V^i(X^{i*}) \subseteq P^i$ , for each  $i$ , there exists  $\pi^i \in P^i$  such that  $\lambda^i \pi^i$  is independent of  $i$ . Hence  $\lambda^i$  and  $\pi^i$  are independent of  $i$  and  $\pi \in \bigcap_i P^i \neq \emptyset$  as was to be proven. To show the second, if there exists a feasible trade  $W^1, \dots, W^n$  fulfilling  $E_\pi(W^i) > 0$  for all  $\pi \in P^i$  and for all  $i$ , then  $V^i(X^i + W^i) > V^i(X^i)$  for all  $i$  contradicting the existence of an efficient allocation ■

**Remark 1** 1. *Assertions 1 and 2 of proposition ?? are weaker than assertions 1 and 2 of theorem ??.*

2. *If  $V^i$  is coherent for any  $i$ , then  $P^i$  is convex compact for any  $i$ . From Samet [?], assertions 1 and 2 of proposition ?? are then equivalent.*

Let us consider the expected utility case with a common prior.



**Corollary 2** *Let  $V^i$  fulfill (??) for each  $i$ . Let  $P^i = \{\pi^i\}$  for all  $i$ . Then there exists an equilibrium if and only if  $\pi^i$  is independent of  $i$ .*

**Proof:** The sufficient condition follows from corollary ?? while the necessary condition follows from proposition ??. ■

## 4 Relation with equilibrium with short-selling

In this section, we provide an alternate proof of existence of an equilibrium using equilibrium with short-selling techniques. We thus recall a number of standard concepts. We first define and characterize for monetary utilities the useful and useless claims. We next define the concept of weak no-arbitrage price as a price giving strictly positive value to any useful and not useless vector. We finally show that HKPE coincides in the case of monetary utilities with the concept of collective absence of arbitrage introduced by Hart [?].

### 4.1 Useful vectors

Let  $C \subseteq \mathbb{R}^k$  be a non-empty convex set. The asymptotic cone of  $C$  is the set

$$\{W \in \mathbb{R}^k \mid X + \lambda W \in C, \text{ for all } X \in C \text{ and } \lambda \geq 0\}.$$

Let  $V$  be a monetary utility and  $X \in \mathbb{R}^k$ . Let

$$\widehat{Q}(X) = \{Y \in \mathbb{R}^k \mid V(Y) \geq V(X)\}$$

be the preferred set at  $X$  and let  $R(X)$  be its asymptotic cone. Since  $V$  is concave, by Rockafellar's theorem 8.7 in [?],  $R(X)$  is independent of  $X$  and will simply be denoted by  $R$ . It is called the set of *useful vectors* for  $V$  in the economic literature. We thus have

$$R = \left\{ W \in \mathbb{R}^k \mid V(\lambda W) \geq V(0), \text{ for all } \lambda \geq 0 \right\}.$$

The lineality space of  $V$  or set of *useless vectors* is defined by

$$L = \{W \in \mathbb{R}^k \mid V(\lambda W) \geq V(0), \text{ for all } \lambda \in \mathbb{R}\} = R \cap (-R).$$

We first characterize  $R$  and  $L$ .

**Proposition 2** *We have*

$$R = \{W \in \mathbb{R}^k \mid E_\pi(W) \geq 0, \text{ for all } \pi \in P\}$$

$$L = \{W \in \mathbb{R}^k \mid E_\pi(W) = 0, \text{ for all } \pi \in P\}$$

**Proof:** Let  $W$  fulfill  $E_\pi(W) \geq 0$  for all  $\pi \in P$ . Then

$$V(\lambda W) = \min_{\pi \in P} E_\pi(\lambda W) + c(\pi) \geq \min_{\pi \in P} c(\pi) = V(0) \text{ for all } \lambda \geq 0$$

which implies that  $W \in R$ . Conversely, let  $W \in R$ . Then

$$V(\lambda W) \geq V(0), \text{ for all } \lambda \geq 0,$$

hence  $E_\pi(\lambda W) + c(\pi) \geq V(0)$ , for all  $\lambda \geq 0$ ,  $\pi \in P$ . For a fixed  $\pi \in P$ , the map from  $\mathbb{R}_+$  into  $\mathbb{R}$ ,  $\lambda \rightarrow \lambda E_\pi(W)$  is bounded below, hence  $E_\pi(W) \geq 0$ . The other assertion is straightforward. ■

In the following subsections,  $R^i$  and  $L^i$  will denote respectively the set of useful and useless vectors for agent  $i$ .

## 4.2 Concepts of absence of arbitrage

A no-arbitrage price for agent  $i$  is a price giving strictly positive value to any useful vector for  $i$ . The existence of a no-arbitrage price for  $i$  being incompatible with that of a useless vector, we use a weaker no-arbitrage concept due to Werner [?].

**Definition 4** *A price vector  $p \in \mathbb{R}^k$  is a "weak no-arbitrage price" for agent  $i$  if  $p \cdot W > 0$  for all  $W \in R^i \setminus L^i$ . A price vector  $p \in \mathbb{R}^k$  is a "weak no-arbitrage price" for the economy if it is a weak no-arbitrage price for each agent.*

The polar of  $A$  is defined by  $A^0 = \{p \in \mathbb{R}^p \mid p \cdot X \leq 0, \text{ for all } X \in A\}$ .

Let  $S_w^i$  denote the set of weak no arbitrage prices for  $i$  and  $\cap_i S_w^i$  the set of weak no arbitrage prices for the economy. We have:

**Proposition 3** *Let  $V^i$  fulfill (??) for each  $i$ . Then*

1.  $S_w^i = \text{ri} - (R^i)^0 = \text{cone ri } P^i$ .
2. *The set of weak no arbitrage prices for the economy is  $\cap_i S_w^i = \text{cone } \cap_i \text{ri } P^i$ .*

**Proof:** From proposition ??,  $R^i = \{W \in \mathbb{R}^k \mid E_\pi(W) \geq 0, \text{ for all } \pi \in P^i\}$ . From Allouch et al, lemma 2,  $S_w^i = \text{ri} - (R^i)^0$ . Therefore

$$S_w^i = \text{ri} - (R^i)^0 = \text{ri cl cone } P^i = \text{ri cone } P^i = \text{cone ri } P^i, \quad (9)$$

the third and fourth equalities following from Rockafellar's [?] theorem 6.3 and corollary 6.6.1. Hence the set of weak no arbitrage prices for the economy is  $\cap_i S_w^i = \cap_i \text{cone}(\text{ri } P^i) = \text{cone } \cap_i \text{ri } P^i$ . ■

We now turn to a concept of *collective absence of arbitrage* introduced by Hart [?] which requires that any utility nondecreasing feasible trade be useless.

**Definition 5** *The economy satisfies the Weak No-Market-Arbitrage condition (WNMA) if  $\sum_i W^i = 0$  and  $W^i \in R^i$  for all  $i$  implies  $W^i \in L^i$ , for all  $i$ .*

The following proposition follows directly from proposition ??.

**Proposition 4** *Let  $V^i$  fulfill (??) for each  $i$ . Then the economy satisfies WNMA if there exists no feasible trades  $W^1, \dots, W^n$  with  $E_\pi(W^i) \geq 0$  for all  $\pi \in P^i$  and all  $i$  with a strict inequality for some  $i$  and  $\pi \in P^i$ .*

HKPE is therefore the same concept as WNMA for monetary utilities.

Let us now prove theorem ?. Assertion 1 implies 5 follows from Allouch et al [?] theorems 4 and 5.

## 5 Constraints on exchanges

### 5.1 The model

Heath and Ku [?], Filipovic and Kupper [?] and Burgert and Rüschenendorf [?] have considered constraints on exchanges when agents use measures of risk.

We now assume that trades are only possible in linear subspaces  $M^i \subseteq \mathbb{R}^k$ ,  $1 \leq i \leq k$ . Agent  $i$  has an endowment  $E^i \in M^i$  of contingent claims. The definitions of attainable, individual rational and Pareto optimal allocations and equilibria are extended by imposing the constraint that  $X^i \in M^i$  for all  $i$ . In particular, the set of constrained useful vectors for  $i$  is defined as

$$R^{M^i} = \{W \in M^i \mid V(\lambda W) \geq V(0), \text{ for all } \lambda \geq 0\}.$$

Therefore  $R^{M^i} = \{W \in M^i \mid E_\pi(W) \geq 0, \text{ for all } \pi \in P\} = R^i \cap M^i$  where  $R^i$  is the unconstrained set of useful vectors for  $i$ .

### 5.2 Weak no-arbitrage prices under constraints

For a subset  $M \subseteq \mathbb{R}^k$ , let  $M^\perp$  be its orthogonal. In order to characterize weak no-arbitrage prices for this new economy, let us first characterize  $R_{M^i}^0$  the polar of the set of constrained useful vectors for  $i$ . From Rockafellar's corollary 16.4.2.,

$$(R^{M^i})^0 = \text{cl}((R^i)^0 + (M^i)^\perp)$$

and from Rockafellar's theorem 6.3 and corollary 6.6.2.

$$\text{ri} (R^{M^i})^0 = \text{ri} \text{cl}((R^i)^0 + (M^i)^\perp) = \text{ri} ((R^i)^0 + (M^i)^\perp) = \text{ri} (R^i)^0 + (M^i)^\perp$$

Using Rockafellar's corollary 6.6.2., we obtain that

$$\cap_i S_w^i = \cap_i \text{ri} (R^{M^i})^0 = \cap_i (\text{ri} \text{cone} P^i + (M^i)^\perp) = \cap_i \text{cone} (\text{ri} P^i + (M^i)^\perp)$$

hence,

$$\cap_i S_w^i \neq \emptyset \text{ iff } \cap_i \text{cone}(\text{ri}P^i + (M^i)^\perp) \neq \emptyset \quad (10)$$

Since (??) is positively homogeneous, let  $H = \{m \in \mathbb{R}^k \mid \sum_j m_j = 1\}$ . The set of weak no-arbitrage price is non empty if and on if there exists  $\mu \in H$  such that

$$\mu = \lambda^i \pi^i + m_\perp^i$$

with  $\pi^i \in \text{ri}P^i$  and  $\lambda^i > 0$  and  $m_\perp^i \in (M^i)^\perp$ . The vector  $\mu$  may be interpreted as a signed measure and we have

$$E_\mu(X^i) = \lambda^i E_{\pi^i}(X^i), \text{ for all } X^i \in M^i \text{ and } i \quad (11)$$

with  $\pi^i \in \text{ri}P^i$ ,  $\lambda^i > 0$ . Hence the restriction of  $\mu$  to  $M^i$  is a non-negative measure proportionnal to a prior in the relative interior of  $P^i$ . Furthermore,

- if agent  $i$  can trade the riskless asset or equivalently if constants belong to  $M^i$ , then  $\lambda^i = \langle \mu, 1 \rangle = 1$ .
- If all agents can trade the riskless asset, then  $\lambda^i$  is independent of  $i$ . (??) may be rewritten as: there exists a signed measure  $\mu$  and probabilities  $\pi^i$  in the relative interior of  $P^i$  for each agent such that

$$E_\mu(X^i) = E_{\pi^i}(X^i), \text{ for all } X^i \in M^i \text{ and } i \quad (12)$$

- If all agents can trade the riskless asset and if  $M^i = \mathbb{R}^k$  for some  $i$ , then  $\mu$  is a probability measure and (??) holds true.

**Remark 2** 1. Condition (??) is equivalent to the WNMA condition: there exists no feasible trade  $W^1, \dots, W^n$ , with  $W^i \in M^i$  for all  $i$  and  $E_\pi(W^i) \geq 0$  for all  $\pi \in P^i$  and all  $i$  with a strict inequality for some  $i$  and  $\pi \in P^i$ .

2. The condition  $\mu = \lambda^i \pi^i + m_\perp^i$  for all  $i$  is very similar to the condition one obtains when writing the no-arbitrage condition for finite financial markets with constraints on portfolios.

Let us summarize the results obtained in a proposition:

**Proposition 5** Let  $V^i$  fulfill (??) and agent's  $i$  trading set be the subspace  $M^i$  for each  $i$ . Then the following are equivalent:

1. there exists a signed measure  $\mu$  and positive constants  $\lambda^i$  and probabilities  $\pi^i \in \text{ri}P^i$  such that (??) holds true,

2. there exists no feasible trade  $W^1, \dots, W^n$ , with  $W^i \in M^i$  for all  $i$  and  $E_\pi(W^i) \geq 0$  for all  $\pi \in P^i$  and all  $i$  with a strict inequality for some  $i$  and  $\pi \in P^i$ .

Any of the previous assertions implies the existence of efficient allocations or of an equilibrium.

The next two examples show that we cannot dispense with the  $\lambda_i$ ,  $i = 1, \dots, m$  in (??) even if some agent is unconstrained contradicting Heath and Ku's proposition 5.1 and corollary 5.1.

### Example 1

There are two states and three agents. Each agent has a unique probability over states: agent 1 has probability  $\pi^1 = (\frac{1}{4}, \frac{3}{4})$ , agent 2 probability  $\pi^2 = (\frac{3}{4}, \frac{1}{4})$  and agent 3, probability  $\pi^3 = (1, 0)$ . Assume that the trading sets are  $M^1 = \{X^1 = (x^1, x^1) \mid x^1 \in \mathbb{R}\}$ ,  $M^2 = \{X^2 = (x^2, -x^2) \mid x^2 \in \mathbb{R}\}$ ,  $M^3 = \{X^3 = (x^3, 0) \mid x^3 \in \mathbb{R}\}$ . HKPE is fulfilled since  $R^{M^1} = \{W^1 = (w, w) \mid w \geq 0\}$ ,  $R^{M^2} = \{W^2 = (w, -w) \mid w \geq 0\}$  and  $R^{M^3} = \{W^3 = (w, 0) \mid w \geq 0\}$  and  $\sum_i W^i = 0$  implies  $W^i = 0$  for all  $i$ . But there exists no solution  $\mu = (\mu_1, \mu_2)$  to the following system:

$$\begin{aligned} E_\mu(X^1) = (\mu_1 + \mu_2)x^1 &= E_{\pi^1}(X^1) = x^1, \text{ for all } x^1 \in \mathbb{R}, \\ E_\mu(X^2) = (\mu_1 - \mu_2)x^2 &= E_{\pi^2}(X^2) = 1/2x^2, \text{ for all } x^2 \in \mathbb{R}, \\ E_\mu(X^3) = \mu_1x^3 &= E_{\pi^3}(X^3) = x^3, \text{ for all } x^3 \in \mathbb{R}. \end{aligned}$$

since the first and the third equations imply  $\mu_1 = 1$ ,  $\mu_2 = 0$  which is incompatible with the second equation.

### Example 2.

There are three agents. The state space and the probabilities are as in Example 1 as well as the trading sets of agents 1 and 2. The trading set of agent 3 is  $M^3 = \mathbb{R}^2$ . Hence,  $R^{M^3} = \{W^3 = (w_1^3, w_2^3) \mid w_1^3 \geq 0\}$ . As in the previous example, HKPE is fulfilled. However, there exists no solution  $\mu = (\mu_1, \mu_2)$  to (??) with  $\lambda^i = 1$  for all  $i$ .

This last example provides a counter-example to Heath and Ku's corollary 5.2.

### Example 3.

The state space and the probabilities are as in Example 1, the trading sets are  $M^1 = M^2 = M^3 = \{(x, x) \mid x \in \mathbb{R}\}$ . Since  $R^{M^1} = R^{M^2} = R^{M^3} = \{(w, w) \mid w \geq 0\}$ , HKPE is fulfilled but the intersection of the sets of priors is empty.

## 5.3 Sup-convolution and constraints

Assuming to simplify that agents can all trade the riskless asset, let us return to the inf-convolution's approach. Define for each  $i$

$$V^{M^i}(X) = \begin{cases} V^i(X) & \text{if } X \in M^i \\ -\infty, & \text{otherwise} \end{cases} \quad (13)$$

The function  $V^{M^i} : \mathbb{R}^k \rightarrow \mathbb{R} \cup \{-\infty\}$  is concave, upper semi-continuous, cash invariant but fails to be monotone. We may still use duality methods but the domain of the conjugate function is larger than the probability simplex. Let  $m \in \mathbb{R}^k$  and

$$c^{M^i}(m) = \sup_{X \in \mathbb{R}^k} V^{M^i}(X) - \langle m, X \rangle = \sup_{X \in M^i} V^{M^i}(X) - \langle m, X \rangle \quad (14)$$

be the conjugate of  $V^{M^i}$ . Clearly we have

$$c^{M^i}(m + m^\perp) = c^{M^i}(m), \text{ for all } m^\perp \in (M^i)^\perp \quad (15)$$

From the cash invariance of  $V^{M^i}$ , we also have

$$c^{M^i}(m) = \sup_{X \in M^i, a \in \mathbb{R}} V^{M^i}(X) - \langle m, X \rangle + a(1 - \langle m, 1 \rangle)$$

therefore  $c^{M^i}(m) = \infty$  if  $1 \neq \langle m, 1 \rangle$ . Defining  $H = \{m \in \mathbb{R}^k \mid \sum_j m_j = 1\}$ , we thus have that  $\text{dom } c^{M^i} \subseteq H$ . For  $m \in P^i$ ,  $c^{M^i}(m) \leq c^i(m) < \infty$ . Hence

$$\text{dom } c^{M^i} = (P^i + (M^i)^\perp) \cap H$$

The function  $\square_i V^{M^i} < \infty$  if and only if  $\bigcap_i \text{dom } c^{M^i} = \bigcap_i (P^i + (M^i)^\perp) \cap H \neq \emptyset$ . In that case, since  $\square_i V^{M^i} > -\infty$  on  $\sum_i M^i$ ,  $\text{dom } \square_i V^{M^i} \neq \emptyset$  and  $\square_i V^{M^i}$  is proper, hence  $\square_i V^{M^i}$  and  $\sum_{i=1}^m c^{M^i}$  are conjugate. From Rockafellar's theorem 16.4 [?], a sufficient condition for existence of a Pareto optimum  $(X^1, \dots, X^m)$  is that

$$\bigcap_i \text{ri dom } c^{M^i} = \bigcap_i (\text{ri } (P^i) + (M^i)^\perp) \cap H \neq \emptyset. \quad (16)$$

We are thus back to the weak no-arbitrage condition (??).

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