GENERALIZED PRINCIPAL EIGENVALUES OF SPACE-TIME PERIODIC, WEAKLY COUPLED, COOPERATIVE, PARABOLIC SYSTEMS

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Abstract. This paper is concerned with generalizations of the notion of principal eigenvalue in the context of space-time periodic cooperative systems. When the spatial domain is the whole space, the Krein–Rutman theorem cannot be applied and this leads to more sophisticated constructions and to the notion of generalized principal eigenvalues. These are not unique in general and we focus on a one-parameter family corresponding to principal eigenfunctions that are space-time periodic multiplicative perturbations of exponentials of the space variable. Besides existence and uniqueness properties of such principal eigenpairs, we also prove various dependence and optimization results illustrating how known results in the scalar setting can, or cannot, be extended to the vector setting. We especially prove an optimization property on minimizers and maximizers among mutation operators valued in the set of bistochastic matrices that is, to the best of our knowledge, new.

Contents

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1. Introduction

In recent years, the study of principal eigenvalues has proved very fruitful, especially (but not exclusively) for the study of several biological phenomena. Indeed, these eigenvalues encode several informations that are crucial in the understanding of population dynamics; we refer to Section 1.3 below for an overview of the motivations. Although the scalar case is now rather well understood, several problems remain open in the case of systems. In this paper, we propose a systematic approach for the case of space-time periodic cooperative systems, and we offer several contributions to their spectral analysis and optimization.

Formally, this paper is concerned with eigenvalues of linear operators of the form

$$\boldsymbol{\mathcal{Q}} : \mathbf{u} \mapsto \operatorname{diag}(\boldsymbol{\mathcal{P}})\mathbf{u} - \mathbf{L}\mathbf{u},$$

where $\mathbf{u} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^N$ is a vector-valued function of size $N \in \mathbb{N}^*$, with a time variable $t \in \mathbb{R}$ and a space variable $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ being the dimension of the underlying space, $\mathbf{L} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{N\times N}$ is a square matrix-valued function of t and x, and each operator of the family $\boldsymbol{\mathcal{P}} = (\mathcal{P}_i)_{i\in[N]}$, where $[N] = \mathbb{N} \cap [1, N]$, has the form

$$\mathcal{P}_i : u \mapsto \partial_t u - \nabla \cdot (A_i \nabla u) + q_i \cdot \nabla u,$$

with $A_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $q_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ functions of t and x, respectively square matrix-valued and vector-valued. The functions A_i, q_i and $\mathbf{L}$ are smooth and periodic (in a sense made precise later on).

The standing assumptions on $\boldsymbol{\mathcal{P}}$ and $\mathbf{L}$ are the following.

(A_1) The family $(A_i)_{i\in[N]}$ is *uniformly elliptic*:

$$0 < \min_{i\in[N]} \min_{y\in\mathbb{S}^{n-1}} \min_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} (y \cdot A_i(t, x)y) .$$

(A_2) The matrix $\underline{\mathbf{L}} \in \mathbb{R}^{N\times N}$, whose entries are

$$\underline{l}_{i,j} = \min_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} l_{i,j}(t, x) \quad \text{for all } (i, j) \in [N]^2,$$

is *essentially nonnegative*: its off-diagonal entries are nonnegative.

(A_3) The matrix $\overline{\mathbf{L}} \in \mathbb{R}^{N\times N}$, whose entries are

$$\bar{l}_{i,j} = \max_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} l_{i,j}(t, x) \quad \text{for all } (i, j) \in [N]^2,$$

is *irreducible*: it does not have a stable subspace of the form $\operatorname{span}(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k})$, where $k \in [N-1]$, $i_1, \dots, i_k \in [N]$ and $\mathbf{e}_i = (\delta_{ij})_{j\in[N]}$. By convention, $[0] = \emptyset$ and 1×1 matrices are irreducible, even if zero.

Under these assumptions, the linear system $\boldsymbol{\mathcal{Q}}\mathbf{u} = \mathbf{0}$ is uniformly parabolic, space-time periodic, weakly coupled (namely, coupled only in the zeroth order term [59]), cooperative and fully coupled [10, 60].

1.1. Organization of the paper.

Organization of the paper. The remainder of Section 1 is devoted to a detailed introduction (notations, motivation, definitions, main results and applications to semilinear systems). Section 2 is devoted to technical preliminaries. Section 3 contains the proofs.

1.2. **Notations.** We fix once and for all $n+1$ positive numbers $T, L_1, \dots, L_n \in \mathbb{R}_+^*$. For the sake of brevity, we use the notations $L = (L_1, \dots, L_n)$, $(0, L) = (0, L_1) \times \dots \times (0, L_n)$ and $|[0, L]| = \prod_{\alpha=1}^n L_\alpha$. Unless specified otherwise, time and space periodicities refer to, respectively, T-periodicity with respect to t and L_α-periodicity with respect to x_α for each $\alpha \in [n]$ (or L-periodicity with respect to x for short). The space-time periodicity cell $(0, T) \times (0, L)$ is denoted Ω_{per} and its volume is $T|[0, L]|$.

Vectors in $\mathbb{R}^N$ and matrices in $\mathbb{R}^{N\times N}$ are denoted in bold font. Functional operators are denoted in calligraphic typeface (bold if they act on functions valued in $\mathbb{R}^N$). Functional spaces, *e.g.* $\mathcal{W}^{1,\infty}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^N)$, are also denoted in calligraphic typeface. A functional space $\mathcal{X}$ denoted with a subscript $\mathcal{X}_{\mathrm{per}}$, $\mathcal{X}_{t-\mathrm{per}}$ or $\mathcal{X}_{x-\mathrm{per}}$ is restricted to functions that are space-time periodic, time periodic or space periodic respectively.

For clarity, Hölder spaces of functions with $k \in \mathbb{N} \cup \{0\}$ derivatives that are all Hölder-continuous with exponent $\alpha \in (0, 1)$ are denoted $\mathcal{C}^{k+\alpha}$; when the domain is $\mathbb{R} \times \mathbb{R}^n$, it should be unambiguously understood that $\mathcal{C}^{k+\alpha,k'+\alpha'}$ denotes the set of functions that have k α-Hölder-continuous derivatives in time and k' α'-Hölder-continuous derivatives in space.

For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, $\mathbf{u} \leq \mathbf{v}$ means $u_i \leq v_i$ for all $i \in [N]$, $\mathbf{u} < \mathbf{v}$ means $\mathbf{u} \leq \mathbf{v}$ together with $\mathbf{u} \neq \mathbf{v}$ and $\mathbf{u} \ll \mathbf{v}$ means $u_i < v_i$ for all $i \in [N]$. If $\mathbf{u} \geq \mathbf{0}$, we refer to $\mathbf{u}$ as *nonnegative*; if $\mathbf{u} > \mathbf{0}$, as *nonnegative nonzero*; if $\mathbf{u} \gg \mathbf{0}$, as *positive*. The sets of all nonnegative, nonnegative nonzero, positive vectors are respectively denoted $[\mathbf{0}, \boldsymbol{\infty})$, $[\mathbf{0}, \boldsymbol{\infty})\backslash\{\mathbf{0}\}$ and $(\mathbf{0}, \boldsymbol{\infty})$. The vector whose entries are all equal to 1 is denoted $\mathbf{1}$ and this never refers to an indicator function. Similar notations and terminologies might be used in other dimensions and for matrices. The identity matrix is denoted $\mathbf{I}$.

Similarly, a function can be nonnegative, nonnegative nonzero, positive. For clarity, a positive function is a function with only positive values.

To avoid confusion between operations in the state space $\mathbb{R}^N$ and operations in the spatial domain $\mathbb{R}^n$, Latin indexes i, j, k are assigned to vectors and matrices of size N whereas Greek indexes α, β, γ are assigned to vectors and matrices of size n. We use mostly subscripts to avoid confusion with algebraic powers, but when both Latin and Greek indexes are involved, we move the Latin ones to a superscript position, *e.g.* $A^i_{\alpha,\beta}(t, x)$. We denote scalar products in $\mathbb{R}^N$ with the transpose operator, $\mathbf{u}^{\mathrm{T}}\mathbf{v} = \sum_{i=1}^N u_i v_i$, and scalar products in $\mathbb{R}^n$ with a dot, $x \cdot y = \sum_{\alpha=1}^n x_\alpha y_\alpha$.

For any vector $\mathbf{u} \in \mathbb{R}^N$, $\mathrm{diag}(\mathbf{u})$, $\mathrm{diag}(u_i)_{i\in[N]}$ or $\mathrm{diag}(u_i)$ for short refer to the diagonal matrix in $\mathbb{R}^{N\times N}$ whose i-th diagonal entry is u_i. These notations can also be used if $\mathbf{u}$ is a function valued in $\mathbb{R}^N$.

Finite dimensional Euclidean norms are denoted $|\cdot|$ whereas the notation $\|\cdot\|$ is reserved for norms in functional spaces.

The notation $\circ$ is reserved in the paper for the Hadamard product (component-wise product of vectors or matrices) and never refers to the composition of functions.

Finally, when the focus of the paper is on the dependence of an eigenvalue on (a parameter of) the underlying operator, and when the context is unambiguous, we write with a slight abuse of notation the eigenvalue as a function of the varying parameter (*e.g.*, an eigenvalue λ of the operator $\boldsymbol{\mathcal{Q}}$ might be denoted $\lambda(\boldsymbol{\mathcal{Q}})$, $\lambda(A_1)$, $\lambda(q_1, \dots, q_n)$, $\lambda(\mathbf{L})$, and so on).

1.3. **Motivation.** The linear parabolic system $\mathcal{Q}\mathbf{u} = \mathbf{0}$ can be understood as the linearization at the homogeneous steady state $\mathbf{0}$ of a semilinear, non-cooperative, space-time periodic, reaction–diffusion system $\operatorname{diag}(\boldsymbol{\mathcal{P}})\mathbf{u}(t,x) = \mathbf{f}(t,x,\mathbf{u}(t,x))$.

From a modeling viewpoint, such systems appear for instance in population dynamics, where each component of $\mathbf{u}$ models a population density for an age class or a phenotypical trait class [20, 21, 28, 33, 39]. In this context, the sign of the principal eigenvalue of the linearization at $\mathbf{0}$ indicates, at least in simple spatio-temporal settings, whether small populations survive and persist or, on the contrary, go extinct.

The population dynamics models we have in mind use a simple growth term of logistic type, also referred to as a Fisher–KPP, or simply KPP, reaction term [29,45], that satisfies $\mathbf{L}(t,x)\mathbf{u}(t,x) = \mathrm{D}_{\mathbf{u}}\mathbf{f}(t,x,\mathbf{0})\mathbf{u} \geq \mathbf{f}(t,x,\mathbf{u})$. Non-cooperative Fisher–KPP systems have been the object of a growing literature in the past few years, especially in the case of two components $N = 2$ (see, *e.g.*, [1, 17, 19, 21, 22, 32–36, 38, 40, 41, 51, 53]). It turns out that for such models population persistence is often equivalent to small population persistence, and this makes the study of the principal eigenvalue even more crucial.

When the underlying model is a population structured with respect to a phenotypical trait, then $\mathbf{L}$ typically takes the form $\mathbf{L} = \operatorname{diag}(r_i) + \mathbf{M}$, where each $r_i > 0$ is an intrinsic growth rate and the matrix $\mathbf{M}$ is a mutation matrix; in the simplest case $\mathbf{M}$ is a discrete Laplacian with Neumann boundary conditions:

$$\mathbf{M} = \mu \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix} \tag{1}$$

where $\mu > 0$ is a mutation rate.

When the underlying model is a population structured with respect to age, then $\mathbf{L}$ is a diagonally perturbed Leslie matrix:

$$\mathbf{L} = -\operatorname{diag}(d_i + a_i) + \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_N \\ a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{N-1} & 0 \end{pmatrix} \tag{2}$$

where each $d_i \geq 0$ is a death rate, each $a_i > 0$ an aging rate and each $b_i \geq 0$ a birth rate with $b_N > 0$.

Each one of these models can be understood as a discretized version of some nonlocal equation [33].

The second example (2) above explains in particular why we do not make any *a priori* assumption on the symmetry of $\mathbf{L}$.

Let us also point out that the periodic cooperative systems we consider find applications in the chemistry of nuclear reactor cores [2, 23]. Due to our long-term goals (*cf.* Section 1.7), in this paper, we favor a population dynamics viewpoint.

1.4. **Generalized principal eigenvalues in space-time periodic media.** In [54], Nadin analyzed the scalar case $N=1$. Following previous efforts [11,13,14,42], he introduced and studied the following quantities:

$$\lambda_1 = \sup\left\{\lambda\in\mathbb{R} \mid \exists u\in\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(0,\infty))\ \mathcal{Q}u\geq\lambda u\right\},$$

$$\lambda_1' = \inf\left\{\lambda\in\mathbb{R} \mid \exists u\in\mathcal{W}^{1,\infty}\cap\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(0,\infty))\ \mathcal{Q}u\leq\lambda u\right\}.$$

These two quantities turn out to be eigenvalues of $\mathcal{Q}$ (in the sense that associated eigenfunctions exist), and are referred to as *generalized principal eigenvalues* (their eigenfunctions are referred to as *generalized principal eigenfunctions*). Due to the lack of compactness in the spatial variable, the existence of these eigenvalues cannot be directly deduced from the Krein–Rutman theorem. However, they can be related with classical Krein–Rutman principal eigenvalues: the first one, λ_1, is the limit of the principal eigenvalues associated with the time periodic problem with Dirichlet boundary conditions in growing balls; the second one, λ_1', coincides with the principal eigenvalue of the space-time periodic problem. Actually, both eigenvalues are related to the family $(\lambda_{1,z})_{z\in\mathbb{R}^n}$ of principal eigenvalues of the space-time periodic problems associated with the operators

$$\mathcal{Q}_z : u\mapsto \mathrm{e}_{-z}\mathcal{Q}(\mathrm{e}_z u)\quad \text{where } \mathrm{e}_{\pm z}: x\mapsto \mathrm{e}^{\pm z\cdot x},$$

which can be expanded as

$$\mathcal{Q}_z u = \mathcal{Q}u - 2z\cdot(A\nabla u) - (z\cdot Az + \nabla\cdot(Az) - q\cdot z)u.$$

Since $\mathcal{Q}(\mathrm{e}_z u) = \lambda_{1,z}\mathrm{e}_z u$, $\lambda_{1,z}$ can be understood as the principal eigenvalue of $\mathcal{Q}$ acting on the set $\mathrm{e}_z.\mathcal{C}^{1,2}_{\mathrm{per}}$ of space-time periodic multiplicative perturbations of the planar exponential e_z. Nadin showed that $\lambda_1' = \lambda_{1,0} \leq \lambda_1 = \max_{z\in\mathbb{R}^n}\lambda_{1,z}$ and subsequently exhibited sufficient conditions for the equality $\lambda_1 = \lambda_1'$ to hold; his study is completed by several dependence and optimization results.

Our aim in this paper is twofold. First, we want to generalize the results of Nadin; second, we want to illustrate the originality of systems compared to scalar equations by means of new results without scalar counterpart. Let us point out that most generalizations of scalar results we consider require work indeed. On one hand, many proofs of [54] rely on algebraic operations that are at least ambiguous, at worst unavailable, in the vector setting, like powers or quotients, and this often leads to counter-examples. On the other hand, the strong coupling assumption that we use to emulate the scalar strong comparison principle, (A_3), is not a pointwise property but rather a global property, and this makes some adaptations quite technical.

We will therefore study the following quantities:

$$\lambda_1 = \sup\left\{\lambda\in\mathbb{R} \mid \exists\mathbf{u}\in\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\infty))\ \boldsymbol{\mathcal{Q}}\mathbf{u}\geq\lambda\mathbf{u}\right\}, \tag{3}$$

$$\lambda_1' = \inf\left\{\lambda\in\mathbb{R} \mid \exists\mathbf{u}\in\mathcal{W}^{1,\infty}\cap\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\infty))\ \boldsymbol{\mathcal{Q}}\mathbf{u}\leq\lambda\mathbf{u}\right\}, \tag{4}$$

as well as the family $(\lambda_{1,z})_{z\in\mathbb{R}^n}$, where:

$$\lambda_{1,z} = \lambda_{1,\mathrm{per}}\left(\boldsymbol{\mathcal{Q}}_z\right), \tag{5}$$

$$\boldsymbol{\mathcal{Q}}_z = \mathrm{e}_{-z}\boldsymbol{\mathcal{Q}}\mathrm{e}_z = \boldsymbol{\mathcal{Q}} - \mathrm{diag}\left((A_i + A_i^{\mathrm{T}})z\cdot\nabla + z\cdot A_i z + \nabla\cdot(A_i z) - q_i\cdot z\right). \tag{6}$$

We admit that the Krein–Rutman theorem can be successfully applied to the operator $\boldsymbol{\mathcal{Q}}_z$ under additional spatial compactness assumptions, so that the *periodic* *principal eigenvalue* $\lambda_{1,z} = \lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}_z)$ is well-defined; similarly, for any nonempty smooth bounded connected open set $\Omega \subset \mathbb{R}^n$, the *Dirichlet principal eigenvalue* $\lambda_{1,\mathrm{Dir}}(\boldsymbol{\mathcal{Q}}_z, \Omega)$ is well-defined. The first one corresponds to the operator $\boldsymbol{\mathcal{Q}}_z$ acting on $\mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n)$, and hereafter we denote $\mathbf{u}_z$ such a positive principal eigenfunction. The second one corresponds to the operator acting on $\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega)\cap\mathcal{C}^1_0(\mathbb{R}\times\overline{\Omega})$, where the subscript 0 denotes functions that vanish on $\partial\Omega$. Eigenfunctions for these principal eigenvalues are unique up to multiplication by a constant. For detailed applications of the Krein–Rutman theory in the Dirichlet case, we refer to Bai–He [10] or Antón–López-Gómez [7].

Let us emphasize once more that the Krein–Rutman theorem cannot be used for λ_1 and λ_1'. This is why the generalized principal eigenproblem is mathematically challenging and has richer outcomes.

Definition 1.1. A *generalized principal eigenfunction associated with* λ_1 is a function $\mathbf{u} \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ such that $\boldsymbol{\mathcal{Q}}\mathbf{u} = \lambda_1\mathbf{u}$.

A *generalized principal eigenfunction associated with* λ_1' is a function $\mathbf{u} \in \mathcal{W}^{1,\infty}\cap \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ such that $\boldsymbol{\mathcal{Q}}\mathbf{u} = \lambda_1'\mathbf{u}$.

1.5. **Results.** Although the theorems and definitions in Section 1.5.1 are completely analogous to the scalar setting [54], the ones in Sections 1.5.2–1.5.6, will require new restrictions specific to the parabolic vector setting and will show how the time structure, the spatial structure and the multidimensional state space interact intricately.

Before stating the results, we precise the standing assumptions on the smoothness and periodicity of the coefficients of $\boldsymbol{\mathcal{Q}}$.

(A_4) There exists $\delta\in(0,1)$ such that $\mathbf{L}\in\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^{N\times N})$ and, for each $i\in[N]$, $A_i\in\mathcal{C}^{\delta/2,1+\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and $q_i\in\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Moreover, $A_i = A_i^{\mathrm{T}}$ for each $i\in[N]$.

We point out that, in such a smooth and generic framework, the symmetry of the diffusion matrices is assumed without loss of generality[1].

1.5.1. *Existence and characterization of generalized principal eigenpairs.*

Theorem 1.1. *The generalized principal eigenvalues λ_1 and λ_1' are well-defined real numbers related to the family $(\lambda_{1,z})_{z\in\mathbb{R}^n}$:*

$$\lambda_1' = \lambda_{1,0}, \quad \lambda_1 = \max_{z\in\mathbb{R}^n}\lambda_{1,z}.$$

The maximum is uniquely achieved.

Consequently, $\lambda_1'\leq\lambda_1$, $\mathbf{u}_0$ is a generalized principal eigenfunction associated with λ_1' and there exists a unique $z^\star\in\mathbb{R}^n$ such that $\mathrm{e}_{z^\star}\mathbf{u}_{z^\star}$ is a generalized principal eigenfunction associated with λ_1.

Furthermore, the following max–min and min–max characterizations hold:

$$\lambda_{1,z} = \max_{\mathbf{u}\in\mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))}\ \min_{i\in[N]}\ \min_{\overline{\Omega}_{\mathrm{per}}}\left(\frac{(\boldsymbol{\mathcal{Q}}_z\mathbf{u})_i}{u_i}\right)\quad \textit{for all } z\in\mathbb{R}^n,$$

[1]Indeed, if A_i is not symmetric, then we can write it as the sum of its symmetric part $A_i^{\mathrm{sym}} = \frac{1}{2}(A_i + A_i^{\mathrm{T}})$ and its skew-symmetric part $A_i^{\mathrm{skew}} = \frac{1}{2}(A_i - A_i^{\mathrm{T}})$. The operator $\nabla\cdot(A_i^{\mathrm{skew}}\nabla)$ acting on the space of functions of class $\mathcal{C}^2$ can be rewritten as an advection operator $a_i\cdot\nabla$, so that $-\nabla\cdot(A_i\nabla)+q_i\cdot\nabla = -\nabla\cdot(A_i^{\mathrm{sym}}\nabla)+(q_i-a_i)\cdot\nabla$ and the operator on the right-hand side has the same structure; has "gained" the symmetry of its diffusion matrix.

$$\lambda_{1,z} = \min_{\mathbf{u}\in\mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))} \max_{i\in[N]} \max_{\overline{\Omega_{\mathrm{per}}}} \left(\frac{(\mathcal{Q}_z\mathbf{u})_i}{u_i}\right) \quad \textit{for all } z\in\mathbb{R}^n,$$

$$\lambda_1 = \max_{\mathbf{u}\in\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))} \min_{i\in[N]} \inf_{\mathbb{R}\times\mathbb{R}^n} \left(\frac{(\mathcal{Q}\mathbf{u})_i}{u_i}\right).$$

By simplicity of the periodic principal eigenvalue, the only non-negative periodic eigenfunctions are periodic principal eigenfunctions. Under assumptions (A_1)–(A_4), if we further impose standard normalisation conditions on the eigenfunction (*e.g.*, $|\mathbf{u}_z(0,0)|=1$) , classical compactness estimates on the family $(\lambda_{1,z},\mathbf{u}_z)$ imply that the spectral elements $(\lambda_{1,z},\mathbf{u}_z)$ are continuous with respect to the coefficients of $\mathcal{Q}_z$. In particular, this show the continuity of λ_1 and λ_1' as functions of the coefficients of $\mathcal{Q}$.

It is well-known that the equality $\lambda_1'=\lambda_1$ can be false: in the scalar case, the differential operator $u\mapsto -u''+u'$ is a classical counter-example. The key to this counter-example is the nonzero advection term that moves the maximum of $\lambda_{1,z}$ away from $z=0$; thus a similar counter-example of fully coupled cooperative parabolic system that does not reduce trivially to an elliptic scalar equation is, in spatial dimension $n=1$, $\mathcal{Q}=\partial_t-\partial_{xx}+\partial_x-(1/8)\mathbf{I}-\mathbf{M}$, where $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{N\times N}$ and $\mathbf{M}$ is the discrete Laplacian defined in (1). By uniqueness of the periodic principal eigenpair and the fact that the coefficients depend neither on time nor space,

$$\lambda_{1,z} = -\lambda_{\mathrm{PF}}\left(-z(1-z)\mathbf{I}+\frac{1}{8}\mathbf{I}+\mathbf{M}\right) = z(1-z)-\frac{1}{8}-\lambda_{\mathrm{PF}}(\mathbf{M}) = z(1-z)-\frac{1}{8},$$

where λ_{PF} denotes the *Perron–Frobenius eigenvalue* of an essentially nonnegative irreducible matrix in $\mathbb{R}^{N\times N}$. Therefore $\lambda_1'=-1/8<\lambda_1=1/8$, and this also confirms that, as in the scalar case, λ_1 and λ_1' need not have the same sign.

In the elliptic scalar setting, the absence of advection implies that $z\mapsto\lambda_{1,z}$ is even, whence the equality $\lambda_1=\lambda_1'$ follows [54, Proposition 3.2]. In the elliptic vector setting, Griette and Matano have very recently proved with a counter-example that this is not the case [37, Proposition 4.1]: the mere asymmetry of $\mathbf{L}(x)$ can induce the strict inequality $\lambda_1'<\lambda_1$. For the sake of completeness, we will recall their counter-example in Remark 3.5.

As in [54], our method of proof actually establishes a few results on λ_1 in arbitrary domains[2]. For any nonempty open connected set $\Omega\subset\mathbb{R}^n$, we define:

$$(7)\quad \lambda_1(\Omega)=\sup\left\{\lambda\in\mathbb{R}\mid \exists\mathbf{u}\in\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega,(\mathbf{0},\boldsymbol{\infty}))\cap\mathcal{C}^1(\mathbb{R}\times\overline{\Omega})\ \mathcal{Q}\mathbf{u}\geq\lambda\mathbf{u}\right\}.$$

Since $\partial\Omega$ is not necessarily smooth, the set $\mathcal{C}^1(\mathbb{R}\times\overline{\Omega})$ is understood here as the set of functions $\mathbf{u}\in\mathcal{C}^1(\mathbb{R}\times\Omega)$ such that both $\mathbf{u}$ and $\nabla\mathbf{u}$ can be continuously extended at any boundary point admitting a strong barrier (*cf.* Berestycki–Nirenberg–Varadhan [13]). The subset $\mathcal{C}^1_0(\mathbb{R}\times\overline{\Omega})$ is the set of functions in $\mathcal{C}^1(\mathbb{R}\times\overline{\Omega})$ vanishing continuously at such boundary points.

[2]In the spirit of Berestycki–Rossi [15], λ_1' can also be defined in an arbitrary domain Ω and further results on $\lambda_1(\Omega)$ and $\lambda_1'(\Omega)$ are likely achievable. We choose not to dig deeper in this direction: the focus of this paper is not on arbitrary domains but rather on the effect of space-time periodicity.

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be a nonempty open connected set. A *generalized principal eigenfunction associated with* $\lambda_1(\Omega)$ is a function $\mathbf{u} \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R} \times \Omega, (\mathbf{0}, \boldsymbol{\infty})) \cap \mathcal{C}^1_0(\mathbb{R} \times \overline{\Omega})$ such that $\mathcal{Q}\mathbf{u} = \lambda_1 \mathbf{u}$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open connected set such that there exists $x_0 \in \Omega$ satisfying $[x_0, x_0 + L] \subset \Omega$. Then the generalized principal eigenvalue $\lambda_1(\Omega)$ is a well-defined real number and there exists an associated generalized principal eigenfunction.*

If $\Omega = \mathbb{R}^n$, $\lambda_1(\Omega) = \lambda_1$. If Ω is bounded and smooth, $\lambda_1(\Omega) = \lambda_{1,\mathrm{Dir}}(\Omega)$.

Furthermore, the following max–min characterization holds true:

$$\lambda_1(\Omega) = \max_{\mathbf{u} \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega,(\mathbf{0},\boldsymbol{\infty}))\cap\mathcal{C}^1(\mathbb{R}\times\overline{\Omega})} \min_{i\in[N]} \inf_{\mathbb{R}\times\Omega} \left(\frac{(\mathcal{Q}\mathbf{u})_i}{u_i} \right).$$

1.5.2. *Monotonic or convex dependence with respect to the coefficients.* As an immediate corollary of the max–min characterization of Theorem 1.1, we already know that the eigenvalues $\lambda_{1,z}$, as functions of the matrix entries $l_{i,j}$, are decreasing: if $l_{i,j} < \tilde{l}_{i,j}$ (*i.e.*, $(t,x) \mapsto \tilde{l}_{i,j}(t,x) - l_{i,j}(t,x)$ is a nonnegative nonzero function), then $\lambda_{1,z}(l_{i,j}) > \lambda_{1,z}(\tilde{l}_{i,j})$. This applies in particular to λ_1 and λ_1', by virtue of the identifications $\lambda_1 = \max \lambda_{1,z}$ and $\lambda_1' = \lambda_{1,0}$.

Our first theorem on coefficient dependence is concerned with the concavity of the eigenvalues $\lambda_{1,z}$ as functions of the entries $l_{i,j}$. It generalizes a well-known result by Nussbaum [58] on matrices in $\mathbb{R}^{N\times N}$ as well as a result by Nadin [54] on the scalar parabolic case.

Theorem 1.3. *Let $z \in \mathbb{R}^n$ and let*

$$(\mathbf{L}[s])_{s\in[0,1]} \in \left(\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})\right)^{[0,1]}$$

be a family of matrices satisfying (A_2), (A_3) *and such that, for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ and $i \in [N]$,*

(1) $s \mapsto l_{i,i}[s](t,x)$ is convex;

(2) for all $j \in [N]\backslash\{i\}$, $s \mapsto l_{i,j}[s](t,x)$ is either identically zero or log-convex.

Then the map

$$s \in [0,1] \mapsto \lambda_{1,\mathrm{per}}(\mathcal{Q}_z[s]),$$

where $\mathcal{Q}_z[s]$ is the operator $\mathcal{Q}_z$ with $\mathbf{L}$ replaced by $\mathbf{L}[s]$, is affine or strictly concave.

It is affine if and only if there exist a constant vector $\mathbf{b} \gg \mathbf{0}$, a function $\mathbf{c} \in \mathcal{C}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ and a function $\mathbf{f} \in \mathcal{C}_{\mathrm{per}}(\mathbb{R}, \mathbb{R}^N)$ satisfying $\int_0^T \mathbf{f} \in \mathrm{span}(\mathbf{1})$ such that the entries of $\mathbf{L}$ have the form:

$$l_{i,j}[s] : (t,x) \mapsto \begin{cases} l_{i,i}[0](t,x) - s f_i(t) & \text{if } i = j, \\ l_{i,j}[0](t,x) \left(\frac{b_j}{c_i(t,x)}\right)^s \mathrm{e}^{s\left(\int_0^t f_j - \frac{t}{T}\int_0^T f_j\right)} & \text{if } i \neq j. \end{cases}$$

Furthermore, if, in addition, $\mathbf{L}[0](t,x)$ is irreducible at all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, then the above necessary and sufficient condition can be strengthened as follows: it is affine if, and only if, there exists a constant vector $\mathbf{b} \gg \mathbf{0}$ and a function $\mathbf{f} \in \mathcal{C}_{\mathrm{per}}(\mathbb{R}, \mathbb{R}^N)$ satisfying $\int_0^T \mathbf{f} \in \mathrm{span}(\mathbf{1})$ such that the entries of $\mathbf{L}$ have the form:

$$l_{i,j}[s] : (t,x) \mapsto \begin{cases} l_{i,i}[0](t,x) - s f_i(t) & \text{if } i = j, \\ l_{i,j}[0](t,x) \left(\frac{b_j}{b_i}\right)^s \mathrm{e}^{s\int_0^t (f_j - f_i)} & \text{if } i \neq j. \end{cases}$$

We explain in Remark 3.2 that the above statement is sharp, in the sense that the necessary and sufficient condition for strict concavity when $\mathbf{L}[0]$ is not pointwise irreducible cannot be improved to obtain the one given for the pointwise irreducible case.

Although Theorem 1.3 directly applies to $\lambda_1' = \lambda_{1,0}$, we are only able to prove a weaker concavity property on the generalized principal eigenvalue λ_1 in arbitrary domains – in bounded and smooth domains, a result exactly analogous to Theorem 1.3 applies, *cf.* Proposition 3.7. Similarly, in the elliptic case with general spatial heterogeneities in $\mathbb{R}^n$, Arapostathis–Biswas–Pradhan [8, Lemma 2.3] proved the concavity of λ_1 with respect to the diagonal entries of $\mathbf{L}$ – they did not consider the off-diagonal entries but, their arguments being the same as ours, their result can be extended accordingly.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open connected set such that there exists $x_0 \in \Omega$ satisfying $[x_0, x_0 + L] \subset \Omega$.*

Let

$$(\mathbf{L}[s])_{s\in[0,1]} \in \left(\mathcal{C}^{\delta/2,\delta}_{\text{per}}(\mathbb{R}\times\Omega, \mathbb{R}^{N\times N})\right)^{[0,1]}$$

be a family of matrices satisfying (A_2), (A_3) and such that, for all $(t,x) \in \mathbb{R}\times\Omega$ and $i \in [N]$,

(1) $s \mapsto l_{i,i}[s](t,x)$ is convex;
(2) for all $j \in [N]\backslash\{i\}$, $s \mapsto l_{i,j}[s](t,x)$ is either identically zero or log-convex.

Then the mapping $s \in [0,1] \mapsto \lambda_1(\Omega, \mathcal{Q}[s])$, where $\mathcal{Q}[s]$ is the operator $\mathcal{Q}$ with $\mathbf{L}$ replaced $\mathbf{L}[s]$, is concave.

Monotonicity or convexity results on the dependence on the diffusion matrices A_i or the advection vectors q_i are in full generality false (in the scalar setting, cases of non-monotonic dependence with respect to the diffusion rate are exhibited in Hutson–Mischaikow–Polacik [43]).

1.5.3. *Asymptotic dependence with respect to the coefficients.* The next theorem shows how the generalized principal eigenvalues $\lambda_{1,z}$ and λ_1 behave close to the boundary where (A_1), (A_2) and (A_4) remain satisfied but the full coupling assumption (A_3) does not[3]. We recall that a nonnegative square matrix can be conjugated into block upper triangular form by a permutation matrix, with each diagonal block an irreducible nonnegative square matrix (recall that 1×1 matrices are by convention referred to as irreducible even if zero). A similar permutation property holds for cooperative linear parabolic systems, and therefore we can assume without loss of generality that the limiting matrix $\mathbf{L}$ is already in block upper triangular form with each block satisfying (A_3).

Theorem 1.5. *Let $\mathbf{L}^{\triangle} \in \mathcal{C}^{\delta/2,\delta}_{\text{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})$ be a block upper triangular essentially nonnegative matrix. Let $N' \in [N]$ and $(N_k)_{k\in[N'-1]}$ such that*

$$N_0 = 0 < 1 \le N_1 \le N_2 \le \cdots \le N_{N'-1} \le N_{N'} = N$$

and such that

$$(l^{\triangle}_{i,j})_{(i,j)\in([N_k]\backslash[N_{k-1}])^2}$$

[3]Of course, if $\mathcal{Q}$ is spatio-temporally homogeneous, then the theorem reduces to the well-known continuity of the dominant eigenvalue in the set of essentially nonnegative square matrices.

is the k-th diagonal block of $\mathbf{L}^{\triangle}$ *(with the convention* $[0]=\emptyset$*). Assume*

$$\left(\max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}} l_{i,j}^{\triangle}(t,x)\right)_{(i,j)\in([N_k]\backslash[N_{k-1}])^2} \quad \textit{is irreducible for all } k\in[N'].$$

Let

$$\boldsymbol{\mathcal{Q}}_k = \mathrm{diag}(\mathcal{P}_i)_{i\in[N_k]\backslash[N_{k-1}]} - (l_{i,j}^{\triangle})_{(i,j)\in([N_k]\backslash[N_{k-1}])^2} \quad \textit{for all } k\in[N'].$$

Then, as $\mathbf{L}\to\mathbf{L}^{\triangle}$ *in* $\mathcal{C}_{\mathrm{per}}^{\delta/2,\delta}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^{N\times N})$*,*

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}) \to \min_{k\in[N']} \lambda_{1,z}\left(\boldsymbol{\mathcal{Q}}_k\right) \quad \textit{for all } z\in\mathbb{R}^n,$$

$$\lambda_1(\boldsymbol{\mathcal{Q}}) \to \max_{z\in\mathbb{R}^n}\min_{k\in[N']} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k) \leq \min_{k\in[N']}\lambda_1(\boldsymbol{\mathcal{Q}}_k).$$

We comment specifically on this important result in Section 1.6.

The next theorem is concerned with concurrently vanishing diffusion and advection rates – the question of vanishing diffusions rates when the advection rates remain nonnegligible is much more difficult, even in the scalar case [49], and is beyond the scope of this paper; for now, it remains open.

Theorem 1.6. *Let* $\delta\in(0,1)$*. For all* $\varepsilon>0$*, let*

$$\mathcal{D}_\varepsilon = \left\{ A\in\mathcal{C}_{\mathrm{per}}^{\delta/2,1+\delta}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^{n\times n}) \mid A=A^{\mathrm{T}},\ 0<\min_{y\in\mathbb{S}^{n-1}} y\cdot Ay,\ \max_{\alpha,\beta\in[n]}\|A_{\alpha,\beta}\|_{\mathcal{C}_{\mathrm{per}}^{0,1}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}\leq\varepsilon^2 \right\},$$

$$\mathcal{A}_\varepsilon = \left\{ q\in\mathcal{C}_{\mathrm{per}}^{\delta/2,\delta}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n) \mid \||q|\|_{\mathcal{C}_{\mathrm{per}}^{0}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}\leq\varepsilon \right\}.$$

Denote (with a slight abuse of notation) $\mathbf{L}(x): t\mapsto\mathbf{L}(t,x)$*. Then, for all* $z\in\mathbb{R}^n$*,*

$$\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}} \sup_{\substack{(A_i)_{i\in[N]}\in\mathcal{D}_\varepsilon^N\\ (q_i)_{i\in[N]}\in\mathcal{A}_\varepsilon^N}} \left| \lambda_{1,z}\left((A_i)_{i\in[N]},(q_i)_{i\in[N]}\right) - \min_{x\in[0,L]}\lambda_{1,\mathrm{per}}\left(\partial_t-\mathbf{L}(x)\right)\right| = 0.$$

Beyond showing that $z\mapsto\lambda_{1,z}$ converges pointwise to a constant in a correctly scaled vanishing diffusion–advection limit[4], the uniform limit above implies the following two limits:

$$\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}} \lambda_1'\left(\partial_t - \mathrm{diag}(\varepsilon^2\nabla\cdot(A_i\nabla)-\varepsilon q_i\cdot\nabla)-\mathbf{L}\right) = \min_{x\in[0,L]}\lambda_{1,\mathrm{per}}\left(\partial_t-\mathbf{L}(x)\right),$$

$$\lim_{\substack{\mathbf{d}\to\mathbf{0}\\ \mathbf{d}\gg\mathbf{0}}} \lambda_1'\left(\partial_t - \mathrm{diag}\left(d_i\left(\nabla\cdot(A_i\nabla)-q_i\cdot\nabla\right)\right)-\mathbf{L}\right) = \min_{x\in[0,L]}\lambda_{1,\mathrm{per}}\left(\partial_t-\mathbf{L}(x)\right).$$

Although the two limits look similar, they do not refer to the same underlying questions.

The first one is related to a slowly varying medium. Indeed, assume, for the sake of simplicity, that all L_α coincide and denote $\varepsilon=L_1^{-1}$. Then the change of variable $x\to\varepsilon x$ changes the $[0,T]\times[0,\varepsilon^{-1}]^n$-periodic operator $\boldsymbol{\mathcal{Q}}$ into the $[0,T]\times[0,1]^n$-periodic operator $\partial_t-\varepsilon^2\,\mathrm{diag}(\nabla\cdot(A_i^\varepsilon\nabla)+\varepsilon\,\mathrm{diag}(q_i^\varepsilon\cdot\nabla)-\mathbf{L}^\varepsilon$, where $A_i^\varepsilon:(t,x)\mapsto A_i(t,\frac{x}{\varepsilon})$, q_i^ε and $\mathbf{L}^\varepsilon$ being defined in a similar way. In the scalar case, the limit $\varepsilon\to0$ has been studied by Nadin [54]. In the vector case with

[4]Actually, since $z\mapsto\lambda_{1,z}$ is concave, this convergence is locally uniform in z, but there is really no hope for uniform convergence since $\lambda_{1,z}\to-\infty$ as $|z|\to+\infty$. Also, in general, $\lambda_1=\max\lambda_{1,z}$ does not converge to the same limit: indeed, even for the spatio-temporally homogeneous one-dimensional operator $\partial_t-\varepsilon^2\partial_{xx}+\varepsilon\partial_x-\mathbf{M}$, with $\mathbf{M}$ the discrete Laplacian, the maximum of $z\mapsto\lambda_{1,z}$ is $1/4$, independently of ε, whereas the pointwise limit as $\varepsilon\to0$ is 0.

temporally homogeneous coefficients and an extra time scaling, it has been studied by Allaire and Hutridurga [2] (parabolic scaling) and by Mirrahimi and Souganidis [52] (hyperbolic scaling).

The second one corresponds for instance to the early stages $t \to t/\varepsilon$ of a fast-reaction system $\mathbf{L} \to \frac{1}{\varepsilon}\mathbf{L}$, when spatial processes (dispersal, transport) are still negligible. In bounded domains with Dirichlet boundary conditions, the singular limit $\varepsilon \to 0$ has been studied by Bai and He [10]. As explained by Lam and Lou in their paper on the Neumannn elliptic case [46], the fact that the vanishing parameter is the vector $\mathbf{d} \in \mathbb{R}^N$ and not an amplitude parameter $\varepsilon \in \mathbb{R}$ is meaningful: the spatial processes for one species may be much faster than for the others (*e.g.*, $d_N = \varepsilon \max_{i\in[N-1]} d_i$), as long as they are all slow compared to the parameter ε measuring the time scale and the speed of the reaction.

By considering a uniform limit, we bring together these two frameworks and prove both limits concurrently. We believe this approach is new.

The next theorem is, on the contrary, concerned with how very large diffusion rates impact the periodic principal eigenvalue $\lambda_{1,\mathrm{per}}$. The large diffusivity limit for the whole family $(\lambda_{1,z}(\mathcal{Q}_{\mathbf{d}}))_{z\in\mathbb{R}^n}$ is an entirely different problem, since the large parameter $\mathbf{d}$ appears also in the zeroth order term which will therefore blow-up as soon as z is nonzero[5]. This problem is beyond our scope and is left open.

The question of very large advection rates, already much more delicate in the scalar case [50], is also beyond our scope.

Theorem 1.7. *Let*

$$\left((\langle A_i\rangle, \langle q_i\rangle)_{i\in[N]}, \langle \mathbf{L}\rangle\right) : t \mapsto \frac{1}{|[0,L]|}\int_{[0,L]} \left((A_i,q_i)_{i\in[N]}, \mathbf{L}\right)(t,x)\mathrm{d}x$$

and, for all $\mathbf{d} \in (\mathbf{0},\boldsymbol{\infty})$*, let* $\mathcal{Q}_{\mathbf{d}}$ *be the operator* $\mathcal{Q}$ *with* $(A_i)_{i\in[N]}$ *replaced by* $(d_iA_i)_{i\in[N]}$*.*

Then, as $\min_{i\in[N]} d_i \to +\infty$*,*

$$\lambda_{1,\mathrm{per}}(\mathcal{Q}_{\mathbf{d}}) \to \lambda_{1,\mathrm{per}}\left(\partial_t - \langle \mathbf{L}\rangle\right).$$

This theorem basically means that, for the periodic principal eigenvalue $\lambda_{1,\mathrm{per}}$, very large diffusion rates tend to replace spatially heterogeneous coefficients by their mean values in space. Again, the fact that the multiplicative coefficients d_i can go to $+\infty$ at different rates is meaningful.

In the scalar case [54], the inequality

$$\lambda_{1,\mathrm{per}}(\partial_t - \langle l_{1,1}\rangle) \geq \min_{x\in[0,L]} \lambda_{1,\mathrm{per}}(\partial_t - l_{1,1}(x))$$

holds, and implies a comparison between the large diffusion asymptotic and the vanishing diffusion asymptotic. In the vector case, this inequality is still true if for instance the periodic principal eigenfunction associated with $\partial_t - \mathbf{L}(x)$ depends neither on t nor on x. Indeed, by integrating the equality it satisfies over $[0,L]$,

$$\lambda_{1,\mathrm{per}}(\partial_t - \langle \mathbf{L}\rangle) = \frac{1}{|[0,L]|}\int_{[0,L]} \lambda_{1,\mathrm{per}}(\partial_t - \mathbf{L}(\cdot)) \geq \min_{x\in[0,L]} \lambda_{1,\mathrm{per}}(\partial_t - \mathbf{L}(x)).$$

[5]More precisely, using $\mathcal{L}^\infty$ bounds on the coefficients of the zeroth order term, we can readily deduce that $-|P_1(z)|\max_{i\in[N]} d_i \leq \lambda_{1,z}(\mathcal{Q}_{\mathbf{d}}) \leq -|P_2(z)|\min_{i\in[N]} d_i$ for some second-order polynomials P_1, P_2. What would then be relevant would be to figure out an asymptotic expansion of $\lambda_{1,z}(\mathcal{Q}_{\mathbf{d}})$.

However it is not true in full generality, as shown by the counter-example of Remark 3.9.

1.5.4. *Dependence on the space or time frequencies.* As explained before, assuming that all spatial periods L_α coincide and changing appropriately the variables t and x, the $\overline{\Omega_{\mathrm{per}}}$-periodic operator $\boldsymbol{\mathcal{Q}}$ becomes the following $[0,1]\times[0,1]^n$-periodic operator:

$$\boldsymbol{\mathcal{Q}}_{T,L_1} = \frac{1}{T}\partial_t - \frac{1}{L_1^2}\,\mathrm{diag}(\nabla\cdot(A_i^\diamondsuit\nabla)) + \frac{1}{L_1}\,\mathrm{diag}(q_i^\diamondsuit\cdot\nabla) - \mathbf{L}^\diamondsuit,$$

where

$$\left(\left(A_i^\diamondsuit, q_i^\diamondsuit\right)_{i\in[N]}, \mathbf{L}^\diamondsuit\right) : (t,x)\mapsto \left((A_i,q_i)_{i\in[N]},\mathbf{L}\right)(Tt, L_1 x).$$

Theorems 1.6 and 1.7 and Remark 3.9 have immediate interpretations in this context, summarized in the following corollary.

Corollary 1.8. *The generalized principal eigenvalues of* $\boldsymbol{\mathcal{Q}}_{T,L_1}$ *satisfy the following properties.*

(1) If $L_1\to+\infty$*, then*

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_{T,L_1}) \to \min_{x\in[0,1]^n}\lambda_{1,\mathrm{per}}\left(\frac{1}{T}\partial_t - \mathbf{L}^\diamondsuit(x)\right) \quad \text{for all } z\in\mathbb{R}^n.$$

(2) If $q_i^\diamondsuit = 0$ *for each* $i\in[N]$ *and* $L_1\to 0$*, then*

$$\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}_{T,L_1}) \to \lambda_{1,\mathrm{per}}\left(\frac{1}{T}\partial_t - \langle\mathbf{L}^\diamondsuit\rangle\right).$$

(3) There exist a choice of $\left(A_i^\diamondsuit\right)_{i\in[N]}$, $\left(q_i^\diamondsuit\right)_{i\in[N]}$ *and* $\mathbf{L}^\diamondsuit$ *such that* $L_1\mapsto \lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}_{T,L_1})$ *is decreasing, a choice such that it is constant and a choice such that it is neither.*

It remains to investigate the effect of the time frequency $1/T$. In the case of a scalar equation in a bounded domain with Dirichlet boundary conditions, this problem was recently studied thoroughly by Liu, Lou, Peng and Zhou [48]. They identified cases where $T\mapsto\lambda_{1,\mathrm{Dir}}(\boldsymbol{\mathcal{Q}}_{T,L_1})$ is constant, cases where it is decreasing and cases where it is neither; additionally, they studied the asymptotics $T\to 0$ and $T\to+\infty$ – reusing the proof of Nadin [54] for the limit $T\to 0$. We will adapt the arguments [48, 54] to prove the following result.

Theorem 1.9. *For all* $\omega\in(0,+\infty)$*, let* $\boldsymbol{\mathcal{Q}}_\omega$ *be the operator* $\boldsymbol{\mathcal{Q}}$ *with* ∂_t *replaced by* $\omega\partial_t$*. Then:*

(1) if $\omega\to 0$*, then for all* $z\in\mathbb{R}^n$*,*

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_\omega) \to \frac{1}{T}\int_0^T \lambda_{1,z}\left(-\,\mathrm{diag}(\nabla\cdot(A_i(t)\nabla) - q_i(t)\cdot\nabla) - \mathbf{L}(t)\right)\mathrm{d}t,$$

$$\lambda_1(\boldsymbol{\mathcal{Q}}_\omega) \to \frac{1}{T}\int_0^T \lambda_1\left(-\,\mathrm{diag}(\nabla\cdot(A_i(t)\nabla) - q_i(t)\cdot\nabla) - \mathbf{L}(t)\right)\mathrm{d}t,$$

where we denote (with a slight abuse of notation) $((A_i(t),q_i(t))_{i\in[N]},\mathbf{L}(t)) : x\mapsto((A_i(t,x),q_i(t,x))_{i\in[N]},\mathbf{L}(t,x))$*;*

(2) *if $\omega \to +\infty$, then for all $z \in \mathbb{R}^n$,*

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_\omega) \to \lambda_{1,z}\left(-\operatorname{diag}(\nabla\cdot(\hat{A}_i\nabla) - \hat{q}_i\cdot\nabla) - \hat{\mathbf{L}}\right),$$

$$\lambda_1(\boldsymbol{\mathcal{Q}}_\omega) \to \lambda_1\left(-\operatorname{diag}(\nabla\cdot(\hat{A}_i\nabla) - \hat{q}_i\cdot\nabla) - \hat{\mathbf{L}}\right),$$

where

$$\left((\hat{A}_i, \hat{q}_i)_{i\in[N]}, \hat{\mathbf{L}}\right) : x \mapsto \frac{1}{T}\int_0^T \left((A_i, q_i)_{i\in[N]}, \mathbf{L}\right)(t,x)\mathrm{d}t.$$

Corollary 1.10. *The generalized principal eigenvalues of $\boldsymbol{\mathcal{Q}}_{T,L_1}$ satisfy the following properties.*

(1) *If $T \to +\infty$, then for all $z \in \mathbb{R}^n$,*

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_{T,L_1}) \to \int_0^1 \lambda_{1,z}\left(-\frac{1}{L_1^2}\operatorname{diag}(\nabla\cdot(A_i^\diamondsuit(t)\nabla)) + \frac{1}{L_1}\operatorname{diag}(q_i^\diamondsuit(t)\cdot\nabla) - \mathbf{L}^\diamondsuit(t)\right)\mathrm{d}t.$$

$$\lambda_1(\boldsymbol{\mathcal{Q}}_{T,L_1}) \to \int_0^1 \lambda_1\left(-\frac{1}{L_1^2}\operatorname{diag}(\nabla\cdot(A_i^\diamondsuit(t)\nabla)) + \frac{1}{L_1}\operatorname{diag}(q_i^\diamondsuit(t)\cdot\nabla) - \mathbf{L}^\diamondsuit(t)\right)\mathrm{d}t.$$

(2) *If $T \to 0$, then for all $z \in \mathbb{R}^n$,*

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_{T,L_1}) \to \lambda_{1,z}\left(-\frac{1}{L_1^2}\operatorname{diag}(\nabla\cdot(\hat{A}_i^\diamondsuit\nabla)) + \frac{1}{L_1}\operatorname{diag}(\hat{q}_i^\diamondsuit\cdot\nabla) - \hat{\mathbf{L}}^\diamondsuit\right).$$

$$\lambda_1(\boldsymbol{\mathcal{Q}}_{T,L_1}) \to \lambda_1\left(-\frac{1}{L_1^2}\operatorname{diag}(\nabla\cdot(\hat{A}_i^\diamondsuit\nabla)) + \frac{1}{L_1}\operatorname{diag}(\hat{q}_i^\diamondsuit\cdot\nabla) - \hat{\mathbf{L}}^\diamondsuit\right).$$

Theorem 1.9 shows that large time frequencies tend to replace time heterogenous coefficients by their mean values in time, whereas small time frequencies tend to replace the parabolic operator by the elliptic operator parametrized by t before averaging the eigenvalue.

1.5.5. *Explicit formulas for operators with space or time homogeneity.* Recall the notations, introduced earlier, $\hat{A}_i$, $\hat{q}_i$, $\hat{\mathbf{L}}$ for the mean values in time and $\langle A_i\rangle$, $\langle q_i\rangle$, $\langle\mathbf{L}\rangle$ for the mean values in space. In this section, for the sake of brevity, we use these notations repeatedly. The combined notation, *e.g.* $\langle\hat{\mathbf{L}}\rangle$, denotes naturally a space-time mean value.

Our next two theorems are concerned with operators whose coefficients depend only on time or only on space, and generalize [54, Propositions 3.1 and 3.2]. In the statements, the Perron–Frobenius eigenvalue of a reducible matrix is defined by continuous extension of the dominant eigenvalue on the set of essentially nonnegative matrices; for the sake of simplicity, its nonnegative eigenvectors are still referred to as Perron–Frobenius eigenvectors, even though they might not be positive and the eigenvalue might not be simple (algebraically and/or geometrically).

Theorem 1.11. *Assume:*

(1) $(A_i)_{i\in[N]}$, $(q_i)_{i\in[N]}$ *and* $\mathbf{L}$ *do not depend on* x,
(2) *there exists a constant positive vector* $\mathbf{u} \in (\mathbf{0}, \boldsymbol{\infty})$ *such that* $\mathbf{u}$ *is a Perron–Frobenius eigenvector of* $\mathbf{L}(t)$ *for all* $t \in \mathbb{R}$.

Let $z \in \mathbb{R}^n$. The equality

$$\lambda_{1,z} = -z \cdot \hat{A}_1 z + \hat{q}_1 \cdot z - \lambda_{\mathrm{PF}}(\hat{\mathbf{L}})$$

is true if $z = 0$ or $(A_1, q_1) = (A_2, q_2) = \cdots = (A_N, q_N)$.

Consequently, if:

(1) $(A_1, q_1) = (A_2, q_2) = \cdots = (A_N, q_N)$,
(2) $\hat{q}_1 = 0$,

then

$$\lambda_1 = \lambda_1' = -\lambda_{\mathrm{PF}}(\hat{\mathbf{L}}).$$

We explain in Remark 3.11 that if the assumption on the existence of a constant positive eigenvector is not satisfied, then the claimed equality on $\lambda_{1,z}$ is false in general. This is striking, since in the scalar setting, the existence of a constant positive eigenvector is not required.

Theorem 1.12. *Assume:*

(1) $(A_i)_{i\in[N]}$ *and* $\mathbf{L}$ *do not depend on* t,
(2) $\mathbf{L}(x)$ *is symmetric for all* $x \in \mathbb{R}^n$,
(3) *there exists* $z \in \mathbb{R}^n$ *and* $Q \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ *such that* $\int_{[0,L]} \nabla Q = 0$ *and*

$$A_1^{-1} q_1 = A_2^{-1} q_2 = \cdots = A_N^{-1} q_N = 2z + \nabla Q.$$

Then

$$\lambda_1 = \lambda_{1,z} = \min_{\mathbf{u}\in\mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n,\mathbb{R}^N)\setminus\{\mathbf{0}\}} \frac{\displaystyle\int_{[0,L]} \left(\sum_{i=1}^N (\nabla u_i \cdot A_i \nabla u_i)^2 - \mathbf{u}^{\mathrm{T}} \mathbf{L}_{Q,z} \mathbf{u} \right)}{\displaystyle\int_{[0,L]} |\mathbf{u}|^2},$$

where

$$\mathbf{L}_{Q,z} = \mathbf{L} + \mathrm{diag}\left(\frac{1}{2} \nabla \cdot (A_i \nabla Q) - \frac{1}{4} \nabla Q \cdot A_i \nabla Q + \nabla \cdot (A_i z) - z \cdot A_i (z + \nabla Q) \right).$$

Furthermore, if there exists a constant positive vector $\mathbf{u} \in (\mathbf{0}, \boldsymbol{\infty})$ *such that* $\mathbf{u}$ *is a Perron–Frobenius eigenvector of* $\mathbf{L}_{Q,z}(x)$ *for all* $x \in \mathbb{R}^n$, *then*

$$\lambda_1 = \lambda_1' \leq -\lambda_{\mathrm{PF}}(\langle \mathbf{L}_{Q,z} \rangle).$$

We will explain in Remark 3.13 that if $\mathbf{L}$ is not symmetric, then, even in the simple case $z = q_1 = q_2 = \cdots = 0$, there are counter-examples where

$$\lambda_1' > \min_{\mathbf{u}\in\mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n,\mathbb{R}^N)\setminus\{\mathbf{0}\}} \frac{\displaystyle\int_{[0,L]} \left(\sum_{i=1}^N (\nabla u_i \cdot A_i \nabla u_i)^2 - \mathbf{u}^{\mathrm{T}} \mathbf{L} \mathbf{u} \right)}{\displaystyle\int_{[0,L]} |\mathbf{u}|^2}.$$

As noted before, we will also recall in Remark 3.5 the counter-example of Griette–Matano [37] where the mere asymmetry of $\mathbf{L}$ breaks the equality $\lambda_1 = \lambda_1'$.

We will also explain in Remark 3.16 that if the assumption on the existence of a constant positive eigenvector is not satisfied, then the inequality between $-\lambda_1'$ and the Perron–Frobenius eigenvalue of $\langle \mathbf{L}_{Q,z} \rangle$ can fail. Again, in the scalar case, this assumption is not required [11].

The following theorem is similar in spirit and requires a line-sum-symmetry assumption ($\mathbf{L}\mathbf{1} = \mathbf{L}^{\mathrm{T}}\mathbf{1}$). Examples of line-sum-symmetric essentially nonnegative matrices are doubly stochastic matrices, essentially nonnegative symmetric matrices and essentially nonnegative circulant matrices. For more details on line-sum-symmetric matrices, refer for instance to Eaves–Hoffman–Rothblum–Schneider [27].

Theorem 1.13. *Assume $\mathbf{L}(t,x)$ is line-sum-symmetric at all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$.*
Let $z \in \mathbb{R}^n$. If, for all $i \in [N]$, $\nabla \cdot (q_i - 2A_i z) = 0$, then

$$\lambda_{1,z} \leq -\frac{1}{N}\left(\sum_{i,j=1}^{N} \langle \hat{l}_{i,j} \rangle + z \cdot \sum_{i=1}^{N} \left(\langle \hat{A}_i \rangle z - \langle \hat{q}_i \rangle \right)\right)$$

and this inequality is an equality if $\mathbf{L} + \mathrm{diag}(\nabla \cdot (A_i z) + z \cdot (A_i z - q_i))$ is irreducible at all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$ with Perron–Frobenius eigenvector $\mathbf{1}$ and constant Perron–Frobenius eigenvalue.

This theorem has several interesting consequences, which we detail in Corollaries 3.28 and 3.29.

Two similar results without line-sum-symmetry follow.

Theorem 1.14. *Let $z \in \mathbb{R}^n$. If, for all $i \in [N]$, $\nabla \cdot (q_i - 2A_i z) = 0$, then*

$$\lambda_{1,z} \leq \lambda_{1,z}\left(\partial_t - \mathrm{diag}(\nabla \cdot (\langle A_i \rangle \nabla) - \langle q_i \rangle) - \mathbf{L}^{\#}\right),$$

where the entries of the matrix $\mathbf{L}^{\#} = \left(l^{\#}_{i,j}\right)_{(i,j)\in[N]^2}$ are defined by:

$$l^{\#}_{i,j} : t \mapsto \begin{cases} \frac{1}{|[0,L]|}\int_{[0,L]} l_{i,i}(t,x)\mathrm{d}x & \text{if } i = j, \\ \exp\left(\frac{1}{|[0,L]|}\int_{[0,L]} \ln l_{i,j}(t,x)\mathrm{d}x\right) & \text{if } i \neq j \text{ and } \min\limits_{(t,x)\in\overline{\Omega_{\mathrm{per}}}} l_{i,j}(t,x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.15. *Let $z \in \mathbb{R}^n$. If $(A_i)_{i\in[N]}$, $(q_i)_{i\in[N]}$ and $\mathbf{L}$ do not depend on x, then*

$$\lambda_{1,z} \leq -\lambda_{\mathrm{PF}}\left(\mathbf{L}^{\flat} + \mathrm{diag}\left(z \cdot \hat{A}_i z - \hat{q}_i \cdot z\right)\right),$$

where the entries of the matrix $\mathbf{L}^{\flat} = \left(l^{\flat}_{i,j}\right)_{(i,j)\in[N]^2}$ are defined by:

$$l^{\flat}_{i,j} = \begin{cases} \frac{1}{T}\int_0^T l_{i,i} & \text{if } i = j, \\ \exp\left(\frac{1}{T}\int_0^T \ln l_{i,j}\right) & \text{if } i \neq j \text{ and } \min\limits_{t\in[0,T]} l_{i,j}(t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The operator introduced in Theorem 1.14 is spatially homogeneous, so that

$$\lambda_{1,z}\left(\partial_t - \mathrm{diag}\left(\nabla \cdot (\langle A_i \rangle \nabla) - \langle q_i \rangle \cdot \nabla\right) - \mathbf{L}^{\#}\right) = \lambda_{1,\mathrm{per}}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{L}^{\#} - \mathrm{diag}(z \cdot \langle A_i \rangle z - \langle q_i \rangle \cdot z)\right).$$

These two theorems show that when comparing heterogeneous environments with averaged environments, heterogeneities tend to decrease the generalized principal eigenvalues, provided the geometric mean is used for the off-diagonal entries of $\mathbf{L}$. This is of course related to the convexity property of Theorem 1.3. This is also related to the asymptotic results of Theorems 1.7 and 1.9, although in these asymptotics the off-diagonal entries are averaged with the arithmetic mean instead of the geometric mean. By comparing the arithmetic and geometric means and using the monotonicity of $\lambda_{1,z}$ with respect to $\mathbf{L}$, we can try to compare these results; however, inequalities are in the wrong sense. For instance, in the simple case $z=0$ with each q_i divergence-free, what we get is:

$$\begin{aligned}\lambda_{1,\mathrm{per}}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathbf{L}^{\#}\right) &\geq \max\left[\lambda_1'(\boldsymbol{\mathcal{Q}}),\lambda_{1,\mathrm{per}}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\langle\mathbf{L}\rangle\right)\right] \\ &= \max\left[\lambda_1'(\boldsymbol{\mathcal{Q}}),\lim_{\min_{i\in[N]} d_i\to+\infty}\lambda_1'(\boldsymbol{\mathcal{Q}}_{\mathbf{d}})\right].\end{aligned}$$

1.5.6. *Optimization.* Our first optimization result is a highly nontrivial generalization of a result on matrices of Neumann–Sze [57]. To the best of our knowledge, in the context of cooperative partial differential operators, it is the first time such a result is stated and proved.

Recall that a doubly stochastic matrix $\mathbf{S}\in\mathbb{R}^{N\times N}$ is a nonnegative matrix such that $\mathbf{S}\mathbf{1}=\mathbf{S}^{\mathrm{T}}\mathbf{1}=\mathbf{1}$. Denote $\boldsymbol{\mathcal{S}}\subset\mathcal{L}^{\infty}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^{N\times N})$ the set of all periodic functions whose values are doubly stochastic matrices almost everywhere and $\boldsymbol{\mathcal{S}}_{\{0,1\}}$ the restriction to functions valued in the set of permutation matrices almost everywhere.

A decomposition $\mathbf{L}=\mathrm{diag}(\mathbf{r})+(\mathbf{S}-\mathbf{I})\,\mathrm{diag}(\boldsymbol{\mu})$ of a given essentially nonnegative matrix $\mathbf{L}$ with $\mathbf{S}$ doubly stochastic and $\boldsymbol{\mu}$ nonnegative exists in many cases (in particular, as soon as $\mathbf{L}(t,x)$ is irreducible) and its uniqueness can be ensured by an appropriate normalization of the pair $(\mathbf{S},\boldsymbol{\mu})$, as explained in Lemma 3.32. The main property of this decomposition is that the so-called mutation part $(\mathbf{S}-\mathbf{I})\,\mathrm{diag}(\boldsymbol{\mu})$ admits $\mathbf{1}$ as left Perron–Frobenius eigenvector, with eigenvalue 0. In other words, summing the lines of the system makes the mutations disappear: if the phenotypes do not differ in intrinsic growth rate (all r_i coincide), then the phenotype distribution has no effect on the growth of the meta-population $\sum_{i=1}^N u_i$. This is indeed under this form that $\mathbf{L}$ appears in several papers on reaction–diffusion models for phenotypically structured populations [21, 38, 53].

Theorem 1.16. *Assume* $\mathbf{L}$ *can be decomposed as* $\mathbf{L}=\mathrm{diag}(\mathbf{r})+(\mathbf{S}-\mathbf{I})\,\mathrm{diag}(\boldsymbol{\mu})$ *with* $\mathbf{S}\in\boldsymbol{\mathcal{S}}$, $\mathbf{r}\in\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^N)$ *and* $\boldsymbol{\mu}\in\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,[\mathbf{0},\boldsymbol{\infty}))$.

Then, for all $z\in\mathbb{R}^n$,

$$\min_{\mathbf{S}\in\boldsymbol{\mathcal{S}}_{\{0,1\}}}\lambda_{1,z}(\mathbf{S})=\min_{\mathbf{S}\in\boldsymbol{\mathcal{S}}}\lambda_{1,z}(\mathbf{S})\leq\max_{\mathbf{S}\in\boldsymbol{\mathcal{S}}}\lambda_{1,z}(\mathbf{S})=\max_{\mathbf{S}\in\boldsymbol{\mathcal{S}}_{\{0,1\}}}\lambda_{1,z}(\mathbf{S}).$$

This theorem does not require the assumption (A_3) (which is not satisfied for some choices of $\mathbf{S}$; in such cases, the generalized principal eigenvalues $\lambda_{1,z}$ are defined by continuous extension, *cf.* Theorem 1.5). In particular, the set of optimal permutation matrices might *a priori* be reduced to the singleton $\{\mathbf{I}\}$. Also, in this theorem, and as usual in optimization problems, we consider $\mathcal{L}^{\infty}$ constraints on $\mathbf{S}$ instead of Hölder-continuity constraints; the optimizers might be for instance "bang-bang" discontinuous piecewise-constant functions. Let us also point out that, as explained in Remark 3.20, the result remains true with any more general decomposition $\mathbf{L}=\mathbf{B}+\mathbf{S}\mathbf{A}$ with $\mathbf{A}$ nonnegative and $\mathbf{B}$ essentially nonnegative.

The modeling viewpoint on this result is natural and enlightening. Say we want to optimize the chances of, for instance, survival of a population, and, for simplicity, that the environment is homogeneous; the phenotypes are labelled as follows: u_1 is the best phenotype when there are no mutations, u_2 is the second best phenotype, and so forth. Intuitively we should select a (reducible) mutation strategy such that the type u_1 is 100% heritable. Thus the first column of $\mathbf{S}$ should be $\mathbf{e}_1$. Since $\mathbf{S}$ is doubly stochastic, its first line is then $\mathbf{e}_1^{\mathrm{T}}$, whence the first phenotype is in fact completely isolated from the others. Subsequently, whatever the mutation strategy for the phenotypes u_2, u_3, etc., is, the periodic principal eigenvalue is optimal and equal to the periodic principal eigenvalue of the scalar equation satisfied by u_1. If u_2 is just as good as u_1, then similarly the pair $\{u_1, u_2\}$ has to be isolated, but apart from this restriction the two blocks of $\mathbf{S}$ can be chosen freely, and in particular they can have the form of permutation matrices. The extension of this intuition to spatio-temporally heterogeneous environments explains why the optimal $\mathbf{S}$ is not in general constant; it has to "switch" as soon as the optimal family of phenotypes changes.

Let us stress that although the set of doubly stochastic matrices is the convex hull of the set of permutation matrices (a classical result known as the Birkhoff–von Neumann theorem), $\mathbf{S} \in \boldsymbol{\mathcal{S}} \mapsto \lambda_{1,z}(\mathbf{S})$ is not concave (*cf.* Theorem 1.3), so that Theorem 1.16 does not follow from mere convexity considerations. Let us also stress that as soon as all $(\mathcal{P}_i, r_i)$ coincide with constant r_i, $\mathbf{S} \mapsto \lambda_{1,z}(\mathbf{S})$ is constant: maximizers and minimizers need not be in $\boldsymbol{\mathcal{S}}_{\{0,1\}}$ and can coincide.

The proof of Theorem 1.16 is in fact quite involved and requires the construction of an explicit rank-one correction of $\mathbf{S}$, localized in space-time.

Our second optimization result, closely related to Theorem 1.16, generalizes a theorem due to Karlin and later generalized by Altenberg [3, 44] which states that, for any irreducible stochastic matrix $\mathbf{S}$ and any diagonal matrix $\mathbf{D}$ with positive diagonal entries, the mapping $\tau \in [0,1] \mapsto \lambda_{\mathrm{PF}}(((1-\tau)\mathbf{I} + \tau\mathbf{S})\mathbf{D})$ is nonincreasing. Karlin's theorem has been interpreted as "greater mixing produces slower growth" and shows how, in a space-time homogeneous setting, mutations reduce the chances of survival.

Theorem 1.17. *Assume* $\mathbf{L}$ *can be decomposed as* $\mathbf{L} = \operatorname{diag}(\mathbf{r}) + (\mathbf{S} - \mathbf{I})\operatorname{diag}(\boldsymbol{\mu})$ *with* $\mathbf{S} \in \boldsymbol{\mathcal{S}}$, $\mathbf{r} \in \mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^N)$ *and* $\boldsymbol{\mu} \in \mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R} \times \mathbb{R}^n, [\mathbf{0}, \boldsymbol{\infty}))$.

Then the mapping

$$s \in (0,1] \mapsto \lambda_1' \left(s \operatorname{diag}(\boldsymbol{\mathcal{P}}) - \operatorname{diag}(\mathbf{r}) - s(\mathbf{S} - \mathbf{I}) \operatorname{diag}(\boldsymbol{\mu}) \right)$$

is nondecreasing, identically zero if $\mathbf{r} = \mathbf{0}$, *and increasing if* $\mathbf{r} \neq \mathbf{0}$ *depends on* x.

Furthermore, this is true for the whole family $(\lambda_{1,z})_{z \in \mathbb{R}^n}$ *if* $\operatorname{diag}(\boldsymbol{\mathcal{P}}) = \mathcal{P}_1\mathbf{I}$ *and* (A_1, q_1) *does not depend on* (t, x).

The proof will establish a more general result that might be of independent interest; we state it in Corollary 3.35.

Note that, in the above statement, the parameter s is not a true mutation rate, as it also multiplies $\operatorname{diag}(\boldsymbol{\mathcal{P}})$. It is unclear to us whether a stronger result, where s only multiplies $(\mathbf{S} - \mathbf{I})\operatorname{diag}(\boldsymbol{\mu})$, can be achieved, apart from of course the very special case where the coefficients of $\boldsymbol{\mathcal{Q}}$ are all space-time homogeneous.

Our last optimization result deals with the spatial distribution in the matrix $\mathbf{L}$ in one dimension of space. In this context, the spatial periodicity cell is then the interval $(0, L_1)$. Our result is a generalization of a result by Nadin [54] and makes use of the periodic rearrangement. We recall that for any scalar L_1-periodic function u there exists a unique L_1-periodic function $u^\dagger$ whose restriction to $[0, L_1]$ is symmetric (with respect to the midpoint $L_1/2$) and non-increasing in $[L_1/2, L_1]$

and that has the same distribution function as u. The distribution function of u is

$$\mu_u : t \mapsto |\{u \geq t\} \cap [0, L_1]|$$

and $u^\dagger$, the periodic rearrangement of u, is the left inverse of μ_u. For a time dependent scalar function u, $u^\dagger$ stands for the function rearranged, at every t, with respect to x. For a time-dependent, vector (respectively matrix) valued function $\mathbf{u}$, the notation $\mathbf{u}^\dagger$ is understood as the vector-valued function with i-th (resp. (i,j)-th) component $u_i^\dagger$ (resp. $u_{i,j}^\dagger$).

Theorem 1.18. *Assume $n = 1$ and $\operatorname{diag}(\boldsymbol{\mathcal{P}}) = \partial_t - \mathbf{D}\Delta$ for some diagonal matrix $\mathbf{D}$ with constant, positive diagonal entries.*

Then

$$\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}) \geq \lambda_{1,\mathrm{per}}(\operatorname{diag}(\boldsymbol{\mathcal{P}}) - \mathbf{L}^\dagger)$$

where $\mathbf{L}^\dagger$ is the entry-wise periodic rearrangement of $\mathbf{L}$.

Note that this theorem optimizes the distribution of each $l_{i,j}$ but does not optimize the distribution of mass in the matrix $\mathbf{L}$. Brenier [18] showed that the rearrangement of a function of x and the polar decomposition of an invertible matrix in $\mathbb{R}^{N\times N}$ are related notions, via the relations $\mu(x) = \mu^\#(u(x))$ ($\mu^\#$ is the spatial rearrangement, u is unitary) and $\mathbf{M} = \mathbf{R}\mathbf{U}$ ($\mathbf{R} = (\mathbf{M}\mathbf{M}^\mathrm{T})^{\frac{1}{2}}$ is symmetric positive definite, $\mathbf{U}$ is orthogonal). In particular, it is well-known that, similarly to $\lambda_{1,\mathrm{per}}(-\Delta - \mu^\#) \leq \lambda_{1,\mathrm{per}}(-\Delta - \mu)$, any essentially nonnegative matrix $\mathbf{M} \in \mathbb{R}^{N\times N}$ satisfies $\lambda_{\mathrm{PF}}(\mathbf{M}) \leq \lambda_{\max}((\mathbf{M}\mathbf{M}^\mathrm{T})^{\frac{1}{2}})$, where $\lambda_{\max}$ denotes the maximal eigenvalue of a real symmetric matrix. In other words, if $\boldsymbol{\mathcal{Q}}$ has only constant coefficients,

$$\lambda_{1,\mathrm{per}}(\operatorname{diag}(\boldsymbol{\mathcal{P}}) - (\mathbf{L}^\mathrm{T}\mathbf{L})^{\frac{1}{2}}) \leq \lambda_{1,\mathrm{per}}(\operatorname{diag}(\boldsymbol{\mathcal{P}}) - \mathbf{L}),$$

where the periodic principal eigenvalue $\lambda_{1,\mathrm{per}}$ on the left-hand side is defined via the spectral theorem for self-adjoint compact operators instead of via the Krein–Rutman theorem – the matrix $(\mathbf{L}^\mathrm{T}\mathbf{L})^{\frac{1}{2}}$ is not, in general, essentially nonnegative. However the proofs of $\lambda_{1,\mathrm{per}}(-\Delta - \mu^\#) \leq \lambda_{1,\mathrm{per}}(-\Delta - \mu)$ and of $\lambda_{\mathrm{PF}}(\mathbf{M}) \leq \lambda_{\max}((\mathbf{M}\mathbf{M}^\mathrm{T})^{\frac{1}{2}})$ differ strongly. The first one typically uses the Hardy–Littlewood inequality, which is false for matrices as showed by Brenier [18]. Therefore it seems that optimizing $\mathbf{L}$ in the spatial sense and in the matrix sense simultaneously is much more difficult and we leave it as a very interesting open problem.

1.6. **Extension to systems with a coupling default.** Theorem 1.5 shows how results on fully coupled cooperative systems (and especially the results of Sections 1.5.1–1.5.6) can be applied to more general cooperative systems, that need not satisfy (A_3), by understanding them as networks of fully coupled subsystems. It also shows that such a perspective is limited regarding λ_1, as we are now going to explain.

Recall that the Perron–Frobenius eigenvalue λ_{PF} can be understood as the restriction to the set of irreducible essentially nonnegative matrices of the dominant eigenvalue, which is a well-defined continuous mapping from the set of essentially nonnegative matrices to $\mathbb{R}$. Therefore it is natural to suggest the following extension of the generalized principal eigenvalues $\lambda_{1,z}$ and $\lambda_1(\Omega)$:

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}) = \min_{k\in[N']} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k) = \min_{k\in[N']} \lambda_{1,\mathrm{per}}(\mathrm{e}_{-z}\boldsymbol{\mathcal{Q}}_k\mathrm{e}_z),$$

$$\lambda_1(\boldsymbol{\mathcal{Q}}, \Omega) = \min_{k\in[N']} \lambda_1(\boldsymbol{\mathcal{Q}}_k, \Omega),$$

where $\boldsymbol{\mathcal{Q}}_k$ denotes as in the statement of Theorem 1.5 the k-th fully coupled block of $\boldsymbol{\mathcal{Q}} = \operatorname{diag}(\boldsymbol{\mathcal{P}}) - \mathbf{L}^{\triangle}$ in block upper triangular form. With these definitions, Theorem 1.5 shows that the extension of each $\lambda_{1,z}$, and in particular that of λ_1', is continuous. However, as explained in Remark 3.7, the inequality

$$\lim_{\mathbf{L}\to\mathbf{L}^{\triangle}} \lambda_1(\boldsymbol{\mathcal{Q}}) = \max_{z\in\mathbb{R}^n} \min_{k\in[N']} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k) \leq \min_{k\in[N']} \lambda_1(\boldsymbol{\mathcal{Q}}_k)$$

is in some cases strict: the extension of λ_1 suggested above is not lower semi-continuous, and *a fortiori* not continuous.

It might be tempting to think that this discontinuity is caused by a wrong choice of generalized definition, and that the correct choice should be continuous. For instance, defining λ_1 as $\max_{z\in\mathbb{R}^n} \lambda_{1,z}$ would give a continuous extension to systems with a coupling default. In view of the literature [8, 13, 14, 54], it is also natural to consider the original definition (3) of λ_1, and since the coupling default induces a weaker maximum principle, it is also natural to consider a relaxed definition with nonnegative nonzero super-solutions instead of positive super-solutions. In order to compare these quantities, let us denote them as follows:

$$\lambda_1^0 = \min_{k\in[N']} \lambda_1(\boldsymbol{\mathcal{Q}}_k),$$

$$\lambda_1^1 = \max_{z\in\mathbb{R}^n} \lambda_{1,z},$$

$$\lambda_1^2 = \sup\left\{\lambda\in\mathbb{R} \mid \exists \mathbf{u}\in\mathcal{C}^{1,2}_{t-\text{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))\ \boldsymbol{\mathcal{Q}}\mathbf{u}\geq\lambda\mathbf{u}\right\},$$

$$\lambda_1^3 = \sup\left\{\lambda\in\mathbb{R} \mid \exists \mathbf{u}\in\mathcal{C}^{1,2}_{t-\text{per}}(\mathbb{R}\times\mathbb{R}^n), [\mathbf{0},\boldsymbol{\infty})),\ \mathbf{u}\neq\mathbf{0},\ \boldsymbol{\mathcal{Q}}\mathbf{u}\geq\lambda\mathbf{u}\right\}.$$

Then we can show[6] that

$$\lambda_1^1 \leq \lambda_1^0 \leq \max_{k\in[N']} \lambda_1(\boldsymbol{\mathcal{Q}}_k) = \lambda_1^3, \quad \lambda_1^2 \leq \lambda_1^0.$$

The inequality $\lambda_1^0 \leq \lambda_1^3$ is strict as soon as two $\lambda_1(\boldsymbol{\mathcal{Q}}_k)$ differ. The equality $\lambda_1^0 = \lambda_1^2$ is easily verified if $\boldsymbol{\mathcal{Q}}$ is block diagonal; although the proof seems to require some work, we believe that it remains true even if $\boldsymbol{\mathcal{Q}}$ is not block diagonal. In any case, since the counter-example of Remark 3.7 is block diagonal, there are block diagonal operators $\boldsymbol{\mathcal{Q}}$ such that $\lambda_1^1 < \lambda_1^0 = \lambda_1^2 < \lambda_1^3$. This shows that reasonable definitions of λ_1 other than λ_1^1 cannot be continuous as (A_3) ceases to be true.

Let us point out that λ_1^1 is indisputably the least natural definition. In particular, having in mind that $\lambda_1 < 0$ should be a criterion for population growth (see Section 1.7 below), then the natural definitions would be either λ_1^0 (growth of at least one population) or λ_1^3 (growth of all populations). In both cases, the default of lower semi-continuity means that populations with vanishingly small couplings might have much stronger chances than decoupled populations. This has strong implications for modeling, as simplifying a vanishingly coupled model into a decoupled one is often tempting. It has been related to the emergence in eco-evolutionary models of unexpectedly large spreading speeds in the vanishing mutation limit. Refer to Elliott–Cornell [28] for the first formal calculations and to Morris–Börger–Crooks [53] for the rigorous analysis.

[6] The proof is voluntarily not detailed, for the sake of brevity.

1.7. **Relation with KPP-type semilinear systems.** In the scalar framework of KPP-type reaction–diffusion equations, $\lambda_1 < 0$ implies the locally uniform convergence of all solutions to the unique periodic and uniformly positive entire solution, whereas $\lambda_1' \geq 0$ implies the uniform convergence of all solutions to 0, as proved by Nadin [56]. The study of entire solutions is much more delicate in the multidimensional setting, simply due to topological freedom [32, 35, 36, 53], and their uniqueness and stability properties cannot in general be inferred from the linearization at $\mathbf{0}$. However, we will show in a sequel that in the multidimensional case, the results of Nadin [56] can be generalized in the following weak form: $\lambda_1 < 0$ implies the locally uniform persistence of all solutions and the existence of a periodic and uniformly positive entire solution, whereas $\lambda_1' \geq 0$ implies the uniform convergence of all solutions to $\mathbf{0}$.

Going toward these results is one of our main motivations for the present work, the other one being the future construction of pulsating traveling waves [55].

2. Preliminaries

Many of our proofs will use a strong maximum principle and a Harnack inequality for parabolic cooperative systems. These already exist in the literature under slightly different forms (refer for instance to [10, 30, 59] or to [6, 9, 16, 24–26, 60] for the elliptic case). For the sake of self-containment and because the parabolic Harnack inequality in [30] is insufficient for our purposes, in this section, we state or prove what we need afterward.

2.1. **Strong maximum principle.** The strong maximum principle for time periodic nonnegative solutions of $\boldsymbol{\mathcal{Q}}\mathbf{u} + K\mathbf{u} = \mathbf{0}$ with large $K > 0$ (actually, $K > \lambda_1$) is established as a side result of the preparation of the application of the Krein–Rutman theorem, just as in Bai–He [10]. In fact, we can repeat the argument of [10, p. 9882] to obtain the strong maximum principle for all values of $K \in \mathbb{R}$, including $K = 0$ (large values of K are required only for the inversion of the operator), and for super-solutions that might not be time periodic but are well-defined in a sufficiently distant past. For clarity, we state this version of the strong maximum principle below.

Proposition 2.1 (Strong maximum principle)**.** *Let* $\mathbf{u} \in \mathcal{C}^{1,2}((0,+\infty)\times\mathbb{R}^n, [\mathbf{0},\boldsymbol{\infty}))\cap \mathcal{C}([0,+\infty)\times\mathbb{R}^N)$ *such that* $\boldsymbol{\mathcal{Q}}\mathbf{u} \geq \mathbf{0}$ *in* $(0,+\infty)\times\mathbb{R}^N$*.*

If there exist $t^\star > T$*,* $x^\star \in \mathbb{R}^n$ *and* $i^\star \in [N]$ *such that* $u_{i^\star}(t^\star, x^\star) = 0$*, then* $\mathbf{u} = \mathbf{0}$ *in* $[0,+\infty)\times\mathbb{R}^N$*.*

A similar property is satisfied in bounded domains. For the sake of simplicity, we only consider smooth boundaries.

Proposition 2.2 (Strong maximum principle in bounded domains)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a nonempty smooth bounded open connected set and* $\mathbf{u} \in \mathcal{C}^{1,2}((0,+\infty)\times\Omega, [\mathbf{0},\boldsymbol{\infty}))\cap \mathcal{C}^{0,1}([0,+\infty)\times\overline{\Omega})$ *such that* $\boldsymbol{\mathcal{Q}}\mathbf{u} \geq \mathbf{0}$ *in* $(0,+\infty)\times\Omega$*.*

Assume that there exists $x_0 \in \Omega$ *such that* $[x_0, x_0 + L] \subset \Omega$*.*

If there exist $t^\star > T$*,* $x^\star \in \Omega$ *and* $i^\star \in [N]$ *such that* $u_{i^\star}(t^\star, x^\star) = 0$*, then* $\mathbf{u} = \mathbf{0}$ *in* $[0,+\infty)\times\overline{\Omega}$*.*

If there exist $t^\star > T$*,* $x^\star \in \partial\Omega$ *and* $i^\star \in [N]$ *such that* $u_{i^\star}(i^\star, x^\star) = \nu(x^\star)\cdot\nabla u_{i^\star}(t^\star, x^\star) = 0$*, where* $\nu(x^\star) \in \mathbb{R}^n$ *is the outward pointing unit normal vector, then* $\mathbf{u} = \mathbf{0}$ *in* $[0,+\infty)\times\overline{\Omega}$*.*

These versions of the strong maximum principle fully exploit the irreducibility assumption (A_3): if one component of $\mathbf{u}$ is zero, then so are the others. Nonnegative super-solutions are either zero or positive. Without (A_3), this alternative is false in general; we refer, for weaker statements applicable to general cooperative systems, to the celebrated book by Protter and Weinberger [59, Chapter 3, Section 8].

2.2. **Harnack inequality.** In this section, we denote by $\sigma > 0$ the smallest positive entry of $\overline{\mathbf{L}}$ and by $K \geq 1$ the smallest positive number such that

$$K^{-1} \leq \min_{i\in[N]} \min_{y\in\mathbb{S}^{n-1}} \min_{(t,x)\in\overline{\Omega_{\text{per}}}} (y \cdot A_i(t,x)y),$$

$$\max_{i\in[N]} \max_{y\in\mathbb{S}^{n-1}} \max_{(t,x)\in\overline{\Omega_{\text{per}}}} (y \cdot A_i(t,x)y) \leq K,$$

$$\max_{i\in[N]} \max_{\alpha\in[n]} \max_{(t,x)\in\overline{\Omega_{\text{per}}}} |q_{i,\alpha}(t,x)| \leq K,$$

$$\max_{i,j\in[N]} \sup_{(t,x)\in\overline{\Omega_{\text{per}}}} |l_{i,j}(t,x)| \leq K.$$

Applying Földes–Poláčik's Harnack inequality [30, Theorem 3.9] to the operator $\mathcal{Q}$, we obtain the following property.

Proposition 2.3. *Let $\theta > 0$. Assume the irreducibility of the matrix*

$$\underline{\mathbf{L}} = \left(\min_{(t,x)\in\overline{\Omega_{\text{per}}}} l_{i,j}(t,x) \right)_{(i,j)\in[N]^2}$$

and denote $\eta > 0$ its smallest positive entry.

There exists a constant $\overline{\kappa}_{\theta,\eta} > 0$, determined only by n, N, η, K and the parameter θ such that, if $\mathbf{u} \in \mathcal{C}([-2\theta, 6\theta] \times [-\frac{3\theta}{2}, \frac{3\theta}{2}]^n, [\mathbf{0}, \boldsymbol{\infty}))$ is a solution of $\mathcal{Q}\mathbf{u} = \mathbf{0}$, then

$$\min_{i\in[N]} \min_{(t,x)\in[5\theta,6\theta]\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_i(t,x) \geq \overline{\kappa}_{\theta,\eta} \max_{i\in[N]} \max_{(t,x)\in[0,2\theta]\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_i(t,x).$$

However, our irreducibility assumption (A_3) is concerned with the matrix

$$\overline{\mathbf{L}} = \left(\max_{(t,x)\in\overline{\Omega_{\text{per}}}} l_{i,j}(t,x) \right)_{(i,j)\in[N]^2}$$

and not with $\underline{\mathbf{L}}$. By continuity and essential nonnegativity, $\overline{\mathbf{L}}$ is irreducible if and only if

$$(T|[0,L]|)^{-1} \int_{\Omega_{\text{per}}} \mathbf{L}(t,x)\mathrm{d}t\mathrm{d}x$$

is itself irreducible. Hence we can understand the assumption (A_3) as "$\mathbf{L}(t,x)$ is irreducible on average". It is known that such an assumption is sufficient, and in some sense necessary, for full coupling of the parabolic or elliptic operator; refer, for instance, to [9, 10, 16, 60].

Since Földes–Poláčik's Harnack inequality requires the pointwise irreducibility of $\mathbf{L}$, which is a much stronger assumption than the irreducibility on average (there are simple examples of matrices that are irreducible on average but reducible pointwise at all (t,x), *cf.*, *e.g.*, Remark 3.11), it is not satisfying for our purposes. Actually, going through the proof of [30, Theorem 3.9], it appears that its adaptation to our setting is not straightforward, as Földes and Poláčik overcome the key obstacle by constructing a nonnegative nonzero sub-solution smaller than η multiplied by some positive constant. Nevertheless, since (A_3) is known to be the optimal assumption for full coupling, it is natural to expect a similar Harnack inequality to hold, provided the parabolic cylinder under consideration is sufficiently larger than the periodicity cell Ω_{per}. This is what we prove below, drawing inspiration from the elliptic case studied in Araposthathis–Ghosh–Marcus [9].

By convenience for future use, we state the result for a zeroth order, diagonal, non-necessarily periodic perturbation of $\boldsymbol{\mathcal{Q}}$. The diffusion and advection terms can be perturbed similarly if needed.

Proposition 2.4 (Fully coupled Harnack inequality)**.** *Let* $\theta \geq \max(T, L_1, \dots, L_n)$, $\mathbf{f} \in \mathcal{L}^\infty \cap \mathcal{C}^{\delta/2,\delta}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^N)$ *with* $\delta \in (0,1)$ *and*

$$F \geq \max_{i\in[N]} \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |f_i(t,x)|.$$

There exists a constant $\overline{\kappa}_{\theta,F} > 0$*, determined only by* n*,* N*,* σ*,* K *and the parameters* θ *and* F *such that, if* $\mathbf{u} \in \mathcal{C}([-2\theta, 6\theta]\times[-\frac{3\theta}{2}, \frac{3\theta}{2}]^n, [\mathbf{0}, \boldsymbol{\infty}))$ *is a solution of* $\boldsymbol{\mathcal{Q}}\mathbf{u} = \operatorname{diag}(\mathbf{f})\mathbf{u}$*, then*

$$\min_{i\in[N]} \min_{(t,x)\in[5\theta,6\theta]\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_i(t,x) \geq \overline{\kappa}_{\theta,F} \max_{i\in[N]} \max_{(t,x)\in[0,2\theta]\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_i(t,x).$$

Proof. Define, for all $i \in [N]$, the $n+1$-dimensional hypercube

$$Q_i = \left(5\theta - \frac{\theta}{2^{i-1}}, 6\theta\right) \times \left(-\frac{\theta}{2} - \frac{\theta}{2^i}, \frac{\theta}{2} + \frac{\theta}{2^i}\right)^n \subset \mathbb{R}\times\mathbb{R}^n.$$

Note the series of compact inclusions

$$Q_1 = (4\theta, 6\theta)\times(-\theta,\theta)^n \supset Q_2 \supset \dots \supset Q_N \supset (5\theta, 6\theta)\times\left(-\frac{\theta}{2}, \frac{\theta}{2}\right)^n.$$

Following carefully the proof of Földes–Poláčik's Harnack inequality [30], we observe that we only have to prove the following claim.

Claim 1: let $k \in [N-1]$. If there exists $I \subset [N]$ of cardinal k and a positive constant κ_k determined only by k, n, N, σ, K, θ and F, such that, for all $j \in I$,

$$\min_{(t,x)\in\overline{Q_k}} u_j(t,x) \geq \kappa_k \max_{(t,x)\in[0,2\theta]\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_1(t,x),$$

then there exists $i \in [N]\backslash I$ and a positive constant $\kappa_{k+1} \leq \kappa_k$ determined only by k, n, N, σ, K, θ and F, such that

$$\min_{(t,x)\in\overline{Q_{k+1}}} u_i(t,x) \geq \kappa_{k+1} \max_{(t,x)\in[0,2\theta]\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_1(t,x).$$

We prove first the following simpler claim, inspired by [9, Lemma 3.6].

Claim 2: Let $k \in [N-1]$, $i \in [N]$ and $g \in \mathcal{C}(\overline{Q_k}, [0,+\infty))$. There exists a positive constant C_k determined only by k, n, K, θ and F, such that, if u is a solution of $\mathcal{P}_i u - l_{i,i}u - f_i u = g$ in Q_k with $u = 0$ on the parabolic boundary $\partial_P Q_k$, then

$$\min_{(t,x)\in\overline{Q_{k+1}}} u(t,x) \geq C_k \max_{(t,x)\in\overline{Q_k}} g(t,x).$$

Proof of Claim 2. When $g = 0$, $u = 0$ as well and the result is obvious (with, say, $C_k = 1$). Therefore we assume without loss of generality that $g > 0$.

Up to dividing u by $\max_{\overline{Q_k}} g$, we assume without loss of generality $\max_{\overline{Q_k}} g = 1$. Since the solution u of the Cauchy–Dirichlet problem with zero data on the parabolic boundary is unique, we only have to prove that this solution has a positive minimum in $\overline{Q_{k+1}}$, and that the infimum of these minima when $\boldsymbol{\mathcal{Q}}$, f and g vary in the correct class is still positive.

The nonnegativity of u is a direct consequence of the (weak) maximum principle. The positivity of its minimum in $\overline{Q_{k+1}}$ is a consequence of the strong maximum principle and the fact that 0 cannot be the solution.

Now, define $\mathcal{U}_k$ as the set of all

$$U = (A, q, l, f, g) \in \mathcal{C}^{\delta/2,1+\delta}(\overline{Q_k}, \mathbb{R}^{n\times n})\times\mathcal{C}^{\delta/2,\delta}(\overline{Q_k}, [-K, K]^n\times[-K, K]\times[-F, F]\times[0, 1])$$

such that $A = A^{\mathrm{T}}$, $\max_{\overline{Q_k}} g = 1$ and

$$K^{-1} \leq \min_{y\in\mathbb{S}^{n-1}} \min_{(t,x)\in\overline{Q_k}} (y \cdot A(t, x)y) \leq \max_{y\in\mathbb{S}^{n-1}} \max_{(t,x)\in\overline{Q_k}} (y \cdot A(t, x)y) \leq K.$$

For all $U \in \mathcal{U}_k$, denote u_U the solution of

$$\begin{cases} \partial_t u - \nabla\cdot(A\nabla u) + q\cdot\nabla u - lu - fu = g & \text{in } Q_k, \\ u = 0 & \text{on } \partial_P Q_k, \end{cases}$$

and denote $m(U) = \min_{\overline{Q_{k+1}}} u_U > 0$. Let us verify that $\inf_{U\in\mathcal{U}_k} m(U) > 0$.

Assume by contradiction $\inf_{U\in\mathcal{U}_k} m(U) = 0$. Then there exists a minimizing sequence $(U_p)_{p\in\mathbb{N}}$ such that $m(U_p) \to 0$ as $p \to +\infty$. By classical compactness and regularity estimates [47], up to extraction, (U_p) converges uniformly to a limit

$$U_\infty = (A_\infty, q_\infty, l_\infty, f_\infty, g_\infty) \in \mathcal{C}^{0,1}(\overline{Q_k}, \mathbb{R}^{n\times n})\times\mathcal{C}(\overline{Q_k}, \mathbb{R}^{n\times n}\times[-K, K]^n\times[-K, K]\times[-F, F]\times[0, 1])$$

such that $A_\infty = A_\infty^{\mathrm{T}}$, $\max_{\overline{Q_k}} g_\infty = 1$ and

$$K^{-1} \leq \min_{y\in\mathbb{S}^{n-1}} \min_{(t,x)\in\overline{Q_k}} (y \cdot A_\infty(t, x)y) \leq \max_{y\in\mathbb{S}^{n-1}} \max_{(t,x)\in\overline{Q_k}} (y \cdot A_\infty(t, x)y) \leq K$$

and $(u_p)_{p\in\mathbb{N}} = \left(u_{U_p}\right)_{p\in\mathbb{N}}$ converges uniformly to the solution u_∞ of

$$\begin{cases} \partial_t u - \nabla\cdot(A_\infty\nabla u) + q_\infty\cdot\nabla u - l_\infty u - f_\infty u = g_\infty & \text{in } Q_k, \\ u = 0 & \text{on } \partial_P Q_k. \end{cases}$$

Moreover, by definition of (U_p), $m(U_\infty) = \lim_{p\to+\infty} m(U_p) = 0$. But then the strong maximum principle yields $u_\infty = 0$, and this contradicts $g_\infty > 0$. Hence $\inf_{U\in\mathcal{U}_k} m(U) > 0$ and Claim 2 is proved with $C_k = \inf_{U\in\mathcal{U}_k} m(U) > 0$. □

Proof of Claim 1. Let $k \in [N-1]$, $I \subset [N]$ of cardinal k,

$$M = \max_{(t,x)\in[0,2\theta]\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_1(t, x),$$

and assume that for all $j \in I$,

$$\min_{(t,x)\in\overline{Q_k}} u_j(t, x) \geq \kappa_k M.$$

By (A_3), there exists $i \in [N]\backslash I$ and $j \in I$ such that $l_{i,j} > 0$, or else $\max_{\overline{\Omega_{\mathrm{per}}}} l_{i,j} > 0$.

Let $\underline{u}$ be the solution of $\mathcal{P}_i\underline{u} - l_{i,i}\underline{u} - f_i\underline{u} = \kappa_k M l_{i,j}$ in Q_k, $\underline{u} = 0$ on $\partial_P Q_k$. Since Q_k contains a translation of $\overline{\Omega_{\mathrm{per}}}$ and $l_{i,j}$ is periodic, applying Claim 1, we get:

$$\min_{\overline{Q_{k+1}}} \underline{u} \geq C_k\kappa_k M \max_{\overline{\Omega_{\mathrm{per}}}} l_{i,j}.$$

Moreover, in Q_k,

$$\mathcal{P}_i u_i - l_{i,i}u_i - f_i u_i = \sum_{k\in[N]\backslash\{i\}} l_{i,k}u_k \geq l_{i,j}u_j \geq l_{i,j} \min_{(t,x)\in\overline{Q_k}} u_j \geq \kappa_k M l_{i,j}.$$

Also, on the parabolic boundary $\partial_P Q_k$, $u_i \gg 0 = \underline{u}$. Therefore, by virtue of the comparison principle, $u_i \geq \underline{u}$ in Q_k, and subsequently, using the definition of σ,

$$\min_{Q_{k+1}} u_i \geq \min_{Q_{k+1}} \underline{u} \geq C_k \kappa_k M \max_{\Omega_{\text{per}}} l_{i,j} \geq C_k \kappa_k M \sigma.$$

Setting $\kappa_{k+1} = C_k \kappa_k \sigma$, we have proved Claim 1. □

This ends the proof. □

Note that, as an immediate corollary, if $\mathbf{u}$ is time periodic, then $\overline{\kappa}_{\theta,F} < 1$ and

$$\min_{i\in[N]} \min_{(t,x)\in\mathbb{R}\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_i(t,x) \geq \overline{\kappa}_{\theta,F} \max_{i\in[N]} \max_{(t,x)\in\mathbb{R}\times[-\frac{\theta}{2},\frac{\theta}{2}]^n} u_i(t,x).$$

If $\mathbf{u}$ is space-time periodic, then an even stronger estimate holds:

$$\min_{i\in[N]} \min_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} u_i(t,x) \geq \overline{\kappa}_{\theta,F} \max_{i\in[N]} \max_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} u_i(t,x).$$

3. Proofs

3.1. **Existence and characterization of generalized principal eigenpairs.**

Most proofs in this subsection are direct adaptations to the vector case of the proofs by Nadin [54], written here only for the paper to be self-contained. The only proofs whose adaptations truly require some care are those of Propositions 3.5 and 3.9.

3.1.1. *The generalized principal eigenvalue λ_1 in arbitrary domains.*

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open connected set such that there exists $x_0 \in \Omega$ satisfying $[x_0, x_0 + L] \subset \Omega$. Then the generalized principal eigenvalue $\lambda_1(\Omega) \in \mathbb{R}$ is well-defined.*

Furthermore, if $\partial\Omega$ is bounded and smooth, then $\lambda_1(\Omega) = \lambda_{1,\text{Dir}}(\Omega)$.

Proof. We begin with the case of bounded smooth domains. The inequality $\lambda_{1,\text{Dir}}(\Omega) \leq \lambda_1(\Omega)$ is obvious by definition of $\lambda_1(\Omega)$, using the Dirichlet principal eigenfunction as test function. The converse inequality is proved by contradiction: assume that $\lambda_{1,\text{Dir}}(\Omega) < \lambda_1(\Omega)$. Then there exists $\mu \in (\lambda_{1,\text{Dir}}(\Omega), \lambda_1(\Omega))$ and $\mathbf{u} \in \mathcal{C}^{1,2}_{t-\text{per}}(\mathbb{R}\times\Omega, (\mathbf{0},\boldsymbol{\infty})) \cap \mathcal{C}^1(\mathbb{R}\times\overline{\Omega})$ such that $\boldsymbol{\mathcal{Q}}\mathbf{u} \geq \mu\mathbf{u}$. By boundedness of the Dirichlet principal eigenfunction $\mathbf{v}$, the quantity

$$\kappa^\star = \inf\{\kappa > 0 \mid \kappa\mathbf{u} - \mathbf{v} \gg \mathbf{0}\}$$

is well-defined in $\mathbb{R}$. The function $\mathbf{w} = \kappa^\star\mathbf{u} - \mathbf{v}$ satisfies

$$\boldsymbol{\mathcal{Q}}\mathbf{w} = \kappa^\star\mu\mathbf{u} - \lambda_{1,\text{Dir}}(\Omega)\mathbf{v} \gg \lambda_{1,\text{Dir}}(\Omega)\mathbf{w} \quad \text{in } \mathbb{R}\times\Omega,$$
$$\mathbf{w} \geq \mathbf{0} \quad \text{in } \mathbb{R}\times\overline{\Omega},$$
$$\mathbf{w} \geq \mathbf{0} \quad \text{on } \mathbb{R}\times\partial\Omega,$$

and there exists $(i^\star, t^\star, x^\star) \in [N]\times[0,T]\times\overline{\Omega}$ such that $w_{i^\star}(t^\star, x^\star) = 0$. If $x^\star \in \Omega$, then by virtue of the strong maximum principle (*cf.* Proposition 2.2), $\mathbf{w}$ is the zero function, which contradicts $\mu > \lambda_{1,\text{Dir}}(\Omega)$. Hence $\mathbf{w} \gg \mathbf{0}$ in $\mathbb{R}\times\Omega$. Since $\mathbf{u} \in \mathcal{C}^1(\mathbb{R}\times\overline{\Omega})$, the normal derivative of $\mathbf{w}$ at any point $(t,x) \in \mathbb{R}\times\partial\Omega$ is well-defined. The optimality of $\kappa^\star$ implies the existence of $(i', t', x') \in [N]\times[0,T]\times\partial\Omega$ such that that both $w_{i'}(t', x')$ and the normal derivative of $w_{i'}$ at (t', x') are zero, which contradicts the boundary version of the strong maximum principle. Hence $\lambda_1(\Omega) \leq \lambda_{1,\text{Dir}}(\Omega)$. This ends the proof in the case of bounded smooth domains.

Then we turn to general, not necessarily bounded and smooth, domains. Let $\nu = -\lambda_{\mathrm{PF}}(\overline{\mathbf{L}}) \in \mathbb{R}$, where the square matrix $\overline{\mathbf{L}}$ is defined in (A_3), and let $\mathbf{u} \in \mathbb{R}^N$ be a positive Perron–Frobenius eigenvector for $\overline{\mathbf{L}}$, namely $\overline{\mathbf{L}}\mathbf{u} = -\nu\mathbf{u}$. Then clearly $\boldsymbol{\mathcal{Q}}\mathbf{u} \geq \nu\mathbf{u}$, which proves that the set

$$\left\{\lambda \in \mathbb{R} \mid \exists \mathbf{u} \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega, (\mathbf{0},\boldsymbol{\infty})) \cap \mathcal{C}^1(\mathbb{R}\times\overline{\Omega})\ \boldsymbol{\mathcal{Q}}\mathbf{u} \geq \lambda\mathbf{u}\right\}$$

is nonempty. Hence its supremum, $\lambda_1(\Omega)$, is well-defined in $\mathbb{R}\times\{\infty\}$.

Next, it follows directly from the definition that $\lambda_1(\Omega) \leq \lambda_1(\Omega')$ for any open set $\Omega' \subset \Omega$. Since Ω is open and contains a periodicity cell $[x_0, x_0+L]$, it contains a bounded smooth connected open set Ω' satisfying $[x_0, x_0+L] \subset \Omega' \subset \overline{\Omega'} \subset \Omega$. Therefore

$$\lambda_1(\Omega) \leq \lambda_1(\Omega') = \lambda_{1,\mathrm{Dir}}(\Omega') < +\infty.$$

This ends the proof. □

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open connected set and let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of nonempty open connected sets such that, for some $x_0 \in \Omega$,*

$$[x_0, x_0+L] \subset \Omega_1, \quad \Omega_k \subset \Omega_{k+1}, \quad \bigcup_{k\in\mathbb{N}} \Omega_k = \Omega.$$

Then $\lambda_1(\Omega_k) \to \lambda_1(\Omega)$ as $k \to +\infty$.

Furthermore, there exists a generalized principal eigenfunction associated with $\lambda_1(\Omega)$.

Proof. In order to work with bounded and smooth domains, we consider a family $(\tilde{\Omega}_k)_{k\in\mathbb{N}}$, nondecreasing and convergent to Ω in the inclusion sense, and such that $\overline{\tilde{\Omega}_k} \subset \Omega_k$ for all $k \in \mathbb{N}$ (with $[x_0, x_0+L] \subset \tilde{\Omega}_1$). Denote $(\mu_k)_{k\in\mathbb{N}} = (\lambda_{1,\mathrm{Dir}}(\Omega_k))_{k\in\mathbb{N}}$, $(\nu_k)_{k\in\mathbb{N}} = (\lambda_{1,\mathrm{Dir}}(\tilde{\Omega}_k))_{k\in\mathbb{N}}$, and note that both sequences converge, with limits satisfying

$$\lambda_1(\Omega) \leq \lim_{k\to+\infty} \mu_k \leq \lim_{k\to+\infty} \nu_k.$$

Let $\nu = \lim \nu_k$. We now aim to prove that $\nu \leq \lambda_1(\Omega)$ by constructing an eigenfunction for the eigenvalue ν of the operator $\boldsymbol{\mathcal{Q}}$ acting on $\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega, (\mathbf{0},\boldsymbol{\infty})) \cap \mathcal{C}^1_0(\mathbb{R}\times\overline{\Omega})$. Since such an eigenfunction will in fact be a generalized principal eigenfunction for the generalized principal eigenvalue $\lambda_1(\Omega)$, this will complete the proof.

Fix $y \in \tilde{\Omega}_1 = \bigcap_{k\in\mathbb{N}} \tilde{\Omega}_k$ and consider the sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ of positive principal eigenfunctions associated with ν_k and normalized by $\max_{i\in[N]} u_{i,k}(0,y) = 1$. Extend these eigenfuctions as functions defined in $\mathbb{R}\times\Omega$ by setting $\mathbf{u}_k = \mathbf{0}$ in $\mathbb{R}\times\Omega\backslash\tilde{\Omega}_k$.

By virtue of the time periodicity of $\mathbf{u}_k$ and of the Harnack inequality of Proposition 2.4, the sequence $\left(\|\mathbf{u}_k\|_{\mathcal{L}^\infty([0,T]\times\tilde{\Omega}_k)}\right)_{k\in\mathbb{N}}$ is bounded. By standard regularity estimates [47], $(\mathbf{u}_k)_{k\in\mathbb{N}}$ converges up to a diagonal extraction to a function $\mathbf{u}_\infty \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega)$ satisfying

$$\boldsymbol{\mathcal{Q}}\mathbf{u}_\infty = \nu\mathbf{u}_\infty \quad \text{in } \mathbb{R}\times\Omega.$$

Moreover, $\mathbf{u}_\infty$ is nonnegative, nonzero at $(t,x) = (0,y)$, and by the maximum principle it is therefore positive in $\mathbb{R}\times\Omega$.

In order to establish $\nu \leq \lambda_1(\Omega)$, it only remains to verify that $\mathbf{u}_\infty \in \mathcal{C}^1_0(\mathbb{R}\times\overline{\Omega})$.

Let
$$C = |\lambda_1(\tilde{\Omega}_1)| \sup_{k\in[N]} \|\mathbf{u}_k\|_{\mathcal{L}^\infty([0,T]\times\tilde{\Omega}_k)}$$
and define $\hat{\mathbf{u}} \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega, (\mathbf{0},\boldsymbol{\infty})) \cap \mathcal{C}^1_0(\mathbb{R}\times\overline{\Omega}, [\mathbf{0},\boldsymbol{\infty}))$ as the time periodic solution of the following (decoupled) system:
$$\begin{cases} \operatorname{diag}(\boldsymbol{\mathcal{P}})\hat{\mathbf{u}} = \mathbf{1} & \text{in } \mathbb{R}\times\Omega, \\ \hat{\mathbf{u}} = \mathbf{0} & \text{on } \mathbb{R}\times\partial\Omega. \end{cases}$$
Then, for any $k \in \mathbb{N}$,
$$\operatorname{diag}(\boldsymbol{\mathcal{P}})((C\hat{\mathbf{u}} - \mathbf{u}_k) \geq C\mathbf{1} - \sup_{k\in[N]} \left(\lambda_1(\tilde{\Omega}_k)\right)\mathbf{u}_k = C\mathbf{1} - \lambda_1(\tilde{\Omega}_1)\mathbf{u}_k \geq \mathbf{0}.$$
This leads to $\mathbf{u}_k \leq C\hat{\mathbf{u}}$ for all $k\in[N]$, and then, passing to the limit, $\mathbf{u}_\infty \leq C\hat{\mathbf{u}}$ in $\mathbb{R}\times\Omega$. Hence $\mathbf{u}_\infty \in \mathcal{C}_0(\mathbb{R}\times\overline{\Omega})$. The continuity of its gradient $\nabla\mathbf{u}_\infty$ on the regular boundary points follows from classical regularity estimates up to the boundary [47]. This ends the proof. □

Remark 3.1. The proof uses the interior Harnack inequality of Proposition 2.4, which, as stated, requires that the domain of definition contains a translation of $[0,3\theta]^n$, with $\theta \geq \max(T, L_1, \dots, L_n)$. This is not optimal and just for convenience of notation; what truly matters for the interior Harnack inequality is that the domain of definition is strictly larger than a closed periodicity cell, as expressed in the preceding statement. We leave the necessary correction of the proof of Proposition 2.4 as an exercise for interested readers.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open connected set such that there exists $x_0 \in \Omega$ satisfying $[x_0, x_0 + L] \subset \Omega$. Then the generalized principal eigenvalue $\lambda_1(\Omega)$ can be characterized as:*
$$\lambda_1(\Omega) = \max_{\mathbf{u}\in\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega,(\mathbf{0},\boldsymbol{\infty}))\cap\mathcal{C}^1(\mathbb{R}\times\overline{\Omega})} \min_{i\in[N]} \inf_{\mathbb{R}\times\Omega} \left(\frac{(\boldsymbol{\mathcal{Q}}\mathbf{u})_i}{u_i}\right).$$

Proof. Testing $\boldsymbol{\mathcal{Q}}$ against a generalized principal eigenfunction (whose existence is guaranteed by Proposition 3.2), we directly find
$$\lambda_1(\Omega) \leq \sup_{\mathbf{u}\in\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\Omega,(\mathbf{0},\boldsymbol{\infty}))\cap\mathcal{C}^1(\mathbb{R}\times\overline{\Omega})} \min_{i\in[N]} \inf_{\mathbb{R}\times\Omega} \left(\frac{(\boldsymbol{\mathcal{Q}}\mathbf{u})_i}{u_i}\right).$$

Next we assume by contradiction that the above inequality is actually strict. Then there exists $\mu > \lambda_1(\Omega)$ and a test function $\mathbf{u}$ such that $\boldsymbol{\mathcal{Q}}\mathbf{u} \geq \mu\mathbf{u}$. This contradicts the definition of $\lambda_1(\Omega)$.

Finally, the existence of a generalized principal eigenfunction shows that the supremum is in fact a maximum, as in the statement. □

3.1.2. *The family of periodic principal eigenvalues $\lambda_{1,z}$.* For any $z\in\mathbb{R}^n$, the existence and uniqueness of the eigenpair $(\lambda_{1,z}, \mathbf{u}_z)$, up to multiplication of the eigenfunction by a constant, follows from the Krein–Rutman theorem. We do not detail the proof of this claim.

Proposition 3.4. *Let $z\in\mathbb{R}^n$. Then the periodic principal eigenvalue $\lambda_{1,z}$ can be characterized as:*
$$\lambda_{1,z} = \max_{\mathbf{u}\in\mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))} \min_{i\in[N]} \min_{\overline{\Omega_{\mathrm{per}}}} \left(\frac{(\boldsymbol{\mathcal{Q}}_z\mathbf{u})_i}{u_i}\right), \tag{8}$$

$$\lambda_{1,z} = \min_{\mathbf{u}\in\mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))} \max_{i\in[N]} \max_{\overline{\Omega_{\mathrm{per}}}} \left(\frac{(\boldsymbol{\mathcal{Q}}_z\mathbf{u})_i}{u_i}\right). \tag{9}$$

Proof. We prove only the max–min characterization, the min–max one being proved quite similarly.

Using the existence of the periodic principal eigenfunction $\mathbf{u}_z$, we immediately obtain

$$\lambda_{1,z} \leq \sup_{\mathbf{u}\in\mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))} \min_{i\in[N]} \min_{\overline{\Omega_{\mathrm{per}}}} \left(\frac{(\boldsymbol{\mathcal{Q}}_z\mathbf{u})_i}{u_i}\right).$$

Next we assume by contradiction that the above inequality is actually strict. Then there exists a test function $\mathbf{u} \in \mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))$ and a real number $\mu > \lambda_{1,z}$ such that $\boldsymbol{\mathcal{Q}}_z\mathbf{u} \geq \mu\mathbf{u}$. Let

$$\kappa^\star = \inf\left\{\kappa > 0 \mid \kappa\mathbf{u} - \mathbf{u}_z \gg \mathbf{0}\right\}.$$

Applying the strong maximum principle to $\kappa^\star\mathbf{u} - \mathbf{u}_z$, just as in the proof of Proposition 3.1, we find a contradiction.

Finally, the existence of $\mathbf{u}_z$ shows that the supremum is in fact a maximum, as in the statement. □

3.1.3. *Concave dependence on z and $\mathbf{L}$.* In order to show later on that $\lambda_1 = \max_{z\in\mathbb{R}^n}\lambda_z$, we need to establish first the strict concavity of $z \mapsto \lambda_z$. Since the proof of Theorem 1.3 on the concavity of $\mathbf{L} \mapsto \lambda_z(\mathbf{L})$ is quite similar, we prove the two results directly together.

Proposition 3.5. *Let $z_1, z_2 \in \mathbb{R}^n$.*

Let

$$(\mathbf{L}[s])_{s\in[0,1]} \in \left(\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^{N\times N})\right)^{[0,1]}$$

a family of matrices satisfying (A_2), (A_3) *and such that, for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ and $i \in [N]$,*

(1) $s \mapsto l_{i,i}[s](t,x)$ is convex;
(2) for all $j \in [N]\backslash\{i\}$, $s \mapsto l_{i,j}[s](t,x)$ is either identically zero or log-convex.

For all $s \in [0,1]$, denote

$$\boldsymbol{\mathcal{Q}}[s] = \mathrm{e}_{-(1-s)z_1-sz_2}(\mathrm{diag}(\mathcal{P}_i) - \mathbf{L}[s])\mathrm{e}_{(1-s)z_1+sz_2}$$

and $\lambda[s] = \lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}[s])$ the associated periodic principal eigenvalue.

Then $s \in [0,1] \mapsto \lambda[s]$ is affine or strictly concave and it is affine if and only if the following conditions are satisfied:

(*Cond.* 1) *$z_1 = z_2$;*
(*Cond.* 2) *there exist a constant vector $\mathbf{b} \gg \mathbf{0}$, a function $\mathbf{c} \in \mathcal{C}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))$ and a function $\mathbf{f} \in \mathcal{C}_{\mathrm{per}}(\mathbb{R},\mathbb{R}^N)$ satisfying $\int_0^T \mathbf{f} \in \mathrm{span}(\mathbf{1})$ such that the entries of $\mathbf{L}$ have the form:*

$$l_{i,j}[s] : (t,x) \mapsto \begin{cases} l_{i,i}[0](t,x) - sf_i(t) & \text{if } i = j, \\ l_{i,j}[0](t,x)\left(\frac{b_j}{c_i(t,x)}\right)^s \mathrm{e}^{s\left(\int_0^t f_j - \frac{t}{T}\int_0^T f_j\right)} & \text{if } i \neq j. \end{cases}$$

Furthermore, if, in addition, $\mathbf{L}[0](t,x)$ is irreducible at all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, then in the above equivalence, (*Cond.* 2) *can be replaced by:*

(*Cond. 2–I*) *there exist a constant vector* $\mathbf{b} \gg \mathbf{0}$ *and a function* $\mathbf{f} \in \mathcal{C}_{\mathrm{per}}(\mathbb{R}, \mathbb{R}^N)$ *satisfying* $\int_0^T \mathbf{f} \in \mathrm{span}(\mathbf{1})$ *such that the entries of* $\mathbf{L}$ *have the form:*

$$l_{i,j}[s] : (t,x) \mapsto \begin{cases} l_{i,i}[0](t,x) - s f_i(t) & \text{if } i = j, \\ l_{i,j}[0](t,x) \left(\frac{b_j}{b_i}\right)^s \mathrm{e}^{s \int_0^t (f_j - f_i)} & \text{if } i \neq j. \end{cases}$$

Proof. We divide the proof into four steps: the concavity of $s \mapsto \lambda[s]$, the alternative between affinity or strict concavity, the characterization of the affinity case, and the characterization of $\mathbf{c}$ under the additional pointwise irreducibility assumption.

Step 1: concavity. Fix $s \in [0,1]$, set $z = (1-s)z_1 + s z_2$ and, for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, define the auxiliary matrix $\tilde{\mathbf{L}}[s](t,x)$ whose entries are:

$$\tilde{l}_{i,j}[s](t,x) = \begin{cases} (1-s) l_{i,i}[0](t,x) + s l_{i,i}[1](t,x) & \text{if } i = j \\ (l_{i,j}[0](t,x))^{1-s} \left(l_{i,j}[1](t,x)\right)^s & \text{if } i \neq j \end{cases}$$

(with $0^0 = 0$ by convention). By construction, and by our convexity assumptions, $\mathbf{L}[s] \leq \tilde{\mathbf{L}}[s]$ in $\mathbb{R} \times \mathbb{R}^n$. Hence, as a direct consequence of the min–max/max–min characterizations of the periodic principal eigenvalue of Proposition 3.4, we get:

$$\lambda[s] \geq \lambda_{1,\mathrm{per}}(\tilde{\boldsymbol{\mathcal{Q}}}[s]), \tag{10}$$

where $\tilde{\boldsymbol{\mathcal{Q}}}[s] = \boldsymbol{\mathcal{Q}}[s] + \mathbf{L}[s] - \tilde{\mathbf{L}}[s]$.

Recall the notation $\mathrm{e}_{z'} : x \mapsto \mathrm{e}^{z' \cdot x}$ and note that, by definition of $\tilde{\boldsymbol{\mathcal{Q}}}[s]$,

$$\frac{\left(\tilde{\boldsymbol{\mathcal{Q}}}[s](\mathrm{e}_{-z}\mathbf{u})\right)_i}{\mathrm{e}_{-z} u_i} = \frac{\mathcal{P}_i u_i - \left(\tilde{\mathbf{L}}[s]\mathbf{u}\right)_i}{u_i} \quad \text{for all } \mathbf{u} \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^n, (\mathbf{0}, \boldsymbol{\infty})).$$

Hence there is a bijection between space-time periodic eigenfunctions of $\tilde{\boldsymbol{\mathcal{Q}}}[s]$ and time periodic eigenfunctions of $\mathrm{diag}(\boldsymbol{\mathcal{P}}) - \tilde{\mathbf{L}}[s]$ whose product with e_z is space periodic.

Let $\mu = \lambda_{1,\mathrm{per}}(\tilde{\boldsymbol{\mathcal{Q}}}[0])$, $\nu = \lambda_{1,\mathrm{per}}(\tilde{\boldsymbol{\mathcal{Q}}}[1])$ and $\mathrm{e}_{-z_1}\mathbf{u}$, $\mathrm{e}_{-z_2}\mathbf{v}$ two respectively associated space-time periodic positive eigenfunctions:

$$\tilde{\boldsymbol{\mathcal{Q}}}[0](\mathrm{e}_{-z_1}\mathbf{u}) = \mu \mathrm{e}_{-z_1}\mathbf{u}, \quad \tilde{\boldsymbol{\mathcal{Q}}}[1](\mathrm{e}_{-z_2}\mathbf{v}) = \nu \mathrm{e}_{-z_2}\mathbf{v},$$

or else

$$\mathrm{diag}(\boldsymbol{\mathcal{P}})\mathbf{u} - \tilde{\mathbf{L}}[0]\mathbf{u} = \mu\mathbf{u}, \quad \mathrm{diag}(\boldsymbol{\mathcal{P}})\mathbf{v} - \tilde{\mathbf{L}}[1]\mathbf{v} = \nu\mathbf{v}.$$

Define $\mathbf{w} = \left(u_i^{1-s} v_i^s\right)_{i \in [N]}$. Since $\mathrm{e}_{-z} w_i = (\mathrm{e}_{-z_1} u_i)^{1-s} (\mathrm{e}_{-z_2} v_i)^s$ for all $i \in [N]$, $\mathrm{e}_{-z}\mathbf{w}$ is space-time periodic and therefore we can use it as test function for $\tilde{\boldsymbol{\mathcal{Q}}}[s]$. Following Nadin [54] for the expansion of the $\mathcal{P}_i$ part and using the uniform ellipticity assumption (A_1), we find:

$$\begin{aligned} \frac{\mathcal{P}_i w_i - \left(\tilde{\mathbf{L}}[s]\mathbf{w}\right)_i}{w_i} &= (1-s)\frac{\mathcal{P}_i u_i}{u_i} + s\frac{\mathcal{P}_i v_i}{v_i} + s(1-s)\left(\frac{\nabla u_i}{u_i} - \frac{\nabla v_i}{v_i}\right) \cdot A_i \left(\frac{\nabla u_i}{u_i} - \frac{\nabla v_i}{v_i}\right) \\ &\quad - (1-s)\tilde{l}_{i,i}[0] - s\tilde{l}_{i,i}[1] - \frac{1}{w_i} \sum_{j \in [N] \setminus \{j\}} (\tilde{l}_{i,j}[0] u_j)^{1-s} (\tilde{l}_{i,j}[1] v_j)^s \\ &\geq (1-s)\frac{\mathcal{P}_i u_i}{u_i} + s\frac{\mathcal{P}_i v_i}{v_i} \\ &\quad - (1-s)\tilde{l}_{i,i}[0] - s\tilde{l}_{i,i}[1] - \frac{1}{w_i} \sum_{j \in [N] \setminus \{j\}} (\tilde{l}_{i,j}[0] u_j)^{1-s} (\tilde{l}_{i,j}[1] v_j)^s. \end{aligned}$$

Following Nussbaum [58] and using the Hölder inequality, the equalities satisfied by $\mathbf{u}$ and $\mathbf{v}$ and the inequality between arithmetic and geometric means, we get

$$\frac{\left(\tilde{\mathcal{Q}}[s](\mathrm{e}_{-z}\mathbf{w})\right)_i}{\mathrm{e}_{-z}w_i} \geq (1-s)\mu + s\nu \quad \text{for all } i \in [N], \tag{11}$$

and, eventually, the max–min characterization yields:

$$\lambda_{1,\mathrm{per}}(\tilde{\mathcal{Q}}[s]) \geq (1-s)\mu + s\nu = (1-s)\lambda_{1,\mathrm{per}}(\tilde{\mathcal{Q}}[0]) + s\lambda_{1,\mathrm{per}}(\tilde{\mathcal{Q}}[1]). \tag{12}$$

Combining (10) and (12) and using the fact that $\mathcal{Q}[s]$ and $\tilde{\mathcal{Q}}[s]$ coincide at $s=0$ and $s=1$, we find indeed the claimed concavity:

$$\lambda_{1,\mathrm{per}}(\mathcal{Q}[s]) \geq (1-s)\lambda_{1,\mathrm{per}}(\mathcal{Q}[0]) + s\lambda_{1,\mathrm{per}}(\mathcal{Q}[1]). \tag{13}$$

□

Step 2: affinity or strict concavity. Assume that $s \mapsto \lambda[s]$ is not strictly concave. This means that there exists $s_0 \in [0,1]$ such that (13) is an equality at $s = s_0$.

The equality in (13) at $s = s_0$ implies the equality in (10) at $s = s_0$, which in turn implies the equality $\mathbf{L}[s_0] = \tilde{\mathbf{L}}[s_0]$ in $\mathbb{R}\times\mathbb{R}^n$. Since all $s \mapsto \tilde{l}_{i,i}[s](t,x)$ are linear and all $s \mapsto l_{i,i}[s](t,x)$ are convex, $l_{i,i}[s](t,x) \leq \tilde{l}_{i,i}[s](t,x)$ together with the equality at $s=0$, $s=s_0$, $s=1$ imply $l_{i,i} = \tilde{l}_{i,i}$ identically for all $i \in [N]$. Similarly, $l_{i,j} = \tilde{l}_{i,j}$ identically for all $i,j \in [N]$. Hence, as functions of (s,t,x), $\mathbf{L} = \tilde{\mathbf{L}}$ identically in $[0,1]\times\mathbb{R}\times\mathbb{R}^n$.

Similarly, the equality in (13) at $s = s_0$ implies the equality in (12) at $s = s_0$, and then the max–min characterization (Proposition 3.4) implies equality in (11) at $s = s_0$ for all $i \in [N]$ in $\mathbb{R}\times\mathbb{R}^n$. Then, this implies, for all $i \in [N]$:

- $\nabla u_i/u_i = \nabla v_i/v_i$, that is there exists a function a_i of the variable t only such that $u_i(t,x) = a_i(t)v_i(t,x)$;
- for all $j \in [N]\backslash\{i\}$, there exists a positive function c_i of t and x such that $\tilde{l}_{i,j}[0]u_j = c_i\tilde{l}_{i,j}[1]v_j$ (equality in the Hölder inequality);
- $\frac{\mathcal{P}_i u_i}{u_i} - \tilde{l}_{i,i}[0] - \mu = \frac{\mathcal{P}_i v_i}{v_i} - \tilde{l}_{i,i}[1] - \nu$ (equality in the inequality between geometric and arithmetic means).

Putting the two together, the equality in (13) at $s = s_0$ implies:

(Cond. 1′) $\mathbf{L} = \tilde{\mathbf{L}}$ identically in $[0,1]\times\mathbb{R}\times\mathbb{R}^n$;
(Cond. 2′) there exists a function a_i of the variable t only such that $u_i(t,x) = a_i(t)v_i(t,x)$;
(Cond. 3′) for all $j \in [N]\backslash\{i\}$, there exists a positive function c_i of t and x such that $\tilde{l}_{i,j}[0]u_j = c_i\tilde{l}_{i,j}[1]v_j$;
(Cond. 4′) $\frac{\mathcal{P}_i u_i}{u_i} - \tilde{l}_{i,i}[0] - \mu = \frac{\mathcal{P}_i v_i}{v_i} - \tilde{l}_{i,i}[1] - \nu$.

These four conditions do not depend on s_0. Going back through Step 1, it appears that under these conditions, all inequalities are equalities. Hence (13) is an equality at all $s \in [0,1]$, or in other words $s \mapsto \lambda[s]$ is affine. It will be useful in the next step to note that this argument precisely shows that (Cond. 1′)–(Cond. 4′) are equivalent to the affinity of $s \mapsto \lambda[s]$.

□

Step 3: necessary and sufficient conditions for affinity. From Step 2, we know that $s \mapsto \lambda[s]$ is affine if and only if (Cond. 1′)–(Cond. 4′). Let us prove that this group of conditions is equivalent to the group (Cond. 1)–(Cond. 2).

Note first that without loss of generality, we can assume that $\mathbf{u}$ and $\mathbf{v}$ are uniquely identified by the following normalizations:

$$\|u_1(0,\cdot)\|_{\mathcal{L}^\infty(\mathbb{R}^n,\mathbb{R})} = 1, \quad \|v_1(0,\cdot)\|_{\mathcal{L}^\infty(\mathbb{R}^n,\mathbb{R})} = 1.$$

To verify that (Cond. 1)–(Cond. 2) imply (Cond. 1′)–(Cond. 4′), it suffices to set

$$a_i : t \mapsto \frac{b_i}{b_1} \exp\left(\int_0^t f_i - \frac{t}{T} \int_0^T f_i \right),$$

and to check $\frac{1}{T}\int_0^T f_i = \mu - \nu$ and $\mathbf{u} = \mathbf{a} \circ \mathbf{v}$. Actually, it can be easily verified that $\tilde{\mathbf{u}} = \mathbf{a} \circ \mathbf{v}$ satisfies

$$\mathcal{P}_i \tilde{u}_i - (\mathbf{L}[0]\tilde{\mathbf{u}})_i = \left(\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}[1]) - \frac{1}{T}\int_0^T f_i \right) \tilde{u}_i.$$

Since $\mathrm{e}_{-z_2}\mathbf{v}$ is space-time periodic, $\mathrm{e}_{-z_1}\tilde{\mathbf{u}} = \mathrm{e}_{-z_2}\mathbf{a} \circ \mathbf{v}$ is also space-time periodic, whence by uniqueness $\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}[1]) - T^{-1}\int_0^T f_i = \lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}[0])$ and $\mathrm{e}_{-z_1}\tilde{\mathbf{u}} \in \mathrm{span}(\mathbf{u}_{z_1})$. This exactly proves the existence of $C > 0$ such that $C\tilde{\mathbf{u}} = \mathbf{u}$, and, in view of the chosen normalizations on $\mathbf{u}$ and $\mathbf{v}$, $C = 1$, or else $\tilde{\mathbf{u}} = \mathbf{u}$.

Now, we prove that (Cond. 1′)–(Cond. 4′) imply (Cond. 1)–(Cond. 2). From $u_i(t,x) = a_i(t)v_i(t,x)$, we deduce $z_1 = z_2$ (recall that $\mathrm{e}_{-z_1}\mathbf{u}$ and $\mathrm{e}_{-z_2}\mathbf{v}$ are both space-time periodic) and $\frac{\mathcal{P}_i u_i}{u_i} = \frac{\mathcal{P}_i v_i}{v_i} + \frac{a_i'}{a_i}$. The equality $\frac{\mathcal{P}_i u_i}{u_i} - \tilde{l}_{i,i}[0] - \mu = \frac{\mathcal{P}_i v_i}{v_i} - \tilde{l}_{i,i}[1] - \nu$ reads $\frac{a_i'}{a_i} = \tilde{l}_{i,i}[0] - \tilde{l}_{i,i}[1] + \mu - \nu$, or in other words there exists $\mathbf{b} \in \mathbb{R}^N$ such that

$$a_i : t \mapsto b_i \exp\left(\int_0^t \left(\tilde{l}_{i,i}[0](t',x) - \tilde{l}_{i,i}[1](t',x) \right) \mathrm{d}t' + (\mu - \nu)t \right).$$

This directly implies that $f_i = \tilde{l}_{i,i}[0] - \tilde{l}_{i,i}[1]$ does not depend on x. Moreover, the positivity of both u_i and v_i implies $b_i > 0$, the normalizations imply $b_1 = 1$, and the time periodicity implies that f_i is periodic with mean value $\nu - \mu$, independent of i.

□

Step 4: characterization of $\mathbf{c}$ *when* $\mathbf{L}$ *is pointwise irreducible.* Assume (Cond. 1) and $\mathbf{L}[0]$ is pointwise irreducible in $\mathbb{R} \times \mathbb{R}^n$ and let us show that (Cond. 2) is equivalent to (Cond. 2–I). The converse implication being obvious, we only have to prove that (Cond. 2) implies (Cond. 2–I). In order to do so, we assume (Cond. 2) and prove that $\mathbf{a} = \mathbf{c}$.

Rewriting the systems satisfied by $\mathbf{u}$ and $\mathbf{v}$,

$$\mathrm{diag}(\boldsymbol{\mathcal{P}})\mathbf{u} - \mathbf{L}[0]\mathbf{u} - \mu\mathbf{u} = \mathbf{0}, \quad \mathrm{diag}(\boldsymbol{\mathcal{P}})\mathbf{v} - \mathbf{L}[1]\mathbf{v} - \nu\mathbf{v} = \mathbf{0},$$

and using the relation $\mathbf{u}(t,x) = \mathbf{a}(t) \circ \mathbf{v}(t,x) = \mathrm{diag}(\mathbf{a}(t))\mathbf{v}(t,x)$, we deduce

$$\mathrm{diag}(\mathbf{a} - \mathbf{c})\hat{\mathbf{L}}[1]\mathbf{v} = \mathbf{0} \quad \text{in } \mathbb{R} \times \mathbb{R}^n,$$

where $\hat{\mathbf{L}}[s] = \mathbf{L}[s] - \mathrm{diag}(l_{i,i}[s])$ is the off-diagonal part of $\mathbf{L}[s]$. Since $\mathbf{L}[0]$ is pointwise irreducible and since the entries of $\mathbf{L}$ satisfy the special form of (Cond. 2), $\mathbf{L}[s]$ is pointwise irreducible for all $s \in [0,1]$, and in particular at $s = 1$. Hence $\hat{\mathbf{L}}[1](t,x)$ is a nonnegative irreducible matrix at any $(t,x) \in \overline{\Omega_{\mathrm{per}}}$, and subsequently $\hat{\hat{\mathbf{L}}}[1]\mathbf{v} \gg \mathbf{0}$ holds true pointwise and implies $\mathbf{c}(t,x) = \mathbf{a}(t)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. □

The proof of the theorem is complete. □

Remark 3.2. It may seem surprising that $\mathbf{c}$ cannot be characterized when (Cond. 1) and (Cond. 2) hold but $\mathbf{L}[0]$ is not pointwise irreducible, especially since pointwise irreducible matrices are dense[7] in the set of admissible matrices (namely, matrices in $\mathcal{C}^{\delta/2,\delta}_{\text{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^{N\times N})$ satisfying (A_2) and (A_3)).

As a matter of fact, we can construct a counter-example where $\mathbf{L}[0]$ is not pointwise irreducible and $\mathbf{c}$ is not uniquely determined. This shows that the statement of Proposition 3.5 is indeed sharp.

Let $h\in\mathcal{C}^{\infty}_{t-\text{per}}(\mathbb{R},\mathbb{R})$ be an even function such that:

$$h_{|[0,T/6]}=1,\quad h_{|[T/6,T/3]}\in[0,1],\quad h_{|[T/3,T/2]}=0,$$

and let $\mathbf{L}[0]$ be the matrix whose entries are all equal to h. Clearly, $\mathbf{L}[0]$ is smooth, periodic and it satisfies (A_2) and (A_3). Let $\mathbf{a}$ be the time periodic positive function constructed in the course of the proof and let $\eta\in\mathcal{C}^{\infty}_{\text{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ be any nonnegative nonzero function such that:

$$\eta_{|([0,T/3]\cup[2T/3,T])\times[0,L]}=0.$$

By construction, $h\eta=0$. Then (Cond. 2) holds true with $\mathbf{c}=\mathbf{a}$ if and only if it holds true with $\mathbf{c}=\mathbf{a}+\eta\mathbf{1}$. Hence $\mathbf{c}$ is not uniquely determined.

Corollary 3.6. *With the notations of Proposition 3.5, if $z_1\neq z_2$, then $s\in[0,1]\mapsto\lambda[s]$ is strictly concave. In particular, $z\mapsto\lambda_{1,z}$ is strictly concave.*

Very minor adaptations of the proof of Proposition 3.5, not detailed here, lead to the following analogous result in the Dirichlet case [8].

Proposition 3.7. *Let $\Omega\subset\mathbb{R}^n$ be a nonempty, bounded, smooth, open, connected set such that there exists $x_0\in\Omega$ satisfying $[x_0,x_0+L]\subset\Omega$.*

Let

$$(\mathbf{L}[s])_{s\in[0,1]}\in\left(\mathcal{C}^{\delta/2,\delta}_{\text{per}}(\mathbb{R}\times\Omega,\mathbb{R}^{N\times N})\right)^{[0,1]}$$

a family of matrices satisfying (A_2), (A_3) and such that, for all $(t,x)\in\mathbb{R}\times\Omega$ and $i\in[N]$,

(1) $s\mapsto l_{i,i}[s](t,x)$ is convex;

(2) for all $j\in[N]\backslash\{i\}$, $s\mapsto l_{i,j}[s](t,x)$ is either identically zero or log-convex.

Then $s\in[0,1]\mapsto\lambda_{1,\text{Dir}}(\Omega,\mathcal{Q}[s])$, where $\mathcal{Q}[s]$ is the operator $\mathcal{Q}$ with $\mathbf{L}$ replaced $\mathbf{L}[s]$, is affine or strictly concave and it is affine if and only if there exist a constant vector $\mathbf{b}\gg\mathbf{0}$, a function $\mathbf{c}\in\mathcal{C}_{t-\text{per}}(\mathbb{R}\times\Omega,(\mathbf{0},\boldsymbol{\infty}))$ and a function $\mathbf{f}\in\mathcal{C}_{t-\text{per}}(\mathbb{R},\mathbb{R}^N)$ satisfying $\int_0^T\mathbf{f}\in\text{span}(\mathbf{1})$ such that the entries of $\mathbf{L}$ have the form:

$$l_{i,j}[s]:(t,x)\mapsto\begin{cases}l_{i,i}[0](t,x)-sf_i(t) & \text{if } i=j,\\ l_{i,j}[0](t,x)\left(\frac{b_j}{c_i(t,x)}\right)^s\mathrm{e}^{s\left(\int_0^t f_j-\frac{t}{T}\int_0^T f_j\right)} & \text{if } i\neq j.\end{cases}$$

Furthermore, if, in addition, $\mathbf{L}[0](t,x)$ is irreducible at all $(t,x)\in\mathbb{R}\times\Omega$, then it is affine if and only if there exist a constant vector $\mathbf{b}\gg\mathbf{0}$ and a function $\mathbf{f}\in$

[7]Just change $\mathbf{L}$ into $\mathbf{L}+\varepsilon\mathbf{1}_{N\times N}$.

[8]The absence of z actually makes the proof shorter.

$\mathcal{C}_{t-\mathrm{per}}(\mathbb{R},\mathbb{R}^N)$ *satisfying* $\int_0^T \mathbf{f} \in \mathrm{span}(\mathbf{1})$ *such that the entries of* $\mathbf{L}$ *have the form:*

$$l_{i,j}[s] : (t,x) \mapsto \begin{cases} l_{i,i}[0](t,x) - sf_i(t) & \text{if } i = j, \\ l_{i,j}[0](t,x) \left(\frac{b_j}{b_i}\right)^s \mathrm{e}^{s\int_0^t (f_j - f_i)} & \text{if } i \neq j. \end{cases}$$

As a corollary, we obtain the concavity of λ_1 in arbitrary domains, namely Theorem 1.4.

Corollary 3.8. *Let* $\Omega \subset \mathbb{R}^n$ *be a nonempty open connected set such that there exists* $x_0 \in \Omega$ *satisfying* $[x_0, x_0 + L] \subset \Omega$.

Let

$$(\mathbf{L}[s])_{s\in[0,1]} \in \left(\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\Omega,\mathbb{R}^{N\times N})\right)^{[0,1]}$$

a family of matrices satisfying (A_2), (A_3) *and such that, for all* $(t,x) \in \mathbb{R}\times\Omega$ *and* $i \in [N]$,

(1) $s \mapsto l_{i,i}[s](t,x)$ *is convex;*
(2) for all $j \in [N]\backslash\{i\}$, $s \mapsto l_{i,j}[s](t,x)$ *is either identically zero or log-convex.*

Then the mapping $s \in [0,1] \mapsto \lambda_1(\Omega, \mathcal{Q}[s])$, *where* $\mathcal{Q}[s]$ *is the operator* $\mathcal{Q}$ *with* $\mathbf{L}$ *replaced* $\mathbf{L}[s]$, *is concave.*

Proof. Just as in the proof of Proposition 3.2, we work with a sequence $(\Omega_k)_{k\in\mathbb{N}}$ of smooth, bounded, nonempty, open, connected subsets of Ω such that

$$[x_0, x_0+L] \subset \Omega_1, \quad \Omega_k \subset \Omega_{k+1}, \quad \bigcup_{k\in\mathbb{N}} \Omega_k = \Omega.$$

By virtue of Proposition 3.7, all $s \in [0,1] \mapsto \lambda_{1,\mathrm{Dir}}(\Omega_k, \mathcal{Q}[s])$ are concave. By virtue of Proposition 3.2, $\lambda_{1,\mathrm{Dir}}(\Omega_k, \mathcal{Q}[s]) \to \lambda_1(\Omega, \mathcal{Q}[s])$ as $k \to +\infty$, for all $s \in [0,1]$.

The pointwise convergence of a sequence of concave functions on the compact set $[0,1]$ is automatically improved as uniform convergence in $[0,1]$, and the limit is concave on $[0,1]$ as well. This ends the proof. □

Remark 3.3. We will establish in the next section that $\lambda_1 = \max_{z\in\mathbb{R}^n} \lambda_z$. However, the maximum of a family of concave functions is in general not a concave function itself, so that this identity cannot be used to prove the concavity of λ_1.

3.1.4. *Relations between* λ_1, λ_1' *and* λ_z.

Proposition 3.9. *There exists* $z \in \mathbb{R}^n$ *such that* $\mathrm{e}_z\mathbf{u}_z$ *is a generalized principal eigenfunction of* $\mathcal{Q}$ *associated with* λ_1 *and* $\lambda_1 = \lambda_{1,z}$.

Proof. From Proposition 3.2, there exists a generalized principal eigenfunction $\mathbf{u} \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ associated with λ_1.

We first prove that there exists $z_1 \in \mathbb{R}$ and a new generalized principal eigenfunction $\mathbf{u}^1$ such that $(t,x) \mapsto \mathrm{e}^{-z_1x_1}\mathbf{u}^1(t,x)$ is L_1-periodic with respect to x_1.

Define the translation $\tau : x \in \mathbb{R}^n \mapsto x + L_1e_1$, where $e_1 = (\delta_{1\alpha})_{\alpha\in[n]}$, and denote $\mathbf{u}^\tau : (t,x) \mapsto \mathbf{u}(t,\tau(x))$ and $\mathbf{v} = (u_i^\tau/u_i)_{i\in[N]}$. By virtue of the fully coupled Harnack inequality of Proposition 2.4 and periodicity of the coefficients of $\mathcal{Q}$, $\mathbf{v}$ is globally bounded. Let

$$z_1 = L_1^{-1} \ln\left(\max_{i\in[N]} \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} v_i(t,x)\right).$$

Recalling that $\mathbf{u}$ and consequently $\mathbf{v}$ are time periodic, there exists $\bar{i} \in [N]$ and $(t_k, x_k)_{k\in\mathbb{N}} \in ([0,T]\times\mathbb{R}^n)^{\mathbb{N}}$ such that $v_{\bar{i}}(t_k,x_k) \to \mathrm{e}^{z_1 L_1}$ as $k\to+\infty$. Moreover, there exists $(y_k)_{k\in\mathbb{N}}$ such that, for all $k\in\mathbb{N}$, $x_k - y_k \in L_1\mathbb{Z}\times\cdots\times L_n\mathbb{Z}$. Up to extraction, we assume that $(t_k,y_k)\to(t_\infty,y_\infty)\in\overline{\Omega_{\mathrm{per}}}$.

Now, define, for all $k\in\mathbb{N}$,

$$\hat{\mathbf{u}}_k : (t,x) \mapsto \frac{1}{u_{\bar{i}}(t_k,x_k)}\mathbf{u}(t+t_k, x+x_k),$$

$$\hat{\mathbf{u}}_k^\tau : (t,x)\mapsto \hat{\mathbf{u}}_k(t,\tau(x)),$$

$$\mathbf{w}_k : (t,x)\mapsto \mathrm{e}^{z_1L_1}\hat{\mathbf{u}}_k - \hat{\mathbf{u}}_k^\tau.$$

Once more by virtue of the Harnack inequality and the periodicity of the coefficients of $\mathcal{Q}$, $(\hat{\mathbf{u}}_k)_{k\in\mathbb{N}}$ is globally bounded. By periodicity of the coefficients of $\mathcal{Q}$, it satisfies:

$$\mathcal{Q}(t+t_k, x+y_k)\hat{\mathbf{u}}_k(t,x) = \lambda_1\hat{\mathbf{u}}_k(t,x) \quad \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R}^n,\ k\in\mathbb{N}.$$

Therefore, by classical regularity estimates [47], $(\hat{\mathbf{u}}_k)_{k\in\mathbb{N}}$ converges up to a diagonal extraction to $\hat{\mathbf{u}}_\infty \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, [\mathbf{0},\boldsymbol{\infty}))$ which satisfies:

$$\mathcal{Q}(t+t_\infty, x+y_\infty)\hat{\mathbf{u}}_\infty(t,x) = \lambda_1\hat{\mathbf{u}}_\infty(t,x) \quad \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R}^n,$$

or else:

$$\mathcal{Q}(t,x)\hat{\mathbf{u}}_\infty(t-t_\infty, x-y_\infty) = \lambda_1\hat{\mathbf{u}}_\infty(t-t_\infty,x-y_\infty) \quad \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R}^n.$$

Moreover, $\hat{u}_{\bar{i},\infty}(0,0)=1$, whence $\hat{\mathbf{u}}_\infty$ is nonzero. By the strong maximum principle (*cf.* Proposition 2.1), it is in fact positive.

We can now extend the family $(\mathbf{w}_k)$ in $\mathbb{N}\cup\{\infty\}$ with $\mathbf{w}_\infty = \mathrm{e}^{z_1L_1}\hat{\mathbf{u}}_\infty - \hat{\mathbf{u}}_\infty^\tau$. Since, for all $k\in\overline{\mathbb{N}}$,

$$\mathbf{w}_k = \hat{\mathbf{u}}_k \circ \left(\mathrm{e}^{z_1L_1}\mathbf{1} - \mathbf{v}_k\right), \quad \text{where } \mathbf{v}_k : (t,x)\mapsto \mathbf{v}(t+t_k, x+x_k),$$

we deduce by definition of z_1 that $\mathbf{w}_\infty \geq \mathbf{0}$ with $w_{\bar{i},\infty}(0,0)=0$. Moreover, $\mathbf{w}_\infty$ satisfies the same equation than $\hat{\mathbf{u}}_\infty$. Therefore, by virtue of the strong maximum principle, $\mathbf{w}_\infty$ is the zero function. This exactly means that $\mathrm{e}^{z_1L_1}\hat{\mathbf{u}}_\infty = \hat{\mathbf{u}}_\infty^\tau$.

It is now clear that $\mathbf{u}^1 : (t,x)\mapsto \hat{\mathbf{u}}_\infty(t-t_\infty, x-y_\infty)$ is positive, time periodic, a solution of $\mathcal{Q}\mathbf{u}^1 = \lambda_1\mathbf{u}^1$, and that the function $(t,x)\mapsto \mathrm{e}^{-z_1x_1}\mathbf{u}^1(t,x)$ L_1-periodic with respect to x_1. The first part of the proof is done.

Next, we iterate this construction, replacing $\mathbf{u}$ by $\mathbf{u}^1$, in order to obtain a new generalized principal eigenfunction $\mathbf{u}^2$ such that $(t,x)\mapsto \mathrm{e}^{-z_1x_1}\mathrm{e}^{-z_2x_2}\mathbf{u}^2(t,x)$ is L_1-periodic with respect to x_1 and L_2-periodic with respect to x_2. Iterating again, we finally obtain $z\in\mathbb{R}^n$ and $\mathbf{u}^n \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ such that $\mathbf{u}^n$ is a generalized principal eigenfunction associated with λ_1 and such that $\mathrm{e}_{-z}\mathbf{u}^n$ is space periodic. The uniqueness of the eigenpair $(\lambda_{1,z},\mathbf{u}_z)$, up to multiplication of $\mathbf{u}_z$ by a constant, yields finally $\lambda_1 = \lambda_{1,z}$ and $\mathrm{e}_{-z}\mathbf{u}^n \in \mathrm{span}(\mathbf{u}_z)$. □

Corollary 3.10. *The generalized principal eigenvalue* λ_1 *satisfies:*

$$\lambda_1 = \max_{z\in\mathbb{R}^n}\lambda_{1,z} \tag{14}$$

and there exists a unique $z\in\mathbb{R}^n$ *such that* $\lambda_1 = \lambda_{1,z}$.

Proof. Proposition 3.9 already shows that λ_1 is in the image of $z \mapsto \lambda_{1,z}$ and Corollary 3.6 already shows that $z \mapsto \lambda_{1,z}$ is strictly concave. Thus it only remains to show $\lambda_1 \geq \sup_{z\in\mathbb{R}^n} \lambda_{1,z}$. This is actually obvious, since the equality $\boldsymbol{\mathcal{Q}}(\mathrm{e}_z \mathbf{u}_z) = \lambda_{1,z}\mathrm{e}_z\mathbf{u}_z$ (which is just the definition of the eigenpair $(\lambda_{1,z}, \mathbf{u}_z)$) directly implies, in view of the definition of λ_1, the inequality $\lambda_1 \geq \lambda_{1,z}$. □

Remark 3.4. Let

$$E = \left\{ \lambda \in \mathbb{R} \mid \exists \mathbf{u} \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))\ \boldsymbol{\mathcal{Q}}\mathbf{u} = \lambda\mathbf{u} \right\}$$

and denote $\Lambda \subset \mathbb{R}$ the image of $z \in \mathbb{R}^n \mapsto \lambda_{1,z}$. From the equality $\boldsymbol{\mathcal{Q}}\mathrm{e}_z\mathbf{u}_z = \lambda_{1,z}\mathrm{e}_z\mathbf{u}_z$, the following set inclusions hold true:

$$\Lambda \subset E \subset \left\{ \lambda \in \mathbb{R} \mid \exists \mathbf{u} \in \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))\ \boldsymbol{\mathcal{Q}}\mathbf{u} \geq \lambda\mathbf{u} \right\}.$$

By strict concavity, $\Lambda = (-\infty, \max \lambda_{1,z}]$, and since $\lambda_1 = \max \lambda_{1,z}$ is by definition the supremum of the larger set above, all inclusions above are actually set equalities.

This shows in particular that the set E of eigenvalues of $\boldsymbol{\mathcal{Q}}$ acting on the set $\mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ is $(-\infty, \lambda_1]$. This is of course in striking contrast with the case of smooth bounded domains, where the Krein–Rutman theorem can be applied and the principal eigenvalue is unique. For the same result in the elliptic case with general spatial heterogeneities, refer to Berestycki–Rossi [14, Theorem 1.4] (scalar setting) and Arapostathis–Biswas–Pradhan [8, Theorem 1.2] (cooperative vector setting).

Proposition 3.11. *The generalized principal eigenvalue* λ_1' *satisfies:*

$$\lambda_1' = \lambda_{1,0}. \tag{15}$$

Proof. Since $\mathbf{u}_0 = \mathrm{e}_0\mathbf{u}_0$ is globally bounded, we can use it as test function in the definition of λ_1' and obtain $\lambda_1' \leq \lambda_{1,0}$.

Now, we assume by contradiction that this inequality is actually strict, so that by definition of λ_1', there exists $\mu \in (\lambda_1', \lambda_{1,0})$ and $\mathbf{u} \in \mathcal{W}^{1,\infty} \cap \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ such that $\boldsymbol{\mathcal{Q}}\mathbf{u} \leq \mu\mathbf{u}$.

We can now define

$$\kappa^\star = \inf\left\{\kappa > 0 \mid \kappa\mathbf{u}_0 - \mathbf{u} \gg \mathbf{0}\right\}$$

and study the sign of $\mathbf{v} = \kappa^\star\mathbf{u}_0 - \mathbf{u}$. This function satisfies

$$\boldsymbol{\mathcal{Q}}\mathbf{v} = (\lambda_{1,0} - \mu)\kappa^\star\mathbf{u}_0 + \mu\mathbf{v},$$

is time periodic and nonnegative, and by optimality there exists $((t_k, x_k))_{k\in\mathbb{N}} \in ([0,T]\times\mathbb{R}^n)^{\mathbb{N}}$ and $\underline{i} \in [N]$ such that

$$v_{\underline{i}}(t_k, x_k) \to 0 \quad \text{as } k \to +\infty.$$

Define

$$\mathbf{v}_k : (t,x) \mapsto \mathbf{v}(t + t_k, x + x_k) \quad \text{for all } k \in [N].$$

By standard regularity estimates [47], $(\mathbf{v}_k)_{k\in\mathbb{N}}$ converges up to a diagonal extraction to a function $\mathbf{v}_\infty \in \mathcal{L}^\infty \cap \mathcal{C}^{1,2}_{t-\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, [\mathbf{0},\boldsymbol{\infty}))$ which satisfies $v_{\underline{i},\infty}(0,0) = 0$ and

$$(\boldsymbol{\mathcal{Q}} - \mu)\mathbf{v}_\infty \geq (\lambda_{1,0} - \mu)\kappa^\star \min_{i\in[N]} \min_{\overline{\Omega_{\mathrm{per}}}} (u_{0,i})\,\mathbf{1} \gg \mathbf{0}.$$

By virtue of the strong maximum principle (*cf.* Proposition 2.1), $\mathbf{v}_\infty = \mathbf{0}$, but then this contradicts the preceding inequality. This ends the proof. □

Remark 3.5. The pair of equalities $\lambda_1 = \max_{z\in\mathbb{R}^n}\lambda_{1,z}$ and $\lambda_1' = \lambda_{1,0}$ together with the concavity of $z\mapsto\lambda_{1,z}$ means in particular that the equality $\lambda_1=\lambda_1'$ is equivalent to the evenness of $z\mapsto\lambda_{1,z}$. The scalar counter-example with constant coefficients $\mathcal{Q}=\partial_t-\partial_{xx}+q\partial_x-l$ shows that both outcomes are possible, since $\lambda_{1,z}=z(q-z)-l$ is even in z if and only if $q=0$. Identifying precise conditions for evenness becomes then one of our main goals. A very recent contribution by Griette and Matano [37, Proposition 4.1] shows that in the vector setting, the absence of advection is not enough.

Their two-dimensional counter-example in one-dimensional space is:

$$\boldsymbol{\mathcal{Q}}=\partial_t-\partial_x\left(\operatorname{diag}\begin{pmatrix}a_1\\a_2\end{pmatrix}\partial_x\right)-\begin{pmatrix}r_1-\frac{1}{\varepsilon}p & \frac{1}{\varepsilon}(1-p)\\ \frac{1}{\varepsilon}p & r_2-\frac{1}{\varepsilon}(1-p)\end{pmatrix}$$

with a_1, a_2, r_1, r_2 and p periodic functions of x. As $\varepsilon\to 0$, locally uniformly with respect to z,

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}})\to\lambda_{1,z}\left(-\partial_x(a\partial_x)+q\partial_x-(r-q')\right)$$

with

$$a=(1-p)a_1+pa_2,\quad r=(1-p)r_1+pr_2,\quad q=(a_1-a_2)p'.$$

Under the condition $\int_0^{L_1} q/a\neq 0$, the limit is not even [37, Appendix A], whence $\lambda_{1,z}(\boldsymbol{\mathcal{Q}})$ is also not even when ε is sufficiently small. For more details, we refer to [37].

Finally, using the cooperativity assumption (A_2), the min–max characterization of Proposition 3.4, the equalities $\lambda_1=\max\lambda_{1,z}$ and $\lambda_1'=\lambda_{1,0}$ of Corollary 3.10 and Proposition 3.11 respectively, and the corresponding scalar results [54], we deduce the following corollary which relates the generalized principal eigenvalues of the operator $\boldsymbol{\mathcal{Q}}$ to the generalized principal eigenvalues of the scalar operators $\mathcal{P}_i-l_{i,i}$.

Corollary 3.12. *For all $z\in\mathbb{R}^n$,*

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}})\leq\max_{i\in[N]}\lambda_{1,z}(\mathcal{P}_i-l_{i,i}).$$

Consequently,

$$\lambda_1(\boldsymbol{\mathcal{Q}})\leq\max_{i\in[N]}\lambda_1(\mathcal{P}_i-l_{i,i})\quad\text{and}\quad\lambda_1'(\boldsymbol{\mathcal{Q}})\leq\max_{i\in[N]}\lambda_1'(\mathcal{P}_i-l_{i,i}).$$

Rougher but explicit estimates can subsequently be derived by considering constant test functions in the min–max characterization of $\lambda_{1,z}(\mathcal{P}_i-l_{i,i})$:

$$\lambda_{1,z}(\mathcal{P}_i-l_{i,i})\leq\underline{l}_{i,i}-\left\|\max_{y\in\mathbb{S}^{n-1}}|A_iy|\right\||z|^2-\sum_{\alpha=1}^n\left\|\sum_{\beta=1}^n|\partial_\alpha A^i_{\alpha,\beta}|^2\right\|^{1/2}|z|-\||q_i|\||z|,$$

where $\underline{\mathbf{L}}$ is defined in (A_2) and the notation $\|\cdot\|$ refers to the norm in the space $\mathcal{L}^\infty(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$.

3.2. **Dependence with respect to the coefficients.** Proposition 3.5 already proves Theorem 1.3. Below, we prove the remaining theorems on coefficient dependence.

3.2.1. *Asymptotic dependence.* We begin with the proof of Theorem 1.5, whose statement is recalled below.

Proposition 3.13. *Let $\mathbf{L}^{\triangle} \in \mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})$ be a block upper triangular essentially nonnegative matrix. Let $N' \in [N]$ and $(N_k)_{k\in[N'-1]}$ such that*

$$N_0 = 0 < 1 \leq N_1 \leq N_2 \leq \cdots \leq N_{N'-1} \leq N_{N'} = N$$

and such that

$$(l^{\triangle}_{i,j})_{(i,j)\in([N_k]\backslash[N_{k-1}])^2}$$

is the k-th diagonal block of $\mathbf{L}^{\triangle}$ (with the convention $[0]=\emptyset$). Assume

$$\left(\max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}} l^{\triangle}_{i,j}(t,x)\right)_{(i,j)\in([N_k]\backslash[N_{k-1}])^2} \quad \text{is irreducible for all } k\in[N'].$$

Let

$$\boldsymbol{\mathcal{Q}}_k = \mathrm{diag}(\mathcal{P}_i)_{i\in[N_k]\backslash[N_{k-1}]} - (l^{\triangle}_{i,j})_{(i,j)\in([N_k]\backslash[N_{k-1}])^2} \quad \text{for all } k\in[N'].$$

Then, as $\mathbf{L}\to\mathbf{L}^{\triangle}$ in $\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N}$,

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}) \to \min_{k\in[N']} \lambda_{1,z}\left(\boldsymbol{\mathcal{Q}}_k\right) \quad \text{for all } z\in\mathbb{R}^n,$$

$$\lambda_1(\boldsymbol{\mathcal{Q}}) \to \max_{z\in\mathbb{R}^n}\min_{k\in[N']} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k) \leq \min_{k\in[N']}\lambda_1(\boldsymbol{\mathcal{Q}}_k).$$

Proof. Step 1: the special case z=0. Let $(\mathbf{L}_p)_{p\in\mathbb{N}}$ be a sequence of matrices satisfying (A_2), (A_3) and that converges to $\mathbf{L}^{\triangle}$ in $\mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})$. Denote $\boldsymbol{\mathcal{Q}}_p = \mathrm{diag}(\mathcal{P}_i) - \mathbf{L}_p$ and $\boldsymbol{\mathcal{Q}}_{k,p} = \mathrm{diag}(\mathcal{P}_i)_{i\in[N_k]\backslash[N_{k-1}]} - (l_{p,i,j})_{(i,j)\in([N_k]\backslash[N_{k-1}])^2}$.

Since

$$0 \leq \mathbf{L}_p \leq \left(\sup_{p\in\mathbb{N}}\max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}} l_{p,i,j}(t,x)\right),$$

we can derive from the max–min and min–max characterizations of $\lambda_1'(\mathbf{L}_p)$ (*cf.* Proposition 3.4) uniform bounds on $(\lambda_1'(\mathbf{L}_p))_{p\in\mathbb{N}}$. Therefore up to extraction this sequence converges to a limit $\lambda\in\mathbb{R}$. Similarly, up to extraction, the associated generalized principal eigenfunction $\mathbf{u}_p \in \mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ with normalization $|\mathbf{u}_p(0,0)|=1$ converges to a nonnegative nonzero limit $\mathbf{u}\in\mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, [\mathbf{0},\boldsymbol{\infty}))$ satisfying the same normalization and satisfying

$$\mathrm{diag}(\mathcal{P}_i)\mathbf{u} - \mathbf{L}^{\triangle}\mathbf{u} = \lambda\mathbf{u}.$$

Note that for each $k\in[N']$,

$$\begin{aligned}\boldsymbol{\mathcal{Q}}_k(u_i)_{i\in[N_k]\backslash[N_{k-1}]} &= \lambda(u_i)_{i\in[N_k]\backslash[N_{k-1}]} + \left(\sum_{j\in[N_{k-1}]\cup[N]\backslash[N_k]} l^{\triangle}_{i,j}u_j\right)_{i\in[N_k]\backslash[N_{k-1}]} \\ &\geq \lambda(u_i)_{i\in[N_k]\backslash[N_{k-1}]}.\end{aligned}$$

Therefore, from the strong maximum principle of Proposition 2.1, either $(u_i)_{i\in[N_k]\backslash[N_{k-1}]} = \mathbf{0}$ or $(u_i)_{i\in[N_k]\backslash[N_{k-1}]} \gg \mathbf{0}$. For all $k\in[N']$ such that $(u_i)_{i\in[N_k]\backslash[N_{k-1}]}\gg\mathbf{0}$, it follows from the characterization of $\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}_k)$ (*cf.* Proposition 3.4) that $\lambda\leq\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}_k)$. Since $\mathbf{u}$ is nonzero, there exists at least one such k. Let $I\subset[N']$ be the set of all such k and let $J=[N']\backslash I$.

If $N' \in I$, then from the special block upper triangular form of $\boldsymbol{\mathcal{Q}}$, $\lambda = \lambda_1'(\boldsymbol{\mathcal{Q}}_{N'})$. Otherwise, there exists $k \in [N'-1] \cap I$. It follows then from a classical inductive argument that there exists indeed $k \in [N']$ such that $\lambda = \lambda_1'(\boldsymbol{\mathcal{Q}}_k)$.

Now, assume by contradiction that there exists $k' \in [N']$ such that $\lambda > \lambda_1'(\boldsymbol{\mathcal{Q}}_{k'})$. Let $\eta = \lambda - \lambda_1'(\boldsymbol{\mathcal{Q}}_{k'}) > 0$. Let $\mathbf{u}_{k'}$ be a periodic principal eigenfunction associated with $\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}_{k'})$. Let $\underline{\mathbf{u}}$ be defined as

$$\underline{u}_i = \begin{cases} u_{k',i-N_{k'-1}} & \text{if } i \in [N_{k'}]\backslash[N_{k'-1}], \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $i \in [N]$,

$$(\boldsymbol{\mathcal{Q}}_p\underline{\mathbf{u}})_i = \begin{cases} ((\boldsymbol{\mathcal{Q}}_{k',p} - \boldsymbol{\mathcal{Q}}_{k'})\underline{\mathbf{u}})_i + \lambda_1'(\boldsymbol{\mathcal{Q}}_{k'})\underline{u}_i & \text{if } i \in [N_{k'}]\backslash[N_{k'-1}], \\ -\displaystyle\sum_{j\in[N_{k'}]\backslash[N_{k'-1}]} l_{p,i,j}\underline{u}_j & \text{otherwise.} \end{cases}$$

On one hand,

$$-\sum_{j\in[N_{k'}]\backslash[N_{k'-1}]} l_{p,i,j}\underline{u}_j \leq 0 = \underline{u}_i \quad \text{for all } i \notin [N_{k'}]\backslash[N_{k'-1}].$$

On the other hand, by convergence of $\mathbf{L}_p$ and the Harnack inequality of Proposition 2.4 applied to the fully coupled operator $\boldsymbol{\mathcal{Q}}_{k'}$, we can assume that $p \in \mathbb{N}$ is so large that, for all $i \in [N_{k'}]\backslash[N_{k'-1}]$,

$$((\boldsymbol{\mathcal{Q}}_{k',p} - \boldsymbol{\mathcal{Q}}_{k'})\underline{\mathbf{u}})_i = ((\mathbf{L}_{k',p} - \mathbf{L}_{k'}^{\triangle})\underline{\mathbf{u}})_i \leq \frac{\eta}{2}\underline{u}_i,$$

where $\mathbf{L}_{k',p} = (l_{p,i,j})_{(i,j)\in([N_{k'}]\backslash[N_{k'-1}])^2}$ and $\mathbf{L}_{k'}^{\triangle} = \left(l_{i,j}^{\triangle}\right)_{(i,j)\in([N_{k'}]\backslash[N_{k'-1}])^2}$. Hence

$$\boldsymbol{\mathcal{Q}}_p\underline{\mathbf{u}} \leq \left(\lambda_1'(\boldsymbol{\mathcal{Q}}_{k'}) + \frac{\eta}{2}\right)\underline{\mathbf{u}} = \left(\lambda - \frac{\eta}{2}\right)\underline{\mathbf{u}}.$$

If $\lambda_1'(\boldsymbol{\mathcal{Q}}_p) > \lambda - \frac{\eta}{2}$, then we can study $\kappa^\star\mathbf{u}_p - \underline{\mathbf{u}}$ with

$$\kappa^\star = \frac{\max_{i\in[N]} \max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}} \underline{u}_i(t,x)}{\min_{i\in[N]} \min_{(t,x)\in\overline{\Omega_{\mathrm{per}}}} u_{p,i}(t,x)} > 0$$

and, by full coupling of $\boldsymbol{\mathcal{Q}}_p$, deduce a contradiction from the strong maximum principle. Hence $\lambda_1'(\boldsymbol{\mathcal{Q}}_p) \leq \lambda - \frac{\eta}{2}$. But now, assuming in addition that p is so large that $\lambda_1'(\boldsymbol{\mathcal{Q}}_p) > \lambda - \frac{\eta}{3}$, we find a contradiction. Therefore, for all $k' \in [N']$, $\lambda \leq \lambda_1'(\boldsymbol{\mathcal{Q}}_{k'})$, or in other words:

$$\lambda \leq \min_{k'\in[N']} \lambda_1'(\boldsymbol{\mathcal{Q}}_{k'}).$$

Combining this with $\lambda = \lambda_1'(\boldsymbol{\mathcal{Q}}_k)$, we deduce that the preceding inequality is an equality.

This argument shows that any convergent subsequence of the sequence $(\lambda_1'(\mathbf{L}_p))_{p\in\mathbb{N}}$ converges to $\min_{k\in[N']} \lambda_1'(\boldsymbol{\mathcal{Q}}_k)$. The conclusion follows. □

Step 2: the general case $z \in \mathbb{R}^n$. In view of

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}) = \lambda_1'(\boldsymbol{\mathcal{Q}}_z) = \lambda_1'\left(\boldsymbol{\mathcal{Q}} - \mathrm{diag}\left((A_i + A_i^{\mathrm{T}})z \cdot \nabla + z \cdot A_i z + \nabla \cdot (A_i z) - q_i \cdot z\right)\right),$$

in order to prove the convergence of $\lambda_{1,z}$ for any $z \in \mathbb{R}^n$, we only have to apply the preceding step to the operator $\boldsymbol{\mathcal{Q}}_z$. □

Step 3: convergence of $\lambda_1 = \max_{z\in\mathbb{R}^n} \lambda_{1,z}$. Since all $z \mapsto \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_p)$, $p \in \mathbb{N}$, are concave, the pointwise convergence is automatically improved to locally uniform convergence.

On one hand, recall from Corollary 3.12 the estimate, valid for all $p \in \mathbb{N}$,

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_p) \leq \max_{i\in[N]} \left[\underline{l}_{p,i,i} - \left\| \max_{y\in\mathbb{S}^{n-1}} |A_i y| \right\| |z|^2 - \sum_{\alpha=1}^{n} \left\| \sum_{\beta=1}^{n} |\partial_\alpha A^i_{\alpha,\beta}|^2 \right\|^{1/2} |z| - \| |q_i| \| |z| \right],$$

where the notation $\|\cdot\|$ refers to the norm in $\mathcal{L}^\infty(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$. It follows from the ellipticity assumption (A_1) that there exist $A>0$, $B\geq 0$ and $C\in\mathbb{R}$, independent of p, such that

$$\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_p) \leq -A|z|^2 - B|z| - C \quad \text{for all } p \in \mathbb{N}.$$

On the other hand, for all $p\in\mathbb{N}$, $\lambda_1(\boldsymbol{\mathcal{Q}}_p) \geq \lambda_1'(\boldsymbol{\mathcal{Q}}_p)$. In particular, $\lambda_1(\boldsymbol{\mathcal{Q}}_p) \geq \inf_{p\in\mathbb{N}} \lambda_1'(\boldsymbol{\mathcal{Q}}_p)$ and this lower bound is finite by virtue of Step 1 above.

Consequently, for all $p\in\mathbb{N}$, the point z_p where the maximum is achieved (which is indeed uniquely defined, *cf.* Corollary 3.10) is necessarily in the set Z defined as:

$$Z = \left\{ z\in\mathbb{R}^n \mid \inf_{p\in\mathbb{N}} \lambda_1'(\boldsymbol{\mathcal{Q}}_p) \leq -A|z|^2 - B|z| - C \right\}.$$

This set is compact.

To conclude, from the already established equality:

$$\lim_{p\to+\infty} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_p) = \min_{k\in[N']} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k),$$

and from the uniform convergence in Z and the concavity in $\mathbb{R}^n$, we deduce

$$\begin{aligned}
\lim_{p\to+\infty} \lambda_1(\boldsymbol{\mathcal{Q}}_p) &= \lim_{p\to+\infty} \max_{z\in Z} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_p) \\
&= \max_{z\in Z} \lim_{p\to+\infty} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_p) \\
&= \max_{z\in\mathbb{R}^n} \lim_{p\to+\infty} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_p) \\
&= \max_{z\in\mathbb{R}^n} \min_{k\in[N']} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k).
\end{aligned}$$

Finally, from the inequality $\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k) \leq \lambda_1(\boldsymbol{\mathcal{Q}}_k)$ for all k and z, it follows that

$$\min_{k\in[N']} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k) \leq \min_{k\in[N']} \lambda_1(\boldsymbol{\mathcal{Q}}_k),$$

whence

$$\max_{z\in\mathbb{R}^n} \min_{k\in[N']} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k) \leq \min_{k\in[N']} \lambda_1(\boldsymbol{\mathcal{Q}}_k).$$

□

This ends the proof. □

Remark 3.6. We will use repeatedly the arguments of Step 2 and Step 3 above in what follows, in order to deduce the convergence of $\lambda_{1,z}$ and λ_1 when the convergence of λ_1' has been established.

Note however that the estimate $\lambda_{1,z}(\boldsymbol{\mathcal{Q}}) \leq -A|z|^2 - B|z| - C$ with $A>0$ and $B\geq 0$ becomes useless when the diffusion matrices A_i vanish. This is consistent with the fact that, in Theorem 1.6, the convergence of λ_1 is in general false.

Remark 3.7. The inequality $\max_{z\in\mathbb{R}^n}\min_{k\in[N']}\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_k) \leq \min_{k\in[N']}\lambda_1(\boldsymbol{\mathcal{Q}}_k)$ is strict in some cases. Consider for instance the following space-time homogeneous, one-dimensional, two-component counter-example:

$$\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}}_\varepsilon = \mathrm{diag}\left(\begin{pmatrix} \partial_t u_1 - \partial_{xx}u_1 \\ \partial_t u_2 - \partial_{xx}u_2 + 2\partial_x u_2 - 1\end{pmatrix}\right) - \varepsilon\begin{pmatrix} -1 & 1 \\ 1 & -1\end{pmatrix},$$

where $\varepsilon > 0$. The operator is diagonal if $\varepsilon = 0$. The two scalar operators on the diagonal, $\boldsymbol{\mathcal{Q}}_1 = \partial_t - \partial_{xx}$ and $\boldsymbol{\mathcal{Q}}_2 = \partial_t - \partial_{xx} + 2\partial_x - 1$, satisfy $\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_1) = -z^2$ and $\lambda_{1,z}(\boldsymbol{\mathcal{Q}}_2) = -(z-1)^2$. In particular, $\lambda_1(\boldsymbol{\mathcal{Q}}_1) = \lambda_1(\boldsymbol{\mathcal{Q}}_2) = 0$. However, the function $z \mapsto \min(-z^2, -(z-1)^2)$ coincides with

$$z \mapsto \begin{cases} -(z-1)^2 & \text{if } z < 1/2, \\ -z^2 & \text{if } z \geq 1/2,\end{cases}$$

whose maximal value is $-1/4 < 0$, which is attained at $1/2$.

Below, $\delta \in (0,1)$ and $\mathbf{L} \in \mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})$ satisfying (A_2) and (A_3) are fixed and we prove Theorem 1.6 on vanishing diffusion and advection rates.

Recall from the statement of Theorem 1.6 the definition of the following sets, parametrized by $\varepsilon > 0$:

$$\mathcal{D}_\varepsilon = \left\{ A \in \mathcal{C}^{\delta/2,1+\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{n\times n}) \mid A = A^{\mathrm{T}}\, 0 < \min_{y\in\mathbb{S}^{n-1}} y\cdot Ay,\ \max_{\alpha,\beta\in[n]} \|A_{\alpha,\beta}\|_{\mathcal{C}^{0,1}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n)} \leq \varepsilon^2 \right\},$$

$$\mathcal{A}_\varepsilon = \left\{ q \in \mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n) \mid \||q|\|_{\mathcal{C}^0_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n)} \leq \varepsilon \right\}.$$

Note that $(A,q) \in \mathcal{D}_\varepsilon \times \mathcal{A}_\varepsilon$ if and only if $(\varepsilon^{-2}A, \varepsilon^{-1}q) \in \mathcal{D}_1 \times \mathcal{A}_1$. Recall also the notation $\mathbf{L}(x) : t \mapsto \mathbf{L}(t,x)$ for a fixed $x \in \mathbb{R}^n$.

For any choice of $\left((A_i)_{i\in[N]}, (q_i)_{i\in[N]}\right) \in \mathcal{D}_1^N \times \mathcal{A}_1^N$ and any $z \in \mathbb{R}^n$, we denote in this subsection

$$\lambda_{1,z}\left(\gamma, (A_i)_{i\in[N]}, (q_i)_{i\in[N]}\right) = \lambda_{1,z}(\partial_t - \mathrm{diag}(\gamma^2\nabla\cdot(A_i\nabla) - \gamma q_i\cdot\nabla) - \mathbf{L})$$

When the context is unambiguous, the notation $\lambda_{1,z}\left(\gamma, (A_i)_{i\in[N]}, (q_i)_{i\in[N]}\right)$ is shortened as $\lambda_{1,z}(\gamma)$. In the special case $z = 0$, we use the notations $\lambda_1'\left(\gamma, (A_i)_{i\in[N]}, (q_i)_{i\in[N]}\right)$ or $\lambda_1'(\gamma)$ for short. Also, we denote

$$\lambda_1'(x) = \lambda_1'(\partial_t - \mathbf{L}(x)) \quad \text{for all } x \in \mathbb{R}^n.$$

In general, (A_3) is not sufficient for the full coupling of the operators of the family $(\partial_t - \mathbf{L}(x))_{x\in[0,L]}$. The principal eigenvalue $\lambda_1'(x)$ can still be defined by continuous extension, and it is a continuous function of x, however its eigenfunctions might not be positive and unique up to a multiplicative constant. Up to replacing $\mathbf{L}$ by $\mathbf{L} - (\min_{x\in[0,L]}\lambda_1'(x))\mathbf{I}$ and up to a spatial translation, we can assume without loss of generality that

$$\min_{x\in[0,L]} \lambda_1'(x) = \lambda_1'(0) = 0.$$

With these notations and simplifications, to prove Theorem 1.6 is to prove that, for any $z \in \mathbb{R}^n$ and any small $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ such that, for any $\gamma \in (0, \gamma_\varepsilon]$ and any $\left((A_i)_{i\in[N]}, (q_i)_{i\in[N]}\right) \in \mathcal{D}_1^N \times \mathcal{A}_1^N$,

$$-\varepsilon \leq \lambda_{1,z}(\gamma) = \lambda_{1,z}\left(\gamma, (A_i)_{i\in[N]}, (q_i)_{i\in[N]}\right) \leq \varepsilon.$$

This is what we will prove. Due to the default of full coupling, some care will be needed. Before starting, we reduce the problem further and introduce some useful quantities.

By virtue of the Hölder continuity of $\mathbf{L}$, there exists a constant $K > 0$ that depends only on $\mathbf{L}$ such that

$$\max_{i,j\in[N]}\max_{(t,x,x')\in[0,T]\times[0,L]\times[0,L]}|l_{i,j}(t,x)-l_{i,j}(t,x')|\leq K|x-x'|^{\delta}.$$

Up to increasing K in a way that depends only on n, the following lemma holds.

Lemma 3.14. *For all* $(A,q)\in\mathcal{D}_1\times\mathcal{A}_1$*,*

$$\max_{y\in\mathbb{S}^{n-1}}\max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}}(y\cdot A(t,x)y)\leq K,$$

$$\max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}}\sum_{\alpha,\beta\in[n]}|\partial_\alpha A_{\alpha,\beta}(t,x)|\leq K,$$

$$\max_{\alpha,\beta\in[n]}\max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}}|A_{\alpha,\beta}(t,x)|\leq K,$$

$$\max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}}|q(t,x)|\leq K.$$

Proof. We only have to verify

$$\sup_{A\in\mathcal{D}_1}\max_{y\in\mathbb{S}^{n-1}}\max_{(t,x)\in\overline{\Omega_{\mathrm{per}}}}(y\cdot A(t,x)y)<+\infty,$$

which follows directly from

$$|y\cdot A(t,x)y|\leq\sum_{\alpha=1}^{n}\left(|y_\alpha|\sum_{\beta=1}^{n}|A_{\alpha,\beta}(t,x)||y_\beta|\right)\leq n^2\max_{\alpha,\beta\in[n]}\|A_{\alpha,\beta}\|_{\mathcal{C}^{0,1}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n)}=n^2.$$

□

We are now in a position to prove Theorem 1.6 in the special case $z=0$, which is the combination of the forthcoming Lemmas 3.15 and 3.16. Since, by definition of $\lambda_{1,z}$, $\lambda_{1,z}(\gamma)$ is the periodic principal eigenvalue of the operator

$$\partial_t-\mathrm{diag}\left(\gamma^2\nabla\cdot(A_i\nabla)-(\gamma q_i-2\gamma^2A_iz)\cdot\nabla\right)-\mathbf{L}-\mathrm{diag}\left(\gamma^2z\cdot A_iz+\gamma^2\nabla\cdot(A_iz)-\gamma q_i\cdot z\right)$$

and since we are only interested in pointwise convergence with respect to z, the general case $z\in\mathbb{R}^n$ is a straightforward consequence of the case $z=0$ applied to the operator above.

Lemma 3.15. *There exists* $\varepsilon_0>0$ *such that, for any* $\varepsilon\in(0,\varepsilon_0)$*, there exists* $\gamma_{\varepsilon,1}>0$ *such that, for any* $\gamma\in(0,\gamma_{\varepsilon,1}]$ *and any* $\left((A_i)_{i\in[N]},(q_i)_{i\in[N]}\right)\in\mathcal{D}_1^N\times\mathcal{A}_1^N$*,*

$$\lambda_1'(\gamma)\leq\varepsilon.$$

Proof. Up to a permutation of the lines of the system, we can assume without loss of generality that there exists $N_0\in[N-1]\cup\{0\}$, $\tilde{N}\in[N-N_0]$ such that:

- $\mathbf{L}(0)$ is a block upper triangular matrix (with only one block if $\tilde{N}=N$) and the diagonal square block whose top left and bottom right entries are $l_{N_0+1,N_0+1}(0)$ and $l_{N_0+\tilde{N},N_0+\tilde{N}}(0)$ respectively, namely the block

$$\tilde{\mathbf{L}}(0):t\in\mathbb{R}\mapsto(l_{i+N_0,j+N_0}(t,0))_{(i,j)\in[\tilde{N}]^2}\in\mathbb{R}^{\tilde{N}\times\tilde{N}},$$

 is such that

$$\left(\max_{t\in[0,T]}\tilde{l}_{i+N_0,j+N_0}(t,0)\right)_{(i,j)\in[\tilde{N}]^2}$$

 is irreducible;

- $\lambda_1'(\partial_t - \tilde{\mathbf{L}}(0)) = 0$ with periodic principal eigenfunction $\tilde{\mathbf{v}} \in \mathcal{C}^1_{\mathrm{per}}(\mathbb{R}, (0,+\infty)^{\tilde{N}})$.

By continuity, there exists $R_0 > 0$ such that, for all $x \in B(0, 2R_0)$,

$$\left(\max_{t\in[0,T]} \tilde{l}_{i+N_0,j+N_0}(t,x) \right)_{(i,j)\in[\tilde{N}]^2}$$

is still irreducible.

Using Földes–Poláčik's Harnack inequality [30] (more precisely, a slight refinement of it for time periodic solutions of time periodic systems fully coupled in the time period $[0,T]$, proved exactly like Proposition 2.4 and therefore not detailed here), define

$$\tilde{\kappa} = \frac{\max_{i\in[\tilde{N}]} \max_{t\in[0,T]} \tilde{v}_i(t)}{\min_{i\in[\tilde{N}]} \min_{t\in[0,T]} \tilde{v}_i(t)}.$$

Let $\varepsilon_0 = R_0^\delta 2KN\tilde{\kappa}$ and fix from now on $\varepsilon \in (0, \varepsilon_0)$.

Define

$$\begin{aligned} R &= \max\left\{ r > 0 \mid \frac{\varepsilon}{K} r^2 - r - n - \frac{\pi}{2} - \left(\frac{\varepsilon}{2KN\tilde{\kappa}}\right)^{\frac{1}{\delta}} = 0 \right\} \\ &= \frac{K}{2\varepsilon}\left(1 + \sqrt{1 + \frac{4\varepsilon}{K}\left(n + \frac{\pi}{2} + \left(\frac{\varepsilon}{2KN\tilde{\kappa}}\right)^{\frac{1}{\delta}}\right)}\right), \end{aligned}$$

$$\gamma_{\varepsilon,1} = \frac{1}{R}\left(\frac{\varepsilon}{2KN\tilde{\kappa}}\right)^{\frac{1}{\delta}}.$$

Note that, by construction, $R\gamma_{\varepsilon,1} < R_0$.

Fix $\left((A_i)_{i\in[N]}, (q_i)_{i\in[N]}\right) \in \mathcal{D}_1^N \times \mathcal{A}_1^N$ and $\gamma \in (0, \gamma_{\varepsilon,1}]$. Denote

$$\boldsymbol{\mathcal{Q}}_\gamma = \partial_t - \mathrm{diag}(\nabla\cdot(A_i^\gamma \nabla)) - q_i^\gamma \cdot \nabla)_{i\in[N]} - \mathbf{L}_\gamma,$$

with $A_i^\gamma(t,x) = A_i(t,\gamma x)$, $q_i^\gamma(t,x) = q_i(t,\gamma x)$, $\mathbf{L}_\gamma(t,x) = \mathbf{L}(t,\gamma x)$ (these new functions are all $[0,T]\times[0,\gamma^{-1}L]$-periodic) and

$$\tilde{\mathbf{L}}_\gamma : (t,x) \mapsto \tilde{\mathbf{L}}(t,\gamma x),$$

$$\tilde{\boldsymbol{\mathcal{Q}}}_\gamma = \partial_t - \mathrm{diag}(\nabla\cdot(A_{i+N_0}^\gamma \nabla - q_{i+N_0}^\gamma \cdot \nabla)_{i\in[\tilde{N}]} - \tilde{\mathbf{L}}_\gamma.$$

In view of the definition of λ_1' (refer to Proposition 3.4), in order to prove $\lambda_1'(\gamma) \leq \varepsilon$, we only have to construct a positive bounded time periodic sub-solution $\underline{\mathbf{u}}$ satisfying $\boldsymbol{\mathcal{Q}}_\gamma \underline{\mathbf{u}} \leq \varepsilon \underline{\mathbf{u}}$. In fact, by comparison and full coupling of $\tilde{\boldsymbol{\mathcal{Q}}}_\gamma$ in $B(0,R)$, the construction of a nonnegative bounded time periodic continuous $\underline{\mathbf{u}}$ whose components $(\underline{u}_{i+N_0})_{i\in[\tilde{N}]}$ are positive in $B(0,R)$ and zero outside and that satisfies $\tilde{\boldsymbol{\mathcal{Q}}}_\gamma(\underline{u}_{i+N_0})_{i\in[\tilde{N}]} \leq \varepsilon(\underline{u}_{i+N_0})_{i\in[\tilde{N}]}$ in $B(0,R)$ is sufficient. Indeed, considering a periodic principal eigenfunction $\mathbf{v}_\gamma$ associated with $\lambda_1'(\gamma)$ and defining

$$\kappa^\star = \inf\left\{ \kappa > 0 \mid (\kappa^\star v_{\gamma,i+N_0} - \underline{u}_{i+N_0})_{i\in[\tilde{N}]} \gg \mathbf{0} \quad \text{in } \mathbb{R}\times\overline{B(0,R)} \right\},$$

the function $\mathbf{w} = (\kappa^\star v_{\gamma,i+N_0} - \underline{u}_{i+N_0})_{i\in[\tilde{N}]}$ satisfies in $\mathbb{R}\times B(0,R)$:

$$\begin{aligned} \tilde{\boldsymbol{\mathcal{Q}}}_\gamma \mathbf{w} &\geq \kappa^\star \lambda_1'(\gamma)(v_{\gamma,i+N_0})_{i\in[\tilde{N}]} + \kappa^\star \left(\sum_{j\in[N], j-N_0\notin[\tilde{N}]} l_{\gamma,i,j} v_{\gamma,j} \right)_{i-N_0\in[\tilde{N}]} - \varepsilon(\underline{u}_{i+N_0})_{i\in[\tilde{N}]} \\ &\geq \lambda_1'(\gamma)\mathbf{w} + (\lambda_\gamma - \varepsilon)(\underline{u}_{i+N_0})_{i\in[\tilde{N}]}, \end{aligned}$$

and subsequently $\lambda_1'(\gamma) > \varepsilon$ contradicts the strong maximum principle since $\tilde{\boldsymbol{\mathcal{Q}}}_\gamma$ is fully coupled in $\mathbb{R} \times B(0,R)$.

We look for a sub-solution of the form

$$\underline{\mathbf{u}} : (t,x) \mapsto \begin{cases} 0 & \text{if } i - N_0 \notin [\tilde{N}], \\ v(x)\tilde{v}_i(t) & \text{if } i - N_0 \in [\tilde{N}]. \end{cases}$$

For any $v \in \mathcal{C}^2(\overline{B(0,R)}, [0,+\infty))$,

$$\tilde{\boldsymbol{\mathcal{Q}}}_\gamma(v\tilde{\mathbf{v}}) = v(\tilde{\mathbf{L}}_\gamma(0) - \tilde{\mathbf{L}}_\gamma)\tilde{\mathbf{v}} + \operatorname{diag}\left(-\nabla\cdot(A_i^\gamma \nabla v) + q_i^\gamma \cdot \nabla v\right)_{i-N_0\in[\tilde{N}]} \tilde{\mathbf{v}} \quad \text{in } [0,T]\times B(0,R).$$

On one hand, for any $v \in \mathcal{C}^2(\overline{B(0,R)}, [0,+\infty))$ and all $(t,x) \in [0,T] \times B(0,R)$,

$$\begin{aligned} v(x)(\tilde{\mathbf{L}}_\gamma(t,0) - \tilde{\mathbf{L}}_\gamma(t,x))\tilde{\mathbf{v}}(t) &= v(x)(\tilde{\mathbf{L}}(t,0) - \tilde{\mathbf{L}}(t,\gamma x))\tilde{\mathbf{v}}(t) \\ &\le v(x)K\left|0-\gamma x\right|^\delta \sum_{i=1}^{\tilde{N}} \tilde{v}_i(t)\mathbf{1} \\ &\le v(x)K\gamma^\delta R^\delta N \max_{i\in[\tilde{N}]} \tilde{v}_i(t)\mathbf{1} \\ &\le K\gamma_{\varepsilon,1}^\delta R^\delta N\tilde{\kappa} v(x)\tilde{\mathbf{v}}(t) \\ &\le \frac{1}{2}\varepsilon v(x)\tilde{\mathbf{v}}(t). \end{aligned}$$

On the other hand, let $\eta > 0$ and $v : x \mapsto \exp\left(\eta\cos\left(\frac{\pi}{2R}|x|\right)\right) - 1$. For any $i \in [N]$ and all $(t,x) \in [0,T] \times B(0,R)\backslash\{0\}$,

$$\nabla v(x) = -\frac{\eta\pi}{2R}\sin\left(\frac{\pi}{2R}|x|\right)(v(x)+1)\frac{x}{|x|},$$

$$\begin{aligned} -\nabla\cdot\left(A_i^\gamma\nabla v\right)(t,x) = &-\frac{\eta^2\pi^4}{16R^4}\frac{\sin\left(\frac{\pi}{2R}|x|\right)^2}{\frac{\pi^2}{4R^2}|x|^2}(v(x)+1)x\cdot A_i^\gamma(t,x)x \\ &+\frac{\eta\pi^2}{4R^2}\cos\left(\frac{\pi}{2R}|x|\right)(v(x)+1)\frac{x}{|x|}\cdot A_i^\gamma(t,x)\frac{x}{|x|} \\ &+\frac{\eta\pi}{2R}\sin\left(\frac{\pi}{2R}|x|\right)(v(x)+1)\gamma\sum_{\alpha,\beta=1}^n \partial_\alpha A_{\alpha,\beta}^i(t,\gamma x)\frac{x_\beta}{|x|} \\ &+\frac{\eta\pi^2}{4R^2}\frac{\sin\left(\frac{\pi}{2R}|x|\right)}{\frac{\pi}{2R}|x|}(v(x)+1)\sum_{\alpha,\beta=1}^n A_{\alpha,\beta}^i(t,\gamma x)\left(\delta_{\alpha\beta}-\frac{x_\alpha x_\beta}{|x|^2}\right) \end{aligned}$$

so that, denoting

$$\underline{\Lambda} = \min_{i\in[N]}\min_{y\in\mathbb{S}^{n-1}}\min_{(t,x)\in\overline{\Omega_{\text{per}}}} y\cdot A_i(t,x)y > 0,$$

we find

$$-\nabla\cdot\left(A_i^\gamma\nabla v\right)(t,x) \le \frac{\eta\pi}{2R}(v(x)+1)\left(-\frac{\underline{\Lambda}\pi\eta|x|^2}{4R^3}+\frac{\pi K}{2R}+\gamma_{\varepsilon,1}K+\frac{Kn}{R}\right)$$

and, by the discrete Cauchy–Schwarz inequality,

$$q_i^\gamma\cdot\nabla v \le \frac{\eta\pi K}{2R}(v+1).$$

Subsequently,

$$\max_{i-N_0\in[\tilde{N}]}\left(-\nabla\cdot(A_i^\gamma\nabla v)+q_i^\gamma\cdot\nabla v\right)(t,x)\leq\frac{\eta\pi}{2R}(v(x)+1)\left(K\left(\frac{2n+\pi}{2R}+\gamma_{\varepsilon,1}+1\right)-\frac{\eta\pi\underline{\Lambda}}{4R^3}|x|^2\right),$$

and, by continuity, this is also true at $x=0$. If $x\in B(0,R/2)$,

$$\begin{aligned}\max_{i-N_0\in[\tilde{N}]}\left(-\nabla\cdot(A_i^\gamma\nabla v)+q_i^\gamma\cdot\nabla v\right)(t,x)&\leq\frac{\eta\pi}{2R}\mathrm{e}^\eta K\left(\frac{2n+\pi}{2R}+\gamma_{\varepsilon,1}+1\right)\\&\leq\frac{\eta\pi}{2R^2}\frac{\mathrm{e}^\eta K\left(n+\frac{\pi}{2}+\gamma_{\varepsilon,1}R+R\right)}{\exp(\eta/\sqrt{2})-1}v(x).\end{aligned}$$

If $x\in B(0,R)\backslash B(0,R/2)$,

$$\max_{i-N_0\in[\tilde{N}]}\left(-\nabla\cdot(A_i^\gamma\nabla v)+q_i^\gamma\cdot\nabla v\right)(t,x)\leq\frac{\eta\pi}{2R}\mathrm{e}^{\eta/\sqrt{2}}\left(K\left(\frac{2n+\pi}{2R}+\gamma_{\varepsilon,1}+1\right)-\frac{\eta\pi\underline{\Lambda}}{16R}\right)$$

and the right-hand side is negative provided $\eta\geq\frac{16KR}{\pi\underline{\Lambda}}\left(\frac{2n+\pi}{2R}+\gamma_{\varepsilon,1}+1\right)$. Hence, in $[0,T]\times B(0,R)$,

$$\max_{i-N_0\in[\tilde{N}]}\left(-\nabla\cdot(A_i^\gamma\nabla v)+q_i^\gamma\cdot\nabla v\right)\leq\frac{\eta\pi}{2R^2}\frac{\mathrm{e}^\eta K\left(n+\frac{\pi}{2}+\gamma_{\varepsilon,1}R+R\right)}{\exp(\eta/\sqrt{2})-1}v.$$

Assuming now that η is indeed large enough and using the definitions of $\gamma_{\varepsilon,1}$ and R, we obtain

$$\frac{\mathrm{e}^{\eta/\sqrt{2}}-1}{\pi\eta\mathrm{e}^\eta}\max_{i-N_0\in[\tilde{N}]}\left(-\nabla\cdot(A_i^\gamma\nabla v)+q_i^\gamma\cdot\nabla v\right)\leq\frac{1}{2}\varepsilon v\quad\text{in }[0,T]\times B(0,R).$$

Therefore, replacing v by $\frac{\mathrm{e}^{\eta/\sqrt{2}}-1}{\pi\eta\mathrm{e}^\eta}v$ and extending continuously the function v as the zero function outside of $B(0,R)$, $\tilde{\boldsymbol{\mathcal{Q}}}_\gamma(v\tilde{\mathbf{v}})\leq\varepsilon v\tilde{\mathbf{v}}$ in $\mathbb{R}\times B(0,R)$ with $v\mathbf{v}=0$ on $\mathbb{R}\times\mathbb{R}^n\backslash B(0,R)$. This ends the proof. □

Lemma 3.16. *Let $\varepsilon>0$. There exists $\gamma_{\varepsilon,2}>0$ such that, for any $\gamma\in(0,\gamma_{\varepsilon,2}]$ and any $\left((A_i)_{i\in[N]},(q_i)_{i\in[N]}\right)\in\mathcal{D}_1^N\times\mathcal{A}_1^N$,*

$$\lambda_1'(\gamma)\geq-\varepsilon.$$

Proof. Since, for any $\nu>0$, $\left((A_i)_{i\in[N]},(q_i)_{i\in[N]}\right)\in\mathcal{D}_1^N\times\mathcal{A}_1^N$, $\gamma>0$, the generalized principal eigenvalue λ_1' of the operator $\boldsymbol{\mathcal{Q}}_\gamma-\nu\mathbf{1}_{N\times N}$ satisfies, by monotonicity of the generalized principal eigenvalue,

$$\lambda_1'(\boldsymbol{\mathcal{Q}}_\gamma-\nu\mathbf{1}_{N\times N})\leq\lambda_1'(\boldsymbol{\mathcal{Q}}_\gamma),$$

it suffices to prove the result in the case where $\mathbf{L}$ is replaced by $\mathbf{L}_\nu=\mathbf{L}+\nu\mathbf{1}_{N\times N}$ for an appropriately small $\nu\in(0,1)$, which is what we do below.

By pointwise irreducibility of $\mathbf{L}_\nu$, we can define a continuous and space periodic function $\overline{\mathbf{u}}$ which associates with $(t,x)\in\mathbb{R}\times\mathbb{R}^n$ the evaluation at t of the unique periodic principal eigenfunction $\mathbf{v}[x]$ of the operator $\partial_t-\mathbf{L}_\nu(x)$ satisfying in addition the normalization

$$\max_{i\in[N]}\max_{t\in[0,T]}v[x]_i(t)=1.$$

Note that, by virtue of the Harnack inequality of Proposition 2.4 and of the *a priori* bounds

$$-\infty<\min_{\nu\in[0,1]}\min_{x\in[0,L]}\lambda_1'(\partial_t-\mathbf{L}_\nu(x))\leq\max_{\nu\in[0,1]}\max_{x\in[0,L]}\lambda_1'(\partial_t-\mathbf{L}_\nu(x))<+\infty,$$

there exists a constant $\overline{\kappa}_\nu$, independent of $x \in \mathbb{R}^n$, such that

$$\frac{1}{\overline{\kappa}_\nu} \leq \min_{i\in[N]} \min_{(t,x)\in\overline{\Omega_{\text{per}}}} v[x]_i(t).$$

Note however that $\overline{\kappa}_\nu \to +\infty$ as $\nu \to 0$, so that we truly need $\nu > 0$ in what follows.

Fix $\varepsilon > 0$. Assume from now on that $\nu > 0$ is fixed and so small that

$$\min_{x\in[0,L]} \lambda_1'(\partial_t - \mathbf{L}_\nu(\gamma x)) \geq -\frac{\varepsilon}{5}.$$

By a classical regularization procedure taking advantage of the fact that $\overline{\kappa}_\nu > 0$ (refer to Bai–He [10, Lemma 4.3] for details), there exists a $\mathcal{C}^{1,2}_{\text{per}}$ approximation $\overline{\mathbf{u}}_\varepsilon$ of $\overline{\mathbf{u}}$ satisfying, for some constant $K_\varepsilon > 0$, for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$,

$$\partial_t \overline{\mathbf{u}}_\varepsilon(t,x) \geq \partial_t \mathbf{v}[x](t) - \frac{\varepsilon}{5}\overline{\mathbf{u}}_\varepsilon(t,x),$$

$$-\mathbf{L}_\nu(t,x)\overline{\mathbf{u}}_\varepsilon(t,x) \geq -\mathbf{L}_\nu(t,x)\mathbf{v}[x](t) - \frac{\varepsilon}{5}\overline{\mathbf{u}}_\varepsilon(t,x),$$

$$\min_{x\in[0,L]} \lambda_1'(\partial_t - \mathbf{L}_\nu(x))\mathbf{v}[x](t) \geq \min_{x\in[0,L]} \lambda_1'(\partial_t - \mathbf{L}_\nu(x))\overline{\mathbf{u}}_\varepsilon(t,x) - \frac{\varepsilon}{5}\overline{\mathbf{u}}_\varepsilon(t,x),$$

$$\left(|\nabla \overline{u}_{\varepsilon,i}(t,x)|\right)_{i\in[N]} \leq K_\varepsilon \overline{\mathbf{u}}_\varepsilon(t,x),$$

$$\left(\sum_{\alpha,\beta\in[n]} |\partial_{\alpha\beta}\overline{u}_{\varepsilon,i}(t,x)|\right)_{i\in[N]} \leq K_\varepsilon \overline{\mathbf{u}}_\varepsilon(t,x).$$

Define

$$\begin{aligned}\gamma_{\varepsilon,2} &= \max\left\{\gamma > 0 \mid (n^2+1)\gamma^2 + \gamma - \frac{\varepsilon}{5KK_\varepsilon} = 0\right\} \\ &= \frac{1}{2(n^2+1)}\left(-1 + \sqrt{1 + \frac{4(n^2+1)\varepsilon}{5KK_\varepsilon}}\right).\end{aligned}$$

Fix $\left((A_i)_{i\in[N]}, (q_i)_{i\in[N]}\right) \in \mathcal{D}_1^N \times \mathcal{A}_1^N$, $\gamma \in (0, \gamma_{\varepsilon,2}]$ and denote

$$\boldsymbol{\mathcal{Q}}_{\gamma,\nu} = \boldsymbol{\mathcal{Q}}_\gamma - \nu\mathbf{1}_{N\times N} = \partial_t - \operatorname{diag}(\gamma^2\nabla\cdot(A_i\nabla) + \gamma q_i\cdot\nabla) - \mathbf{L}_\nu.$$

By definition of $\gamma_{\varepsilon,2}$,

$$\begin{aligned}(-\gamma^2\nabla\cdot(A_i\nabla\overline{u}_{\varepsilon,i}) + \gamma q_i\cdot\nabla\overline{u}_{\varepsilon,i})_{i\in[N]} &= \gamma\left(-\gamma\nabla\cdot(A_i\nabla\overline{u}_{\varepsilon,i}) + q_i\cdot\nabla\overline{u}_{\varepsilon,i}\right)_{i\in[N]} \\ &\geq -\gamma_\varepsilon KK_\varepsilon(\gamma_{\varepsilon,2}n^2 + \gamma_{\varepsilon,2} + 1)\overline{\mathbf{u}}_\varepsilon \\ &\geq -\frac{\varepsilon}{5}\overline{\mathbf{u}}_\varepsilon\end{aligned}$$

so that, for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, using the assumptions on $\overline{\mathbf{u}}_\varepsilon$,

$$\begin{aligned}\boldsymbol{\mathcal{Q}}_{\gamma,\nu}\overline{\mathbf{u}}_\varepsilon(t,x) &\geq \partial_t\mathbf{v}[x](t) - \mathbf{L}_\nu(t,x)\mathbf{v}[x](t) - \frac{3\varepsilon}{5}\overline{\mathbf{u}}_\varepsilon(t,x) \\ &\geq \min_{x\in[0,L]}\lambda_1'(\partial_t - \mathbf{L}_\nu(x))\mathbf{v}[x](t) - \frac{3\varepsilon}{5}\overline{\mathbf{u}}_\varepsilon(t,x) \\ &\geq \min_{x\in[0,L]}\lambda_1'(\partial_t - \mathbf{L}_\nu(x))\overline{\mathbf{u}}_\varepsilon(t,x) - \frac{4\varepsilon}{5}\overline{\mathbf{u}}_\varepsilon(t,x) \\ &\geq -\varepsilon\overline{\mathbf{u}}_\varepsilon(t,x).\end{aligned}$$

By the min–max characterization of Proposition 3.4, in order to prove $\lambda_1'(\mathcal{Q}_{\gamma,\nu}) \geq -\varepsilon$, it is sufficient to prove that for any $\eta < -\varepsilon$ and any $\underline{\mathbf{u}} \in \mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}, [\mathbf{0}, \boldsymbol{\infty}))$ such that $\mathcal{Q}_{\gamma,\nu}\underline{\mathbf{u}} \leq \eta\underline{\mathbf{u}}$, $\underline{\mathbf{u}} = \mathbf{0}$.

Let $\eta \leq -\varepsilon$ and $\underline{\mathbf{u}} \in \mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}, [\mathbf{0}, \boldsymbol{\infty}))$ such that $\mathcal{Q}_{\gamma,\nu}\underline{\mathbf{u}} \leq \eta\underline{\mathbf{u}}$. Since $\overline{\mathbf{u}}_\varepsilon \gg \mathbf{0}$, we can define

$$\kappa^\star = \max_{i\in[N]} \max_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} \frac{\underline{u}_i(t,x)}{\overline{u}_{\varepsilon,i}(t,x)} \geq 0.$$

The function $\mathbf{w} = \kappa^\star\overline{\mathbf{u}}^\gamma_\varepsilon - \underline{\mathbf{u}}$ is nonnegative and periodic in $\mathbb{R}\times\mathbb{R}^n$, satisfies $w_{i^\star}(t^\star, x^\star) = 0$ for some $(i^\star, t^\star, x^\star) \in [N]\times[0,T]\times[0,L]$, and satisfies

$$\mathcal{Q}_{\gamma,\nu}\mathbf{w} \geq -\varepsilon\kappa^\star\overline{\mathbf{u}}_\varepsilon - \eta\underline{\mathbf{u}} \geq -\varepsilon\mathbf{w} \quad \text{in } \mathbb{R}\times\mathbb{R}^n.$$

By the strong maximum principle of Proposition 2.1, $\mathbf{w} = \mathbf{0}$ in $\mathbb{R}\times\mathbb{R}^n$, or in other words $\kappa^\star\overline{\mathbf{u}}_\varepsilon = \underline{\mathbf{u}}$ in $\mathbb{R}\times\mathbb{R}^n$. But then

$$\mathbf{0} \geq (\eta+\varepsilon)\underline{\mathbf{u}} \geq \mathcal{Q}_{\gamma,\nu}\underline{\mathbf{u}} + \varepsilon\underline{\mathbf{u}} = \kappa^\star(\mathcal{Q}_{\gamma,\nu}\overline{\mathbf{u}}_\varepsilon + \varepsilon\overline{\mathbf{u}}_\varepsilon) \geq \mathbf{0}$$

which implies that all inequalities are equalities. Recalling that $\eta < -\varepsilon$, we deduce $\underline{\mathbf{u}} = \mathbf{0}$ (and $\kappa^\star = 0$).

This ends the proof. $\square$

We now prove Theorem 1.7.

Proposition 3.17. *Let*

$$\left((\langle A_i\rangle, \langle q_i\rangle)_{i\in[N]}, \langle\mathbf{L}\rangle\right) : t \mapsto \frac{1}{|[0,L]|}\int_{[0,L]} \left((A_i, q_i)_{i\in[N]}, \mathbf{L}\right)(t,x)\mathrm{d}x$$

and, for all $\mathbf{d} \in (\mathbf{0}, \boldsymbol{\infty})$, *let* $\mathcal{Q}_\mathbf{d}$ *be the operator* $\mathcal{Q}$ *with* $(A_i)_{i\in[N]}$ *replaced by* $(d_iA_i)_{i\in[N]}$.

Then

$$\lim_{\min_{i\in[N]} d_i \to +\infty} \lambda_{1,\mathrm{per}}(\mathcal{Q}_\mathbf{d}) = \lambda_{1,\mathrm{per}}\left(\partial_t - \langle\mathbf{L}\rangle\right).$$

Proof. Let $\mathbf{d} \gg \mathbf{0}$ and $\mathbf{u}_\mathbf{d}$ be the periodic principal eigenfunction associated with $\lambda_1'(\mathcal{Q}_\mathbf{d})$ and normalized by

$$\frac{1}{T|[0,L]|}\int_{\overline{\Omega_{\mathrm{per}}}} |\mathbf{u}_\mathbf{d}|^2 = 1.$$

Multiplying $(\mathcal{Q}_\mathbf{d}\mathbf{u}_\mathbf{d})_i - \lambda_1'(\mathcal{Q}_\mathbf{d})u_{\mathbf{d},i}$ by $u_{\mathbf{d},i}$ and then integrating over $\overline{\Omega_{\mathrm{per}}}$, we find for each $i \in [N]$:

$$d_i\int_{\overline{\Omega_{\mathrm{per}}}} \nabla u_{\mathbf{d},i}\cdot A_i\nabla u_{\mathbf{d},i} = \int_{\overline{\Omega_{\mathrm{per}}}}\left(\left(\frac{\nabla\cdot q_i}{2} + \lambda_1'(\mathcal{Q}_\mathbf{d})\right)u^2_{\mathbf{d},i} + \sum_{i=1}^N l_{i,j}u_{\mathbf{d},i}u_{\mathbf{d},j}\right).$$

Since $\overline{\mathbf{L}} \geq \mathbf{L} \geq \underline{\mathbf{L}}$,

$$-\lambda_{\mathrm{PF}}(\overline{\mathbf{L}}) \leq \lambda_1'(\mathcal{Q}_\mathbf{d}) \leq -\lambda_{\mathrm{PF}}(\underline{\mathbf{L}}),$$

($\underline{\mathbf{L}}$ might not be irreducible but its Perron–Frobenius eigenvalue is still well-defined by continuous extension and it admits nonnegative nonzero eigenvectors that can be used as sub-solutions), whence there exists a constant $K > 0$ independent of $\mathbf{d}$ such that

$$0 \leq \sum_{i=1}^N \int_{\overline{\Omega_{\mathrm{per}}}} \nabla u_{\mathbf{d},i}\cdot A_i\nabla u_{\mathbf{d},i} \leq \frac{K}{\min_{i\in[N]} d_i}.$$

Let $\langle \mathbf{u_d} \rangle : t \mapsto \frac{1}{|[0,L]|}\int_{[0,L]} \mathbf{u_d}(t,x)\mathrm{d}x$ and $\mathbf{v_d} = \mathbf{u_d} - \langle \mathbf{u_d} \rangle$. By the Poincaré inequality, there exists another constant $K' > 0$ such that

$$\sum_{i=1}^{N} \int_{[0,L]} \nabla u_{\mathbf{d},i} \cdot A_i \nabla u_{\mathbf{d},i} = \sum_{i=1}^{N} \int_{[0,L]} \nabla v_{\mathbf{d},i} \cdot A_i \nabla v_{\mathbf{d},i} \geq K' \int_{[0,L]} |\mathbf{v_d}|^2.$$

Since the mean value in $[0,T]$ of the nonnegative function on the left-hand side converges to 0 as $\min_{i\in[N]} d_i \to +\infty$, so does the mean value in $[0,T]$ of $\int_{[0,L]} |\mathbf{v_d}|^2$, whence $\mathbf{v_d}$ itself converges to $\mathbf{0}$ almost everywhere.

Since, for each $i \in [N]$,

$$\begin{aligned}\int_0^T \langle \mathbf{u_d} \rangle_i &= |[0,L]|^{-1} \int_{\Omega_{\mathrm{per}}} u_{\mathbf{d},i} \\ &\leq (T|[0,L]|)^{-1/2} \left(\int_{\Omega_{\mathrm{per}}} u_{\mathbf{d},i}^2 \right)^{1/2} \\ &\leq (T|[0,L]|)^{-1/2} \left(\int_{\Omega_{\mathrm{per}}} \sum_{i=1}^{N} u_{\mathbf{d},i}^2 \right)^{1/2} = 1,\end{aligned}$$

$\langle \mathbf{u_d} \rangle_i$ is bounded in $\mathcal{L}^1([0,T])$ uniformly with respect to $\mathbf{d}$.

Integrating $\mathcal{Q}_{\mathbf{d}}\mathbf{u_d} = \lambda_1'(\mathcal{Q}_{\mathbf{d}})\mathbf{u_d}$ over $[0,L]$ and dividing by $|[0,L]|$, we find:

$$\begin{aligned}\partial_t \langle \mathbf{u_d} \rangle &= |[0,L]|^{-1} \int_{[0,L]} (\mathrm{diag}(\nabla \cdot q_i)\mathbf{u_d}) + |[0,L]|^{-1} \int_{[0,L]} (\mathbf{L}\mathbf{u_d}) + \lambda_1'(\mathcal{Q}_{\mathbf{d}})\langle \mathbf{u_d} \rangle \\ &= |[0,L]|^{-1} \int_{[0,L]} \mathrm{diag}(\nabla \cdot q_i)\langle \mathbf{u_d} \rangle + |[0,L]|^{-1} \int_{[0,L]} \mathbf{L}\langle \mathbf{u_d} \rangle + \lambda_1'(\mathcal{Q}_{\mathbf{d}})\langle \mathbf{u_d} \rangle \\ &\quad + |[0,L]|^{-1} \int_{[0,L]} ((\mathrm{diag}(\nabla \cdot q_i) + \mathbf{L})\,\mathbf{v_d}) \\ &= \langle \mathbf{L} \rangle \langle \mathbf{u_d} \rangle + \lambda_1'(\mathcal{Q}_{\mathbf{d}})\langle \mathbf{u_d} \rangle + |[0,L]|^{-1} \int_{[0,L]} ((\mathrm{diag}(\nabla \cdot q_i) + \mathbf{L})\,\mathbf{v_d}).\end{aligned}$$

Recalling that the last term converges to 0 in $\mathcal{L}^1([0,T])$, we deduce that each component of $\partial_t \langle \mathbf{u_d} \rangle$ is bounded in $\mathcal{L}^1([0,T])$ uniformly with respect to $\mathbf{d}$. Hence each component of $\langle \mathbf{u_d} \rangle$ is bounded uniformly in $\mathcal{W}^{1,1}([0,T])$, and then via the fundamental theorem of calculus it is bounded uniformly in $\mathcal{L}^\infty([0,T])$, whence it is bounded uniformly in the space of functions of bounded variation $\mathcal{BV}([0,T])$. By compactness of the embedding $\mathcal{W}^{1,1} \hookrightarrow \mathcal{L}^1$, each component of $\langle \mathbf{u_d} \rangle$ converges up to extraction in $\mathcal{L}^1([0,T])$. Using the equation and assuming up to another extraction that $\lambda_1'(\mathcal{Q}_{\mathbf{d}}) \to \lambda \in \mathbb{R}$, so does each component of $\partial_t \langle \mathbf{u_d} \rangle$. Denoting by $\mathbf{u}_\infty$ the limit of $\langle \mathbf{u_d} \rangle$, we deduce that the limit of $\partial_t \langle \mathbf{u_d} \rangle$ is, in distributional sense, the derivative of $\mathbf{u}_\infty$, so that $\mathbf{u}_\infty$ satisfies

$$\partial_t \mathbf{u}_\infty = \langle \mathbf{L} \rangle \mathbf{u}_\infty + \lambda \mathbf{u}_\infty \quad \text{in } (\mathcal{L}^\infty)'(\mathbb{R}).$$

Again by virtue of the fundamental theorem of calculus, each component of $\mathbf{u}_\infty$ is actually in $\mathcal{L}^\infty([0,T])$, and now from the equation it appears that so does each component of $\partial_t \mathbf{u}_\infty$. Therefore $\mathbf{u}_\infty$ is in fact Lipschitz-continuous, and using again the equation it is $\mathcal{C}^1$. Since it is periodic, nonnegative (by almost everywhere convergence, up to another extraction) and nonzero (if on the contrary it was zero, then $\mathbf{u_d}$ would converge to 0 almost everywhere and this would contradict the normalization on $\mathbf{u_d}$) and since the operator $\partial_t - \langle \mathbf{L} \rangle$ is fully coupled in $[0,T]$ by (A_3), we deduce by uniqueness of the classical solution that

$$\lambda = \lambda_{1,\mathrm{per}}(\partial_t - \langle \mathbf{L} \rangle).$$

By uniqueness, the sequence $(\lambda_1'(\mathcal{Q}_\mathbf{d}))_{\mathbf{d}\gg\mathbf{0}}$ has a unique closure point and thus:

$$\lim_{\min_{i\in[N]} d_i \to +\infty} \lambda_1'(\mathcal{Q}_\mathbf{d}) = \lambda_{1,\mathrm{per}}(\partial_t - \langle \mathbf{L} \rangle).$$

□

Remark 3.8. Contrarily to what was claimed by Nadin in [54], the large diffusion limit of the family $(\lambda_{1,z})_{z\in\mathbb{R}^n}$ and of the generalized principal eigenvalue λ_1 cannot be directly deduced from the above proof. Indeed, the large parameter $\mathbf{d}$ appears in the zeroth order term of the operator $\mathcal{Q}_z$ and makes the eigenvalue $\lambda_{1,z}$ blow-up to $-\infty$ as $\min d_i \to +\infty$.

Remark 3.9. Contrarily to the scalar setting [54] or the special cases where $\partial_t - \mathbf{L}(x)$ admits a space-time homogeneous periodic principal eigenfunction, it is in general false that for systems without diffusion and advection, spatial average and periodic principal eigenvalue commute, namely

$$\lambda_{1,\mathrm{per}}(\partial_t - \langle \mathbf{L} \rangle) \neq \langle x \mapsto \lambda_{1,\mathrm{per}}(\partial_t - \mathbf{L}(x)) \rangle.$$

In fact, even the inequality

$$\lambda_{1,\mathrm{per}}(\partial_t - \langle \mathbf{L} \rangle) \geq \min_{x\in[0,L]} \lambda_{1,\mathrm{per}}(\partial_t - \mathbf{L}(x))$$

is false in general, as shown by the following very simple time-homogeneous one-dimensional counter-example

$$\mathbf{L} : (t,x) \mapsto \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{if } x \in [0, L_1/2] + L_1\mathbb{Z}, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{if } x \in [L_1/2, L_1] + L_1\mathbb{Z}. \end{cases}$$

In a time homogeneous setting,

$$\lambda_{1,\mathrm{per}}(\partial_t - \langle \mathbf{L} \rangle) = -\lambda_{\mathrm{PF}}(\langle \mathbf{L} \rangle), \quad \min_{x\in[0,L]} \lambda_{1,\mathrm{per}}(\partial_t - \mathbf{L}(x)) = -\max_{x\in[0,L]} \lambda_{\mathrm{PF}}(\mathbf{L}(x)).$$

With the counter-example above, these two quantities turn out to be respectively $-\frac{3}{2}$ and -1: the averaged matrix has a larger Perron–Frobenius eigenvalue than the matrix at any point in space. In other words, considering for instance the operator $\partial_t - d\Delta - \mathbf{L}$, the limit $d \to 0$ of the periodic principal eigenvalue is larger than the limit $d \to +\infty$. This is in sharp contrast with the variational formula of Theorem 1.12, which indicates a nondecreasing dependence on d but does not apply here due to the asymmetry of $\mathbf{L}$. Of course $\mathbf{L}$ in this counter-example is not continuous and therefore does not satisfy (A_4); however, any smooth sufficiently precise approximation of $\mathbf{L}$ will give the same conclusion, by continuity of the Perron–Frobenius eigenvalue. As a side result, this counter-example also shows that the variational formula of Theorem 1.12 does not hold if only $\langle \mathbf{L} \rangle$ is symmetric, namely if the pointwise symmetry assumption is replaced by an assumption of symmetry on average.

Now we turn to the proof of Theorem 1.9. Denoting by $\boldsymbol{\mathcal{Q}}_\omega$ the operator $\boldsymbol{\mathcal{Q}}$ with ∂_t replaced by $\omega\partial_t$, we first prove the small frequency limit $\omega \to 0$ in Proposition 3.18, then the high frequency one $\omega \to +\infty$ in Proposition 3.19.

Proposition 3.18. *For all $z \in \mathbb{R}^n$,*

$$\lim_{\substack{\omega>0\\ \omega\to 0}} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_\omega) = \frac{1}{T}\int_0^T \lambda_{1,z}\left(-\operatorname{diag}(\nabla\cdot(A_i(t)\nabla) - q_i(t)\cdot\nabla) - \mathbf{L}(t)\right)\mathrm{d}t,$$

$$\lim_{\substack{\omega>0\\ \omega\to 0}} \lambda_1(\boldsymbol{\mathcal{Q}}_\omega) = \frac{1}{T}\int_0^T \lambda_1\left(-\operatorname{diag}(\nabla\cdot(A_i(t)\nabla) - q_i(t)\cdot\nabla) - \mathbf{L}(t)\right)\mathrm{d}t,$$

where, with a slight abuse of notation, for all $t \in [0,T]$,

$$((A_i(t), q_i(t))_{i\in[N]}, \mathbf{L}(t)) : x \mapsto ((A_i(t,x), q_i(t,x))_{i\in[N]}, \mathbf{L}(t,x)).$$

Proof. It is sufficient to prove only the case $z = 0$, since we can deduce the general case for $\lambda_{1,z}$ by applying the result to the operator $\boldsymbol{\mathcal{Q}}_z$, and then we can deduce the result for λ_1 by applying the same argument as in the proof of Proposition 3.13, using Corollary 3.12 and the strict concativity of $z \mapsto \lambda_{1,z}$.

The proof requires two steps.

Step 1: the pointwise irreducibility of $\mathbf{L}$ can be assumed without loss of generality. Assume the limit has been proved provided $\mathbf{L}(t,x)$ is irreducible at all $(t,x) \in \overline{\Omega_{\text{per}}}$.

Define

$$\mathbf{L} : s \in [0,+\infty) \mapsto \mathbf{L} + (\mathrm{e}^s - 1)\mathbf{1}_{N\times N} - (\mathrm{e}^s - 1)\mathbf{I}.$$

Obviously, $\mathbf{L}(0) = \mathbf{L}$ and, for all $s \in (0,+\infty)$, $\mathbf{L}(s,t,x)$ is irreducible at all $(t,x) \in \overline{\Omega_{\text{per}}}$. Moreover, by virtue of Propositions 3.5, 3.4 and 3.13, the periodic principal eigenvalue $\lambda_1'(\omega,s)$ associated with the operator

$$\boldsymbol{\mathcal{Q}}_{\omega,s} = \omega\partial_t - \operatorname{diag}(\nabla\cdot(A_i\nabla) - q_i\cdot\nabla) - \mathbf{L}(s)$$

is, as a function of s, continuous in $[0,+\infty)$, decreasing in $[0,+\infty)$, strictly concave in $[0,+\infty)$.

Let

$$K = \inf_{\omega\in(0,1]} \lim_{\substack{s<1\\ s\to 1}} \frac{\lambda(s,\omega) - \lambda(1,\omega)}{s-1}.$$

By concavity, the one-sided derivatives of $s \mapsto \lambda_1'(\omega,s)$ are well-defined at any $s \in [0,1]$, for all $\omega \in (0,1]$. Thus $K \in (-\infty,0)$ and, by monotonicity of $s \mapsto \lambda_1'(\omega,s)$, all functions $s \in [0,1] \mapsto \lambda_1'(\omega,s)$ are $|K|$-Lipschitz-continuous. Therefore the family $(s \in [0,1] \mapsto \lambda_1'(\omega,s))_{\omega\in(0,1]}$ is equicontinuous. By virtue of the Arzelà–Ascoli theorem, it is relatively compact in $\mathcal{C}([0,1])$.

Let $\lambda \in \mathcal{C}([0,1])$ be any closure point of the family as $s \to 0$. Since we assumed the pointwise convergence

$$\lim_{\substack{\omega>0\\ \omega\to 0}} \lambda_1'(\boldsymbol{\mathcal{Q}}_{\omega,s}) = \frac{1}{T}\int_0^T \lambda_1'\left(-\operatorname{diag}(\nabla\cdot(A_i(t)\nabla) - q_i(t)\cdot\nabla) - \mathbf{L}(s,t)\right)\mathrm{d}t,$$

it follows that λ coincides in $(0,1]$ with

$$s \mapsto \frac{1}{T}\int_0^T \lambda_{1,z}\left(-\operatorname{diag}(\nabla\cdot(A_i(t)\nabla) - q_i(t)\cdot\nabla) - \mathbf{L}(s,t)\right)\mathrm{d}t.$$

By continuity of λ and of the above function (due to Proposition 3.13), they also coincide at $s = 0$. Hence there is a unique closure point for the sequence, whence the whole family $(s \mapsto \lambda_1'(\omega, s))_{\omega\in(0,1]}$ converges uniformly to the above function as $\omega \to 0$. This implies the pointwise convergence at $s = 0$, and this ends the proof of this step. □

In the following step we assume, without loss of generality, that $\mathbf{L}(t, x)$ is indeed irreducible at all $(t, x) \in \overline{\Omega_{\mathrm{per}}}$.

Step 2: the proof in the pointwise irreducible case. Let $\omega > 0$, $\varepsilon > 0$ and

$$\lambda : t \mapsto \lambda_1' \left(-\operatorname{diag}(\nabla\cdot(A_i(t)\nabla) - q_i(t)\cdot\nabla) - \mathbf{L}(t)\right).$$

By virtue of the pointwise irreducibility of $\mathbf{L}$, which implies the full coupling of all operators in the family

$$\left(-\operatorname{diag}(\nabla\cdot(A_i(t)\nabla) - q_i(t)\cdot\nabla) - \mathbf{L}(t)\right)_{t\in[0,T]},$$

the function λ is continuous and periodic.

By positivity of the principal eigenfunction associated with $\lambda(t)$ and by a regularization procedure similar to that of Proposition 3.16, there exists $\mathbf{v}_\varepsilon \in \mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0}, \boldsymbol{\infty}))$ and $K_\varepsilon > 0$ satisfying in $\overline{\Omega_{\mathrm{per}}}$:

$$-\frac{\varepsilon}{2}\mathbf{v}_\varepsilon + \lambda\mathbf{v}_\varepsilon \le -\operatorname{diag}(\nabla\cdot(A_i\nabla) - q_i\cdot\nabla)\mathbf{v}_\varepsilon - \mathbf{L}\mathbf{v}_\varepsilon \le \frac{\varepsilon}{2}\mathbf{v}_\varepsilon + \lambda\mathbf{v}_\varepsilon,$$
$$-K_\varepsilon\mathbf{v}_\varepsilon \le \partial_t\mathbf{v}_\varepsilon \le K_\varepsilon\mathbf{v}_\varepsilon,$$

so that

$$\left(-K_\varepsilon\omega - \frac{\varepsilon}{2} + \lambda\right)\mathbf{v}_\varepsilon \le \boldsymbol{\mathcal{Q}}_\omega\mathbf{v}_\varepsilon \le \left(K_\varepsilon\omega + \frac{\varepsilon}{2} + \lambda\right)\mathbf{v}_\varepsilon.$$

Provided $\omega \le \frac{\varepsilon}{2K_\varepsilon}$,

$$(-\varepsilon + \lambda)\,\mathbf{v}_\varepsilon \le \boldsymbol{\mathcal{Q}}_\omega\mathbf{v}_\varepsilon \le (\varepsilon + \lambda)\,\mathbf{v}_\varepsilon.$$

Let

$$v : t \mapsto \exp\left(\frac{1}{\omega}\left(\frac{t}{T}\int_0^T \lambda(t')\mathrm{d}t' - \int_0^t \lambda(t')\mathrm{d}t'\right)\right)$$

which is positive, periodic and satisfies $\omega v' = \left(\frac{1}{T}\int_0^T \lambda - \lambda\right) v$. Then

$$\left(-\varepsilon + \frac{1}{T}\int_0^T \lambda\right) v\mathbf{v}_\varepsilon \le \omega v'\mathbf{v}_\varepsilon + v\boldsymbol{\mathcal{Q}}_\omega\mathbf{v}_\varepsilon \le \left(\varepsilon + \frac{1}{T}\int_0^T \lambda\right) v\mathbf{v}_\varepsilon.$$

Since $\omega v'\mathbf{v}_\varepsilon + v\boldsymbol{\mathcal{Q}}_\omega\mathbf{v}_\varepsilon = \boldsymbol{\mathcal{Q}}_\omega(v\mathbf{v}_\varepsilon)$, this shows that $v\mathbf{v}_\varepsilon$ can be used both as a super-solution and as a sub-solution to derive the following inequalities:

$$\frac{1}{T}\int_0^T \lambda(t)\mathrm{d}t - \varepsilon \le \lambda_1'(\boldsymbol{\mathcal{Q}}_\omega) \le \frac{1}{T}\int_0^T \lambda(t)\mathrm{d}t + \varepsilon.$$

This ends the proof. □

□

Remark 3.10. Proposition 3.18 turns out to be surprisingly difficult to prove directly, *i.e.* without resorting to pointwise irreducible approximations of $\mathbf{L}$.

The proof that we used in the pointwise irreducible case, which is a direct adaptation of the proof in the scalar case [48], cannot be applied to the general case, since it uses the pointwise positivity of the mapping which associates with (t, x) the value at x of the principal eigenfunction of the elliptic problem at t. This property is not satisfied in general. In fact, the following simple counter-example shows that the aforementioned mapping might have components that oscillate between

0 and a positive value and also components that are identically zero, despite the full coupling of the parabolic problem. Consider the matrix $\mathbf{L}$ defined in $\overline{\Omega_{\text{per}}}$ as follows:

$$\mathbf{L}(t,x) = \begin{cases} \sin\left(\frac{2\pi}{T}t\right)\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \text{if } t \in [0, T/2], \\ -\sin\left(\frac{2\pi}{T}t\right)\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \text{if } t \in [T/2, T]. \end{cases}$$

It is indeed Hölder-continuous (this is the only role of the sinusoidal prefactor), cooperative and fully coupled, and its Perron–Frobenius eigenvectors are the periodic principal eigenfunctions of the operator, say, $-\Delta - \mathbf{L}(t)$. The Perron–Frobenius eigenvector is, up to a multiplicative constant, $(1,0,0)^{\mathrm{T}}$ in $(0,T/2)$ and $(0,0,1)^{\mathrm{T}}$ in $(T/2,T)$.

Another possible strategy of proof would be a time rescaling and a study of the limiting parabolic problem, as in the proof of the vanishing viscosity limit in the scalar case [54]. However, the idea of this strategy is to identify the limiting problem as the principal elliptic problem at the time around which the rescaling was performed, and this identification uses the fact that any nonnegative nonzero eigenfunction of the elliptic operator is a principal eigenfunction. Again, this is true if $\mathbf{L}$ is pointwise irreducible but not in general, as shown by the following counter-example:

$$\mathbf{L}(t,x) = \begin{cases} \sin\left(\frac{3\pi}{T}t\right)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } t \in [0, T/3], \\ -\sin\left(\frac{3\pi}{T}t\right)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t \in [T/3, 2T/3], \\ \sin\left(\frac{3\pi}{T}t\right)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{if } t \in [2T/3, T]. \end{cases}$$

A third possibility would be to try to adapt the proof of the vanishing viscosity limit in the present paper, *cf.* Lemmas 3.16 and 3.15. However the sub-solution there was localized in space. The only nonnegative sub-solution satisfying $v' \leq Cv$ and $v = 0$ at both ends of a time interval is zero. Moreover, any localized nonzero sub-solution would likely fail to capture correctly the expected limit, which is an average in time and therefore requires global knowledge.

Next we prove the limit $\omega \to +\infty$.

Proposition 3.19. *For all* $z \in \mathbb{R}^n$*,*

$$\lim_{\omega\to+\infty} \lambda_{1,z}(\boldsymbol{\mathcal{Q}}_\omega) = \lambda_{1,z}\left(-\operatorname{diag}(\nabla\cdot(\hat{A}_i\nabla) - \hat{q}_i\cdot\nabla) - \hat{\mathbf{L}}\right),$$

$$\lim_{\omega\to+\infty} \lambda_1(\boldsymbol{\mathcal{Q}}_\omega) = \lambda_1\left(-\operatorname{diag}(\nabla\cdot(\hat{A}_i\nabla) - \hat{q}_i\cdot\nabla) - \hat{\mathbf{L}}\right),$$

where

$$\left((\hat{A}_i, \hat{q}_i)_{i\in[N]}, \hat{\mathbf{L}}\right) : x \mapsto \frac{1}{T}\int_0^T \left((A_i, q_i)_{i\in[N]}, \mathbf{L}\right)(t,x)\mathrm{d}t.$$

Proof. Similarly to the proof of Proposition 3.18, it is sufficient to prove only the case $z = 0$.

Since $\overline{\mathbf{L}} \geq \mathbf{L} \geq \underline{\mathbf{L}}$,

$$-\lambda_{\mathrm{PF}}(\overline{\mathbf{L}}) \leq \lambda_1'(\boldsymbol{\mathcal{Q}}_\omega) \leq -\lambda_{\mathrm{PF}}(\underline{\mathbf{L}}).$$

Hence there exists a sequence $(\omega_k)_{k\in\mathbb{N}}$ and $\lambda_\infty \in \mathbb{R}$ such that, as $k \to +\infty$, $\omega_k \to +\infty$ and $\lambda_k = \lambda_1'(\boldsymbol{\mathcal{Q}}_{\omega_k}) \to \lambda_\infty$.

Let $\mathbf{u}_k \in \mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, (\mathbf{0},\boldsymbol{\infty}))$ be the unique generalized principal eigenfunction associated with λ_k satisfying the normalization $\fint_{\overline{\Omega_{\mathrm{per}}}} |\mathbf{u}_k|^2 = 1$.

Multiplying $(\boldsymbol{\mathcal{Q}}_{\omega_k}\mathbf{u}_k)_i - \lambda_k u_{k,i}$ by $u_{k,i}$, integrating by parts over $\overline{\Omega_{\mathrm{per}}}$, and using the Cauchy–Schwarz inequality $\fint_{\overline{\Omega_{\mathrm{per}}}} u_{k,j}u_{k,i} \leq \|u_{k,j}\|_{\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})}\|u_{k,i}\|_{\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})}$, we obtain the uniform boundedness of $(\nabla u_{k,i})_{k\in\mathbb{N}}$ in $\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})$ for each $i \in [N]$, just as in Nadin [54, Proof of Theorem 3.10].

Multiplying $(\boldsymbol{\mathcal{Q}}_{\omega_k}\mathbf{u}_k)_i - \lambda_k u_{k,i}$ by $\partial_t u_{k,i}$, integrating by parts over $\overline{\Omega_{\mathrm{per}}}$ and using the symmetry of each A_i, we deduce that for each $i \in [N]$,

$$\omega_k \fint_{\overline{\Omega_{\mathrm{per}}}} (\partial_t u_{k,i})^2 = \frac{1}{2}\fint_{\overline{\Omega_{\mathrm{per}}}} \nabla u_{k,i}\cdot(\partial_t A_i \nabla u_{k,i}) - \fint_{\overline{\Omega_{\mathrm{per}}}} (q_i\cdot\nabla u_{k,i})\partial_t u_{k,i} + \sum_{j=1}^N \fint_{\overline{\Omega_{\mathrm{per}}}} l_{i,j}u_{k,j}\partial_t u_{k,i}.$$

By the Cauchy–Schwarz inequality, there exists $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$, that only depends on $\mathcal{L}^\infty$ bounds on $(A_i)_{i\in[N]}$, $(q_i)_{i\in[N]}$ and $\mathbf{L}$, such that, for each $i \in [N]$,

$$\omega_k\|\partial_t u_{k,i}\|^2 \leq A\|\nabla u_{k,i}\|^2 + B\|\partial_t u_{k,i}\|\|\nabla u_{k,i}\| + C\|\partial_t u_{k,i}\|, \tag{16}$$

where $\|\cdot\|$ denotes the norm in $\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})$. Therefore, if $M = \sup_{k\in\mathbb{N}}\|\nabla u_{k,i}\|$, then $M < +\infty$ and $X = \|\partial_t u_{k,i}\|$ satisfies $\omega_k X^2 - (BM + C)X - AM^2 \leq 0$, whence

$$\|\partial_t u_{k,i}\| \leq \frac{1}{2\omega_k}\left(BM + C + \sqrt{(BM+C)^2 + 4AM^2\omega_k}\right).$$

The bound on the right-hand side is uniformly bounded with respect to $k \in \mathbb{N}$ and actually converges to 0 as $k \to +\infty$. Back to the estimate (16), we deduce that $\|\omega_k\partial_t u_{k,i}\|$ is also uniformly bounded with respect to $k \in \mathbb{N}$.

Hence, for each $i \in [N]$, $(u_{k,i})_k$, $(\partial_t u_{k,i})_k$, $(\omega_k\partial_t u_{k,i})_k$ and $(\nabla u_{k,i})_k$ are all uniformly bounded in $\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})$, with $\|\partial_t u_{k,i}\|_{\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})} \to 0$ as well. Therefore, up to extraction of a subsequence, $u_{k,i}$ converges in $\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})$ to a limit $u_{\infty,i}$, $\nabla u_{k,i}$, $\partial_t u_{k,i}$, $\omega_k\partial_t u_{k,i}$ converge weakly in $\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})$ to limits $\nabla u_{\infty,i}$, $\partial_t u_{\infty,i}$, v_i respectively. By weak lower-semicontinuity of the norm in $\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})$, the convergence $\partial_t u_{k,i} \to 0$ occurs in fact in the sense of the strong convergence in $\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})$.

Let $\hat{\mathbf{u}}_k : x \mapsto \frac{1}{T}\int_0^T \mathbf{u}_k(t,x)\mathrm{d}t$ and $\mathbf{v}_k = \mathbf{u}_k - \hat{\mathbf{u}}_k$. By the Poincaré inequality, there exists a constant $K > 0$ such that, for each $i \in [N]$,

$$\int_0^T (\partial_t u_{k,i})^2 = \int_0^T (\partial_t v_{k,i})^2 \geq K\int_0^T v_{k,i}^2.$$

Since the mean value in $[0,L]$ of the nonnegative function on the left-hand side converges to 0, so does the mean value of the right-hand side. Also, since

$$\begin{aligned}\int_{[0,L]} |\hat{u}_{k,i} - u_{\infty,i}| &= \int_{[0,L]} \left| \frac{1}{T}\int_0^T u_{k,i} - \frac{1}{T}\int_0^T u_{\infty,i} \right| \\ &\leq \frac{1}{T}\int_{\overline{\Omega_{\mathrm{per}}}} |u_{k,i} - u_{\infty,i}| \\ &\leq \sqrt{\frac{|[0,L]|}{T}} \left(\int_{\overline{\Omega_{\mathrm{per}}}} (u_{k,i} - u_{\infty,i})^2 \right)^{1/2}\end{aligned}$$

for each $i \in [N]$, $\hat{\mathbf{u}}_k$ converges to $\mathbf{u}_\infty$ in $\mathcal{L}^1([0,L])$. Similarly, for any test function $\varphi \in \mathcal{L}^2_{\mathrm{per}}(\mathbb{R}^n)$,

$$\left| \int_{[0,L]} (\nabla \hat{u}_{k,i} - \nabla u_{\infty,i})\varphi \right| \leq \left| \int_{\overline{\Omega_{\mathrm{per}}}} (\nabla u_{k,i} - \nabla u_{\infty,i})\varphi \right|,$$

so that $\nabla \hat{\mathbf{u}}_k \rightharpoonup \nabla \mathbf{u}_\infty$ in $\mathcal{L}^2_{\mathrm{per}}(\mathbb{R}^n)$.

Integrating for $k \in \mathbb{N}$ $(\mathcal{Q}_{\omega_k}\mathbf{u}_k)_i - \lambda_k u_{k,i}$ in $[0,T]$ and dividing by T, we deduce

$$\begin{aligned} 0 &= \frac{1}{T}\nabla \cdot \left(\int_0^T (A_i \nabla u_{k,i}) \right) - \frac{1}{T}\int_0^T (q_i \cdot \nabla u_{k,i}) + \frac{1}{T}\sum_{j=1}^N \int_0^T l_{i,j} u_{k,j} + \lambda_k \hat{u}_{k,i} \\ &= \nabla \cdot (\hat{A}_i \nabla \hat{u}_{k,i}) - \hat{q}_i \cdot \nabla \hat{u}_{k,i} + \sum_{i=1}^N \hat{l}_{i,j}\hat{u}_{k,j} + \lambda_k \hat{u}_{k,i} \\ &\quad + \frac{1}{T}\nabla \cdot \left(\int_0^T (A_i \nabla v_{k,i}) \right) - \frac{1}{T}\int_0^T (q_i \cdot \nabla v_{k,i}) + \frac{1}{T}\sum_{j=1}^N \int_0^T l_{i,j} v_{k,j}. \end{aligned}$$

Testing against a test function in $\mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n)$ and using the convergence of $(\mathbf{v}_k)_k$ to $\mathbf{0}$ in $\mathcal{L}^2(\overline{\Omega_{\mathrm{per}}})$ as well as the convergence of $(\hat{\mathbf{u}}_k)_k$ to $\mathbf{u}_\infty$ in $\mathcal{L}^1(\overline{\Omega_{\mathrm{per}}})$, we deduce that $x \mapsto \mathbf{u}_\infty(x)$ is a weak solution in the dual of $\mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n)$ of

$$\mathrm{diag}(\nabla \cdot (\hat{A}_i \nabla) - \hat{q}_i \cdot \nabla)\mathbf{u}_\infty + \hat{\mathbf{L}}\mathbf{u}_\infty + \lambda \mathbf{u}_\infty = \mathbf{0}.$$

By density, this remains true with test functions in $\mathcal{H}^1_{\mathrm{per}}(\mathbb{R}^n)$, or in other words $\mathbf{u}_\infty$ is a weak solution on $\mathcal{H}^{-1}_{\mathrm{per}}(\mathbb{R}^n)$. By elliptic regularity [31], $\mathbf{u}_\infty \in \mathcal{H}^1_{\mathrm{per}}(\mathbb{R}^n)$ is in fact a classical solution, in $\mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n)$. Since $\mathbf{u}_\infty$ is nonnegative and satisfies the normalization $\int_{\overline{\Omega_{\mathrm{per}}}} |\mathbf{u}_\infty|^2 = |[0,L]| \int_0^T |\mathbf{u}_\infty|^2 = 1$, it is nonnegative nonzero, and then positive by the maximum principle (the elliptic operator under consideration is fully coupled in $[0,L]$), whence it is a generalized principal eigenfunction associated with $\lambda_1'(-\mathrm{diag}(\nabla\cdot(\hat{A}_i\nabla) - \hat{q}_i\cdot\nabla) - \hat{\mathbf{L}})$. Thus $\lambda = \lambda_1'(-\mathrm{diag}(\nabla\cdot(\hat{A}_i\nabla) - \hat{q}_i\cdot\nabla) - \hat{\mathbf{L}})$. As a conclusion, the limit point of $(\lambda_k)_{k\in\mathbb{N}}$ is unique and therefore the whole sequence converges. □

3.2.2. *Explicit formulas for operators with space or time homogeneity.* Recall the notations $\hat{A}_i$, $\hat{q}_i$, $\hat{\mathbf{L}}$ for the mean values in time, $\langle A_i \rangle$, $\langle q_i \rangle$, $\langle \mathbf{L} \rangle$ for the mean values in space and $\langle \hat{A}_i \rangle$, $\langle \hat{q}_i \rangle$, $\langle \hat{\mathbf{L}} \rangle$ for the mean values in space-time.

Proposition 3.20. *Let $z \in \mathbb{R}^n$. If*

(1) A_1, q_1 and $\mathbf{L}$ do not depend on x,

(2) there exists a constant positive vector $\mathbf{u} \in (\mathbf{0}, \infty)$ *such that* $\mathbf{u}$ *is a Perron–Frobenius eigenvector of* $\mathbf{L}(t)$ *for all* $t \in \mathbb{R}$*,*

(3) either $z = 0$ *or* $(A_1, q_1) = (A_2, q_2) = \cdots = (A_N, q_N)$*,*

then

$$\lambda_{1,z} = -z \cdot \hat{A}_1 z + \hat{q}_1 \cdot z - \lambda_{\mathrm{PF}}(\hat{\mathbf{L}}).$$

Proof. First, writing the equality satisfied by $\mathbf{u}$ and taking the mean value in time, we obtain $\frac{1}{T}\int_0^T \lambda_{\mathrm{PF}}(\mathbf{L}(t))\mathrm{d}t = \lambda_{\mathrm{PF}}(\hat{\mathbf{L}})$. Note that $\hat{\mathbf{L}}$ is irreducible.

Next, let $f : t \mapsto -z \cdot A_1(t)z + q_1(t) \cdot z - \lambda_{\mathrm{PF}}(\mathbf{L}(t))$. By uniqueness of the periodic principal eigenpair of $\boldsymbol{\mathcal{Q}}$, it suffices to verify that the space-independent function

$$(t, x) \mapsto \exp\left(-\int_0^t f(t')\mathrm{d}t' + \frac{t}{T}\int_0^T f\right)\mathbf{u}$$

is a $\mathcal{C}^{1,2}$, periodic, positive eigenfunction of $\boldsymbol{\mathcal{Q}}_z$ associated with the eigenvalue $\frac{1}{T}\int_0^T f(t)\mathrm{d}t$. The continuity of $t \mapsto \lambda_{\mathrm{PF}}(\mathbf{L}(t))$ follows from (A_4) and the continuity of the Perron–Frobenius eigenvalue as function of the entries of the matrix. □

Corollary 3.21. *If*

(1) A_1*,* q_1 *and* $\mathbf{L}$ *do not depend on* x*,*

(2) there exists a constant positive vector $\mathbf{u} \in (\mathbf{0}, \infty)$ *such that* $\mathbf{u}$ *is a Perron–Frobenius eigenvector of* $\mathbf{L}(t)$ *for all* $t \in \mathbb{R}$*,*

then

$$\lambda_1' = -\lambda_{\mathrm{PF}}(\hat{\mathbf{L}}).$$

Furthermore, if $(A_1, q_1) = (A_2, q_2) = \cdots = (A_N, q_N)$*, then* $\lambda_1 = \lambda_1'$ *if and only if* $\hat{q}_1 = 0$*.*

Remark 3.11. Although we do not know if the second condition in the statement is truly optimal, we know that the first condition alone cannot be sufficient. Indeed, simple counter-examples exist.

For instance, consider in dimension $N = 2$ the matrix

$$\mathbf{L} : t \mapsto \begin{pmatrix} 0 & \eta(t) \\ \eta(t - T/2) & 0 \end{pmatrix}$$

where η is the continuous T-periodic function that coincides on $[0, T]$ with $t \mapsto \max\left(\sin\left(\frac{2\pi}{T}t\right), 0\right)$.

Even though $\mathbf{L}(t)$ is actually always reducible, its Perron–Frobenius eigenvalue, understood as the continuous extension of the Perron–Frobenius eigenvalue to essentially nonnegative matrices, is 0 for all $t \in [0, T]$, it is always a geometrically simple eigenvalue and its unit Perron–Frobenius eigenvector is $(1, 0)^{\mathrm{T}}$ in $(0, T/2)$ and $(0, 1)^{\mathrm{T}}$ in $(T/2, T)$. The matrix $\hat{\mathbf{L}}$ is symmetric and admits $\mathbf{1}$ as Perron–Frobenius eigenvector and $1/\pi$ as Perron–Frobenius eigenvalue.

Due to the uniqueness of the periodic principal eigenfunction and the symmetries of $\mathbf{L}$, the periodic principal eigenfunction necessarily has the form $\mathbf{u} : t \mapsto (u(t), u(t - T/2))^{\mathrm{T}}$. Moreover, we can choose to normalize it with $u(0) = 1$. It follows that u satisfies:

$$u(t)\mathrm{e}^{-\lambda_1' t} = 1 + \int_0^t \mathrm{e}^{-\lambda_1' t'} \sin\left(\frac{2\pi}{T}t'\right) u(t' - T/2)\mathrm{d}t' \quad \text{for all } t \in [0, T/2],$$

$$u(t)\mathrm{e}^{-\lambda_1'(t - T/2)} = u(T/2) \quad \text{for all } t \in [T/2, T].$$

Since $u(t'-T/2) = u(0) = 1$ for all $t' \in [0, T/2]$ by periodicity, the first equality is simplified as $u(t) = \mathrm{e}^{\lambda_1' t} + \int_0^t \mathrm{e}^{\lambda_1'(t-t')} \sin\left(\frac{2\pi}{T}t'\right) \mathrm{d}t'$. This gives an expression for $u(T/2)$. Plugging this expression in the second equality and rewriting the periodicity condition $u(T) = u(0)$, we find:

$$\mathrm{e}^{-\lambda_1' T} = 1 + \int_0^{T/2} \mathrm{e}^{-\lambda_1' t} \sin\left(\frac{2\pi}{T}t\right) \mathrm{d}t = 1 + \frac{2\pi T(\mathrm{e}^{-\lambda_1' T/2}+1)}{(\lambda_1' T)^2 + 4\pi^2}.$$

Solving this equality for λ_1' is tedious, however it is easily checked that neither 0 nor $-1/\pi$ are solutions. More precisely, on one hand, the impossibility of $\lambda_1' = 0$ is direct, and on the other hand, assuming $\lambda_1' = -1/\pi$, then the equality reads:

$$\left(\frac{T^2}{\pi^2} + 4\pi^2\right)\mathrm{e}^{T/\pi} = 2\pi T \mathrm{e}^{T/(2\pi)} + \frac{T^2}{\pi^2} + 4\pi^2 + 2\pi T,$$

or else $f(T/\pi) = 0$ where $f : \tau \mapsto (\tau^2 + 4\pi^2)\mathrm{e}^{\tau} - 2\pi^2\tau\mathrm{e}^{\tau/2} - \tau^2 - 4\pi^2 - 2\pi^2\tau$. The function f has an obvious zero at $\tau = 0$, diverges to $+\infty$ as $\tau \to +\infty$, and its derivative is, at any $\tau \geq 0$, $f'(\tau) = (\tau^2+2\tau+4\pi^2)\mathrm{e}^{\tau} - \pi^2(\tau+2)\mathrm{e}^{\tau/2} - 2\tau - 2\pi^2$. Thus $\tau = 0$ is a critical point of f, and provided there is no critical point in $(0, +\infty)$, then f is increasing in $(0, +\infty)$ and therefore $f(T/\pi) = 0$ with $T > 0$ is a contradiction. Differentiating twice more, it appears that f' is strictly convex in $(0, +\infty)$ with $f''(0) = 2\pi^2 > 0$. Hence $f'' > 0$ in $(0, +\infty)$, whence $f' > 0$ in $(0, +\infty)$, whence there is no critical point of f in $(0, +\infty)$ indeed.

Therefore this counter-example shows that in general, $\lambda_{1,\mathrm{per}}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{L}\right)$ coincides neither with $-\frac{1}{T}\int_0^T \lambda_{\mathrm{PF}}(\mathbf{L})$ nor with $-\lambda_{\mathrm{PF}}(\hat{\mathbf{L}})$.

Remark 3.12. In the preceding counter-example, we can show that $\lambda_1(\partial_t - \Delta - \mathbf{L}) = \lambda_1'(\partial_t - \Delta - \mathbf{L}) < 0$. Indeed, assume by contradiction that $\lambda_1 > 0$; then by definition, there exist $\varepsilon > 0$ and a time periodic positive super-solution $\overline{\mathbf{u}}$ satisfying $\overline{\mathbf{u}}' \geq \mathbf{L}\mathbf{u} + \varepsilon\overline{\mathbf{u}}$. By positivity of $\mathbf{L}$, $\overline{\mathbf{u}}$, ε and integration on $[0, T]$, we deduce $\int_0^T \mathbf{L}\overline{\mathbf{u}} = \varepsilon\int_0^T \overline{\mathbf{u}} = \mathbf{0}$, which is obviously impossible. Similarly, if $\lambda_1 = 0$, then $\lambda_1' = 0$ and using the periodic principal eigenfunction $\mathbf{u}$ and the exact same reasoning, we find again a contradiction. Therefore $\lambda_1' < 0$.

Now, consider a diagonal perturbation of $\mathbf{L}$ of the form $\mathbf{L}_\nu = \mathbf{L} - \nu\mathbf{I}$ with $\nu > 0$. Then, clearly, λ_1' is replaced by $\lambda_1' + \nu$ and the pointwise equality $\lambda_{\mathrm{PF}}(\mathbf{L}(t)) = 0$ is replaced by $\lambda_{\mathrm{PF}}(\mathbf{L}_\nu(t)) = \lambda_{\mathrm{PF}}(\mathbf{L}(t) - \nu\mathbf{I}) = -\nu < 0$. Therefore, at each time $t_0 \in [0, T]$, all eigenvalues of $\mathbf{L}_\nu(t_0)$ have negative real parts, yet $\lambda_1 + \nu = \lambda_1' + \nu$ remains negative provided $\nu > 0$ is small enough: although the trajectories of the "frozen in time" dynamical system $\mathbf{u}'(t) = \mathbf{L}_\nu(t_0)\mathbf{u}(t)$ converge exponentially fast to $\mathbf{0}$, the nonnegative nonzero solutions of the nonautonomous dynamical system $\mathbf{u}'(t) = \mathbf{L}_\nu(t)\mathbf{u}(t)$ diverge from $\mathbf{0}$ exponentially fast. Hence the stability properties of the two systems are completely unrelated.

This fact should not surprise readers familiar with nonautonomous dynamical systems, since the existence of such counter-examples, relying strongly upon the non-symmetry, is classical. Yet other readers might not be aware of this quite interesting possibility and this is why we highlight it here.

The following property concerns variational formulas in the self-adjoint case.

Proposition 3.22. *If*

(1) $(A_i)_{i\in[N]}$ *and* $\mathbf{L}$ *do not depend on* t,

(2) $\mathbf{L}(x)$ *is symmetric for all* $x \in \mathbb{R}^n$*,*
(3) $q_1 = q_2 = \cdots = q_N = 0$*,*

then the periodic principal eigenvalue satisfies:

$$\lambda_1' = \min_{\mathbf{u}\in\mathcal{C}^2_{\text{per}}(\mathbb{R}^n,\mathbb{R}^N)\setminus\{\mathbf{0}\}} \frac{\int_{[0,L]}\left(\sum_{i=1}^N (\nabla u_i\cdot A_i\nabla u_i)^2 - \mathbf{u}^{\mathrm{T}}\mathbf{L}\mathbf{u}\right)}{\int_{[0,L]}|\mathbf{u}|^2}.$$

and, for any nonempty bounded smooth open set Ω*, the Dirichlet principal eigenvalue satisfies:*

$$\lambda_1(\Omega) = \min_{\mathbf{u}\in\mathcal{C}^1_0(B_R,\mathbb{R}^N)\setminus\{\mathbf{0}\}} \frac{\int_{B_R}\left(\sum_{i=1}^N (\nabla u_i\cdot A_i\nabla u_i)^2 - \mathbf{u}^{\mathrm{T}}\mathbf{L}\mathbf{u}\right)}{\int_{B_R}|\mathbf{u}|^2}.$$

It is classical and we admit it, at least in the elliptic case; to prove it for the parabolic operator $\boldsymbol{\mathcal{Q}}$, it suffices to remark that, by uniqueness of the periodic principal eigenpair, respectively Dirichlet principal eigenpair, $\lambda_1' = \lambda_{1,\text{per}}(-\boldsymbol{\mathcal{L}})$, respectively $\lambda_1(\Omega) = \lambda_{1,\text{Dir}}(-\boldsymbol{\mathcal{L}},\Omega)$, where $\boldsymbol{\mathcal{L}} = \operatorname{diag}(\nabla\cdot(A_i\nabla)) + \mathbf{L}$ is the underlying self-adjoint elliptic operator.

Note that the diffusion matrices A_i are invertible (due to (A_1), (A_4)), with periodic inverses A_i^{-1}.

Recalling that

$$\boldsymbol{\mathcal{Q}}_z = \boldsymbol{\mathcal{Q}} - \operatorname{diag}(2A_iz\cdot\nabla + z\cdot A_iz + \nabla\cdot(A_iz) - q_i\cdot z),$$

it is clear that $\boldsymbol{\mathcal{Q}}_z$ is self-adjoint if and only if $\mathbf{L}$ is symmetric and $q_i = 2A_iz$ for all $i\in[N]$. This last condition is satisfied for one z if $A_i^{-1}q_i$ is spatio-temporally homogeneous and independent of i, and none otherwise. Therefore, up to changing $(q_i)_{i\in[N]}$ and $\mathbf{L}$ into $(q_i - 2A_iz)_{i\in[N]}$ and $\mathbf{L} + \operatorname{diag}(z\cdot A_iz + \nabla\cdot(A_iz) - q_i\cdot z)$ respectively, we can assume $z=0$ without loss of generality. Therefore, we first focus on the case $z=0$.

Proposition 3.23. *If*

(1) $(A_i)_{i\in[N]}$ *and* $\mathbf{L}$ *do not depend on* t*,*
(2) $\mathbf{L}(x)$ *is symmetric for all* $x\in\mathbb{R}^n$*,*
(3) there exists $Q\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ *such that* $\int_{[0,L]}\nabla Q = 0$ *and*

$$A_1^{-1}q_1 = A_2^{-1}q_2 = \cdots = A_N^{-1}q_N = \nabla Q,$$

then

$$\lambda_1 = \lambda_1' = \min_{\mathbf{u}\in\mathcal{C}^2_{\text{per}}(\mathbb{R}^n,\mathbb{R}^N)\setminus\{\mathbf{0}\}} \frac{\displaystyle\int_{[0,L]}\left(\sum_{i=1}^N (\nabla u_i\cdot A_i\nabla u_i)^2 - \mathbf{u}^{\mathrm{T}}\mathbf{L}_Q\mathbf{u}\right)}{\displaystyle\int_{[0,L]}|\mathbf{u}|^2},$$

where

$$\mathbf{L}_Q = \mathbf{L} + \frac{1}{4}\operatorname{diag}\left(2\nabla\cdot q_i - A_i^{-1}q_i\cdot q_i\right) = \mathbf{L} + \frac{1}{4}\operatorname{diag}(2\nabla\cdot(A_i\nabla Q) - \nabla Q\cdot A_i\nabla Q).$$

Proof. Let $\boldsymbol{\mathcal{L}} = \operatorname{diag}(\nabla\cdot(A_i\nabla) - q_i\cdot\nabla) + \mathbf{L}$.

Step 1: the case $q_1 = q_2 = \cdots = q_N = 0$*, i.e.* Q *constant.* By uniqueness of the periodic principal eigenpair,

$$\lambda_{1,z} = \lambda_{1,\text{per}}(-\mathrm{e}_{-z}\boldsymbol{\mathcal{L}}\mathrm{e}_z) \quad \text{for all } z\in\mathbb{R}^n$$

(in particular, $\lambda_1(\boldsymbol{\mathcal{Q}}) = \lambda_1(-\boldsymbol{\mathcal{L}})$ and $\lambda_1'(\boldsymbol{\mathcal{Q}}) = \lambda_1'(-\boldsymbol{\mathcal{L}})$) and space-time periodic principal eigenfunctions of $\boldsymbol{\mathcal{Q}}_z$ are space periodic principal eigenfunctions of $-\mathrm{e}_{-z}\boldsymbol{\mathcal{L}}\mathrm{e}_z$. Therefore we only have to prove the statement for the elliptic operator $\boldsymbol{\mathcal{L}}$. Also, we already know that $\lambda_1' = \lambda_{1,0} \leq \lambda_1 = \max_{z\in\mathbb{R}^n}\lambda_{1,z}$.

Following Berestycki–Rossi [14], we consider an even function $\chi \in \mathcal{C}^\infty(\mathbb{R},[0,1])$ such that $\chi = 0$ in $\mathbb{R}\backslash[-1,1]$ and $\chi(0) = 1$. Next, we construct a family of radial smooth cut-off functions $(\chi_R)_{R>1}$ such that, for each $R > 1$, $\chi_R = 1$ in B_{R-1} and $\chi_R(x) = \chi(|x| - (R-1))$ if $x \in \mathbb{R}^n\backslash B_{R-1}$, where B_{R-1} is the open ball of center 0 and radius $R-1$. By construction, the family $\left(\|\chi_R\|_{\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})}\right)_{R>1}$ is bounded.

Let $R > 1$ and denote μ_R the Dirichlet principal eigenvalue $\lambda_{1,\mathrm{Dir}}(-\boldsymbol{\mathcal{L}}, B_R)$. By Proposition 3.22,

$$\mu_R = \min_{\mathbf{u}\in\mathcal{C}_0^1(B_R,\mathbb{R}^N)\backslash\{\mathbf{0}\}} \frac{\int_{B_R}\left(\sum_{i=1}^N (\nabla u_i \cdot A_i\nabla u_i)^2 - \mathbf{u}^{\mathrm{T}}\mathbf{L}\mathbf{u}\right)}{\int_{B_R}|\mathbf{u}|^2}.$$

Taking $\chi_R\mathbf{u}_0$ as test function (recall that $\mathbf{u}_0$ is a positive periodic principal eigenfunction of $\boldsymbol{\mathcal{L}}$) and using the equality $-\boldsymbol{\mathcal{L}}\mathbf{u}_0 = \lambda_{1,0}\mathbf{u}_0$ satisfied pointwise in B_{R-1}, we find

$$\begin{aligned}
\mu_R &\leq \lambda_{1,0} - \frac{\lambda_{1,0}\int_{B_R\backslash B_{R-1}}\chi_R^2|\mathbf{u}_0|^2 + \int_{B_R\backslash B_{R-1}}\chi_R\mathbf{u}_0^{\mathrm{T}}\boldsymbol{\mathcal{L}}(\chi_R\mathbf{u}_0)}{\int_{B_R}\chi_R^2|\mathbf{u}_0|^2} \\
&\leq \lambda_{1,0} + \frac{|\lambda_{1,0}|\int_{B_R\backslash B_{R-1}}\chi_R^2|\mathbf{u}_0|^2 + \int_{B_R\backslash B_{R-1}}|\chi_R\mathbf{u}_0^{\mathrm{T}}\boldsymbol{\mathcal{L}}(\chi_R\mathbf{u}_0)|}{\int_{B_{R-1}}|\mathbf{u}_0|^2} \\
&\leq \lambda_{1,0} + \frac{|\lambda_{1,0}|\|\chi_R\mathbf{u}_0\|^2 + \|\chi_R\mathbf{u}_0\|\|\boldsymbol{\mathcal{L}}(\chi_R\mathbf{u}_0)\|}{\min_{x\in\overline{\Omega_{\mathrm{per}}}}|\mathbf{u}_0(x)|^2}\frac{\int_{B_R\backslash B_{R-1}}1}{\int_{B_{R-1}}1}.
\end{aligned}$$

where the norm $\|\ \|$ is defined as $\|\mathbf{v}\| = \sup_{x\in\mathbb{R}^n}|\mathbf{v}(x)|$ (appropriate for $\mathcal{C}(\mathbb{R}^n,\mathbb{R}^N)$). Thus, from the boundedness of the operator $\boldsymbol{\mathcal{L}} : \mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^N) \to \mathcal{C}(\mathbb{R}^n,\mathbb{R}^N)$ and the boundedness in $\mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^N)$ of the family $(\chi_R\mathbf{u}_0)_{R>1}$, there exists a constant $K > 0$, independent of R, such that

$$\mu_R \leq \lambda_{1,0} + K\frac{R^{n-1}}{(R-1)^n},$$

and, passing to the limit $R \to +\infty$, we deduce finally $\lambda_1 \leq \lambda_{1,0}$ (the proof of the convergence of Dirichlet principal eigenvalues in growing balls to the periodic principal eigenvalue for the elliptic operator $-\boldsymbol{\mathcal{L}}$ is done exactly as in the parabolic case, *cf.* Proposition 3.2).

The variational formula in this case is a mere consequence of Proposition 3.22. □

Step 2: the general case. From now on, for all $i \in [N]$, $q_i = A_i\nabla Q$ with $\int_{[0,L]}\nabla Q = 0$. Following Berestycki–Hamel–Rossi [12], the idea is to change variables to reduce this case to the previous one.

Preliminarily, we check that $Q \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ is necessarily space periodic. Fix $\alpha \in [n]$. The function $x \mapsto Q(x+L_\alpha e_\alpha) - Q(x)$, where $e_\alpha = (\delta_{\alpha\beta})_{\beta\in[n]}$, is constant, since

$$\nabla(Q(x+L_\alpha e_\alpha) - Q(x)) = A_1^{-1}(x+L_\alpha e_\alpha)q_1(x+L_\alpha e_\alpha) - A_1^{-1}(x)q_1(x) = 0.$$

Then

$$\begin{aligned} Q(L_\alpha e_\alpha) - Q(0) &= (|[0,L]|)^{-1} \int_{[0,L]} Q(x + L_\alpha e_\alpha) - Q(x)\mathrm{d}x \\ &= (|[0,L]|)^{-1} \int_0^{L_\alpha} \int_{[0,L]} \frac{\partial Q}{\partial x_\alpha}(x + s e_\alpha)\mathrm{d}x\mathrm{d}s \\ &= 0 \end{aligned}$$

Hence Q is indeed periodic with respect to x_α, and then with respect to x.

Then, introducing the transformation $\mathbf{v}(x) = \exp(Q(x)/2)\mathbf{u}(x))$ and following [12], we get:

$$\begin{aligned} -(\boldsymbol{\mathcal{L}}\mathbf{v})_i &= \mathrm{e}^{Q/2}\left[-(\boldsymbol{\mathcal{L}}\mathbf{u})_i - \frac{1}{2}\left(u_i \nabla \cdot (A_i \nabla Q) + 2\nabla u_i \cdot (A_i \nabla Q) + \frac{1}{2} u_i \nabla Q \cdot (A_i \nabla Q) - u_i q_i \cdot \nabla Q\right)\right] \\ &= \mathrm{e}^{Q/2}\left[-(\boldsymbol{\mathcal{L}}\mathbf{u})_i - \frac{1}{2}\left(u_i \nabla \cdot q_i + 2\nabla u_i \cdot q_i - \frac{1}{2} u_i \nabla Q \cdot q_i\right)\right] \\ &= \mathrm{e}^{Q/2}\left[-\nabla \cdot (A_i \nabla u_i) - (\mathbf{L}\mathbf{u})_i - \frac{1}{2}\left(\nabla \cdot q_i - \frac{1}{2}\nabla Q \cdot q_i\right) u_i\right]. \end{aligned}$$

Therefore $\mathbf{v}$ is an eigenfunction of $-\boldsymbol{\mathcal{L}}$ if and only if $\mathbf{u}$ is an eigenfunction of the new periodic elliptic operator:

$$-\boldsymbol{\mathcal{L}}_Q = -\operatorname{diag}\left(\nabla \cdot (A_i \nabla)\right) - \mathbf{L}_Q,$$

and the periodic and Dirichlet principal eigenvalues coincide: for instance, with $\mathbf{u}$ a periodic principal eigenfunction of $-\boldsymbol{\mathcal{L}}_Q$, $\mathbf{v}$ satisfies $-\boldsymbol{\mathcal{L}}\mathbf{v} = \lambda_1'(-\boldsymbol{\mathcal{L}}_Q)\mathbf{v}$, and since $\mathbf{v} = \mathrm{e}^{Q/2}\mathbf{u}$ is periodic, it is then (by uniqueness) a periodic principal eigenfunction of $-\boldsymbol{\mathcal{L}}$, whence $\lambda_1'(-\boldsymbol{\mathcal{L}}) = \lambda_1'(-\boldsymbol{\mathcal{L}}_Q)$. Using the characterization of λ_1 as the limit of Dirichlet eigenvalues in growing balls and the variational formula for the operator $-\boldsymbol{\mathcal{L}}_Q$, the proof is ended. □

□

Remark 3.13. The symmetry assumption on $\mathbf{L}$ is crucial, both for the equality between λ_1 and λ_1' (as explained above in Remark 3.5) and for the equality between λ_1' and the minimized integral.

Denote

$$R = \min_{\mathbf{u} \in \mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n, \mathbb{R}^N) \setminus \{\mathbf{0}\}} \frac{\int_{[0,L]} \left(\sum_{i=1}^N |\nabla u_i|^2 - \mathbf{u}^{\mathrm{T}}\mathbf{L}\mathbf{u}\right)}{\int_{[0,L]} |\mathbf{u}|^2},$$

which is the quotient appearing in the variational formula in the special case $q_i = 0$ and $A_i = \mathrm{Id}$ for each $i \in [N]$.

It is well-known that for a general non-symmetric square matrix, the maximum of the Rayleigh quotient needs not coincide with the dominant eigenvalue. More precisely, the maximum of the Rayleigh quotient of a matrix $\mathbf{L}$ coincides with the dominant eigenvalue of the symmetric part $\frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}})$. Similarly, R is the periodic principal eigenvalue of the symmetrized operator $-\Delta - \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}})$.

Therefore, using a constant irreducible non-symmetric matrix

$$\mathbf{L} = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} \quad \text{with } \varepsilon > 0,$$

we obtain a counter-example of the equality between $\lambda_1' = -1-\sqrt{\varepsilon}$ and $R = -(3+\varepsilon)/2$.

Subsequently, replacing $(q_i)_{i\in[N]}$ by $(q_i - 2A_i z)_{i\in[N]}$ and $\mathbf{L}$ by $\mathbf{L}+\operatorname{diag}(z\cdot A_i z + \nabla\cdot(A_i z) - q_i\cdot z)$, we obtain the following corollary.

Corollary 3.24. *If*

(1) $(A_i)_{i\in[N]}$ *and* $\mathbf{L}$ *do not depend on* t*,*
(2) $\mathbf{L}(x)$ *is symmetric for all* $x\in\mathbb{R}^n$*,*
(3) there exists $z\in\mathbb{R}^n$ *and* $Q\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ *such that* $\int_{[0,L]}\nabla Q = 0$ *and*

$$A_1^{-1}q_1 = A_2^{-1}q_2 = \cdots = A_N^{-1}q_N = 2z+\nabla Q,$$

then

$$\lambda_1 = \lambda_{1,z} = \min_{\mathbf{u}\in\mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n,\mathbb{R}^N)\setminus\{\mathbf{0}\}} \frac{\displaystyle\int_{[0,L]}\left(\sum_{i=1}^N (\nabla u_i\cdot A_i\nabla u_i)^2 - \mathbf{u}^{\mathrm{T}}\mathbf{L}_{Q,z}\mathbf{u}\right)}{\displaystyle\int_{[0,L]}|\mathbf{u}|^2},$$

where

$$\mathbf{L}_{Q,z} = \mathbf{L}_Q + \operatorname{diag}\left(\nabla\cdot(A_i z) - z\cdot A_i(z+\nabla Q)\right)$$

and $\mathbf{L}_Q$ *is defined as in the statement of Proposition 3.23.*

With no symmetry assumption on $\mathbf{L}$ and more general advection terms, we can still compare $\lambda_{1,z}$ with the variational formula.

Corollary 3.25. *Let* $z\in\mathbb{R}^n$*. If*

(1) $(A_i,q_i)_{i\in[N]}$ *and* $\mathbf{L}$ *do not depend on* t*,*
(2) for all $i\in[N]$*,* $\nabla\cdot(q_i-2A_i z)\leq 0$*,*
(3) for all $i\in[N]$*,* $q_i\cdot z\geq 0$*,*

then

$$\lambda_{1,z} \geq \lambda_{1,z}(\partial_t - \operatorname{diag}(\nabla\cdot(A_i\nabla)) - \frac{1}{2}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})).$$

Proof. By time homogeneity of the coefficients, the periodic principal eigenfunction of the parabolic operator $\boldsymbol{\mathcal{Q}}_z$ is time homogeneous. Taking the scalar product between the periodic principal eigenfunction $\mathbf{u}_z\in\mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))$ associated with $\lambda_{1,z}$ and $\boldsymbol{\mathcal{Q}}_z\mathbf{u}_z$ and then integrating in $[0,L]$, we get immediately:

$$\begin{aligned}\lambda_{1,z}\int_{[0,L]}|\mathbf{u}_z|^2 &= \sum_{i=1}^N\int_{[0,L]}(\nabla u_{z,i}\cdot A_i\nabla u_{z,i})^2 - \int_{[0,L]}\mathbf{u}_z^{\mathrm{T}}\mathbf{L}\mathbf{u}_z \\ &\quad - \sum_{i=1}^N\int_{[0,L]}\left(\frac{1}{2}\nabla\cdot(q_i-2A_i z) + (z\cdot A_i z) + \nabla\cdot(A_i z) - (q_i\cdot z)\right)u_{z,i}^2.\end{aligned}$$

The conclusion follows from $\mathbf{u}_z^{\mathrm{T}}\mathbf{L}\mathbf{u}_z = \mathbf{u}_z^{\mathrm{T}}\frac{1}{2}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})\mathbf{u}_z$, the sign assumptions on $\nabla\cdot(q_i-2A_i z)$ and $q_i\cdot z$ and the variational formula for $\lambda_{1,z}(\partial_t - \operatorname{diag}(\nabla\cdot(A_i\nabla)) - \frac{1}{2}(\mathbf{L}+\mathbf{L}^{\mathrm{T}}))$. □

Remark 3.14. In the nonnegative square matrix context, the inequality $\lambda_{\mathrm{PF}}(\mathbf{L}) \leq \lambda_{\mathrm{PF}}\left(\frac{1}{2}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})\right)$ is a consequence of a theorem by Levinger which states that $t\in[0,1]\mapsto\lambda_{\mathrm{PF}}\left(t\mathbf{L}+(1-t)\mathbf{L}^{\mathrm{T}}\right)$ is nondecreasing in $[0,1/2]$, nonincreasing in

$[1/2, 1]$, and that the function is constant if and only if the unit Perron–Frobenius eigenvectors of $\mathbf{L}$ and $\mathbf{L}^{\mathrm{T}}$ coincide. There are many works on this theorem and on its extension to Banach spaces. We refer for instance to the recent paper of Altenberg–Cohen [4] and references therein.

Remark 3.15. The second and third assumptions are obviously satisfied if q_i is divergence-free and $z = 0$, but it is also interesting to consider for instance the case $z \neq 0$ with shear flows $q_i : x \mapsto (\alpha_i(x_2, \dots, x_n), 0, \dots, 0)^{\mathrm{T}}$ with α_i of constant sign. In biological applications (climate change at constant speed towards the north, fish populations living in a river, etc.) or when studying planar spreading, such shear flows appear naturally.

Taking as a test function in the variational quotient any constant eigenvector of $\mathbf{L}_{Q,z}(x)$, we obtain that the mean value of the corresponding eigenvalue is smaller than or equal to $-\lambda_1 = -\lambda_{1,z}$. In particular, noting that a constant Perron–Frobenius eigenvector implies $\frac{1}{|[0,L]|}\displaystyle\int_{[0,L]} \lambda_{\mathrm{PF}}(\mathbf{L}_{Q,z}(x))\mathrm{d}x = \lambda_{\mathrm{PF}}(\langle \mathbf{L}_{Q,z}\rangle)$, the following corollary holds.

Corollary 3.26. *If*

(1) $(A_i)_{i\in[N]}$ *and* $\mathbf{L}$ *do not depend on* t*,*
(2) $\mathbf{L}(x)$ *is symmetric for all* $x \in \mathbb{R}^n$*,*
(3) there exists $z \in \mathbb{R}^n$ *and* $Q \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ *such that* $\int_{[0,L]} \nabla Q = 0$ *and*

$$A_1^{-1}q_1 = A_2^{-1}q_2 = \cdots = A_N^{-1}q_N = 2z + \nabla Q,$$

(4) there exists a constant positive vector $\mathbf{u} \in (\mathbf{0}, \boldsymbol{\infty})$ *such that* $\mathbf{u}$ *is a Perron–Frobenius eigenvector of* $\mathbf{L}_{Q,z}(x)$ *for all* $x \in \mathbb{R}^n$*,*

then

$$\lambda_1 = \lambda_{1,z} \leq -\lambda_{\mathrm{PF}}(\langle \mathbf{L}_{Q,z}\rangle).$$

Remark 3.16. Again, denote

$$R = \min_{\mathbf{u}\in\mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n,\mathbb{R}^N)\backslash\{\mathbf{0}\}} \frac{\int_{[0,L]}\left(\sum_{i=1}^N |\nabla u_i|^2 - \mathbf{u}^{\mathrm{T}}\mathbf{L}\mathbf{u}\right)}{\int_{[0,L]}|\mathbf{u}|^2}.$$

Let us construct a counter-example where all the conditions of the statement are satisfied but where, due to heterogeneities in $\mathbf{L}(x)$,

$$R < -\lambda_{\mathrm{PF}}(\langle \mathbf{L}\rangle) = -\frac{1}{|[0,L]|}\int_{[0,L]} \lambda_{\mathrm{PF}}\left(\mathbf{L}(x)\right)\mathrm{d}x.$$

(The existence of such counter-examples in the scalar setting is well-known, we provide a vector counter-example just for the sake of completeness.)

In a spirit similar to that of Remark 3.11, we set

$$\mathbf{L} : x \mapsto \begin{pmatrix} 1 & \eta(x_1) \\ \eta(x_1) & 1 \end{pmatrix}$$

where η is the continuous L_1-periodic function that coincides on $[0, L_1]$ with $x_1 \mapsto \max(L_1/4 - |x_1 - L_1/4|, 0)$.

For all $x \in [0, L]$, $\lambda_{\mathrm{PF}}(\mathbf{L}(x)) = 1 + \eta(x_1)$ with constant eigenvector $\mathbf{1}$, whence

$$R \leq -\frac{1}{L_1}\int_0^{L_1} (1 + \eta(x_1))\,\mathrm{d}x_1 = -1 - \frac{L_1}{16}.$$

Considering test functions of the form $\mathbf{u}(x) = u(x_1)\mathbf{1}$, we get

$$R \leq \min_{u \in \mathcal{C}^2_{\mathrm{per}}(\mathbb{R}), \|u\|_{\mathcal{L}^2}=1} \int_0^{L_1} \left(|u'(x_1)|^2 - (1+\eta(x_1))\, u(x_1)^2 \right) \mathrm{d}x_1$$

Testing against (a $\mathcal{C}^2$ approximation of) $u : x_1 \mapsto \frac{8\sqrt{3}}{L_1^{3/2}}\eta(x_1)$, we find:

$$\begin{aligned} R &\leq \frac{96}{L_1^3} \int_0^{L_1} \left(1 - \eta(x_1)^2 - \eta(x_1)^3 \right) \mathrm{d}x_1 \\ &= -1 + \frac{96}{L_1^2} - \frac{3L_1}{16} \\ &< -1 - \frac{L_1}{16} \quad \text{if } L_1 > 768^{1/3}. \end{aligned}$$

Remark 3.17. More interestingly, in the vector setting, the inequality

$$-R \geq \frac{1}{|[0,L]|} \int_{[0,L]} \lambda_{\mathrm{PF}}(\mathbf{L})$$

might not be satisfied if the fourth assumption of the statement, regarding the existence of a constant positive eigenvector, fails.

To verify this claim, we consider the counter-example $\mathcal{Q}\mathbf{u} = \partial_t \mathbf{u} - \Delta \mathbf{u} - \mathbf{L}\mathbf{u}$, where

$$\mathbf{L} : x \mapsto \frac{1}{1+\eta(x_1) + \eta\left(x_1 - \frac{L}{3}\right) + \eta\left(x_1 - \frac{2L}{3}\right)} \begin{pmatrix} 1+\eta(x_1 - L/3) & \eta(x_1) \\ \eta(x_1) & 1+\eta(x_1 - 2L/3) \end{pmatrix},$$

where, this time, η is the continuous L_1-periodic function that coincides on $[-L_1/2, L_1/2]$ with $x_1 \mapsto \max(L_1/6 - |x_1|, 0)$. The Perron–Frobenius eigenvalue of $\mathbf{L}(x)$ is 1, for all $x \in [0, L]$. The unique one-dimensional left-continuous unit Perron–Frobenius eigenvector is

$$\mathbf{u}_{\mathrm{PF}} : x \mapsto \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } x_1 \in \left(0, \frac{L_1}{6}\right] \cup \left(\frac{5L_1}{6}, L_1\right] + \mathbb{Z}L_1, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } x_1 \in \left(\frac{L_1}{6}, \frac{L_1}{2}\right] + \mathbb{Z}L_1, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } x_1 \in \left(\frac{L_1}{2}, \frac{5L_1}{6}\right] + \mathbb{Z}L_1. \end{cases}$$

All other unit Perron–Frobenius eigenvectors coincide with this one at all continuity points.

Now, let $\mathbf{u} \in \mathcal{C}^2_{\mathrm{per}}(\mathbb{R}^n, [\mathbf{0}, \boldsymbol{\infty}))$ and $\lambda \leq -1$ such that $-\Delta\mathbf{u} - \mathbf{L}\mathbf{u} \leq \lambda \mathbf{u}$. Taking the scalar product in $\mathbb{R}^2$ with $\mathbf{u}$ and integrating by parts in $[0, L]$, we obtain:

$$0 \leq \int_{[0,L]} \sum_{i=1}^{2} |\nabla u_i|^2 \leq \int_{[0,L]} \mathbf{u}^{\mathrm{T}}(\mathbf{L} - \mathbf{I})\mathbf{u}.$$

Since, at all $x \in [0, L]$, $\mathbf{L}(x) - \mathbf{I}$ is a symmetric matrix with nonpositive eigenvalues, $\mathbf{u}(x)^{\mathrm{T}}(\mathbf{L}(x) - \mathbf{I})\mathbf{u}(x) \leq 0$. Therefore all inequalities above are actually equalities, and in particular $\mathbf{u}$ is a constant vector satisfying $\mathbf{u}^{\mathrm{T}}(\mathbf{L}(x) - \mathbf{I})\mathbf{u} = 0$. Since no Perron–Frobenius eigenvector is constant, necessarily $\mathbf{u} = \mathbf{0}$.

Therefore no $\lambda \leq -1$ can satisfy $-\Delta \mathbf{u} - \mathbf{L}\mathbf{u} = \lambda \mathbf{u}$ for some positive periodic $\mathcal{C}^2$ eigenfunction $\mathbf{u}$, and this directly implies that $\lambda_1' > -1$, or in other words

$$-R < \frac{1}{|[0,L]|} \int_{[0,L]} \lambda_{\mathrm{PF}}(\mathbf{L}).$$

In a similar spirit, the following property requires a line-sum-symmetry assumption ($\mathbf{L}\mathbf{1} = \mathbf{L}^{\mathrm{T}}\mathbf{1}$) and uses the property described in Eaves–Hoffman–Rothblum–Schneider [27, Corollary 3].

Proposition 3.27. *Let $z \in \mathbb{R}^n$. Assume:*

1. *for all $i \in [N]$, $\nabla \cdot (q_i - 2A_i z) = 0$,*
2. *$\mathbf{L}(t,x)$ is line-sum-symmetric for all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$.*

Then

$$\lambda_{1,z} \leq -\frac{1}{N}\left(\sum_{i,j=1}^{N} \langle \hat{l}_{i,j} \rangle + z \cdot \sum_{i=1}^{N} \left(\langle \hat{A}_i \rangle z - \langle \hat{q}_i \rangle \right)\right)$$

and this inequality is an equality if $\mathbf{L} + \mathrm{diag}(\nabla \cdot (A_i z) + z \cdot (A_i z - q_i))$ is irreducible at all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$ with Perron–Frobenius eigenvector $\mathbf{1}$ and constant Perron–Frobenius eigenvalue.

Proof. Denote, for all $i \in [N]$, $q_i - 2A_i z = b_i$ and recall

$$\begin{aligned}\boldsymbol{\mathcal{Q}}_z &= \boldsymbol{\mathcal{Q}} - \mathrm{diag}\left(2A_i z \cdot \nabla + z \cdot A_i z + \nabla \cdot (A_i z) - q_i \cdot z\right) \\ &= \mathrm{diag}\left(\partial_t - \nabla \cdot (A_i \nabla) + b_i \cdot \nabla - \nabla \cdot (A_i z) - z \cdot (A_i z - q_i)\right) - \mathbf{L}\end{aligned}$$

Denote $\mathbf{u} = \mathbf{u}_z$ the unit periodic principal eigenfunction associated with $\lambda_{1,z}$. Taking the scalar product in $\mathbb{R}^N$ between $\mathbf{u}^{\circ -1} = (1/u_i)_{i\in[N]}$ and $\boldsymbol{\mathcal{Q}}_z \mathbf{u} - \lambda_{1,z}\mathbf{u}$, integrating by parts in $\overline{\Omega_{\mathrm{per}}}$, using the fact that all b_i are divergence-free and using

$$\sum_{i=1}^{N} \frac{(\mathbf{L}\mathbf{u})_i}{u_i} \geq \sum_{i,j=1}^{N} l_{i,j} \quad \text{and} \quad \int_{\overline{\Omega_{\mathrm{per}}}} \frac{\nabla u_i}{u_i} \cdot A_i \frac{\nabla u_i}{u_i} \geq 0, \tag{17}$$

we get

$$\lambda_{1,z} \leq -\frac{1}{NT|[0,L]|}\left(\sum_{i,j=1}^{N} \int_{\overline{\Omega_{\mathrm{per}}}} l_{i,j} + z \cdot \sum_{i=1}^{N} \int_{\overline{\Omega_{\mathrm{per}}}} (A_i z - q_i)\right).$$

From the equality case in (17), we deduce that this inequality is an equality if $\mathbf{u}_z \in \mathrm{span}(\mathbf{1})$ and $\mathbf{L}(t,x)$ is irreducible at all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$. These conditions are satisfied if and only if $\mathbf{L} + \mathrm{diag}(\nabla \cdot (A_i z) + z \cdot (A_i z - q_i))$ is irreducible at all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$ with Perron–Frobenius eigenvector $\mathbf{1}$ and Perron–Frobenius eigenvalue $\lambda_{1,z}$, both constant. Finally, by uniqueness of the periodic principal eigenvalue, the assumption that the Perron–Frobenius eigenvalue is $\lambda_{1,z}$ can be replaced without loss of generality by the assumption that the Perron–Frobenius eigenvalue is constant. □

Remark 3.18. Circulant matrices and doubly stochastic matrices are line-sum-symmetric and always admit $\mathbf{1}$ as eigenvector. Hence all inequalities on $\lambda_{1,z}$ are equalities if:

1. all A_i are constant and coincide and all q_i are constant and coincide,
2. $\mathbf{L}$ is, at all (t,x), irreducible and either circulant or doubly stochastic,

(3) its Perron–Frobenius eigenvalue $\lambda_{\mathrm{PF}}(\mathbf{L}(t,x)) = \sum_{j=1}^{N} l_{1,j}(t,x)$ is constant (this condition being automatically satisfied in the doubly stochastic case).

This shows in particular that the inequalities can all be equalities even if $\mathbf{L}$ is not spatio-temporally constant.

The following two corollaries are concerned with special cases.

Corollary 3.28. *Assume:*

(1) for all $i \in [N]$, $\nabla \cdot q_i = 0$,
(2) $\mathbf{L}(t,x)$ is line-sum-symmetric for all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$.

Then

$$\lambda_1' \leq -\frac{1}{N}\sum_{i,j=1}^{N}\langle \hat{l}_{i,j}\rangle = -\frac{1}{N}\mathbf{1}^{\mathrm{T}}\langle\hat{\mathbf{L}}\rangle\mathbf{1}$$

and this inequality is an equality if $\mathbf{L}$ is irreducible at all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$ with Perron–Frobenius eigenvector $\mathbf{1}$ and constant Perron–Frobenius eigenvalue.

Corollary 3.29. *Assume:*

(1) for all $i \in [N]$, q_i and each column of A_i are divergence-free,
(2) $\mathbf{L}(t,x)$ is line-sum-symmetric for all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$.

Denote

$$[A] = \frac{1}{N}\sum_{i=1}^{N}\langle \hat{A}_i\rangle, \quad [q] = \frac{1}{N}\sum_{i=1}^{N}\langle \hat{q}_i\rangle.$$

Then

$$\lambda_1 \leq -\frac{1}{N}\mathbf{1}^{\mathrm{T}}\langle\hat{\mathbf{L}}\rangle\mathbf{1} + \frac{1}{4}[q]\cdot[A][q]$$

with equality if

$$\mathbf{L} + \frac{1}{2}\operatorname{diag}\left(\nabla\cdot(A_i[A]^{-1}[q] + [A]^{-1}[q]\cdot\left(\frac{1}{2}A_i[A]^{-1}[q] - q_i\right)\right)$$

is irreducible at all $(t,x) \in \overline{\Omega_{\mathrm{per}}}$ with Perron–Frobenius eigenvector $\mathbf{1}$ and constant Perron–Frobenius eigenvalue.

Proof. The assumption that all q_i and all columns of all A_i are divergence-free implies that, for all $i \in [N]$ and $z \in \mathbb{R}^n$, $q_i - 2A_i z$ is divergence-free (and actually the converse implication is also true: consider the special cases $z = 0, e_1, \dots, e_n$, where $e_\alpha = (\delta_{\alpha\beta})_{\beta\in[n]}$).

By (A_1), $[A]$ is invertible, so that the inequality of Proposition 3.27 reads

$$\lambda_{1,z+\frac{1}{2}[A]^{-1}[q]} \leq -\frac{1}{N}\mathbf{1}^{\mathrm{T}}\langle\hat{\mathbf{L}}\rangle\mathbf{1} - z\cdot[A]z + \frac{1}{4}[q]\cdot[A][q].$$

The inequality on $\lambda_1 = \max_{z\in\mathbb{R}^n}\lambda_{1,z}$ and the associated equality case follow directly. $\square$

Proposition 3.30. *Let $z \in \mathbb{R}^n$. If, for all $i \in [N]$, $\nabla\cdot(q_i - 2A_i z) = 0$, then*

$$\lambda_{1,z} \leq \lambda_{1,\mathrm{per}}\left(\partial_t - \mathbf{L}^{\#} - \operatorname{diag}\left(z\cdot(\langle A_i\rangle z - \langle q_i\rangle)\right)\right),$$

where the entries of the matrix $\mathbf{L}^{\#} = \left(l^{\#}_{i,j}\right)_{(i,j)\in[N]^2}$ *are defined by:*

$$l^{\#}_{i,j} = \begin{cases} \frac{1}{|[0,L]|}\int_{[0,L]} l_{i,i} & \text{if } i=j, \\ \exp\left(\frac{1}{|[0,L]|}\int_{[0,L]} \ln l_{i,j}\right) & \text{if } i\neq j \text{ and } \min_{(t,x)\in\overline{\Omega_{\mathrm{per}}}} l_{i,j}(t,x)>0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is quite similar to that of Proposition 3.27, we only sketch it.

Multiply each line of $\boldsymbol{\mathcal{Q}}_z\mathbf{u}-\lambda_{1,z}\mathbf{u}$ by $1/u_i$, integrate by parts in $[0,L]$, divide by $|[0,L]|$, define $J_i=\{j\in[N]\backslash\{i\} \mid \min_{\overline{\Omega_{\mathrm{per}}}} l_{i,j}>0\}$, use the Jensen inequality:

$$\begin{aligned}\frac{1}{|[0,L]|}\int_{[0,L]}\sum_{j\in[N]\backslash\{i\}}\frac{l_{i,j}u_j}{u_i} &\geq \frac{1}{|[0,L]|}\int_{[0,L]}\sum_{j\in J_i}\mathrm{e}^{-\ln(u_i)+\ln(l_{i,j})+\ln(u_j)} \\ &\geq \sum_{j\in J_i}\mathrm{e}^{\frac{1}{|[0,L]|}\int_{[0,L]}(-\ln(u_i)+\ln(l_{i,j})+\ln(u_j))} \\ &= \sum_{j\in J_i}\frac{\mathrm{e}^{\frac{1}{|[0,L]|}\int_{[0,L]}\ln(l_{i,j})}\mathrm{e}^{\frac{1}{|[0,L]|}\int_{[0,L]}\ln(u_j)}}{\mathrm{e}^{\frac{1}{|[0,L]|}\int_{[0,L]}\ln(u_i)}},\end{aligned}$$

define the positive function $\mathbf{v}:t\mapsto\left(\exp\left(\frac{1}{|[0,L]|}\int_{[0,L]}\ln(u_i(t,x))\mathrm{d}x\right)\right)_{i\in[N]}$, find

$$\mathbf{v}'-\left[\mathbf{L}^{\#}+\operatorname{diag}\left(z\cdot\frac{1}{|[0,L]|}\int_{[0,L]}(A_iz-q_i)\right)\right]\mathbf{v}\geq\lambda_{1,z}\mathbf{v},$$

and subsequently use the min–max formula for the periodic principal eigenvalue. The result follows. □

Repeating the exact same procedure but this time with mean values in $[0,T]$, we also find the following estimate, that can be directly combined with the previous one.

Proposition 3.31. *Let* $z\in\mathbb{R}^n$*. If* $(A_i)_{i\in[N]}$, $(q_i)_{i\in[N]}$ *and* $\mathbf{L}$ *do not depend on* x*, then*

$$\lambda_{1,z}\leq-\lambda_{\mathrm{PF}}\left(\mathbf{L}^{\flat}+\operatorname{diag}\left(z\cdot\hat{A}_iz-\hat{q}_i\cdot z\right)\right),$$

where the entries of the matrix $\mathbf{L}^{\flat}=\left(l^{\flat}_{i,j}\right)_{(i,j)\in[N]^2}$ *are defined by:*

$$l^{\flat}_{i,j} = \begin{cases} \frac{1}{T}\int_0^T l_{i,i} & \text{if } i=j, \\ \exp\left(\frac{1}{T}\int_0^T \ln l_{i,j}\right) & \text{if } i\neq j \text{ and } \min_{t\in[0,T]} l_{i,j}(t)>0, \\ 0 & \text{otherwise.} \end{cases}$$

3.3. **Optimization.**

3.3.1. *Optimization of the mutation matrix.* In this section we prove Theorem 1.16 and Theorem 1.17.

We recall that a doubly stochastic matrix $\mathbf{S}\in\mathbb{R}^{N\times N}$ is a nonnegative matrix such that $\mathbf{S1}=\mathbf{S}^{\mathrm{T}}\mathbf{1}=\mathbf{1}$. Denote $\boldsymbol{\mathcal{S}}\subset\mathcal{L}^{\infty}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^{N\times N})$ the set of all periodic functions whose values are doubly stochastic matrices almost everywhere and $\boldsymbol{\mathcal{S}}_{\{0,1\}}$ the subset of all functions valued almost everywhere in the set of permutation matrices.

Although we assumed until now that the zeroth order term $\mathbf{L}$ of $\boldsymbol{\mathcal{Q}}$ is Hölder-continuous, it is readily verified that the family $(\lambda_{1,z})_{z\in\mathbb{R}^n}$ can still be defined if $\mathbf{L}$ has only an $\mathcal{L}^\infty$ regularity, using for instance a standard regularization procedure not detailed here.

We begin with the following decomposition lemma.

Lemma 3.32. *Let $\mathbf{A} \in \mathbb{R}^{N\times N}$ be an essentially nonnegative matrix. Assume $\mathbf{A} - \operatorname{diag}\left(\sum_{j=1}^N a_{j,i}\right)$ admits a positive Perron–Frobenius eigenvector $\mathbf{v} \in (\mathbf{0},\boldsymbol{\infty})$.*

Then a decomposition $\mathbf{A} = \operatorname{diag}(\mathbf{r}) + (\mathbf{S}-\mathbf{I})\operatorname{diag}(\boldsymbol{\mu})$, with $\mathbf{r}\in\mathbb{R}^N$, $\mathbf{S}\in\mathbb{R}^{N\times N}$ doubly stochastic, $\boldsymbol{\mu}\in[\mathbf{0},\boldsymbol{\infty})$, exists.

Furthermore, for any $\gamma > 0$ such that $\mathbf{S}_\gamma = \mathbf{I}+\gamma(\mathbf{S}-\mathbf{I})$ is still doubly stochastic, $\operatorname{diag}(\mathbf{r}) + (\mathbf{S}_\gamma - \mathbf{I})\operatorname{diag}\left(\frac{\boldsymbol{\mu}}{\gamma}\right)$ is another decomposition and, with the normalization $\min_{i\in[N]} s_{i,i} = 0$, the decomposition is unique.

Proof. The vector $\mathbf{r}\in\mathbb{R}^N$ is defined as

$$\mathbf{r} = \mathbf{A}^{\mathrm{T}}\mathbf{1} = \left(\sum_{j=1}^N a_{j,i}\right)_{i\in[N]}.$$

The vector $\boldsymbol{\mu}\in(\mathbf{0},\boldsymbol{\infty})$ is chosen in $\operatorname{span}\left(\left(v_i^{-1}\right)_{i\in[N]}\right)$, with a norm $|\boldsymbol{\mu}|$ so large that $(\mathbf{A}-\operatorname{diag}(\mathbf{r}))\operatorname{diag}(\boldsymbol{\mu})^{-1}+\mathbf{I}$ is doubly stochastic – it can be easily verified that this matrix is essentially nonnegative, in $[0,1]^{N\times N}$ provided $|\boldsymbol{\mu}|$ is large enough, and admits $\mathbf{1}$ as Perron–Frobenius eigenvector with Perron–Frobenius eigenvalue 1. Finally, the matrix $\mathbf{S}$ is defined as $\mathbf{S} = (\mathbf{A}-\operatorname{diag}(\mathbf{r}))\operatorname{diag}(\boldsymbol{\mu})^{-1}+\mathbf{I}$. By construction, the triplet $(\mathbf{r},\boldsymbol{\mu},\mathbf{S})$ gives indeed a suitable decomposition.

Note at this point that $\mathbf{r}$ is uniquely defined and that $\boldsymbol{\mu}/|\boldsymbol{\mu}|$ is uniquely defined as well (even though $\mathbf{A}$ might be reducible, its positive Perron–Frobenius eigenvectors are necessarily all equal up to multiplicative constants). Also, once $|\boldsymbol{\mu}|$ is fixed, $\mathbf{S}$ is uniquely defined. On the contrary, once an admissible value $m > 0$ of $|\boldsymbol{\mu}|$ is identified, any choice $|\boldsymbol{\mu}| \geq m$ leads to a new, different decomposition.

By construction of $\mathbf{r}$, $a_{i,i} - r_i < 0$ and $|a_{i,i} - r_i| \geq a_{i,j} \geq 0$ for all $i\in[N]$ and $j\in[N]\backslash\{i\}$. Therefore the N inequalities $(a_{i,i}-r_i)/\mu_i \geq -1$ together form a necessary and sufficient condition for $\mathbf{S}$ to be doubly stochastic. Fixing $|\boldsymbol{\mu}|$ exactly so that at least one of these inequalities is an equality, the pair $(\boldsymbol{\mu},\mathbf{S})$ is now uniquely defined. □

Remark 3.19. Clearly, the only true question lies in the existence of the pair $(\boldsymbol{\mu},\mathbf{S})$. As a by-product of the proof, the assumption of the statement (satisfied, *e.g.*, if $\mathbf{A}$ is irreducible) actually ensures the positivity of all μ_i, whereas only their nonnegativity was required. In fact, whenever such a decomposition exists, the positivity of all μ_i is equivalent with the positivity of the Perron–Frobenius eigenvector of $\mathbf{A}-\operatorname{diag}(\mathbf{r})$. On the contrary, when $\mathbf{A}$ is reducible and $\mathbf{A}-\operatorname{diag}(\mathbf{r})$ has Perron–Frobenius eigenvectors only in $\partial(\mathbf{0},\boldsymbol{\infty})$, the existence of such a decomposition can be both a true or a false statement, as shown by the following two examples:

$$\mathbf{A}-\operatorname{diag}(\mathbf{r}) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \left(\begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix} - \mathbf{I}\right)\operatorname{diag}(0,2,2)$$

$$\mathbf{A} - \operatorname{diag}(\mathbf{r}) = \begin{pmatrix} -1 & 1 & 1/3 \\ 1 & -1 & 2/3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proposition 3.33. *Let* $\mathbf{r} \in \mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^N)$*,* $\boldsymbol{\mu} \in \mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, [\mathbf{0}, \boldsymbol{\infty}))$ *and assume* $\mathbf{L}$ *has the form* $\mathbf{L} = \operatorname{diag}(\mathbf{r}) + (\mathbf{S}-\mathbf{I})\operatorname{diag}(\boldsymbol{\mu})$ *with* $\mathbf{S}\in\boldsymbol{\mathcal{S}}$*.*

Then, for all $z \in \mathbb{R}^n$*,*

$$\min_{\mathbf{S}\in\boldsymbol{\mathcal{S}}_{\{0,1\}}} \lambda_{1,z}(\mathbf{S}) = \inf_{\mathbf{S}\in\boldsymbol{\mathcal{S}}} \lambda_{1,z}(\mathbf{S}) \le \sup_{\mathbf{S}\in\boldsymbol{\mathcal{S}}} \lambda_{1,z}(\mathbf{S}) = \max_{\mathbf{S}\in\boldsymbol{\mathcal{S}}_{\{0,1\}}} \lambda_{1,z}(\mathbf{S}).$$

Proof. We prove only that the minimum is attained in $\boldsymbol{\mathcal{S}}_{\{0,1\}}$, the property on the maximum being proved similarly. Also, it is sufficient to prove only the case $z = 0$.

Step 1: strengthening the full coupling. To begin with, we show that, denoting $\mathbf{B}(\eta) = \operatorname{diag}(\mathbf{r}-\boldsymbol{\mu}) + \eta\mathbf{1}_{N\times N}$, it is sufficient to prove the claim for the matrix $\mathbf{L}$ where $\mathbf{B}(0)$ is replaced by $\mathbf{B}(\eta)$ with $\eta > 0$. Indeed, assume by contradiction that the result holds true for any $\eta > 0$ but not for $\eta = 0$. Then there exists $\mathbf{S}^\star \in \boldsymbol{\mathcal{S}}\backslash\boldsymbol{\mathcal{S}}_{\{0,1\}}$ such that

$$\lambda_1'(\mathbf{B}(0) + \mathbf{S}^\star\operatorname{diag}(\boldsymbol{\mu})) < \inf_{\mathbf{S}\in\boldsymbol{\mathcal{S}}_{\{0,1\}}} \lambda_1'(\mathbf{B}(0) + \mathbf{S}\operatorname{diag}(\boldsymbol{\mu})). \tag{18}$$

Denoting $\varepsilon > 0$ the gap between these two real numbers and using the uniform continuity of $\mathbf{L} \mapsto \lambda_1'(\mathbf{L})$, there exists a small $\eta > 0$ such that, for any $\mathbf{S} \in \boldsymbol{\mathcal{S}}_{\{0,1\}}$,

$$0 \le \lambda_1'(\mathbf{B}(0) + \mathbf{S}\operatorname{diag}(\boldsymbol{\mu})) - \lambda_1'(\mathbf{B}(\eta) + \mathbf{S}\operatorname{diag}(\boldsymbol{\mu})) \le \frac{\varepsilon}{2}, \tag{19}$$

Combining (18) and (19), we deduce that for all $\mathbf{S} \in \boldsymbol{\mathcal{S}}_{\{0,1\}}$,

$$\lambda_1'(\mathbf{B}(\eta) + \mathbf{S}\operatorname{diag}(\boldsymbol{\mu})) - \lambda_1'(\mathbf{B}(0) + \mathbf{S}^\star\operatorname{diag}(\boldsymbol{\mu})) \ge \frac{\varepsilon}{2},$$

which implies

$$\begin{aligned}\inf_{\mathbf{S}\in\boldsymbol{\mathcal{S}}} \lambda_1'(\mathbf{B}(\eta) + \mathbf{S}\operatorname{diag}(\boldsymbol{\mu})) &\le \lambda_1'(\mathbf{B}(\eta) + \mathbf{S}^\star\operatorname{diag}(\boldsymbol{\mu})) \\ &< \lambda_1'(\mathbf{B}(0) + \mathbf{S}^\star\operatorname{diag}(\boldsymbol{\mu})) \\ &< \inf_{\mathbf{S}\in\boldsymbol{\mathcal{S}}_{\{0,1\}}} \lambda_1'(\mathbf{B}(\eta) + \mathbf{S}\operatorname{diag}(\boldsymbol{\mu})),\end{aligned}$$

which contradicts the assumption that the claim is true for any small nonzero η. □

From now on, we fix a small $\eta > 0$, denote $\mathbf{B} = \mathbf{B}(\eta)$ and show that a minimizer of $\lambda_1'(\mathbf{S}) = \lambda_1'(\mathbf{B} + \mathbf{S}\operatorname{diag}(\boldsymbol{\mu}))$ is indeed valued in the set of permutation matrices.

Step 2: exhibiting a minimizer in $\boldsymbol{\mathcal{S}}$*.* The closed and bounded set

$$\boldsymbol{\mathcal{S}} = \left\{\mathbf{S} \in \mathcal{L}^\infty(\Omega_{\mathrm{per}}, \mathbb{R}^{N\times N}) \mid \mathbf{S} \ge \mathbf{0},\ \mathbf{S1} = \mathbf{S}^{\mathrm{T}}\mathbf{1} = \mathbf{1} \text{ a.e.}\right\}$$

is, by virtue of the Banach–Alaoglu theorem, compact in the weak-$\star$ topology of $\mathcal{L}^\infty(\Omega_{\mathrm{per}}) = (\mathcal{L}^1(\Omega_{\mathrm{per}}))'$. Hence a minimizing sequence $(\mathbf{S}_k)_{k\in\mathbb{N}}$ converges, up to extraction, to a weak-$\star$ limit $\mathbf{S}_\infty \in \mathcal{L}^\infty(\Omega_{\mathrm{per}}, \mathbb{R}^{N\times N})$. Extending periodically $\mathbf{S}_\infty \in \mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})$, it only remains to verify $\mathbf{S} \in \boldsymbol{\mathcal{S}}$ and $\lambda_1'(\mathbf{S}_\infty) = \lim_{k\to+\infty} \lambda_1'(\mathbf{S}_k)$.

The nonnegativity of $\mathbf{S}_\infty$ in the sense of linear forms is immediate, testing the convergence against arbitrary nonnegative functions in $\mathcal{L}^1(\Omega_{\mathrm{per}})$. Subsequently, testing for all $(i,j) \in [N]^2$ against $\mathbf{e}_j$ multiplied by the indicator of $\{s_{\infty,i,j} < 0\}\cap\overline{\Omega_{\mathrm{per}}}$, we deduce the nonnegativity almost everywhere. Testing for any $(t_0, x_0) \in$

$\overline{\Omega_{\text{per}}}$, $\rho > 0$, against $\mathbf{1}$ multiplied by the indicator of $B((t_0, x_0), \rho)$ and divided by $|B((t_0, x_0), \rho)|$, we deduce by virtue of the Lebesgue differentiation theorem $\mathbf{S}_\infty \mathbf{1} = \mathbf{1}$ almost everywhere. Next, testing the convergence against all $(\mathbf{e}_j)_{j \in [N]}$, we find that all entries $s_{k,i,j}$ converge in the weak-$\star$ topology of $\mathcal{L}^\infty(\Omega_{\text{per}}, \mathbb{R})$, so that $(\mathbf{S}_k^{\mathrm{T}})_{k \in \mathbb{N}}$ also converges. Similarly, $\mathbf{S}_\infty^{\mathrm{T}} \mathbf{1} = \mathbf{1}$ almost everywhere. Therefore $\mathbf{S}_\infty \in \boldsymbol{\mathcal{S}}$ indeed.

Let $\mathbf{u}_k \in \mathcal{C}^{1,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^n, (\mathbf{0}, \boldsymbol{\infty}))$ be a periodic principal eigenfunction associated with $\mathbf{S}_k$, $k \in \mathbb{N} \cup \{\infty\}$ and normalized so that $\int_{\overline{\Omega_{\text{per}}}} |\mathbf{u}_k|^2 = 1$. Up to extraction, $\mathbf{u}_k$ converges as $k \to +\infty$ to a limit $\mathbf{v} \in \mathcal{C}^{1,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^n, [\mathbf{0}, \boldsymbol{\infty}))$. Taking the scalar product in $\mathbb{R}^N$ between $(\operatorname{diag}(\mathcal{P}_i) - \mathbf{B} - \mathbf{S}_k \operatorname{diag}(\boldsymbol{\mu}))\mathbf{u}_k$ and an arbitrary test function $\mathbf{w} \in \mathcal{C}^{1,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^n)$ and integrating over $\overline{\Omega_{\text{per}}}$, we find

$$\lambda_1'(\mathbf{S}_k) \int_{\overline{\Omega_{\text{per}}}} \mathbf{w}^{\mathrm{T}} \mathbf{u}_k = \int_{\overline{\Omega_{\text{per}}}} \mathbf{w}^{\mathrm{T}} \operatorname{diag}(\mathcal{P}_i)\mathbf{u}_k - \mathbf{w}^{\mathrm{T}} \mathbf{B} \mathbf{u}_k - \mathbf{w}^{\mathrm{T}} \mathbf{S}_k \operatorname{diag}(\boldsymbol{\mu})\mathbf{u}_k.$$

Since

$$\int_{\overline{\Omega_{\text{per}}}} \mathbf{w}^{\mathrm{T}} \mathbf{S}_k \operatorname{diag}(\boldsymbol{\mu})\mathbf{u}_k = \int_{\overline{\Omega_{\text{per}}}} \mathbf{w}^{\mathrm{T}} \mathbf{S}_k \operatorname{diag}(\boldsymbol{\mu})\mathbf{v} + \int_{\overline{\Omega_{\text{per}}}} \mathbf{w}^{\mathrm{T}} \mathbf{S}_k \operatorname{diag}(\boldsymbol{\mu})(\mathbf{u}_k - \mathbf{v})$$

and since the first term on the right-hand side converges, by weak-$\star$ convergence of $\mathbf{S}_k$, to $\int_{\overline{\Omega_{\text{per}}}} \mathbf{w}^{\mathrm{T}} \mathbf{S}_\infty \operatorname{diag}(\boldsymbol{\mu})\mathbf{v}$ whereas the second term converges, by boundedness of $\mathbf{S}_k$ and uniform convergence of $\mathbf{u}_k - \mathbf{v}$, to 0, we deduce:

$$\lim_{k \to +\infty} \lambda_1'(\mathbf{S}_k) \int_{\overline{\Omega_{\text{per}}}} \mathbf{w}^{\mathrm{T}} \mathbf{v} = \int_{\overline{\Omega_{\text{per}}}} \mathbf{w}^{\mathrm{T}} \operatorname{diag}(\mathcal{P}_i)\mathbf{v} - \mathbf{w}^{\mathrm{T}} \mathbf{B} \mathbf{v} - \mathbf{w}^{\mathrm{T}} \mathbf{S}_\infty \operatorname{diag}(\boldsymbol{\mu})\mathbf{v}.$$

This holds true for any test function $\mathbf{w} \in \mathcal{C}^{1,2}_{\text{per}}(\mathbb{R} \times \mathbb{R}^n)$, whence $\mathbf{v}$ is a weak solution of

$$\operatorname{diag}(\mathcal{P}_i)\mathbf{v} - \mathbf{B}\mathbf{v} - \mathbf{S}_\infty \operatorname{diag}(\boldsymbol{\mu})\mathbf{v} = \lim_{k \to +\infty} \lambda_1'(\mathbf{S}_k)\mathbf{v},$$

and then by standard parabolic regularity [47], it is in fact a classical solution. It is nonnegative and, by virtue of the normalization $\int_{\overline{\Omega_{\text{per}}}} |\mathbf{u}_k|^2 = 1$, it is nonzero. By uniqueness up to a multiplicative constant of the nonnegative periodic eigenfunction, we deduce $\mathbf{v} = \mathbf{u}_\infty$ and $\lim_{k \to +\infty} \lambda_1'(\mathbf{S}_k) = \lambda_1'(\mathbf{S}_\infty)$. Therefore $\mathbf{S}_\infty \in \boldsymbol{\mathcal{S}}$ is indeed a minimizer of λ_1' in the strong topology of $\mathcal{L}^\infty_{\text{per}}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^{N \times N})$. □

Define the function:

$$\begin{array}{llll} \Phi : & \boldsymbol{\mathcal{S}} \times \mathbb{R} \times \mathbb{R}^n & \to & [N] \\ & (\mathbf{S}, t, x) & \mapsto & \#\{(i,j) \in [N]^2 \mid s_{i,j}(t,x) = 1\} \end{array}$$

and remark that, for any $\mathbf{S} \in \boldsymbol{\mathcal{S}}$,

$$\mathbf{S} \in \boldsymbol{\mathcal{S}}_{\{0,1\}} \iff |\{(t,x) \in \overline{\Omega_{\text{per}}} \mid \Phi(\mathbf{S}, t, x) = N\}| = |\overline{\Omega_{\text{per}}}| = T|[0,L]|.$$

In other words, defining $\Omega_{\text{per}}(\mathbf{S}, N_0) = \{(t,x) \in \overline{\Omega_{\text{per}}} \mid \Phi(\mathbf{S}, t, x) = N_0\}$,

$$\mathbf{S} \in \boldsymbol{\mathcal{S}}_{\{0,1\}} \iff \forall N_0 \in [N-1] \quad |\Omega_{\text{per}}(\mathbf{S}, N_0)| = 0.$$

Let $\mathbf{S}^\wedge \in \boldsymbol{\mathcal{S}}$ be a minimizer. Assume $\mathbf{S}^\wedge \notin \boldsymbol{\mathcal{S}}_{\{0,1\}}$. This means that there exists $N_0 \in [N-1]$ such that $\Omega_{N_0} = \Omega_{\text{per}}(\mathbf{S}^\wedge, N_0)$ has a positive measure. We are now going to correct this minimizer step by step.

Step 3: correcting the minimizer in a large subset of Ω_{N_0}. Let $\mathbf{u},\mathbf{v}\in\mathcal{C}^{1,2}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,(\mathbf{0},\boldsymbol{\infty}))$ be respectively a periodic principal eigenfunction of $\boldsymbol{\mathcal{Q}}$ and a periodic principal eigenfunction of the adjoint operator

$$\boldsymbol{\mathcal{Q}}^\star = -\partial_t - \mathrm{diag}(\nabla\cdot(A_i\nabla)+q_i\cdot\nabla+\nabla\cdot q_i) - \mathbf{B}^{\mathrm{T}} - \mathrm{diag}(\boldsymbol{\mu})(\mathbf{S}^\wedge)^{\mathrm{T}}.$$

By irreducibility of $\mathbf{B}$, both operators are fully coupled in $\overline{\Omega_{\mathrm{per}}}$ and $\mathbf{u}$ and $\mathbf{v}$ are positive. With the normalizations $\int_{\overline{\Omega_{\mathrm{per}}}}|\mathbf{u}|^2 = \int_{\overline{\Omega_{\mathrm{per}}}}\mathbf{v}^{\mathrm{T}}\mathbf{u} = 1$ (the second one is possible because, by positivity, $\mathbf{u}$ and $\mathbf{v}$ cannot be orthogonal), $\mathbf{u}$ and $\mathbf{v}$ are uniquely defined.

Let $(t,x)\in\Omega_{N_0}$. There exist two permutation matrices $\mathbf{P}(t,x)\in\boldsymbol{\mathcal{S}}_{\{0,1\}}$ and $\mathbf{Q}(t,x)\in\boldsymbol{\mathcal{S}}_{\{0,1\}}$ such that:

(1) the matrix $\tilde{\mathbf{S}}^\wedge(t,x)=\mathbf{P}(t,x)\mathbf{S}^\wedge(t,x)\mathbf{Q}(t,x)^{\mathrm{T}}$ is doubly stochastic and has a block diagonal form:

$$\begin{pmatrix}\tilde{\mathbf{S}}^\wedge_{\mathrm{top}}(t,x) & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{S}}^\wedge_{\mathrm{bottom}}(t,x)\end{pmatrix}$$

with $\tilde{\mathbf{S}}^\wedge_{\mathrm{bottom}}(t,x)\in\{0,1\}^{N_0\times N_0}$ empty (if $N_0=0$) or a permutation matrix (if $N_0>0$) and all entries in $\tilde{\mathbf{S}}^\wedge_{\mathrm{top}}(t,x)$ smaller than 1;

(2) for all $i\in[N-N_0]$,

$$\mathbf{e}_i^{\mathrm{T}}\mathbf{P}(t,x)\mathbf{v}(t,x)\leq\mathbf{e}_1^{\mathrm{T}}\mathbf{P}(t,x)\mathbf{v}(t,x);$$

(3) for all $j\in[N-N_0]$,

$$\mathbf{e}_j^{\mathrm{T}}\mathbf{Q}(t,x)\,\mathrm{diag}(\boldsymbol{\mu}(t,x))\mathbf{u}(t,x)\leq\mathbf{e}_1^{\mathrm{T}}\mathbf{Q}(t,x)\,\mathrm{diag}(\boldsymbol{\mu}(t,x))\mathbf{u}(t,x).$$

The three properties together imply that,

$$(\tilde{s}^\wedge_{i,1}(t,x))^{\mathrm{T}}_{i\in[N]}\mathbf{P}(t,x)\mathbf{v}(t,x)\leq\mathbf{e}_1^{\mathrm{T}}\mathbf{P}(t,x)\mathbf{v}(t,x),\tag{20}$$

$$(\tilde{s}^\wedge_{1,j}(t,x))^{\mathrm{T}}_{j\in[N]}\mathbf{Q}(t,x)\,\mathrm{diag}(\boldsymbol{\mu}(t,x))\mathbf{u}(t,x)\leq\mathbf{e}_1^{\mathrm{T}}\mathbf{Q}(t,x)\,\mathrm{diag}(\boldsymbol{\mu}(t,x))\mathbf{u}(t,x).\tag{21}$$

Next, define, for the same $(t,x)\in\Omega_{N_0}$,

$$\mathbf{a}(t,x)=-\mathbf{e}_1+(\tilde{s}^\wedge_{i,1}(t,x))_{i\in[N]}=\begin{pmatrix}-(1-\tilde{s}^\wedge_{1,1}(t,x))\\ \tilde{s}^\wedge_{2,1}(t,x)\\ \vdots\\ \tilde{s}^\wedge_{N-N_0,1}(t,x)\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$\mathbf{b}(t,x)=-\mathbf{e}_1+(\tilde{s}^\wedge_{1,j}(t,x))_{j\in[N]}=\begin{pmatrix}-(1-\tilde{s}^\wedge_{1,1}(t,x))\\ \tilde{s}^\wedge_{1,2}(t,x)\\ \vdots\\ \tilde{s}^\wedge_{1,N-N_0}(t,x)\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$\mathbf{T}^{\wedge}(t,x) = \frac{1}{1-\tilde{s}^{\wedge}_{1,1}(t,x)}\mathbf{a}(t,x)\mathbf{b}(t,x)^{\mathrm{T}}.$$

Let us verify that $\tilde{\mathbf{S}}^{\wedge} + \mathbf{T}^{\wedge}$ is doubly stochastic at (t,x). Since $\tilde{\mathbf{S}}^{\wedge}(t,x)$ is doubly stochastic, we only have to verify that $\tilde{\mathbf{S}}^{\wedge}(t,x) + \mathbf{T}^{\wedge}(t,x)$ is nonnegative and that $\mathbf{a}(t,x)\mathbf{b}(t,x)^{\mathrm{T}}\mathbf{1} = \mathbf{b}(t,x)\mathbf{a}(t,x)^{\mathrm{T}}\mathbf{1} = \mathbf{0}$. Both properties turn out to be obvious. By construction, $\Phi(\tilde{\mathbf{S}}^{\wedge} + \mathbf{T}^{\wedge}, t, x) = N_0 + 1$. Indeed, $\tilde{\mathbf{S}}^{\wedge}(t,x) + \mathbf{T}^{\wedge}(t,x)$ has N_0 entries equal to 1 in its bottom right block and its upper left entry satisfies

$$\tilde{s}^{\wedge}_{1,1}(t,x) + t^{\wedge}_{1,1}(t,x) = \tilde{s}^{\wedge}_{1,1}(t,x) + \frac{1}{1-\tilde{s}^{\wedge}_{1,1}(t,x)}(1-\tilde{s}^{\wedge}_{1,1}(t,x))^2 = 1.$$

Let $\omega \subset \Omega_{N_0}$ be a measurable subset. Setting

$$\mathbf{T}^{\wedge}_{\omega} : (t,x) \in \overline{\Omega_{\mathrm{per}}} \mapsto \begin{cases} \mathbf{T}^{\wedge}(t,x) & \text{if } (t,x) \in \omega, \\ \mathbf{0} & \text{if } (t,x) \in \overline{\Omega_{\mathrm{per}}}\backslash\omega, \end{cases}$$

extending $\mathbf{T}^{\wedge}_{\omega}$ periodically in $\mathbb{R}\times\mathbb{R}^n$ and verifying routinely that $\mathbf{T}^{\wedge}_{\omega}$ is measurable, we are now in a position to verify that this construction does not modify the periodic principal eigenvalue λ_1', namely $\tilde{\mathbf{S}}^{\wedge} + \mathbf{T}^{\wedge}_{\omega} \in \mathcal{L}^{\infty}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})$ is still a minimizer, provided ω is appropriately chosen.

Denote, for any $\alpha \in [0,1]$,

$$\boldsymbol{\mathcal{Q}}_{\alpha} = \mathrm{diag}(\mathcal{P}_i) - \mathbf{B} - \mathbf{P}^{\mathrm{T}}\left(\alpha\mathbf{T}^{\wedge}_{\omega} + \tilde{\mathbf{S}}^{\wedge}\right)\mathbf{Q}\,\mathrm{diag}(\boldsymbol{\mu}),$$

$\lambda(\alpha) = \lambda_1'(\boldsymbol{\mathcal{Q}}_{\alpha})$, and let $\mathbf{u}_{\alpha}$ and $\mathbf{v}_{\alpha}$ be two positive periodic principal eigenfunctions of respectively $\boldsymbol{\mathcal{Q}}_{\alpha}$ and of the adjoint operator

$$\boldsymbol{\mathcal{Q}}^{\star}_{\alpha} = -\partial_t - \mathrm{diag}(\nabla\cdot(A_i\nabla) + q_i\cdot\nabla + \nabla\cdot q_i) - \mathbf{B}^{\mathrm{T}} - \mathrm{diag}(\boldsymbol{\mu})\mathbf{Q}^{\mathrm{T}}\left(\alpha\mathbf{T}^{\wedge}_{\omega} + \tilde{\mathbf{S}}^{\wedge}\right)^{\mathrm{T}}\mathbf{P},$$

normalized so that $\fint_{\Omega_{\mathrm{per}}}|\mathbf{u}_{\alpha}|^2 = \fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\alpha}^{\mathrm{T}}\mathbf{u}_{\alpha} = 1$. For any $\alpha,\beta \in [0,1]$, $\alpha \neq \beta$,

$$\begin{aligned}
\frac{\lambda(\beta)-\lambda(\alpha)}{\beta-\alpha} &= \frac{1}{(\beta-\alpha)\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\mathbf{u}_{\alpha}}\left(\fint_{\Omega_{\mathrm{per}}}\mathbf{u}_{\alpha}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}^{\star}_{\beta}\mathbf{v}_{\beta} - \fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_{\alpha}\mathbf{u}_{\alpha}\right) \\
&= \frac{1}{(\beta-\alpha)\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\mathbf{u}_{\alpha}}\left(\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_{\beta}\mathbf{u}_{\alpha} - \fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_{\alpha}\mathbf{u}_{\alpha}\right) \\
&= \frac{1}{(\beta-\alpha)\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\mathbf{u}_{\alpha}}\left(\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}(\boldsymbol{\mathcal{Q}}_{\beta} - \boldsymbol{\mathcal{Q}}_{\alpha})\mathbf{u}_{\alpha}\right) \\
&= \frac{1}{(\beta-\alpha)\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\mathbf{u}_{\alpha}}\left(\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}((\beta-\alpha)\boldsymbol{\mathcal{Q}}_1 - (\beta-\alpha)\boldsymbol{\mathcal{Q}}_0)\mathbf{u}_{\alpha}\right) \\
&= -\frac{1}{\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\mathbf{u}_{\alpha}}\left(\fint_{\Omega_{\mathrm{per}}}(\mathbf{P}\mathbf{v}_{\beta})^{\mathrm{T}}\mathbf{T}^{\wedge}_{\omega}\mathbf{Q}\,\mathrm{diag}(\boldsymbol{\mu})\mathbf{u}_{\alpha}\right) \\
&= -\frac{1}{\fint_{\Omega_{\mathrm{per}}}\mathbf{v}_{\beta}^{\mathrm{T}}\mathbf{u}_{\alpha}}\int_{\omega}\frac{(\mathbf{a}^{\mathrm{T}}\mathbf{P}\mathbf{v}_{\beta})(\mathbf{b}^{\mathrm{T}}\mathbf{Q}\,\mathrm{diag}(\boldsymbol{\mu})\mathbf{u}_{\alpha})}{1-\tilde{s}^{\wedge}_{1,1}}.
\end{aligned}$$

Taking the limit $\beta \to \alpha$, this leads to

$$\lambda'(\alpha) = -\int_{\omega}\frac{(\mathbf{a}^{\mathrm{T}}\mathbf{P}\mathbf{v}_{\alpha})(\mathbf{b}^{\mathrm{T}}\mathbf{Q}\,\mathrm{diag}(\boldsymbol{\mu})\mathbf{u}_{\alpha})}{1-\tilde{s}^{\wedge}_{1,1}}.$$

In view of this equality and of (20)–(21), $\lambda'(0)\leq 0$. We claim that in fact $\lambda'(0)=0$. Indeed, if this is not the case, then there exists a small $\alpha>0$ such that $\lambda(\alpha)<\lambda(0)$. Then the minimality of $\lambda(0)=\lambda_1'(\mathbf{P}^{\mathrm{T}}\tilde{\mathbf{S}}^{\wedge}\mathbf{Q})=\lambda_1'(\mathbf{S}^{\wedge})$ in $\boldsymbol{\mathcal{S}}$ is contradicted[9].

Since $\lambda'(0)=0$ for any choice of ω, using the Lebesgue differentiation theorem, we obtain:

$$-\frac{\left(\mathbf{a}^{\mathrm{T}}\mathbf{P}\mathbf{v}\right)\left(\mathbf{b}^{\mathrm{T}}\mathbf{Q}\operatorname{diag}\left(\boldsymbol{\mu}\right)\mathbf{u}\right)}{1-\tilde{s}^{\wedge}_{1,1}}=0\quad\text{almost everywhere in }\Omega_{N_0}.$$

Subsequently, for almost every $(t,x)\in\Omega_{N_0}$,

$$\mathbf{T}^{\wedge}_{\Omega_{N_0}}(t,x)\mathbf{Q}(t,x)\operatorname{diag}(\boldsymbol{\mu}(t,x))\mathbf{u}(t,x)=\mathbf{0}\quad\text{or}\quad\mathbf{T}^{\wedge}_{\Omega_{N_0}}(t,x)^{\mathrm{T}}\mathbf{P}(t,x)\mathbf{v}(t,x)=\mathbf{0}.$$

Let

$$\omega_{\mathbf{u}}=\left\{(t,x)\in\Omega_{N_0}\mid\mathbf{T}^{\wedge}_{\Omega_{N_0}}(t,x)\mathbf{Q}(t,x)\operatorname{diag}(\boldsymbol{\mu}(t,x))\mathbf{u}(t,x)=\mathbf{0}\right\},$$

$$\omega_{\mathbf{v}}=\left\{(t,x)\in\Omega_{N_0}\mid\mathbf{T}^{\wedge}_{\Omega_{N_0}}(t,x)^{\mathrm{T}}\mathbf{P}(t,x)\mathbf{v}(t,x)=\mathbf{0}\right\}.$$

The subsets $\omega_{\mathbf{u}}$ and $\omega_{\mathbf{v}}\backslash\omega_{\mathbf{u}}$ are measurable, disjoint and satisfy $|\omega_{\mathbf{u}}\cup(\omega_{\mathbf{v}}\backslash\omega_{\mathbf{u}})|=|\Omega_{N_0}|$. One of the two, denoted below ω, satisfies $|\omega|\geq\frac{1}{2}|\Omega_{N_0}|$. Choosing this ω in the definition of $\mathbf{T}^{\wedge}_{\omega}$, we deduce directly that the corresponding eigenvector ($\mathbf{u}_\alpha$ if $\omega=\omega_{\mathbf{u}}$, $\mathbf{v}_\alpha$ if $\omega=\omega_{\mathbf{v}}$) does not depend anymore on α. Then, using again the same calculations, we deduce $\lambda(\alpha)=\lambda(0)$ for any $\alpha\in[0,1]$. In particular,

$$\lambda_1'(\mathbf{S}^{\wedge})=\lambda(0)=\lambda(1)=\lambda_1'(\mathbf{S}^{\wedge}+\mathbf{P}^{\mathrm{T}}\mathbf{T}^{\wedge}_{\omega}\mathbf{Q}).$$

□

Step 4: correcting the minimizer in Ω_{N_0} almost everywhere. Let $\mathbf{S}^{\wedge}_1=\mathbf{S}^{\wedge}+\mathbf{P}^{\mathrm{T}}\mathbf{T}^{\wedge}_{\omega}\mathbf{Q}$ and $\Omega_{N_0,1}=\Omega_{\mathrm{per}}(\mathbf{S}^{\wedge}_1,N_0)$. Note that $\omega=\Omega_{N_0}\backslash\Omega_{N_0,1}$ up to a negligible set. The new minimizer $\mathbf{S}^{\wedge}_1$ satisfies, by construction:

- $|\Omega_{\mathrm{per}}(\mathbf{S}^{\wedge}_1,N_0+1)|=|\omega|+|\Omega_{\mathrm{per}}(\mathbf{S}^{\wedge},N_0+1)|$,
- $(\mathbf{S}^{\wedge}_1)_{|\overline{\Omega_{\mathrm{per}}}\backslash\Omega_{N_0}}=(\mathbf{S}^{\wedge})_{|\overline{\Omega_{\mathrm{per}}}\backslash\Omega_{N_0}}$,
- $|\Omega_{N_0,1}|\leq\frac{1}{2}|\Omega_{N_0}|$,
- $\Omega_{N_0,1}\subset\Omega_{N_0}$.

Iterating the construction, we obtain a sequence $(\mathbf{S}^{\wedge}_k)_{k\in\mathbb{N}}\in\boldsymbol{\mathcal{S}}^{\mathbb{N}}$ of minimizers and a sequence of measurable sets $(\Omega_{N_0,k})_{k\in\mathbb{N}}$ such that, for each $k\geq 2$,

(1) $\Omega_{\mathrm{per}}(\mathbf{S}^{\wedge}_k,N_0)=\Omega_{N_0,k}$,
(2) $|\Omega_{\mathrm{per}}(\mathbf{S}^{\wedge}_k,N_0+1)|=|\Omega_{N_0,k-1}\backslash\Omega_{N_0,k}|+|\Omega_{\mathrm{per}}(\mathbf{S}^{\wedge}_{k-1},N_0+1)|$,
(3) $(\mathbf{S}^{\wedge}_k)_{|\overline{\Omega_{\mathrm{per}}}\backslash\Omega_{N_0,k-1}}=(\mathbf{S}^{\wedge}_{k-1})_{|\overline{\Omega_{\mathrm{per}}}\backslash\Omega_{N_0,k-1}}$,
(4) $|\Omega_{N_0,k}|\leq\frac{1}{2}|\Omega_{N_0,k-1}|$,
(5) $\Omega_{N_0,k}\subset\Omega_{N_0,k-1}$.

In particular,

$$0\leq|\Omega_{N_0,k}|\leq\frac{1}{2^k}|\Omega_{N_0}|,\quad|\Omega_{\mathrm{per}}(\mathbf{S}^{\wedge}_k,N_0+1)|=|\Omega_{N_0}|-|\Omega_{N_0,k}|+|\Omega_{\mathrm{per}}(\mathbf{S}^{\wedge},N_0+1)|,$$

---

[9]Note that we cannot in general extend λ on the left of $\alpha=0$, since for $\alpha<0$, the matrix $\alpha\mathbf{T}^{\wedge}+\tilde{\mathbf{S}}^{\wedge}$ might loose the crucial property of essential nonnegativity. Thus the minimizer of $\lambda(\alpha)$, $\alpha=0$, is not in general an interior critical point and $\lambda'(0)=0$ cannot be deduced only from the first-order optimality condition. The role played by (20)–(21) is indeed crucial. By reversing one of the two inequalities, we obtain the proof of the complementary result on maximizers.

whence, as $k \to +\infty$,

$$|\Omega_{N_0,k}| \to 0, \quad |\Omega_{\mathrm{per}}(\mathbf{S}_k^\wedge, N_0+1)| \to |\Omega_{N_0}| + |\Omega_{\mathrm{per}}(\mathbf{S}^\wedge, N_0+1)|.$$

Let

$$\mathbf{S}_\infty^\wedge : (t,x) \in \overline{\Omega_{\mathrm{per}}} \mapsto \begin{cases} \mathbf{S}^\wedge(t,x) & \text{if } (t,x) \in \overline{\Omega_{\mathrm{per}}} \backslash \Omega_{N_0}, \\ \mathbf{S}_1^\wedge(t,x) & \text{if } (t,x) \in \Omega_{N_0} \backslash \Omega_{N_0,1}, \\ \mathbf{S}_2^\wedge(t,x) & \text{if } (t,x) \in \Omega_{N_0,1} \backslash \Omega_{N_0,2}, \\ \vdots & \\ \mathbf{I} & \text{if } (t,x) \in \bigcap_{k\in\mathbb{N}} \Omega_{N_0,k}. \end{cases}$$

and extend it periodically in $\mathbb{R}\times\mathbb{R}^n$, so that $\mathbf{S}_\infty^\wedge \in \boldsymbol{\mathcal{S}}$. Note that $|\bigcap_{k\in\mathbb{N}} \Omega_{N_0,k}| = 0$.

Then the sequence $(\mathbf{S}_k^\wedge)_{k\in\mathbb{N}}$ converges almost everywhere, and in any $\mathcal{L}^p_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})$ with $p \in [1,+\infty)$, to $\mathbf{S}_\infty^\wedge$. Moreover, by continuity of the mapping $\mathbf{S} \mapsto \lambda_1'(\mathbf{S})$ with respect to the topology of, say, $\mathcal{L}^2_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^{N\times N})$, $\mathbf{S}_\infty^\wedge$ is still a minimizer. Finally, it satisfies

$$|\Omega_{\mathrm{per}}(\mathbf{S}_\infty^\wedge, N_0)| = 0 \quad \text{and} \quad |\Omega_{\mathrm{per}}(\mathbf{S}_\infty^\wedge, N_0+1)| = |\Omega_{\mathrm{per}}(\mathbf{S}^\wedge, N_0)| + |\Omega_{\mathrm{per}}(\mathbf{S}^\wedge, N_0+1)|.$$

□

Step 5: correcting the minimizer in all possible sets Ω_{N_0}. Performing the construction of Steps 3 and 4, first for

$$N_0^\wedge = \min\{N_0 \in [N-1] \mid |\Omega_{\mathrm{per}}(\mathbf{S}^\wedge, N_0)| > 0\},$$

and then for $N_0^\wedge + 1$, etc., up to $N-1$, we obtain in the end a new minimizer whose restriction to $\overline{\Omega_{\mathrm{per}}}$ is valued in the set of permutation matrices almost everywhere, that is a new minimizer in $\boldsymbol{\mathcal{S}}_{\{0,1\}}$. □

This ends the proof. □

Remark 3.20. Consistently with Neumann–Sze [57], the decomposition $\mathbf{L} = \mathrm{diag}(\mathbf{r}) + (\mathbf{S}-\mathbf{I})\,\mathrm{diag}(\boldsymbol{\mu})$ can be replaced by a more general decomposition $\mathbf{L} = \mathbf{B} + \mathbf{S}\mathbf{A}$ with $\mathbf{A}$ nonnegative and $\mathbf{B}$ essentially nonnegative. The generalization of the proof is straightforward.

Remark 3.21. From Proposition 3.33 and the fact that $\max_z$ and $\max_{\mathbf{S}}$ commute, we can deduce a similar result on the maximizers of $\max_{z\in\mathbb{R}^n} \lambda_{1,z}(\mathbf{S})$. Yet we do not insist on it, for two reasons:

(1) from the discussion in Section 1.6, we know that $\max_{z\in\mathbb{R}^n} \lambda_{1,z}(\mathbf{S})$ is not a satisfying generalization of $\lambda_1(\mathbf{S})$ when $\mathbf{S}$ ceases to satisfy (A_3), and clearly there are many $\mathbf{S} \in \mathbf{S}_{\{0,1\}}$ that do not satisfy (A_3);
(2) since there is no reason why $\min_{\mathbf{S}}$ and $\max_z$ should commute (in particular, $(\mathbf{S}, z) \mapsto \lambda_{1,z}(\mathbf{S})$ is not convex–concave), the argument does not apply to minimizers.

The problem of optimizing $\mathbf{S} \in \boldsymbol{\mathcal{S}} \mapsto \lambda_1(\mathbf{S})$, that needs both a unambiguous definition of λ_1 when $\mathbf{L}$ ceases to satisfy (A_3) and a new method of proof that applies to minimizers, remains therefore open.

Next we prove Theorem 1.17. The proof relies on a dual convexity lemma of Altenberg [3, Lemma 1] whose statement is recalled below.

Lemma 3.34 (Altenberg's dual convexity lemma)**.** *Let $f : (0,+\infty)\times[0,+\infty)\to\mathbb{R}$ be a function of two variables r and s, positively homogeneous of degree 1, and convex with respect to its second variable s.*

Then:

1. *f is convex with respect to its first variable r; furthermore, $r \mapsto f(r,s)$ is strictly convex if $s \neq 0$ and if the convexity with respect to s is strict;*
2. *for all $(r,s) \in (0,+\infty)\times[0,+\infty)$, $z \mapsto f(r,s) + zf(1,0) - f(r+z,s)$ is either identically zero or positive; furthermore it is positive if $s \neq 0$ and if f is strictly convex with respect to s;*
3. *for all $r \in (0,+\infty)$,*

$$\lim_{\substack{r'\to r\\ r'<r}} \frac{f(r',s)-f(r,s)}{r'-r} \leq \lim_{\substack{r'\to r\\ r'>r}} \frac{f(r',s)-f(r,s)}{r'-r} \leq f(1,0) \quad \text{for all } s \in [0,+\infty),$$

and the first inequality is an equality except possibly at a countable number of values of r.

In our context, this lemma brings forth the following result, which is more general than Theorem 1.17 and might be of independent interest.

Corollary 3.35. *Let $z \in \mathbb{R}^n$, $s \in [0,1]$ and $\mathbf{r} \in \mathcal{C}^{\delta/2,\delta}_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^N)$.*

Then the mapping

$$\lambda : r \in (0,1] \mapsto \lambda_{1,z}\left(r\boldsymbol{\mathcal{Q}} - s\,\mathrm{diag}(\mathbf{r})\right)$$

is concave, strictly if $\mathbf{r}$ depends on x and $s > 0$, and, for all $r \in (0,1]$,

$$\lambda'(r) \geq \lambda_{1,z}(\boldsymbol{\mathcal{Q}}),$$

with strict inequality if λ is strictly concave.

Proof. It suffices to consider the case $\mathbf{r} \neq \mathbf{0}$. Fixing $z \in \mathbb{R}^n$, we apply Lemma 3.34 to the function

$$f : (r,s) \mapsto -\lambda_{1,z}\left(r\boldsymbol{\mathcal{Q}} - s\,\mathrm{diag}(\mathbf{r})\right)$$

which is, by virtue of Theorem 1.3, convex with respect to s, strictly if $\mathbf{r}$ depends on x, and which is of class $\mathcal{C}^1$ in (r,s) away from $r=0$. □

As a direct corollary, replacing $\mathbf{L}$ by $(\mathbf{S}-\mathbf{I})\,\mathrm{diag}(\boldsymbol{\mu})$ and noting that, for instance, $\lambda_1'(\mathrm{diag}(\boldsymbol{\mathcal{P}}) - (\mathbf{S}-\mathbf{I})\,\mathrm{diag}(\boldsymbol{\mu})) = 0$ with left-eigenfunction $\mathbf{1}$, we obtain Theorem 1.17.

3.3.2. *Optimization of the spatial distribution of* $\mathbf{L}$ *in dimension* 1*.* In this section, we prove Theorem 1.18.

First, we investigate a Talenti inequality for cooperative elliptic systems, as such estimates are milestones to proving spectral comparison [54, Theorem 3.9]. Let us recall that the core idea underlying these estimates is to compare some $\mathcal{L}^p$ norms (here, the $\mathcal{L}^\infty$ norms) of the solution of an elliptic problem with that of a related equation, the coefficients of which have been replaced by their symmetrization. It should be noted that our results would also hold for boundary conditions of Dirichlet type in the spatial domain $B(0,R)$, $R>0$.

Remark 3.22. In what follows, we will use a few specific notations.

It will be convenient to identify $[0,L_1]$ with $\left[-\frac{L_1}{2},\frac{L_1}{2}\right]$; this amounts to translating the functions, and has the advantage of having 0 as a symmetry point. In this context, let us recall the fundamental ordering on the set of functions: for two functions $\mathbf{f}, \mathbf{g} \in \mathcal{L}^2([0, L_1], [\mathbf{0}, \boldsymbol{\infty}))$, the notation $\mathbf{f} \prec \mathbf{g}$ stands for:

$$\int_{-\frac{r}{2}}^{\frac{r}{2}} \mathbf{f} \leq \int_{-\frac{r}{2}}^{\frac{r}{2}} \mathbf{g} \quad \text{for all } r \leq \frac{L_1}{2}.$$

In particular, these inequalities hold component wise.

For any non-negative scalar function, we may identify its rearrangement $u^\dagger$ with a non-increasing mapping $\overline{u} : [0, L_1/2] \to \mathbb{R}$. For any non-negative $\mathbf{u}$, we denote by $\mathbf{u}^\dagger$ its periodic rearrangement.

The first step in the proof of Theorem 1.18 is the following comparison result for elliptic systems.

Proposition 3.36. *Assume* $\operatorname{diag}(\boldsymbol{\mathcal{P}}) = \partial_t - \mathbf{D}\Delta$ *for some diagonal matrix* $\mathbf{D}$ *with constant, positive diagonal entries and assume that* $\mathbf{L}$ *is nonnegative and depends only on* x*.*

Let $c > 0$ *and* $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathcal{L}^2(\left[-\frac{L_1}{2}, \frac{L_1}{2}\right], [\mathbf{0}, \boldsymbol{\infty}))$ *such that* $\boldsymbol{\phi} \prec \boldsymbol{\psi}$*. Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be the (unique) solutions of*

$$\begin{cases} -\mathbf{D}\Delta\mathbf{u} + c\mathbf{u} = \mathbf{L}\boldsymbol{\phi} & \text{in } \left[-\frac{L_1}{2}, \frac{L_1}{2}\right], \\ \mathbf{u} \in \mathcal{W}^{1,2}_{\text{per}} \end{cases}$$

and

$$\begin{cases} -\mathbf{D}\Delta\mathbf{v} + c\mathbf{v} = \mathbf{L}^\dagger\boldsymbol{\psi}^\dagger & \text{in } \left[-\frac{L_1}{2}, \frac{L_1}{2}\right] \\ \mathbf{v} \in \mathcal{W}^{1,2}_{\text{per}} \end{cases}$$

Then $\mathbf{u} \prec \mathbf{v}$*.*

Proof. First, let us verify that if $\boldsymbol{\phi}, \boldsymbol{\psi}$ are nonnegative and satisfy $\boldsymbol{\phi} \prec \boldsymbol{\psi}$, then

$$\mathbf{L}^\dagger\boldsymbol{\phi}^\dagger \prec \mathbf{L}^\dagger\boldsymbol{\psi}^\dagger. \tag{22}$$

First of all, for any $s \in \left[0, \frac{L_1}{2}\right]$ and any $j \in [N]$,

$$\chi_{[-s,s]}\phi_j^\dagger \prec \chi_{[-s,s]}\psi_j^\dagger,$$

where χ denotes the characteristic function of a set. This property is stable by addition and multiplication by a nonnegative constant [5]. As a consequence, since any nonnegative nonincreasing function can be approximated from below by a nonnegative step function, (22) follows from the monotone convergence theorem.

For the sake of simplicity, assume the level sets of each u_i have zero Lebesgue measure- should this not be the case, we can argue exactly as in [61]. Let $\tau \geq 0$ be a fixed real number and let $i \in [N]$. Integrating the i-th equation on the level set $\{u_i \geq \tau\}$, we get

$$d_i \int_{\{u_i=\tau\}} |\nabla u_i| = -c \int_{\{u_i \geq \tau\}} u_i + \sum_{j=1}^{N} \int_{\{u_i \geq \tau\}} l_{i,j}\phi_j.$$

Since u and $u^\dagger$ are equimeasurable by the definition of the periodic rearrangement, there holds

$$c \int_{\{u_i \geq \tau\}} u_i = c \int_{\{u_i^\dagger \geq \tau\}} u_i^\dagger.$$

By (22),

$$\sum_{j=1}^{N}\int_{\left\{u_i^\dagger\geq\tau\right\}} l_{i,j}^\dagger\phi_j^\dagger \leq \sum_{j=1}^{N}\int_{\left\{u_i^\dagger\geq\tau\right\}} l_{i,j}^\dagger\psi_j^\dagger.$$

At this point, we have obtained

$$d_i\int_{\{u_i=\tau\}}|\nabla u_i| \leq -c\int_{\left\{u_i^\dagger\geq\tau\right\}} u_i^\dagger + \sum_{j=1}^{N}\int_{\left\{u_i^\dagger\geq\tau\right\}} l_{i,j}^\dagger\phi_j^\dagger.$$

We introduce the distribution function μ_i of u_i,

$$\mu_i(\tau) = |\{u_i > \tau\}|.$$

From the co-area formula,

$$-\mu_i'(\tau) = \int_{\{u_i=\tau\}}\frac{1}{|\nabla u_i|}.$$

Since the periodic rearrangement decreases the perimeter of level-sets, we have

$$\operatorname{Per}\left(\{u_i^\dagger=\tau\}\right) \leq \operatorname{Per}\left(\{u_i=\tau\}\right).$$

From the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\operatorname{Per}\left(\{u_i^\dagger=\tau\}\right)^2 &\leq \operatorname{Per}\left(\{u_i=\tau\}\right)^2\\ &\leq \int_{\{u_i=\tau\}}\frac{1}{|\nabla u_i|}\int_{\{u_i=\tau\}}|\nabla u_i|\\ &\leq -\mu_i'(\tau)\int_{\{u_i=\tau\}}|\nabla u_i| \leq -\frac{\mu_i'(\tau)}{d_i}\left(-c\int_{\left\{u_i^\dagger\geq\tau\right\}} u_i^\dagger + \sum_{j=1}^{N}\int_{\left\{u_i^\dagger\geq\tau\right\}} l_{i,j}^\dagger\phi_j^\dagger\right).\end{aligned}$$

Since we are working in one dimension,

$$4 \leq \operatorname{Per}\left(\{u_i^\dagger=\tau\}\right)^2.$$

Furthermore, by definition of the rearrangement,

$$\int_{\{u_i>\tau\}} u_i = \int_0^{\mu_i(\tau)}\overline{u}_i.$$

We define, for $i\in[N]$,

$$k_i^{\mathbf{u}}:\xi\in[0,\frac{L_1}{2}]\mapsto\int_0^\xi\overline{u_i}.$$

From this definition, we obtain

$$(k_i^{\mathbf{u}})''(\mu_i(\tau)) = \overline{u_i}'(\mu_i(\tau)) = \frac{1}{\mu_i'(\tau)}.$$

With these notations, we obtain the following differential inequality: for any $\xi\in[0,L_1]$,

$$-4\left(k_i^{\mathbf{u}}\right)''(\xi) \leq -\frac{c}{d_i}k_i^{\mathbf{u}}(\xi) + \sum_{j=1}^{N}\int_0^\xi l_{i,j}^\dagger\phi_j^\dagger.$$

Furthermore,

$$k_i^{\mathbf{u}}(0) = 0.$$

Working with $\boldsymbol{\psi}$ instead of $\boldsymbol{\phi}$ and with $\mathbf{v}$ instead of $\mathbf{u}$, all the previous inequalities are equalities. Thus, with transparent notations, $k_i^{\mathbf{v}}$ solves the differential equation

$$-4\left(k_i^{\mathbf{v}}\right)''(\xi) \leq -\frac{c}{d_i}k_i^{\mathbf{v}}(\xi) + \sum_{j=1}^{N}\int_0^{\xi} l_{i,j}^{\dagger}\psi_j^{\dagger}.$$

Similarly,

$$k_i^{\mathbf{v}}(0) = 0.$$

Hence, from (22) the vector $\mathbf{K} := k^{\mathbf{u}} - k^{\mathbf{v}}$ satisfies, component-wise,

$$-4\mathbf{K}_i'' + \frac{c}{d_i}\mathbf{K}_i \leq 0\,, \mathbf{K}_i(0) = \mathbf{0}.$$

Finally, integrating both equations in $\mathbf{u}$ and $\mathbf{v}$ on the domain we obtain

$$\int_{[0,L_1]} \mathbf{u}_i = \frac{1}{c}\int_{[0,L_1]} (\mathbf{L}\boldsymbol{\phi})_i \leq \frac{1}{c}\int_{[0,L_1]} (\mathbf{L}\boldsymbol{\psi})_i = \int_{[0,L_1]} \mathbf{v}_i,$$

so that

$$\mathbf{K}(L_1) \leq \mathbf{0}.$$

From the maximum principle,

$$\mathbf{K} \leq \mathbf{0} \text{ in } (0, L_1).$$

However, this is exactly the desired conclusion. □

We now apply Proposition 3.36 to derive a comparison principle.

Proposition 3.37. *Assume* $\operatorname{diag}(\boldsymbol{\mathcal{P}}) = \partial_t - \mathbf{D}\Delta$ *for some diagonal matrix* $\mathbf{D}$ *with constant, positive diagonal entries.*

Let $\mathbf{u}_0 \in \mathcal{L}^{\infty}_{\text{per}}(\mathbb{R}, [\mathbf{0}, \boldsymbol{\infty}))$ *and let* $\mathbf{u}, \mathbf{v}$ *be the respective space-periodic solutions of*

$$\begin{cases} \boldsymbol{\mathcal{Q}}\mathbf{u} = \mathbf{0} & \text{in } (0,T)\times\mathbb{R}, \\ \mathbf{u} = \mathbf{u}_0 & \text{on } \{0\}\times\mathbb{R} \end{cases}$$

and

$$\begin{cases} \operatorname{diag}(\boldsymbol{\mathcal{P}})\mathbf{v} - \mathbf{L}^{\dagger}\mathbf{v} = \mathbf{0} & \text{in } (0,T)\times\mathbb{R}, \\ \mathbf{v} = \mathbf{u}_0^{\dagger} & \text{on } \{0\}\times\mathbb{R}. \end{cases}$$

Then, for all $t \in [0,T]$, $\mathbf{u}(t,\cdot) \prec \mathbf{v}(t,\cdot)$.

Proof. Following [5], we use a classical time discretization of the system.

Let $c > 0$ so large that $l_{i,i} + c \geq 0$ for all $i \in [N]$. Let $K \in \mathbb{N}$ and $\delta = \frac{T}{K} > 0$. We define, for any $\omega \in [K]$,

$$\mathbf{L}_{\omega} = K\int_{(\omega-1)\delta}^{\omega\delta}(\mathbf{L}(\tau,\cdot) + c\mathbf{I})\mathrm{d}\tau, \quad \mathbf{L}_{\omega,\dagger} = K\int_{(\omega-1)\delta}^{\omega\delta}(\mathbf{L}^{\dagger}(\tau,\cdot) + c\mathbf{I})\mathrm{d}\tau.$$

Clearly $\mathbf{L}_{\omega} + K\mathbf{I} \prec \mathbf{L}_{\omega,\dagger} + K\mathbf{I}$ and both are nonnegative and only depend on space. We set $\mathbf{u}^0 = \mathbf{u}_0$, $\mathbf{v}^0 = \mathbf{u}_0^{\dagger}$ and consider, for any $\omega \in [K]$, the space-periodic solutions of the elliptic systems

$$(K+c)\mathbf{u}^{\omega} - \mathbf{D}\Delta\mathbf{u}^{\omega} = K\mathbf{u}^{\omega-1} + \mathbf{L}_{\omega}\mathbf{u}^{\omega-1},$$

$$(K+c)\mathbf{v}^{\omega} - \mathbf{D}\Delta\mathbf{v}^{\omega} = K\mathbf{v}^{\omega-1} + \mathbf{L}_{\omega,\dagger}\mathbf{v}^{\omega-1},$$

By Proposition 3.36, for any $\omega \in [K]$, $\mathbf{u}^{\omega} \prec \mathbf{v}^{\omega}$. Passing to the limit $K \to \infty$ ends the proof. □

Proposition 3.38. *Assume* $\mathrm{diag}(\boldsymbol{\mathcal{P}}) = \partial_t - \mathbf{D}\Delta$ *for some diagonal matrix* $\mathbf{D}$ *with constant, positive diagonal entries.*

Then

$$\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}) \geq \lambda_{1,\mathrm{per}}(\mathrm{diag}(\boldsymbol{\mathcal{P}}) - \mathbf{L}^\dagger)$$

where $\mathbf{L}^\dagger$ *is the entry-wise periodic rearrangement of* $\mathbf{L}$*.*

Proof. The proof relies on Proposition 3.37.

We proceed as in [54] and introduce, for some $c > 0$ so large that $\mathbf{L} + c\mathbf{I} \geq \mathbf{0}$, the Poincaré mapping

$$\mathcal{G}_{\mathbf{L}+c\mathbf{I}} : \begin{cases} \mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}, \mathbb{R}^N) & \to \mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}, \mathbb{R}^N) \\ \mathbf{u}_0 & \mapsto \mathbf{u}(\mathbf{u}_0, \mathbf{L}+c\mathbf{I}; T, \cdot) \end{cases}$$

where $(t,x) \mapsto \mathbf{u}(\mathbf{u}_0, \mathbf{L}+c\mathbf{I}; t, x)$ is the solution of $\mathrm{diag}(\boldsymbol{\mathcal{P}})\mathbf{u} + c\mathbf{u} = (\mathbf{L}+c\mathbf{I})\mathbf{u}$ with initial condition $\mathbf{u}_0$.

We define $r(\mathbf{L}+c\mathbf{I})$ as the principal eigenvalue of the operator $\mathcal{G}_{\mathbf{L}+c\mathbf{I}}$. As is classical, this eigenvalue can be obtained as

$$r(\mathbf{L}+c\mathbf{I}) = \lim_{k\to\infty} \left\| \mathcal{G}^k_{\mathbf{L}+c\mathbf{I}} \right\|^{\frac{1}{k}},$$

where the notation $\| \ \|$ stands for the norm on the vector space of linear mappings from $\mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}, \mathbb{R}^N)$ into itself (for the $\mathcal{L}^\infty - \mathcal{L}^\infty$ norm), and the two quantities $r(\mathbf{L}+c\mathbf{I})$ and $\lambda_{1,\mathrm{per}}(\mathrm{diag}(\boldsymbol{\mathcal{P}}) - \mathbf{L})$ are immediately related through $r(\mathbf{L}+c\mathbf{I}) = -\frac{1}{T}\lambda_{1,\mathrm{per}}(\mathrm{diag}(\boldsymbol{\mathcal{P}}) - \mathbf{L})$. As a consequence, in order to obtain the required comparison result, it suffices to establish that, for any $\mathbf{u}_0 \in \mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}, \mathbb{R}^N)$,

$$\|\mathcal{G}_{\mathbf{L}+c\mathbf{I}}\mathbf{u}_0\|_{\mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R},\mathbb{R}^N)} \leq \|\mathcal{G}_{\mathbf{L}^\dagger+c\mathbf{I}}\mathbf{u}_0^\dagger\|_{\mathcal{L}^\infty_{\mathrm{per}}(\mathbb{R}\times\mathbb{R},\mathbb{R}^N)}.$$

Yet this is a direct consequence of Proposition 3.37. □

Remark 3.23. The previous result established the optimality of periodically rearranged coefficients when all the coefficients are rearranged one by one independently, namely when the coefficients are not correlated. However this optimality can fail if the coefficients are correlated, as the following counter-example shows.

We consider, in the one-dimensional case $n = 1$ with $L_1 = 2$, a space-periodic function χ whose restriction to $[-1,1]$ is the indicator function of $(-y,y)$, $0 < y < 1$, a real number η, and the matrix

$$\mathbf{L} : x \mapsto \begin{pmatrix} -1+\chi(x) & 1+\chi(x-\eta) \\ 1+\chi(x-\eta) & -1+\chi(x) \end{pmatrix}.$$

The vector $\mathbf{1}$ is a Perron–Frobenius eigenvector of $\mathbf{L}(x)$ with Perron–Frobenius eigenvalue $\mu_\eta(x) = \chi(x) + \chi(x-\eta)$. Let $\mathbf{u}$ be a periodic principal eigenfunction of $\boldsymbol{\mathcal{Q}} = \partial_t - \Delta - \mathbf{L}$. The function $u_\eta = \mathbf{1}^{\mathrm{T}}\mathbf{u}$ is positive, time homogeneous, space periodic and solves

$$-\Delta u_\eta = \mu_\eta u_\eta + \lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}})u_\eta.$$

Therefore $\lambda_{1,\mathrm{per}}(\boldsymbol{\mathcal{Q}}) = \lambda_{1,\mathrm{per}}(-\Delta - \mu_\eta)$, where the last operator is a scalar space-periodic elliptic operator. Note that χ and any spatial translation of it are invariant by periodic rearrangement: $\mathbf{L} = \mathbf{L}^\dagger$. Nonetheless, $\mathbf{L}$ is not optimal as soon as $\eta \neq 0$.

Indeed,

$$\mu_\eta(x) = \begin{cases} 0 & \text{if } x \in (-1, -y) \cup (y+\eta, 1), \\ 1 & \text{if } x \in (-y, -y+\eta) \cup (y, y+\eta), \\ 2 & \text{if } x \in (-y+\eta, y). \end{cases}$$

Hence all $(\mu_\eta)_{\eta\in\mathbb{R}}$ are piecewise-constant, space periodic functions of total mass equal to $4y > 0$. It is well-known, *cf.* [54], that among these the one that minimizes $\lambda_{1,\mathrm{per}}(-\Delta - \mu_\eta)$ is that corresponding to $\eta = 0$.

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