A note on the rearrangement of functions in time and on the parabolic Talenti inequality

Idriss Mazari*

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Abstract

Talenti inequalities are a central feature in the qualitative analysis of PDE constrained optimal control as well as in calculus of variations. The classical parabolic Talenti inequality states that if we consider the parabolic equation $\frac{\partial u}{\partial t} - \Delta u = f = f(t,x)$ then, replacing, for any time t, $f(t,\cdot)$ with its Schwarz rearrangement $f^\#(t,\cdot)$ increases the concentration of the solution in the following sense: letting v be the solution of $\frac{\partial v}{\partial t} - \Delta v = f^\#$ in the ball, then the solution u is less concentrated than v. This property can be rephrased in terms of the existence of a maximal element for a certain order relationship. It is natural to try and rearrange the source term not only in space but also in time, and thus to investigate the existence of such a maximal element when we rearrange the function with respect to the two variables. In the present paper we prove that this is not possible.

Keywords: Optimisation, Optimal Control of PDEs, Rearrangement of functions, Talenti Inequality.

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1 Introduction and motivation

1.1 Scope of the paper and mathematical context

In this paper we want to address some qualitative questions related to the time-rearrangement of functions in the context of optimal control and Talenti inequalities. Roughly speaking, it has been known since the seminal paper [19] that the spatial rearrangement (*i.e.* the Schwarz rearrangement) of source terms in elliptic equations improved "concentration"-like properties. Before we make this statement more precise let us note that this work of Talenti has sparked an immense interest from the calculus of variations and optimisation community, leading to major developments, whether in calculus of variations, in optimal control or in fine comparison relations for parabolic and elliptic partial differential equations [1, 2, 3, 6, 5, 4, 7, 8, 9, 12, 11, 13, 15, 16, 17, 18, 21, 22]. For the time being we refer to the monograph [10] and to the survey of Talenti himself [20]. In general these comparison principles are expressed in terms of *concentration of solutions*, using the

*CEREMADE, UMR CNRS 7534, Université Paris-Dauphine, Université PSL, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France, (mazari@ceremade.dauphine.fr).

order relation $\prec$ defined as follows: for a domain Ω, for any non-negative functions $f\,,g\in L^1(\Omega)$, we say that

$$f\ \prec\ g \text{ if, and only if, for any } V\in(0;\mathrm{Vol}(\Omega)),\quad \sup_{E\subset\Omega\,,\mathrm{Vol}(E)=V}\int_E f\ \leqslant\ \sup_{E\subset\Omega\,,\mathrm{Vol}(E)=V}\int_E g. \tag{1.1}$$

This relation can be expressed using the Schwarz rearrangement, see Definitions 1-2 below. The content of any Talenti-type inequality is that if we consider a parabolic or an elliptic equation of the form $\mathcal{L}u=f$ then we can compare the solution u with a solution $\tilde{u}$ of a related equation $\tilde{\mathcal{L}}\tilde{u}=\tilde{f}$ in the ball, where the tilde $\tilde{}$ simply means that certain coefficients of the equation were symmetrised.

While most of the works we cited above deal with rearrangements in space (*i.e.* for certain criteria is it better to have symmetric in space source terms/advection matrices?) it is interesting to investigate the influence of *time-rearrangement* of functions: if we are working with a parabolic equation, is there a *good* way to rearrange the source term both in time and space? In this paper, we prove that the answer to this question is *no* and that rearranging source terms in time can not yield as strong concentration results as rearranging source terms in space.

1.2 Rearrangement and order relation

To fix notations, let a dimension $d\in\mathbb{N}\backslash\{0\}$ and a radius $R>0$ be fixed, and consider the ball $\Omega:=\mathbb{B}(0;R)$ in $\mathbb{R}^d$. We will use the notation $\mathscr{C}^\infty(\Omega)$ to denote the set of infinitely differentiable functions in Ω.

Definition 1. *For any non-negative function $g\in L^2(\Omega)$, there exists a unique radially symmetric, non-negative, non-increasing function $g^\#\in L^2(\Omega)$ that has the same distribution function as g* i.e.

$$\forall t\geqslant 0\,,\mathrm{Vol}\left(\{g\geqslant t\}\right)=\mathrm{Vol}\left(\{g^\#\geqslant t\}\right).$$

$g^\#$ is called the Schwarz rearrangement of g.

There are two famous inequalities that are related to the Schwarz rearrangement:

1. First, the Pólya-Szegö inequality, which states that, if $f\in W^{1,2}(\Omega)$ is a non-negative function , then $f^\#\in W^{1,2}(\Omega)$ and, furthermore, that we have
$$\int_\Omega \left|\nabla f^\#\right|^2\leqslant\int_\Omega|\nabla f|^2. \tag{1.2}$$
2. Second, the Hardy-Littlewood inequality: it states that, if $f\,,g\in L^1(\Omega)$ are non-negative functions then
$$\int_\Omega fg\leqslant\int_\Omega f^\# g^\#. \tag{1.3}$$

The Schwarz rearrangement allows to reformulate the comparison relation (1.1):

Definition 2. *For any non-negative $f,g\in L^2(\Omega)$, we say that g dominates f, and we write $f\prec g$ if, and only if*

$$\forall r\in(0;R)\,,\int_{\mathbb{B}(0;r)}f^\#\leqslant\int_{\mathbb{B}(0;r)}g^\#.$$

It is easily checked that this definition is equivalent to (1.1) since one can check that, by equimeasurability of f and of its Schwarz rearrangement, and since $f^\#$ is radially non-increasing, there holds

$$\forall V\in(0;\mathrm{Vol}(\Omega))\,,\ \sup_{E\subset\Omega\,,\mathrm{Vol}(V)}\int_\Omega f=\int_{\mathbb{B}(0;r_V)}f^\#\text{ with }\mathrm{Vol}(\mathbb{B}(0;r_V))=V.$$

1.3 Parabolic model, problem under scrutiny and main result

The model under scrutiny in this paper is a linear heat equation: for any $f \in L^\infty((0;T)\times\Omega)$, we let u_f be the only solution of the linear heat equation

$$\begin{cases} \frac{\partial u_f}{\partial t} - \Delta u_f = f & \text{in } (0;T)\times\Omega\,, \\ u_f(t,\cdot) = 0 & \text{on } (0;T)\times\partial\Omega\,, \\ u_f(0,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{1.4}$$

The classical isoperimetric parabolic inequality [17, 22] asserts the following: denoting, for a given $f = f(t,x)$ the spatially rearranged function $f^\#$ as

$$f^\# : (0;T)\times\Omega \ni (t,x) \mapsto (f(t,\cdot))^\#(x)$$

we have

$$\forall t \in [0;T]\,, u_f(t,\cdot) \prec u_{f^\#}(t,\cdot).$$

This quite naturally leads to the question: can such estimates be reached when we rearrange f not only in space, but also in time? Note that this Talenti inequality implies the existence of a maximal element for the order relation $\prec$ in the following sense: let $\delta : [0;T] \to (0;\mathrm{Vol}(\Omega))$ be a function that models a time-dependent volume constraint and consider the set

$$\mathcal{F}_\delta := \left\{ f \in L^\infty((0;T)\times\Omega) : 0 \leqslant f \leqslant 1 \text{ a.e., and for a.e. } t \in [0;T]\,, \int_\Omega f(t,\cdot) = \delta_t \right\}.$$

Let $\overline{f}_\delta$ be defined as

$$\overline{f}_\delta : (t,x) \mapsto \mathbb{1}_{\mathbb{B}(0;r_{\delta(t)})}(x) \text{ where } r_{\delta(t)} \text{ is chosen so that } \mathrm{Vol}(\mathbb{B}(0;r_{\delta(t)})) = \delta(t).$$

If we define

$$\mathcal{H}_\delta(T) := \{u_f(T,\cdot)\,, f \in \mathcal{F}_\delta\} \subset L^2(\Omega)$$

then the parabolic Talenti inequality implies that, for any $T > 0$, $u_{\overline{f}_\delta}$ is a $\prec$-maximal element in $\mathcal{H}_\delta$.

Our question here is the following: can we obtain maximal elements in a wider class of source terms where, unlike in the definition of $\mathcal{F}_\delta$, we do not impose, for every time, a volume constraint? Let us thus introduce, for a given volume constraint $V_0 \in (0;\mathrm{Vol}((0;T)\times\Omega))$, the class of admissible controls

$$\mathcal{F} := \left\{ f \in L^\infty((0;T)\times\Omega)\,, 0 \leqslant f \leqslant 1 \text{ a.e.}, \iint_{(0;T)\times\Omega} f = V_0 \right\}. \tag{\textbf{Adm}}$$

Our question is then: defining, for any $T > 0$,

$$\mathcal{H}(T) := \{u_f(T,\cdot)\,, f \in \mathcal{F}\}\,,$$

does there exist a $\prec$-maximal element in $\mathcal{H}(T)$? In other words, does there exists a $f^* \in \mathcal{F}$ such that:

$$\forall f \in \mathcal{F}\,, u_f(T,\cdot) \prec u_{f^*}(T,\cdot)? \tag{1.5}$$

Here, the answer is no:

Theorem I. *There exists no $f^* \in \mathcal{F}$ such that (1.5) holds.*

2 Proof of theorem I

Strategy of proof and auxiliary problems To prove the result we will argue by contradiction and assume that there exists $f^* \in \mathcal{F}$ such that (1.5) holds for a certain time horizon $T > 0$. By the parabolic Talenti inequality, we may assume that $f^* = (f^*)^\#$ so that $u_{f^*} = u_{f^*}^\#$. By definition of f^* we know that, for any $f \in \mathcal{F}$ and any $r \in [0, R]$,

$$\int_0^r \xi^{d-1} u_f^\#(t,\xi) d\xi \leqslant \int_0^r \xi^{d-1} u_{f^*}(t,\xi) d\xi.$$

In particular, for any $r \in [0; R]$, f^* is a solution of the optimisation problem

$$\max_{f \in \mathcal{F}} \left(\max_{E \subset \Omega\,, \mathrm{Vol}(E)=\omega_d r^d} \int_E u_f \right), \tag{$P(r)$}$$

where $\omega_d = \mathrm{Vol}(\mathbb{B}(0,1))$.

To prove Theorem I, it suffices to show that no $f^* \in \mathcal{F}$ can solve ($P(r)$) for all $r \in [0; R]$.

Proof of Theorem I Following the discussion above we prove the following result:

Lemma 3. *Let $f^* \in \mathcal{F}$ be such that, for any $r \in (0, R)$, f^* is a solution of ($P(r)$). Then, for any radially symmetric, non-increasing, non-negative function $\varphi \in \mathscr{C}^\infty(\Omega)$, f^* is a solution of*

$$\max_{f \in \mathcal{F}} \int_\Omega u_f(T, \cdot)\varphi.$$

Proof of lemma 3. Let us fix φ in the conditions of the lemma. We can approximate φ by an increasing sequence of radially symmetric step-functions $\{\phi_k\}_{k\in\mathbb{N}}$ as follows: define, for an integer $k \geqslant 1$,

$$r_{k,j} := \frac{j}{k}R \quad (j = 0, \dots, k)\,, \alpha_{k,j} := \varphi(r_{k,j+1}) \quad (j = 0, \dots, k-1)$$

and set

$$\phi_k := \sum_{j=0}^{k-1} \alpha_{k,j} \mathbb{1}_{\mathbb{B}(0;r_{k,j+1})\setminus\mathbb{B}(0;r_{k,j})}.$$

However, from this decomposition it appears that we may rewrite ϕ_k as

$$\phi_k = \sum_{j=1}^{k} \beta_{k,j} \mathbb{1}_{\mathbb{B}(0;r_{k,j})} \text{ where, for any } j \in \{0, \dots, k\},\ \beta_{k,j} \geqslant 0.$$

Indeed it suffices to define the coefficients $\beta_{k,j}$ as

$$\beta_{k,k} := \alpha_{k,k-1} \text{ and, for any } j \in \{1, \dots, k-1\},\ \beta_{k,j} := \alpha_{k,j-1} - \alpha_{k,j} \geqslant 0$$

where the last inequality comes from the fact that φ is non-increasing. Consequently, for any $k \in \mathbb{N}$ and any $j \leqslant k$,

$$\beta_{k,j} \int_{\mathbb{B}(0,r_{k,j})} u_f(T,\cdot) \leqslant \beta_{k,j} \int_{\mathbb{B}(0,r_{k,j})} u_{f^*}(T,\cdot)$$

by the definition of f^*. Passing to the limit $k \to \infty$ yields the result.

□

We single out the following optimisation problem defined for any $\varphi \in \mathscr{C}^\infty(\Omega)$:

$$\max_{f \in \mathcal{F}} \int_\Omega u_f(T, \cdot)\varphi. \tag{P_φ}$$

To prove Theorem I, we will need to characterise the optimisers of (P_φ) in certain cases. Such a characterisation can be obtained by studying the optimality conditions for (P_φ) which is what we now set out to do.

Optimality conditions for (P_φ):

Define p_φ as the unique solution of the backward heat equation

$$\begin{cases} \frac{\partial p_\varphi}{\partial t} + \Delta p_\varphi = 0 & \text{in } (0;T) \times \Omega\,, \\ p_\varphi(t, \cdot) = 0 & \text{on } [0;T] \times \partial\Omega\,, \\ p_\varphi(T, \cdot) = \varphi & \text{in } \Omega\,, \end{cases} \tag{2.1}$$

Multiplying (1.4) by p_φ and integrating by parts we obtain

$$\forall f \in \mathcal{F}\,, \int_\Omega u_f(T, \cdot)\varphi = \iint_{(0;T)\times\Omega} f p_\varphi. \tag{2.2}$$

The function p_φ encodes the optimality conditions for (P_φ). To further characterise optimisers we need some information on the level sets of the function p_φ. Such information is given in the following lemma:

Lemma 4. *Assume $\varphi \in \mathscr{C}^\infty(\Omega) \cap W_0^{1,2}(\Omega)$, $\varphi = \varphi^\#$, $\varphi \geqslant 0$ and φ is not constant. Then, for any $t \in [0;T)$ and for any $\tau \in \left(0; \|\varphi\|_{L^\infty(\Omega)}\right)$ the level set $\{p_\varphi(t, \cdot) = \tau\}$ is a $(d-1)$-dimensional sphere.*

Proof of Lemma 4. Since $\varphi \in \mathscr{C}^\infty(\Omega)$, standard parabolic estimates imply that $p_\phi \in \mathscr{C}^\infty((0;T) \times \Omega)$. Since φ is radially symmetric, so is p_φ. By the maximum principle, for any $t \in [0;T]$,

$$\frac{\partial p_\varphi}{\partial \nu}(t, \cdot) \leqslant 0 \text{ on } \partial\Omega.$$

Let $q_\varphi := \frac{\partial p_\varphi}{\partial r}$. We already know that

$$\forall t \in [0;T]\,, q_\varphi(t, \cdot) \leqslant 0 \text{ on } \partial\Omega.$$

Furthermore, at $t = T$, since φ is not constant and radially symmetric, non-increasing,

$$q_\varphi(T, \cdot) \leqslant 0\,, q_\varphi(T, \cdot) \neq 0.$$

Differentiating (2.1) with respect to r we get the following equation

$$\begin{cases} \frac{\partial q_\varphi}{\partial t} + \Delta q_\varphi = 0 & \text{in } (0;T) \times \Omega\,, \\ q_\varphi \leqslant 0 & \text{on } [0;T] \times \partial\Omega\,, \\ q_\varphi(T, \cdot) \leqslant 0\,, q_\varphi(T, \cdot) \neq 0 & \text{in } \Omega. \end{cases}$$

By the strong maximum principle it follows that

$$\forall t < T\,, q_\varphi(t, \cdot) < 0 \text{ in } \Omega.$$

Thus, for any $t \in [0;T)$, $p_\varphi(t, \cdot)$ is radially decreasing. In particular, its level sets have zero Lebesgue measure and coincide with spheres. □

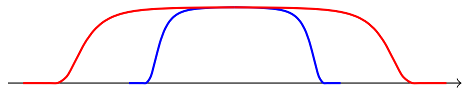

Figure 1: In blue, the graph of φ. In red, the graph of ψ.

Now let us turn back to the optimality conditions for (P_φ): let f_φ be a solution of (P_φ). From the bathtub principle [14, Theorem 1.14] and the fact that p_φ only has level sets of measure zero, it follows that there exists a Lagrange multiplier $c_\varphi \in \mathbb{R}$ such that, up to negligible sets,

1. $\{(t,x) \in (0;T) \times \Omega : \, f_\varphi(t,x) = 1\} = \{(t,x) \in (0;T) \times \Omega : \, p_\varphi(t,x) > c_\varphi\}$,
2. $\{(t,x) \in (0;T) \times \Omega : \, f_\varphi(t,x) = 0\} = \{(t,x) \in (0;T) \times \Omega : \, p_\varphi(t,x) < c_\varphi\}$,
3. $\{(t,x) \in (0;T) \times \Omega : \, 0 < f_\varphi(t,x) < 1\}$ has Lebesgue measure zero.

The constant c_φ appearing is dubbed the *Lagrange multiplier associated with* φ. We emphasise that it is a constant that depends neither on space nor on time. These conditions define f_φ univocally. Furthermore, as the (time-space) dependent level-set satisfies $\mathrm{Vol}(\{f_\varphi = 1\}) \in (0; V_0)$ the maximum principle implies

$$0 < c_\varphi < \|p_\varphi\|_{L^\infty((0;T)\times\Omega)} \leqslant \|\varphi\|_{L^\infty(\Omega)}. \tag{2.3}$$

The following lemma essentially contains the proof of Theorem I:

Lemma 5. *There exist two radially symmetric, decreasing and non-negative functions* $\varphi, \psi \in \mathscr{C}^\infty(\Omega)$ *such that* (P_φ) *and* (P_ψ) *do not have the same solutions.*

Proof of Lemma 5. **Construction of φ, ψ such that $f_\varphi \neq f_\psi$** Let φ be a cut-off function; in other words, φ satisfies:

- $\varphi \in \mathscr{C}^\infty(\Omega, \mathbb{R}_+)$ is a radially symmetric, non-increasing function.
- $\varphi \equiv 1$ on $\mathbb{B}(0, R/8)$ and is radially decreasing on $\mathbb{B}(0, R/4) \backslash \mathbb{B}(0, R/8)$.
- $\varphi \equiv 0$ on $\mathbb{B}(0, R) \backslash \mathbb{B}(0, R/4)$.

Similarly we pick ψ that satisfies

- $\psi \in \mathscr{C}^\infty(\Omega, \mathbb{R}_+)$ is a radially symmetric, non-increasing function.
- $\psi \equiv 1$ on $\mathbb{B}(0, R/2)$ and is radially decreasing on $\mathbb{B}(0, R/2) \backslash \mathbb{B}(0, 3R/4)$.
- $\psi \equiv 0$ on $\mathbb{B}(0, R) \backslash \mathbb{B}(0, 3R/4)$.

We claim that for this ψ and this φ we have $f_\varphi \neq f_\psi$.

To see why $f_\varphi \neq f_\psi$, let c_φ, c_ψ be the Lagrange multipliers associated, respectively, with φ and ψ. Recall that (2.3) gives

$$0 < c_\varphi, c_\psi < \max\left(\|\varphi\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega)}\right) = 1.$$

Define, for any $r > 0$, $\mathbb{S}(0; r)$ as the $(d-1)$-dimensional sphere or radius r (*i.e.* $\mathbb{S}(0; r) = \{x \in \mathbb{R}^d, \|x\| = r\}$) and let, for any $t \in (0;T)$, $r_\varphi(t)$ (resp. $r_\psi(t)$) be such that

$$\{p_\varphi(t, \cdot) = c_\varphi\} = \mathbb{S}(0, r_\varphi(t)) \text{ (resp. } \{p_\psi(t, \cdot) = c_\psi\} = \mathbb{S}(0, r_\psi(t))).$$

As $c_\varphi\,,c_\psi > 0$ we have

$$\sup_{t\in[0;T]} r_\varphi(t)\,,r_\psi(t) < R.$$

As we also have

$$c_\varphi\,,c_\psi < 1$$

we deduce

$$\inf_{t\in[0;T]} r_\varphi(0)\,,r_\psi(0) > 0.$$

Since, by parabolic regularity, p_φ and p_ψ are $\mathscr{C}^\infty$ and radially decreasing in the sense that $\partial_r p_\psi\,,\partial_r p_\varphi < 0$ in $(0;T)\times\Omega$, r_φ and r_ψ are continuous[1] in $[0;T]$.

Finally, since $\varphi = 0$ on $\mathbb{B}(0,R)\backslash\mathbb{B}(0,R/4)$ we have $r_\varphi(T) < R/4$. Similarly, since $\psi \equiv 1$ on $\mathbb{B}(0,R/2)$ we have $r_\psi(T) > R/2$. Consequently, $r_\varphi \neq r_\psi$ in a neighbourhood of T. But now recall that from the optimality conditions of (P_φ)-(P_ψ), we have

$$f_\varphi(t,x) = \mathbb{1}_{\mathbb{B}(0;r_\varphi(t))}(x)\,, f_\psi(t,x) = \mathbb{1}_{\mathbb{B}(0;r_\psi(t))}(x).$$

As $r_\varphi \neq r_\psi$ in a neighbourhood of T, $f_\varphi \neq f_\psi$. This concludes the proof of the Theorem.

$\square$

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