Long Term Risk

Lectures at Collège de France

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Objective

In financial economics risk-return tradeoffs show how expected rates of return and consequently asset prices are altered in response to changes in the exposure to the underlying shocks that impinge in the economy. In these lectures we will:

(a) Present some of the recent literature that is concerned with the effect of long run risk on returns and prices.

(b) Develop an analytical structure that reveals the long-run risk-return relationship in nonlinear continuous time Markov environments.

Motivation

- Evaluation of economic models of preferences and technologies using asset prices.
  - How long run-risks affect short term returns.
    - 6% equity premium
    - low risk-free rate
    - Equity market volatility of 19% per annum

- What are the long run implications of a model
  - Market microstructure, transaction costs... may make it hard to evaluate these models using short run data.
  - Behavioral biases
  - Long run risk-return frontier

- How risk averse agents value the risks of permanent shocks.
- How should we discount risky projects.

Utility

- C a consumption process in $[0, \infty]$

1. Standard additive utility

   - $$U_s = E_s[\int_s^\infty e^{b(s-t)} u(C_t) dt], \quad b > 0$$

   - $$u(C) = \frac{C^{1-a} - 1}{1-a}, \quad a > 0$$

     – If $a = 1$, $U(C) = \log C$.

2. Habit Formation

- An external (to the individual) habit process H
-

$$U_s = E_s[\int_s^{\infty} e^{b(s-t)ds} u(C_t, H_t)dt], \quad b > 0$$

- Campbell and Cochrane (1999), Menzly, Santos and Veronesi (2004)

$$u(C, H) = \frac{(C - H)^{1-a}}{1-a}, \quad a > 0$$

3. Kreps-Porteus utility

- A probability space with a d dimensional Brownian motion and associated (completed) filtration $\mathcal{F}_s$ and $E_s(\cdot) = E(\cdot|\mathcal{F}_s)$
- $\bar{W}^\gamma$ the continuation utility for a consumption path c from t on (conditional on current information)
- $\bar{W}^\gamma$ solves

$$\bar{W}^\gamma_s = E_s[\int_s^T \bar{f}^\gamma(C_t, \bar{W}^\gamma_t)dt + \frac{1}{2}A^\gamma(\bar{W}^\gamma_t)||\sigma_\gamma(t)||^2 + \bar{W}^\gamma_T]$$

- + Transversality condition

  – A^γ is a variance multiplier applying a penalty to the volatility of continuation utility σ_γ.

- $\gamma = (\rho, a),\ \rho \leq 1,\ a \geq 0,\ a \neq 1.$
- $\bar{f}^\gamma(C, x) = \frac{\beta}{\rho}\frac{C^\rho - x^\rho}{x^{\rho-1}},\ A^\gamma(x) = \frac{-a}{x}$
- a measures risk-aversion, $(1-\rho)^{-1}$ is the elasticity of intertemporal substitution.

- $\rho = 1 - a$ standard additive utility
- Transformation $W^\gamma = \phi^\gamma(\bar{W}^\gamma)$ of Duffie-Lions (1991) to eliminate variance multiplier A
- $$W_s^\gamma = E_s[\int_s^T f^\gamma(C_t, W_t^\gamma)dt + W_T^\gamma]$$
- $$f^\gamma(C, W) = \frac{b}{\rho} \frac{C^\rho - [(a-1)W]^{\frac{\rho}{a-1}}}{(a-1)^{\frac{\rho}{a-1}} W^{(\frac{\rho}{a-1} - 1)}}$$
- For $\rho = 0$,
$$f^{0,a}(C, x) = -bx\{(a-1)\log C + \log[(1-a)x]\}$$
- Duffie-Epstein (1992).

Stochastic discount factor

- $\{X_t : t \geq 0\}$ a Markov process in $(\Omega, \mathcal{F}, Pr)$ and $\mathcal{F}_t$ the associated (completed) filtration.
- A *Stochastic Discount Factor* S is a strictly positive adapted process with $S_0 = 1$ such that if $s \leq t$
$$\frac{E\left[S_t \Pi_t | \mathcal{F}_s\right]}{S_s} \tag{1}$$
is the price at time s of a claim to the payoff Π_t at t.
- $$\mathbb{S}_t \psi(x) = E\left[S_t \psi(X_t) | X_0 = x\right],$$
is the time-zero price of payoff $\psi(X_t)$.
- Law of one price with intermediate trading dates
- $\mathbb{S}_0 = \mathbb{I}$ and $\mathbb{S}_{t+u} = \mathbb{S}_t \mathbb{S}_u$
- Garman (1984), Rogers (1998)

- Let $(\theta_t X)_s = X_{t+s}$
- S_u is a function of the realization of the Markov process X that only depends on the history of X between dates 0 and u. Thus $S_u(\theta_t)$ only depends on the history of X between dates t and $t+u$.
- Consider payoffs at $t+u$ that are indicator functions of sets of histories observable at $t+u$, i.e. sets $B \in \mathcal{F}_{t+u}$, and again using intermediate trading dates and the law of one price one obtains:

$$E[S_{t+u}\mathbf{1}_B|X_0] = E[S_t E[S_u(\theta_t)\mathbf{1}_B|\mathcal{F}_t]|X_0] = E[S_t S_u(\theta_t)\mathbf{1}_B|X_0]$$

- $S_0 = 1$ and $S_{t+u} = S_t S_u(\theta_t)$.
- S_t is a **multiplicative functional.**

Example

- $$\begin{aligned} dX_t^f &= \xi_f(\bar{x}_f - X_t^f)dt + \sqrt{X_t^f}\sigma_f dB_t^f, \\ dX_t^o &= \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o \end{aligned}$$

  with $\xi_i > 0$, $\bar{x}_i > 0$ for $i = f, o$ and $2\xi_f \bar{x}_f \geq \sigma_f^2$ where $B = (B^f, B^o)$ is a bivariate standard Brownian motion.

- Per-capita consumption

  $$dc_t = X_t^o dt + \sqrt{X_t^f}\vartheta_f dB_t^f + \vartheta_o dB_t^o$$

  where $c_t = \log(C_t)$

- Interesting case:
  - $\sigma_o > 0,\ \vartheta_0 \geq 0$ (positive B^o's are unambiguously good)
  - $\sigma_f < 0,\ \vartheta_f > 0$ (positive B^f's are unambiguously good)

Breeden model

- Representative investor preferences are given by:

$$E \int_0^{\infty} \exp(-\mathsf{b}t) \frac{{C_t}^{1-\mathsf{a}} - 1}{1 - \mathsf{a}}$$

  for a and b strictly positive.

- In the standard case $S_t = \frac{e^{-\mathsf{b}t} u'(C_t)}{u'(C_0)}$

- Ito's Lemma implies that the stochastic discount factor in the Breeden model in this example is $S_t = \exp(A_t^s)$ where

$$A_t^s = -\mathsf{a} \int_0^t X_s ds - \mathsf{b}t - \mathsf{a} \int_0^t \sqrt{X_s^f} \vartheta_f dB_s^f - \mathsf{a} \int_0^t \vartheta_o dB_s^o.$$

Kreps-Porteus with $\rho = 0$

- Utility aggregator satisfies:

$$f^{0,a}(C, W) = -bW\{(a-1)\log C + \log[(1-a)W]\}$$

.

- Supress dependence of W on a
- Guess a continuation value process of the form:

$$W_t = \frac{1}{1-\mathsf{a}} \exp\left[(1-\mathsf{a})(\mathsf{w}_f X_t^f + \mathsf{w}_o X_t^o + \log C_t + \bar{\mathsf{w}})\right]$$

- The coefficients satisfy:

$$\begin{aligned} -\xi_f \mathsf{w}_f + \frac{(1-\mathsf{a})\sigma_f^2}{2}(\mathsf{w}_f)^2 + (1-\mathsf{a})\vartheta_f \sigma_f \mathsf{w}_f + \frac{(1-\mathsf{a})\vartheta_f^2}{2} &= \mathsf{b}\mathsf{w}_f \\ -\xi_o \mathsf{w}_o + 1 &= \mathsf{b}\mathsf{w}_o \\ \xi_f \bar{x}_f \mathsf{w}_f + \xi_o \bar{x}_o \mathsf{w}_o + \frac{(1-\mathsf{a})\sigma_o^2}{2}(\mathsf{w}_o)^2 + (1-\mathsf{a})\vartheta_o \sigma_o \mathsf{w}_o + \frac{(1-\mathsf{a})\vartheta_o^2}{2} &= \mathsf{b}\bar{\mathsf{w}}. \end{aligned}$$

- $\mathsf{w}_0 > 0$, w_f real if either $\vartheta_f \sigma_f \geq 0$ or $\vartheta_f \sigma_f < 0$ and

$\xi_f + \mathsf{b} \geq 2(\mathsf{a}-1)|\vartheta_f \sigma_f|$. In either case $\mathsf{w}_f < 0$ and we are interested in root with smallest absolute value.

- From Duffie-Epstein

$$S_t = \frac{e^{[\int_0^t f_W(C_s,W_s)ds]} f_C(C_t,W_t)}{f_C(C_0,W_0)}$$

- The stochastic discount factor is the product of two functionals. One is the exponential of:

$$A_t^B = -\int_0^t X_s^o ds - \mathsf{b}t - \int_0^t \sqrt{X_s^f}\vartheta_f dB_s^f - \int_0^t \vartheta_o dB_s^o.$$

The other is a martingale that is the exponential of:

$$\begin{aligned} A_t^{KP} &= (1-\mathsf{a})\left[\int_0^t \sqrt{X_s^f}(\vartheta_f + \mathsf{w}_f\sigma_f)dB_s^f + \int_0^t (\vartheta_o + \mathsf{w}_o\sigma_o)dB_s^o\right] \\ &- \frac{(1-\mathsf{a})^2}{2}\int_0^t X_s^f(\vartheta_f + \mathsf{w}_f\sigma_f)^2 ds - \frac{(1-\mathsf{a})^2(\vartheta_o + \mathsf{w}_o\sigma_o)^2}{2}t \end{aligned}$$

- For $\rho \neq 0$ expand the continuation value for ρ small
  - Expand stochastic discount factor for ρ small (Hansen *et al.* 2008)
  - For instance,

$$A_t^B = -(1-\rho)[\int_0^t X_s^o ds + \mathsf{b}t + \int_0^t \sqrt{X_s^f}\vartheta_f dB_s^f + \int_0^t \vartheta_o dB_s^o].$$

Heterogeneity

- Production.
- When consumers are heterogeneous individual consumption is determined in equilibrium.
- With complete markets, first welfare theorem guarantees that stochastic discount factor can be calculated using an "artificial" consumer who is a weighted average of actual consumers.
- Case where tradeable claims are a subfield of $\mathcal{F}_t$.
  - There are tradeable claims on aggregate uncertainty but not on individual risk.
- Heterogeneous beliefs.*

The basic Markov process

- $\{X_t : t \geq 0\}$ be a continuous time Markov process on a state space $D_0 \subseteq \mathbb{R}^n$. The sample paths of $\{X_t : t \geq 0\}$ are continuous from the right and with left limits. Let $\mathcal{F}_t$ be completion of the sigma algebra generated by $\{X_u : 0 \leq u \leq t\}$.
- Often treat the case where X is stationary.
- Semimartingale $X = X^c + X^j$
- An $\mathcal{F}_t$ n-dimensional Brownian motion $\{B_t\}$
- $X_t^c = X_0 + \int_0^t \xi(X_u)du + \int_0^t \sigma(X_u)dB_u$
- $(\xi, \Sigma, \eta), \quad \Sigma = \sigma\sigma'$

- X^j with a finite number of jumps in any finite interval and compensator $\eta[dy|x]dt$.
  -
    $$dX_t^j = \int_{\mathbb{R}^n} y\zeta(dy, dt)$$

    where $\zeta = \zeta(\cdot, \cdot; \omega)$ is a random counting measure. That is, for each ω, $\zeta(b, [0, t]; \omega)$ gives the total number of jumps in $[0, t]$ with a size in the Borel set b in the realization ω.
  - Finite number of jumps implies there is a finite measure $\eta(dy|x)dt$ that is the *compensator* of the random measure ζ.
  - The (unique) predictable random measure, such that for each predictable stochastic function $f(x, t; \omega)$, the process $\int_0^t \int_{\mathbb{R}^n} f(y, s; \omega)\zeta(dy, ds; \omega) - \int_0^t \int_{\mathbb{R}^n} f(y, s; \omega)\eta[dy|X_{s-}(\omega)]ds$ is a martingale

Building non-stationary processes

- **A functional**: A real-valued process $\{M_t : t \geq 0\}$ adapted, (with a version that is) right continuous with left limits.
- The functional $\{M_t : t \geq 0\}$ is **multiplicative** if $M_0 = 1$, and $M_{t+u} = M_u(\theta_t)M_t$.
- Product of multiplicative processes is multiplicative.
- If M strictly positive $\log(M)$ will satisfy an additive property.
- A functional is **additive** if $A_0 = 0$ and $A_{t+u} = A_u(\theta_t) + A_t$, for each nonnegative t and u.
- Parameterize $\log(M)$

Multiplicative semigroups

- A family of linear operators $\{\mathbb{T}_t : t \geq 0\}$ in a Banach space L is a one-parameter **semigroup** if $\mathbb{T}_0 = \mathbb{I}$ and $\mathbb{T}_{t+s} = \mathbb{T}_t \mathbb{T}_s$ for all $s, t \geq 0$.

Proposition 1. *Let M be a a multiplicative functional such that for each $\psi \in L$, $E\left[M_t \psi(X_t) | X_0 = x\right] \in L$. Then*

$$\mathbb{M}_t \psi(x) = E\left[M_t \psi(X_t) | X_0 = x\right]$$

is a semigroup in L.

Proof. For $\psi \in L$, $\mathbb{M}_0 \psi = \psi$ and:

$$\begin{aligned}
\mathbb{M}_{t+u}\psi(x) &= E\left[E\left[M_{t+u}\psi(X_{t+u})|\mathcal{F}_t\right]|X_0 = x\right] \\
&= E\left[E\left[M_t M_u(\theta_t)\psi[\theta_t X_u]|\mathcal{F}_t\right]|X_0 = x\right] \\
&= E\left[M_t E\left[M_u(\theta_t)\psi[\theta_t X_u]|X_t\right]|X_0 = x\right] \\
&= E\left[M_t \mathbb{M}_u \psi(X_t)|X_0 = x\right] = \mathbb{M}_t \mathbb{M}_u \psi(x).
\end{aligned}$$

□

Parameterization of multiplicative processes

- (β, γ, κ) that satisfies:

  a) $\beta : D_0 \to \mathbb{R}$ and $\int_0^t \beta(X_u)du < \infty$ for every positive t;

  b) $\gamma : D_0 \to \mathbb{R}^m$ and $\int_0^t |\gamma(X_u)|^2 du < \infty$ for every positive t;

  c) $\kappa : D_0 \times D_0 \to \mathbb{R}$, $\kappa(x, x) = 0$ for all $x \in D_0$.

$$A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u) \cdot dB_u + \sum_{0 \leq u \leq t} \kappa(X_u, X_{u-})$$

- $A_t = \psi(X_t) - \psi(X_0)$
- Exponential of additive processes (strictly positive multiplicative functionals).
  - Parameterized by the additive process (β, γ, κ)

Valuation functionals

- A *valuation functional* $\{V_t : t \geq 0\}$ is a multiplicative functional such that $\{V_t S_t : t \geq 0\}$ is a martingale.
  - Example: result of continuously reinvesting the payouts of an investment
- If V parameterized by $(\beta_v, \gamma_v, \kappa_v)$ is strictly positive the implied net return evolution is:

$$\frac{dV_t}{V_{t-}} = \left[\beta_v(X_{t-}) + \frac{|\gamma_v(X_{t-})|^2}{2}\right] dt + \gamma_v(X_{t-})' dB_t + e^{\kappa_v(X_t, X_{t-})} - 1$$

- The expected net rate of return is:

$$\varepsilon_v \doteq \beta_v + \frac{|\gamma_v|^2}{2} + \int \left(\exp\left[\kappa_v(y,\cdot)\right] - 1\right)\eta(dy|\cdot).$$

- VS is the exponential of an additive process parameterized by: $\beta = \beta_v + \beta_s,\ \gamma = \gamma_v + \gamma_s,\ \kappa = \kappa_v + \kappa_s.$

Proposition 2. *A valuation process parameterized by* $(\beta_v, \gamma_v, \kappa_v)$ *satisfies the pricing restriction:*

$$\beta_v + \beta_s = -\frac{|\gamma_v + \gamma_s|^2}{2} - \int \left(e^{[\kappa_v(y,\cdot)+\kappa_s(y,\cdot)]} - 1\right)\eta(dy|\cdot).$$

Corollary 1. *The required mean rate of return for the risk exposure* (γ_v, κ_v) *is* $\varepsilon_v =$

$$-\beta_s - \gamma_v \cdot \gamma_s - \frac{|\gamma_s|^2}{2} - \int \left(e^{[\kappa_v(y,\cdot)+\kappa_s(y,\cdot)]} - e^{\kappa_v(y,\cdot)}\right)\eta(dy,\cdot)$$

- Instantaneous run risk-return frontier

$$(\gamma_v, e^{\kappa_v}) \rightsquigarrow \varepsilon_v,$$

in particular, $-\gamma_s$ is the price of Brownian risk.

- Self-financing and multiplicative property.

Example

- Breeden model
  - $-\gamma_s = (\mathsf{a}\sqrt{X_s^f}\vartheta_f, \mathsf{a}\vartheta_o)$.
  - $dc_s = X_s^o ds + \sqrt{X_s^f}\vartheta_f dB_s^f + \vartheta_o dB_s^o$.
  - a is ratio between risk prices and consumption volatility.
  - Instantaneous risk-free rate: $b + aX_s^o - \frac{\mathsf{a}^2(X_s^f {\vartheta_f}^2 + \vartheta_o^2)}{2}$
- Kreps-Porteus model ($\rho = 0$)
  - $-\gamma_s = (\mathsf{a}\sqrt{X_s^f}\vartheta_f + (\mathsf{a}-1)\sqrt{X_s^f}\mathsf{w}_f\sigma_f, \mathsf{a}\vartheta_o + (\mathsf{a}-1)\mathsf{w}_o\sigma_o)$
  - Vol of rate of growth of consumption and vol of vol matter.
  - Even when $\vartheta_o = 0$, B^o is priced.
    - Since $\mathsf{w}_o = \frac{1}{b+\xi_0}$, if $a > 1$ risk-price increases more the less X^o means revert and the less the future is discounted.
  - Instantaneous risk-free rate: $b + X_s^o - \frac{(X_s^f {\vartheta_f}^2 + \vartheta_o^2)}{2} - (\mathsf{a}-1)X_s^f\vartheta_f(\vartheta_f + \mathsf{w}_f\sigma_f) - (\mathsf{a}-1)\vartheta_o(\vartheta_o + \mathsf{w}_o\sigma_o)$

Habit

- Modify power utility as:

$$E\int_0^\infty e^{-\mathsf{b}t}\frac{(C_t - H_t)^{1-\mathsf{a}} - 1}{1-\mathsf{a}}$$

- $S_t = e^{-\mathsf{b}t}\frac{(C_t-H_t)^{-\mathsf{a}}}{(C_0-H_0)^{-\mathsf{a}}}$

- $S_t = S_t^{\mathsf{B}}\frac{X_t}{X_0},\ X_t = \frac{(C_t)^{\mathsf{a}}}{(C_t-H_t)^{\mathsf{a}}},\ S_t^{\mathsf{B}} = e^{-\mathsf{b}t}\frac{C_t^{-\mathsf{a}}}{C_0^{-\mathsf{a}}}$

- Menzly, Santos and Veronesi
  - $dX_t = \xi(\mu_x - X_t)dt - \kappa(X_t - \lambda)dB_t,\ \mu_x > \lambda,\ X_0 > \lambda,\ \kappa > 0.$
  - $dc_t = u_c dt + \vartheta_o dB_t$
  - Risk-price: $\mathsf{a}\vartheta_o + \frac{\kappa(X_t-\lambda)}{X_t} > \mathsf{a}\vartheta_o$, but a does no longer corresponds to risk-aversion (with respect to consumption or wealth gambles).

- Campbell and Cochrane
  - $Y_t = \frac{1}{\mathsf{a}} \log X_t$
  - $dY_t = \xi_y(\mu_y - Y_t)dt + \frac{\lambda(Y_t)}{|\vartheta_o|}\vartheta_o dB_t$
  - $\lambda(0) = 0$ (Guarantees $0 \leq \frac{H_t}{C_t} \leq 1$)
  - Short rate of interest $r_t = r^*$
  - $\lambda(y) = |\vartheta_o| - \sqrt{\vartheta_o^2 - 2\xi y/\mathsf{a}}$

Some notation introduced previously

- $X_t = X_t^c + X_t^j$ a Markov process in $[0, \infty)$, $\mathcal{F}_t$.
- An $\mathcal{F}_t$ n-dimensional Brownian motion $\{B_t\}$
- $X_t^c = X_0 + \int_0^t \xi(X_u)du + \int_0^t \sigma(X_u)dB_u$
- A compensator for X^j, η.
- (ξ, σ, η).
- A multiplicative functional is $\mathcal{F}_t$ measurable and satisfies $M_0 = 1$ and $M_{t+u} = M_t M_u(\theta_t)$.
  - Examples:
    1. $M_t^1 = e^{[\int_0^t \beta(X_s)ds]}$
       * Since, $\int_0^{t+u} \beta(X_s)ds = \int_0^t \beta(X_s)ds + \int_0^u \beta(X_{t+s})ds$
    2. $M_t^2 = e^{[\int_0^t \gamma(X_s)dB_s]}$
    3. $M_t^3 = e^{[\sum_{0 \le s \le t} \kappa(X_s, X_{s-})]}$, with $\kappa(x, x) = 0$.
    4. $M = M^1 \times M^2 \times M^3$, $M = (\beta, \gamma, \kappa)$.

- $\mathbb{M}_t\psi(x) = E[M_t\psi(X_t)|X_0 = x]$ is a semigroup.
  - $\mathbb{M}_0 = I$
  - $\mathbb{M}_{t+u} = \mathbb{M}_t(\mathbb{M}_u)$
- S_t a stochastic discount factor, that is $S_0 = 1$ and, for any ($\mathcal{F}_t$-measurable) payoff Π_t, $\frac{E[S_t\Pi_t|\mathcal{F}_s]}{S_s}$ is the price at time s of a claim to the payoff Π_t at t.
- V is a valuation functional if V is multiplicative and VS is a martingale
- If V is a valuation functional parameterized by $(\beta_v, \gamma_v, \kappa_v)$ the martingale restriction gives β_v as a function of (γ_v, κ_v).
- Restricts expected return ϵ_v of V.
- Mapping $(\gamma_v, \kappa_v) \hookrightarrow \epsilon_v$ is instantaneous risk-return relationship.

Long run risk

- Develop "long-run" counterpart of risk-return tradeoff
- Slope of the term-structure of "risk-prices."
- Which aspects of an economic model have transient as opposed to permanent effects.

Growth in payouts

- Risk-premia depend on risk exposure and price of that exposure.
- G a growth process: G adapted, $G_0 = 1$ and $G_{t+u} = G_t G_u(\theta_t)$
- Parametrization $G = e^{(A^g)}$

$$A_t^g = \int_0^t \beta^g(X_u)du + \int_0^t \gamma^g(X_u) \cdot dB_u + \sum_{0 \le u \le t} \kappa^g(X_u, X_{u-})$$

- $M = SG$ is also multiplicative
- $$\mathbb{M}_t \psi(x) = E\left[M_t \psi(X_t) | X_0 = x\right],$$

  is the time-zero price of payoff $G\psi(X_t)$.
- $\mathbb{M}_0 = \mathbb{I}$ and $\mathbb{M}_{t+u} = \mathbb{M}_t \mathbb{M}_u$

Objective: Multiplicative decomposition

- Establish decomposition for multiplicative functionals:

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{\varphi(X_0)}{\varphi(X_t)}\right]$$

  where

  - ρ is a deterministic growth rate;
  - $\hat{M}_t$ is a multiplicative martingale;
  - φ is a strictly positive function of the Markov state;

- If X is stationary, $\frac{\varphi(X_0)}{\varphi(X_t)}$ stationary component, $\hat{M}$ the martingale component of M, and ρ its growth rate.
  - Not entirely correct because of possible correlation between stationary and martingale components.

Implications of multiplicative decomposition

- If $\hat{M}$ is a martingale for $F \in \mathcal{F}_t$

$$\hat{P}r(F) = E[\hat{M}_t \mathbf{1}_F]$$

- X remains Markovian.

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$$E\left[M_t \psi(X_t) | X_0 = x\right] = \exp(\rho t)\phi(x) E_{\hat{P}r}\left[\frac{\psi(X_t)}{\phi(X_t)} | X_0 = x\right]$$

- $\exp(-\rho t)\phi(X_t)$ as a *numeraire*. Applicable when the multiplicative process does not define a price.

- Suppose there exists a stationary distribution $\hat{\varsigma}$ for X under $\hat{P}r$.

- If in addition to stationarity, "stochastic stability" holds

$$\lim_{t\to\infty} E_{\hat{P}r}\left[\psi(X_t)|X_0 = x\right] = \int \psi(X_t) d\hat{\varsigma},$$

whenever $\int \psi(X_t) d\hat{\varsigma}$ is well defined, then

$$\begin{aligned}
\lim_{t\to\infty} e^{-\rho t} E[M_t \psi(X_t) \quad &| \quad X_0 = x] \\
&= \lim_{t\to\infty} E\left(\hat{M}_t \left[\frac{\psi(X_t)}{\phi(X_t)}\right] |X_0 = x \right) \phi(x) \\
&= (\int \frac{\psi(X_t)}{\phi(X_t)} d\hat{\varsigma}) \phi(x)
\end{aligned}$$

  – ρ is the (deterministic) growth rate
  – All state dependence is given by the eigenfunction ϕ
  – Mapping "risk in M" to ρ is a long-run risk-return frontier
  – Approximation is "good" for ψ with $|\int \frac{\psi(X_t)}{\phi(X_t)} d\hat{\varsigma}| < \infty$

Long term bonds

- S a stochastic discount factor

$$S_t = \exp(\rho t)\hat{M}_t \frac{\phi(X_0)}{\phi(X_t)}$$

- Prices of long term discount bonds:

$$\exp(-\rho t) E\left(S_t | X_0 = x\right) \approx c\phi(x)$$

- Alvarez and Jerman [2005] estimate the volatility of $\frac{\hat{M}_{t+1}}{\hat{M}_t}$ as a proportion of volatility of $\frac{S_{t+1}}{S_t}$. (around 75-100%.). Transitory component has low estimated conditional volatility.

To do

- Strategy to establish decomposition (1)
  - Perron-Frobenius
- Uniqueness (2)
- Existence
- Stationarity, recurrence
- Some applications (3)

Generators

- Associate to each ψ a function χ such that $M_t \chi(X_t)$ is the "expected time derivative" of $M_t \psi(X_t)$.
- A Borel function ψ is in the domain of the **extended generator** $\mathbb{A}$ of the multiplicative functional M_t if for a Borel function χ, $N_t = M_t \psi(X_t) - \psi(X_0) - \int_0^t M_s \chi(X_s) ds$ is a local martingale with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$. The extended generator assigns χ to ψ and we write $\chi = \mathbb{A}\psi$.
- Extended generator associated with Markov Process (*e.g.* Revuz and Yor)
- If ψ smooth, Ito's lemma.
- Example: If $M \equiv 1$, $X_t \in R$, and X described by (ξ, σ^2), $\mathbb{A}\psi = \xi\psi' + \frac{1}{2}\sigma^2\psi''$
- Example: If $M = e^{bt}$, $\mathbb{A}\psi = b\psi + \xi\psi' + \frac{1}{2}\sigma^2\psi''$

- X described by (ξ, σ), M_t the exponential of (β, γ)
  $\mathbb{A}\psi(x) =$
  $[\beta(x) + \frac{|\gamma(x)|^2}{2}]\psi(x) + [\xi(x) + \sigma(x)\gamma(x)]\frac{\partial\psi(x)}{\partial x} + \frac{1}{2}\text{trace}(\sigma\sigma'\frac{\partial^2\psi(x)}{\partial x\partial x'})$
- X described by (ξ, σ, η), M_t the exponential of (β, γ, κ)
  $\mathbb{A}\psi(x) = [\beta(x) + \frac{|\gamma(x)|^2}{2} + \int (\exp[\kappa(y,x)] - 1)\,\eta(dy,x)]\psi(x) + [\xi(x) + \sigma(x)\gamma(x)]\frac{\partial\psi(x)}{\partial x} + \frac{1}{2}\text{trace}(\sigma\sigma'\frac{\partial^2\psi(x)}{\partial x\partial x'}) + \int_{R^n - \{x\}}[\psi(y) - \psi(x)]\exp[\kappa(y,x)]\,\eta(dy|x)$

Eigenfunctions and martingales

- A Borel function ϕ is an **eigenfunction** of the extended generator (with **eigenvalue** ρ) if $\mathbb{A}\phi = \rho\phi$.
- $N_t = M_t\phi(X_t) - \phi(X_0) - \rho \int_0^t M_s\phi(X_s)ds$ is a $\mathcal{F}_t$ local martingale.
- Set $Y_t = M_t\phi(X_t)$. Since $dN_t = dY_t - \rho Y_{t-}dt$, integration by parts yields:

$$\begin{aligned} \exp(-\rho t)Y_t - Y_0 &= -\int_0^t \rho \exp(-\rho s)Y_{s-}ds + \int_0^t \exp(-\rho s)dY_s \\ &= \int_0^t \exp(-\rho s)dN_s. \end{aligned}$$

- $\exp(-\rho t)M_t\phi(X_t)$ is a local martingale.

- A **principal eigenfunction** of the extended generator is an eigenfunction that is strictly positive.
- If ϕ is a principal eigenfunction,

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{\phi(X_0)}{\phi(X_t)}\right].$$

where $\hat{M}_t = \exp(-\rho t)M_t \frac{\phi(X_t)}{\phi(X_0)}$ is a multiplicative local martingale.

- $\hat{M}_t \geq 0$ and hence a super-martingale ($E[\hat{M}_t|\mathcal{F}_s] \leq \hat{M}_s$).

Eigenfunctions of the semigroup

Proposition 3. *If ϕ is a principal eigenfunction with eigenvalue ρ for the extended generator of the multiplicative functional M, then for each $t \geq 0$, $\exp(\rho t)\phi \geq \mathbb{M}_t\phi$. If, in addition, $\hat{M}$ is a martingale then, for each $t \geq 0$*

$$\mathbb{M}_t\phi = \exp(\rho t)\phi. \tag{2}$$

Conversely, if ϕ is strictly positive, $\mathbb{M}_t\phi$ is well defined for $t \geq 0$, and (2) holds, then $\hat{M}$ is a martingale.

-
  $$1 \geq E[\hat{M}_t|X_0 = x] = \frac{\exp(-\rho t)}{\phi(x)} E[M_t\phi(X_t)|X_0 = x],$$

  with equality when $\hat{M}$ is a martingale, that is (2) . Conversely, using (2) and the multiplicative property of M one obtains, $E[\exp(-\rho t)M_t\phi(X_t)|\mathcal{F}_s] = \exp(-\rho t)M_s E[M_{t-s}(\theta_s)\phi(X_t)|X_s] = \exp(-\rho s)M_s\phi(X_s)$.

Markov diffusion example continued

- $M = \exp(A)$ where:

$$A_t = \bar{\beta}t + \int_0^t \beta_f X_s^f ds + \int_0^t \beta_o X_s^o ds + \int_0^t \sqrt{X_s^f}\gamma_f dB_s^f + \int_0^t \gamma_o dB_s^o,$$

$$\begin{aligned} dX_t^f &= \xi_f(\bar{x}_f - X_t^f)dt + \sqrt{X_t^f}\sigma_f dB_t^f, \\ dX_t^o &= \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o \end{aligned}$$

- X described by (ξ, σ), M_t the exponential of (β, γ); $\mathbb{A}\psi(x)$ $= [\beta(x) + \frac{|\gamma(x)|^2}{2}]\psi(x) + [\xi(x) + \sigma(x)\gamma(x)]\frac{\partial\psi(x)}{\partial x} + \frac{1}{2}\mathrm{trace}(\sigma\sigma'\frac{\partial^2\psi(x)}{\partial x\partial x'})$
- Guess an eigenfunction : $\exp(\mathsf{c}_f x_f + \mathsf{c}_o x_o)$.

$$\begin{aligned} \rho &= \bar{\beta} + \beta_f x_f + \beta_o x_o + \frac{\gamma_f^2}{2}x_f + \frac{\gamma_o^2}{2} \\ &\quad + \mathsf{c}_f[\xi_f(\bar{x}_f - x_f) + x_f\gamma_f\sigma_f] + \mathsf{c}_o[\xi_o(\bar{x}_o - x_o) + \gamma_o\sigma_o] \\ &+ (\mathsf{c}_f)^2 x_f \frac{\sigma_f^2}{2} + (\mathsf{c}_o)^2\frac{\sigma_o^2}{2} \end{aligned}$$

- $$\begin{aligned} 0 &= \beta_f + \frac{\gamma_f^2}{2} + \mathsf{c}_f(\gamma_f \sigma_f - \xi_f) + (\mathsf{c}_f)^2 \frac{\sigma_f^2}{2} \\ 0 &= \beta_o - \mathsf{c}_o \xi_o. \end{aligned}$$

- $$\mathsf{c}_f = \frac{(\xi_f - \gamma_f \sigma_f) \pm \sqrt{(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 \left(2\beta_f + \gamma_f^2\right)}}{(\sigma_f)^2} \tag{3}$$

  provided that $(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 \left(2\beta_f + \gamma_f^2\right) \geq 0.$

- $$\mathsf{c}_o = \frac{\beta_o}{\xi_o}. \tag{4}$$

- The resulting eigenvalue is:

$$\rho = \bar{\beta} + \frac{\gamma_o^2}{2} + \mathsf{c}_f \xi_f \bar{x}_f + \mathsf{c}_o(\xi_o \bar{x}_o + \gamma_o \sigma_o) + (\mathsf{c}_o)^2 \frac{\sigma_o^2}{2}. \tag{5}$$

Write

$$\hat{M}_t = \exp(-\rho t) M_t \frac{\exp(\mathsf{c}_f X_t^f + \mathsf{c}_o X_t^o)}{\exp(\mathsf{c}_f X_0^f + \mathsf{c}_o X_0^o)},$$

- Verify: $\hat{M}_t = \exp(\hat{A}_t)$ where, $\hat{A}_t = \int_0^t \sqrt{X_s^f}(\gamma_f + \mathsf{c}_f \sigma_f) dB_s^f + \int_0^t (\gamma_o + \mathsf{c}_o \sigma_o) dB_s^o - \frac{(\gamma_f + \mathsf{c}_f \sigma_f)^2}{2} \int_0^t X_s^f ds - \frac{(\gamma_o + \mathsf{c}_o \sigma_o)^2}{2} t.$
- $\hat{M}$ is always a martingale. Use $\hat{M}$ to change probability.
- $\tilde{B}_t^f = B_t^f - \int_0^t \sqrt{X_s^f}(\gamma_f + \mathsf{c}_f \sigma_f) dt$ and $\tilde{B}_t^o = B_t^o - (\gamma_o + \mathsf{c}_o \sigma_o) t$ form a Browian in the new probability space.
- The resulting (twisted) drifts are:

$$\xi_o(\bar{x}_o - x_o) + \sigma_o(\gamma_o + \mathsf{c}_o \sigma_o).$$

$$\xi_f(\bar{x}_f - x_f) + x_f \sigma_f(\gamma_f + \mathsf{c}_f \sigma_f),$$

- Only the negative root produces mean reversion and a stationary distribution.

- Suppose that X^o solves instead:

$$dX_t^o = \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o + dZ_t$$

where Z is a pure jump process whose jumps have a fixed probability distribution ν on $\mathbb{R}$ and arrive with intensity $\varpi_1 x_f + \varpi_2$ with $\varpi_1 \geq 0$, $\varpi_2 \geq 0$. Suppose that the additive functional A has an additional jump term modeled using $\kappa(y, x) \doteq \bar{\kappa}(y_o - x_o)$ for $y \neq x$ and $\int \exp\left[\bar{\kappa}(z)\right] d\nu(z) < \infty$.

- Add to the generator $\mathbb{A}$ an extra term:

$$(\varpi_1 x_f + \varpi_2) \int \left[\phi(x_f, x_o + z) - \phi(x_f, x_o)\right] \exp\left[\bar{\kappa}(z)\right] d\nu(z).$$

- $\phi(x) = \exp(\mathsf{c}_f x_f + \mathsf{c}_o x_o)$ the extra term reduces to:

$$(\varpi_1 x_f + \varpi_2) \exp(\mathsf{c}_f x_f + \mathsf{c}_o x_o) \int \left[\exp(\mathsf{c}_o z) - 1\right] \exp\left[\bar{\kappa}(z)\right] d\nu(z).$$

- $$\mathsf{c}_o = \frac{\beta_o}{\xi_o}$$

- c_f must solve:

  $0 = \beta_f + \frac{\gamma_f^2}{2} + \mathsf{c}_f(\gamma_f \sigma_f - \xi_f) + (\mathsf{c}_f)^2 \frac{\sigma_f^2}{2} +$
  $\varpi_1 \int \left[\exp\left(\frac{\beta_o}{\xi_o} z\right) - 1\right] \exp\left[\bar{\kappa}(z)\right] d\nu(z).$

- The resulting eigenvalue is $\rho = \bar{\beta} + \frac{\gamma_o^2}{2} + \mathsf{c}_f \xi_f \bar{x}_f + \mathsf{c}_o(\xi_o \bar{x}_o +$
  $\gamma_o \sigma_o) + (\mathsf{c}_o)^2 \frac{\sigma_o^2}{2} + \varpi_2 \int \left[\exp\left(\frac{\beta_o}{\xi_o} z\right) - 1\right] \exp\left[\bar{\kappa}(z)\right] d\nu(z).$

Uniqueness

- Suppose the multiplicative process M is strictly positive and ϕ is a principal eigenfunction of the extended generator $\mathbb{A}$ for which the associated process $\{\hat{M}_t : t \geq 0\}$ is a martingale.
- Suppose there exists an invariant probability measure $\hat{\varsigma}$ for X in $(\Omega, \hat{P}r)$.
  - $\hat{\mathbb{A}}$ the generator associated with the multiplicative process $\hat{M}$. Suppose there exists probability measure $\hat{\varsigma}$ such that

    $$\int \hat{\mathbb{A}}\psi d\hat{\varsigma} = 0$$

    for all ψ in the L^∞ domain of the generator $\hat{\mathbb{A}}$.
- Let $\hat{E}$ and $\hat{\mathcal{P}r}$ denote the expectation operator and the probability measure associated with $\hat{M}$ and $\hat{\varsigma}$. The process $\hat{M}$ determines the distorted transition probabilities and $\hat{\varsigma}$ is the initial distribution.

- Suppose that for some $\Delta > 0$ the discrete time process $\hat{X}_{\Delta j}$, $j = 1, 2...$, satisfies "stochastic stability".
  -
    $$\lim_{j \to \infty} \hat{E}\left[\psi(X_{\Delta_j})|X_0 = x\right] = \int \psi(y) d\hat{\varsigma},$$

    whenever $\int \psi(y) d\hat{\varsigma}$ is well defined.
- Then the associated eigenvalue ρ is the smallest eigenvalue associated with a principal eigenfunction. Furthermore, ϕ is the unique eigenfunction (up to scale and $\hat{\varsigma}$ a.s.) associated with ρ.

- Proof: Consider another principal eigenfunction ϕ^* with associated eigenvalue ρ^*. Then:

$$\begin{aligned} \mathbb{M}_t \phi(x) &= \exp(\rho t)\phi(x) \\ \mathbb{M}_t \phi^*(x) &\leq \exp(\rho^* t)\phi^*(x). \end{aligned}$$

If $\hat{M}$ is the martingale associated with the eigenvector ϕ, then $E\left[\hat{M}_t \frac{\phi^*(X_t)}{\phi(X_t)} | X_0 = x\right] = \exp[-\rho t] E\left[M_t \phi^*(X_t) | X_0 = x\right] \leq \exp[(\rho^* - \rho)t] \frac{\phi^*(x)}{\phi(x)}$. Since the discrete-time process $\hat{M}$ satisfies stochastic stability the left-hand side converges to $\int \frac{\phi^*(y)}{\phi(y)} d\hat{\varsigma}$ for $t = \Delta j$ as the integer j tends to ∞. While the limit could be $+\infty$, it must be strictly positive, implying, that $\rho \leq \rho^*$. If $\rho^* = \rho$ then for each x,

$$\int \frac{\phi^*(y)}{\phi(y)} d\hat{\varsigma} \leq \frac{\phi^*(x)}{\phi(x)}.$$

Hence the ratio of the two eigenfunctions is constant ($\hat{\varsigma}$ a.s.).

Some notation introduced previously

- $X_t = X_t^c + X_t^j$ a Markov process in $[0, \infty)$, $\mathcal{F}_t$.
- An $\mathcal{F}_t$ n-dimensional Brownian motion $\{B_t\}$
- $X_t^c = X_0 + \int_0^t \xi(X_u)du + \int_0^t \sigma(X_u)dB_u$
- A compensator for X^j, η.
- (ξ, σ, η).
- A multiplicative functional is $\mathcal{F}_t$ measurable and satisfies $M_0 = 1$ and $M_{t+u} = M_t M_u(\theta_t)$.
  - Examples:
    1. $M_t^1 = e^{[\int_0^t \beta(X_s)ds]}$
       * Since, $\int_0^{t+u} \beta(X_s)ds = \int_0^t \beta(X_s)ds + \int_0^u \beta(X_{t+s})ds$
    2. $M_t^2 = e^{[\int_0^t \gamma(X_s)dB_s]}$
    3. $M_t^3 = e^{[\sum_{0 \le s \le t} \kappa(X_s, X_{s-})]}$, with $\kappa(x, x) = 0$.
    4. $M = M^1 \times M^2 \times M^3$, $M = (\beta, \gamma, \kappa)$.

- $\mathbb{M}_t\psi(x) = E[M_t\psi(X_t)|X_0 = x]$ is a semigroup.
  - $\mathbb{M}_0 = I$
  - $\mathbb{M}_{t+u} = \mathbb{M}_t(\mathbb{M}_u)$
- Decompose multiplicative functionals as:

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{\phi(X_0)}{\phi(X_t)}\right]$$

- $\hat{M}$ a multiplicative martingale
- $\hat{P}r(F) = E[\hat{M}_t \mathbf{1}_F]$
-

$$E\left[M_t\psi(X_t)|X_0 = x\right] = \exp(\rho t)\phi(x)E_{\hat{P}r}\left[\frac{\psi(X_t)}{\phi(X_t)}|X_0 = x\right]$$

- Suppose there exists a stationary distribution $\hat{\varsigma}$ for X under $\hat{P}r$.

- If in addition to stationarity, "stochastic stability" holds

$$\lim_{t\to\infty} E_{\hat{P}r}\left[\psi(X_t)|X_0 = x\right] = \int \psi(y) d\hat{\varsigma},$$

whenever $\int \psi(X_t) d\hat{\varsigma}$ is well defined, then

$$\begin{aligned} \lim_{t\to\infty} e^{-\rho t} E[M_t \psi(X_t)|X_0 = x] &= \\ \lim_{t\to\infty} E\left(\hat{M}_t \left[\frac{\psi(X_t)}{\phi(X_t)}\right] |X_0 = x\right) \phi(x) &= \left(\int \frac{\psi(y)}{\phi(y)} d\hat{\varsigma}\right)\phi(x) \end{aligned}$$

  - ρ is the (deterministic) growth rate
  - All state dependence is given by the eigenfunction ϕ
  - Mapping "risk in M" to ρ is a long-run risk-return frontier
  - Approximation is "good" for ψ with $|\int \frac{\psi(y)}{\phi(y)} d\hat{\varsigma}| < \infty$
- Perron-Frobenius, ϕ a positive eigenfunction and ρ eigenvalue.
- ρ smallest eigenvalue associated with a positive eigenfunction of the extended generator of $\mathbb{A}$ of M.

Plan for remaining two lectures

- Examples
  - Long run risk-return
- Ideas behind proofs
- Heterogeneous beliefs.

Example

- $dX_t = -\xi X_t dt + \sigma dW_t,\ \xi \neq 0.$
- $M \equiv 1$
- $\hat{M} = M,\ \rho = 0,\ \phi \equiv 1.$
- $\phi(x) = e^{\frac{\xi x^2}{\sigma^2}},\ \rho = \xi,\ \hat{M}_t = e^{\frac{\xi(X_t^2 - X_0^2)}{\sigma} - \xi t} = e^{\int_0^t \frac{2\xi X_s}{\sigma} dW_s - \int_0^t \frac{2\xi^2 X_s^2}{\sigma^2} ds}$
- If $\xi > 0$, $\rho = 0$ is smaller.
- If $\xi < 0$ then $\rho = \xi < 0$ and Girsanov states that $\hat{W}_t = W_t - \int_0^t \frac{2\xi X_s}{\sigma} ds$, is a Brownian motion on the probability space obtained when we change measure using $\hat{M}$.

- In any case, after the change in measure we obtain $dX_t = -|\xi| X_t dt + \sigma d\hat{W}_t$
- There always exists a stationary density $\hat{\varsigma} \sim e^{-\frac{|\xi| x^2}{\sigma^2}}$.
- Stochastic stability obtains

$$\lim_{t \to \infty} e^{-\rho t} E[\psi(X_t) | X_0 = x] = (\int \frac{\psi(y)}{\phi(y)} d\hat{\varsigma}) \phi(x)$$

  - If $\xi > 0$, $\rho = 0$ and approximation is good for ψ such that $| \int \psi(y) d\hat{\varsigma}| < \infty$.
  - If $\xi < 0$, $\rho = \xi$ and approximation is good for ψ such that $| \int \psi(y) dy| < \infty$.

Example

- $$\begin{aligned} dX_t^f &= \xi_f(\bar{x}_f - X_t^f)dt + \sqrt{X_t^f}\sigma_f dB_t^f, \\ dX_t^o &= \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o \end{aligned}$$

  with $\xi_i > 0$, $\bar{x}_i > 0$ for $i = f, o$ and $2\xi_f \bar{x}_f \geq \sigma_f^2$ where $B = (B^f, B^o)$ is a bivariate standard Brownian motion.

- Per-capita consumption

  $$dc_t = X_t^o dt + \sqrt{X_t^f}\vartheta_f dB_t^f + \vartheta_o dB_t^o$$

  where $c_t = \log(C_t)$

- Interesting case:
  - $\sigma_o > 0$, $\vartheta_0 \geq 0$ (positive B^o's are unambiguously good)
  - $\sigma_f < 0$, $\vartheta_f > 0$ (positive B^f's are unambiguously good)

Breeden model

- Representative investor preferences are given by:

$$E\int_0^\infty \exp(-\mathsf{b}t)\frac{C_t^{\,1-\mathsf{a}}-1}{1-\mathsf{a}}$$

for a and b strictly positive.

- S_t a stochastic discount factor, that is $S_0 = 1$ and, for any ($\mathcal{F}_t$-measurable) payoff Π_t, $\frac{E[S_t\Pi_t|\mathcal{F}_s]}{S_s}$ is the price at time s of a claim to the payoff Π_t at t.

- In the additive utility case $S_t = \frac{e^{-\mathsf{b}t}u'(C_t)}{u'(C_0)}$

- Ito's Lemma implies that the stochastic discount factor in the Breeden model in this example is $S_t = \exp(A_t^s)$ where

$$A_t^s = -\mathsf{a}\int_0^t X_s^o ds - \mathsf{b}t - \mathsf{a}\int_0^t \sqrt{X_s^f}\vartheta_f dB_s^f - \mathsf{a}\int_0^t \vartheta_o dB_s^o.$$

- V is a valuation functional if V is multiplicative and VS is a martingale
  - Example: result of continuously reinvesting the payouts of an investment
- If V is a strictly positive valuation functional parameterized by (β_v, γ_v), the expected net rate of return is:

$$\varepsilon_v \doteq \beta_v + \frac{|\gamma_v|^2}{2}$$

- If V is a valuation functional parameterized by (β_v, γ_v) the martingale restriction gives β_v as a function of γ_v.
  - $\beta_v - \mathsf{a} X_t^o - b = -\frac{|\gamma_v - (\mathsf{a}\sqrt{X_s^f}\vartheta_f, \mathsf{a}\vartheta_o)|^2}{2}$
- Restricts expected return ϵ_v of V.
  - $\varepsilon_v = \mathsf{a} X_t^o + b + \gamma_v \cdot (\mathsf{a}\sqrt{X_s^f}\vartheta_f, \mathsf{a}\vartheta_o) - \frac{|(\mathsf{a}\sqrt{X_s^f}\vartheta_f, \mathsf{a}\vartheta_o)|^2}{2}$

- Mapping $\gamma_v \hookrightarrow \epsilon_v$ is instantaneous risk-return relationship.
- $(\mathsf{a}\sqrt{X_s^f}\vartheta_f, \mathsf{a}\vartheta_o)$ is vector of (instantaneous) Brownian risk-prices.
- $\vartheta_o = 0 \Rightarrow$ risk B^o is not priced.

Long rung risk-return

- $M = V$
  - V the cumulated results of a self-financing stategy.
  - Suppose V is parameterized by (γ^v, κ^v)
  - β^v results from martingale restriction.
  - $\rho(\gamma^v, \kappa^v)$ gives the equilibrium long run rate of return as a function of the risk-structure of V
  - Long run price of Brownian risks is $\frac{\partial \rho}{\partial \gamma^v}$.

- $D_t = G_t \psi(X_t)$, G a growth process.
- $M = GS$
  - Negative of eigenvalue is the rate of decay in market value of a cash flow paid in the far future.
  - If γ^g parameterizes the sensitivity of G to the Brownian motions, long-run price of risk for exposure to the Brownian risk is $\frac{\partial(-\rho)}{\partial \gamma^g}$.

- G multiplicative, $G = \exp(A_t^g)$
-
$$A_t^g = \delta t + \int_0^t \sqrt{X_s^f}\gamma_f^g dB_s^f + \int_0^t \gamma_o^g dB_s^o - \int_0^t \frac{X_s^f(\gamma_f^g)^2 + (\gamma_o^g)^2}{2} ds.$$
- γ_f^g parameterizes B_s^f risk of cash flow, γ_o^g parameterizes B^o risk.
- δ is the rate of growth of dividends.
  - In the example, X is stationary.

- $S_t = \exp(A_t^s)$ where

$$A_t^s = \bar{\beta}^s t + \int_0^t \beta_f^s X_s^f ds + \int_0^t \beta_o^s X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f^s dB_s^f + \int_0^t \gamma_o^s dB_s^o$$

  - Breeden: $\bar{\beta}^s = -\mathsf{b}$, $\beta_f^s = 0$, $\beta_o^s = -\mathsf{a}$, $\gamma_s^f = -\mathsf{a}\vartheta_f$, $\gamma_s^o = -\mathsf{a}\vartheta_o$.
  - Kreps-Porteus

- $A = A^s + A^g$ is given by

$$A_t = \bar{\beta} t + \int_0^t \beta_f X_s^f ds + \int_0^t \beta_o X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f dB_s^f + \int_0^t \gamma_o dB_s^o$$

- $\bar{\beta} = \delta - \frac{(\gamma_o^g)^2}{2} + \bar{\beta}^s$, $\beta_f = -\frac{(\gamma_f^g)^2}{2} + \beta_f^s$, $\beta_o = \beta_o^s$, $\gamma_f = \gamma_f^g + \gamma_f^s$ and $\gamma_o = \gamma_o^g + \gamma_o^s$.

- Eigenfunction $\phi = \exp(\mathsf{c}_f x_f + \mathsf{c}_o x_o)$.

$$\begin{aligned}\rho &= \bar{\beta} + \beta_f x_f + \beta_o x_o + \frac{\gamma_f^2}{2} x_f + \frac{\gamma_o^2}{2} \\ &\quad + \mathsf{c}_f[\xi_f(\bar{x}_f - x_f) + x_f \gamma_f \sigma_f] + \mathsf{c}_o[\xi_o(\bar{x}_o - x_o) + \gamma_o \sigma_o] \\ &+ (\mathsf{c}_f)^2 x_f \frac{\sigma_f^2}{2} + (\mathsf{c}_o)^2 \frac{\sigma_o^2}{2}\end{aligned}$$

- $\mathsf{c_f}$ and $\mathsf{c_o}$ chosen such that the process X is stationary when we change probability of $F \in \mathcal{F}_t$ to $E[\hat{M}_t 1_F] \equiv E[e^{-\rho t} M_t \phi(X_t)]$.
  - unique
  - $\mathsf{c_o} = \frac{\beta_\mathsf{o}}{\xi_\mathsf{o}}$
  - $\mathsf{c}_f = \frac{(\xi_f - \gamma_f \sigma_f) - \sqrt{(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 \left(2\beta_f + \gamma_f^2\right)}}{(\sigma_f)^2}$
  - $\mathsf{c}_f = \mathsf{c}_f(\gamma_f^g)$
- Asymptotic risk adjusted rate of return is $-(\rho - \delta)$.

- Long-run risk price for the exposure to the B^o risk is:

$$-\frac{d\rho}{d\gamma_o^g} = -\gamma_o^s - \frac{\beta_o^s}{\xi_o}\sigma_o$$

  - Breeden: $\mathsf{a}\vartheta_\mathsf{o} + \frac{\mathsf{a}}{\xi_\mathsf{o}}\sigma_\mathsf{o}$.
  - Difference from local risk price, $\frac{\mathsf{a}}{\xi_o}\sigma_o$
    * increases with risk aversion
    * increases with the volatility of the rate of growth of consumption
    * decreases with the strength of mean reversion of the rate of growth of consumption.
    * No difference if rate of growth of consumption is constant.
  - Long run risk price is positive even when $\vartheta_o = 0$.

- Long run risk price for exposure to B^f depends on γ_f^g and is state independent (stationarity).
  - Local risk price is independent of γ_f^g and is proportional to $\sqrt{X_t^f}$.

Sochastic Stability

- Assume $\hat{M}$ is a martingale
- M is positive with probability one.
- $\hat{\mathbb{A}}\psi = \frac{\mathbb{A}(\phi\psi)}{\phi} - \rho\psi$

Assumption 1. *There exists a probability measure $\hat{\varsigma}$ such that*

$$\int \hat{\mathbb{A}}\psi d\hat{\varsigma} = 0$$

for all ψ in the L^∞ domain of the generator $\hat{\mathbb{A}}$.

- This guarantees that $\hat{\varsigma}$ is a stationary distribution for X under $\hat{P}r$.
- $\hat{E}$ and $\hat{\mathcal{P}}r$ denote the expectation operator and the probability measure associated with $\hat{M}$ and $\hat{\varsigma}$. The process $\hat{M}$ determines the distorted transition probabilities and $\hat{\varsigma}$ is the initial distribution.

- Let $\hat{\Delta} > 0$ and consider the discrete time Markov process obtained by sampling the process at $\hat{\Delta}j$ for $j = 0, 1, ...$ *skeleton.*

  Assumption 2. *There exists a* $\hat{\Delta} > 0$ *such that the discretely sampled process* $\{X_{\hat{\Delta}j} : j = 0, 1, ...\}$ *is* **irreducible**. *That is, for any Borel set* Λ *of the state space* $\mathcal{D}_0$ *with* $\hat{\varsigma}(\Lambda) > 0$,

$$\hat{E}\left[\sum_{j=0}^{\infty} \mathbf{1}_{\{X_{\hat{\Delta}j} \in \Lambda\}} | X_0 = x\right] > 0$$

  for all $x \in \mathcal{D}_0$.

Assumption 3. *The process X is* **Harris recurrent** *under the measure $\hat{P}r$. That is, for any Borel set Λ of the state space $\mathcal{D}_0$ with positive $\hat{\varsigma}$ measure,*

$$\hat{P}r\left\{\int_0^\infty \mathbf{1}_{\{X_t \in \Lambda\}} = \infty | X_0 = x\right\} = 1$$

for all $x \in \mathcal{D}_0$.

– Among other things, this assumption guarantees that the stationary distribution $\hat{\varsigma}$ is unique.

- These assumptions guarantee the stochastic stability of skeletons and as a consequence, for any ψ for which $\int(|\psi(y)|/\phi(y))d\hat{\varsigma} < +\infty$

$$\lim_{j\to\infty} \exp(-\rho\Delta j)E[M_{\Delta j}\psi(X_{\Delta j})|X_0 = x] = \phi(x)\int \frac{\psi(y)}{\phi(y)}d\hat{\varsigma}$$

for almost all $(\hat{\varsigma})$ x.

Proposition 4. *Suppose that $\hat{M}$ satisfies Assumptions 1-3, and let* $\Delta > 0$.

a. *For any ψ for which $\int (|\psi|/\phi) d\hat{\varsigma} < \infty$*

$$\lim_{j\to\infty} \exp(-\rho\Delta j)\mathbb{M}_{\Delta j}\psi = \phi \int \frac{\psi}{\phi} d\hat{\varsigma}$$

for almost all ($\hat{\varsigma}$) x.

b. *For any ψ for which ψ/ϕ is bounded,*

$$\lim_{t\to\infty} \exp(-\rho t)\mathbb{M}_t\psi = \phi \int \frac{\psi}{\phi} d\hat{\varsigma}$$

for $x \in \mathcal{D}_0$.

Proof: Note that

$$\exp(-\rho t)\mathbb{M}_t\psi(x) = \hat{\mathbb{M}}_t \left(\frac{\psi}{\phi}\right) \phi(x).$$

Theorem 6.1 of Meyn and Tweedie 1993 states that Harris recurrence plus irreducibility of some skeleton chain implies that,

$$\lim_{t\to\infty} \sup_{0\leq\psi\leq\phi} \left| \hat{\mathbb{M}}_t \left(\frac{\psi}{\phi} \right) - \int \frac{\psi}{\phi} d\hat{\varsigma} \right| = 0,$$

which proves (b). Consider any sample interval $\Delta > 0$. Then

$$\lim_{j\to\infty} \sup_{0\leq\psi\leq\phi} \left| \hat{\mathbb{M}}_{\Delta j} \left(\frac{\psi}{\phi} \right) - \int \frac{\psi}{\phi} d\hat{\varsigma} \right| = 0.$$

From Proposition 6.3 of Nummelin, the sampled process $\{X_{\Delta j} : j = 0, 1, ...\}$ is aperiodic and Harris recurrent with stationary density $\hat{\varsigma}$. Hence if $\int \left| \frac{\psi}{\phi}(x) \right| d\hat{\varsigma}(x) < \infty$,

$$\lim_{j\to\infty} \hat{\mathbb{M}}_{\Delta j} \left(\frac{\psi}{\phi} \right) = \int \frac{\psi}{\phi} d\hat{\varsigma}$$

for almost all ($\hat{\varsigma}$) x, which proves (a). (See for example, Theorem 5.2 of Meyn and Tweedie 1992)

- If X is a Feller process, these assumptions can be verifyed by using Lyapunov function arguments for $\mathbb{A}$.
  - A continuous function V is called **norm-like** if the set $\{x : V(x) \leq r\}$ is precompact for each $r > 0$.
  - A sufficient condition for the existence of a stationary distribution and for Harris recurrence is that there exists a norm-like function V for which

$$\frac{\mathbb{A}(\phi V)}{\phi} - \rho V = \hat{\mathbb{A}} V \leq -1$$

    outside a compact subset of the state space.
  - Meyn and Tweedie

Existence

- Nummelin (1984), Kontoyiannis and Meyn (2003, 2005)
  - R-Recurrence of kernels
- Liapunov functions
- Construction guarantees that ϕ is an eigenfunction of semigroup $\mathbb{M}$.
- $\hat{M}$ is a martingale.

Modelling Heterogeneous beliefs

Overconfidence and Speculative Bubbles (Jose Scheinkman and Wei Xiong, *J.P.E.*, December 03)

Equilibrium Portfolio Strategies in the Presence of Sentiment Risk and Excess Volatility (B. Dumas, A. Kurshev and R. Uppal, preprint 2008.)

Some motivating observations

- Difficulty of explaining prices with fundamentals
- Bubbles frequently accompanied by increased trading activity
  - Trading volume (internet stocks: 6% of capital, 20% of trading, Average of spin offs reported in Lamont and Thaler (2001) 38% daily.)
  - China's A and B shares.

Some earlier literature

- A static models with heterogeneous beliefs and short-sale constraints Miller (1977)
- With short-sales constraints, pessimistic investors stay out of the market. Sufficient heterogeneity + short-sales constraints $\Rightarrow$ asset prices are higher than in the absence of short-sale constraints.
- Harrison and Kreps (1978): Heterogeneous beliefs and short sale constraints induces speculative behavior.

Overconfidence as source of heterogeneous beliefs

- Overconfidence = tendency of people to overestimate the precision of their knowledge.
- Psychology studies suggest that people are overconfident.
  - Alpert and Raiffa - The 98% confidence interval cover only 60% of realizations.
  - Overconfidemce more pronounced if questions more difficult.
  - Illusion of knowledge: Disagreement more polarized when given arguments that serve both sides. (Lord, Ross and Lepper,1979)

Consequences of overconfidence in a dynamic model of pricing and trading.

- Elements of the model
- Individuals receive signals on the value of an asset.
- Different groups of individuals have common information but excess confidence on distinct signals.
- This creates two phenomena:
  - divergence of opinions.
  - buyer knows than in the future others may value the asset to such an extent that a trade will occur.(Option)
- How do bubbles appear and vanish?

- Henry Blodget quoted by Michael Lewis *"We all have the same information, and we're just making different conclusions about what the future will hold."*

- Michael Lewis quoted by Michael Lewis *"I had been to a Merrill Lynch conference (them again!) that featured Exodus Communications, and the story Henry Blodget and a few other people told was so good that I figured that even if Exodus Communications didn't wind up being a big success, enough people would believe in the thing to drive the stock price even higher and allow me to get out with a quick profit."*

Outline

- Two groups of Agents A and B.
- Model as linear as possible
- Filtering.
- Buyer acquires option to sell asset.
- Option is American.
- Optimal stopping time.

- Value that buyer is willing to pay today depends on prices he forecasts for the future.
- Equilibrium.
- Solve for infinite horizon model, but can accommodate finite horizon securities.

Payoffs and information

- Cumulative dividend process D_t:

$$dD_t = f_t dt + \sigma_D dZ_t^D$$

$$df_t = -\lambda(f_t - \bar{f})dt + \sigma_f dZ_t^f$$

- Two extra signals:

$$\begin{aligned} ds_t^A &= f_t dt + \sigma_s dZ_t^A \\ ds_t^B &= f_t dt + \sigma_s dZ_t^B \end{aligned}$$

- Group A agents believe:

$$ds_t^A = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1-\phi^2} dZ_t^A$$

Group B agents believe :

$$ds_t^B = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1-\phi^2} dZ_t^B$$

- $0 \leq \phi \leq 1$, agents views on correlations of innovations is public information.

Filtering

- Agents use signals and D to forecast f.
- Stationary variance,

$$\gamma \equiv [\sqrt{(\lambda+\phi\sigma_f/\sigma_s)^2+(1-\phi^2)(2\sigma_f^2/\sigma_s^2+\sigma_f^2/\sigma_D^2)} \\ -(\lambda+\phi\sigma_f/\sigma_s)]/(\frac{1}{\sigma_D^2}+\frac{2}{\sigma_s^2})$$

γ decreases with ϕ

- $$\begin{aligned} d\hat{f}^A &= -\lambda(\hat{f}^A - \bar{f})dt + \frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s^2}(ds^A - \hat{f}^A dt) \\ &\quad + \frac{\gamma}{\sigma_s^2}(ds^B - \hat{f}^A dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^A dt) \end{aligned}$$

  $$\begin{aligned} d\hat{f}^B &= -\lambda(\hat{f}^B - \bar{f})dt + \frac{\gamma}{\sigma_s^2}(ds^A - \hat{f}^B dt) \\ &\quad + \frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s^2}(ds^B - \hat{f}^B dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^B dt) \end{aligned}$$

- “Beliefs”

Difference in beliefs

- To A investors: $g^A = \hat{f}^B - \hat{f}^A$,

$$dg^A = -\rho g^A dt + \sigma_g dW_g^A$$

- Difference in beliefs is a state variable.
- Larger ϕ increases volatility and decreases mean-reversion.

Trading

- Cost c per trade.
- All agents are risk neutral.
- Fixed rate of interest r.
- Finite supply, no short sales and large number of agents of each type guarantee that buyers pay reservation price.

$$
\begin{aligned}
p_t^o \quad &= \quad \max_{\tau \geq 0} \mathrm{E}_t^o \Big\{ \int_t^{t+\tau} \mathrm{e}^{-r(s-t)} [\bar{f} + \mathrm{e}^{-\lambda(s-t)} (\hat{f}_t^o - \bar{f})] ds \\
&+ \quad \mathrm{e}^{-r\tau} (p_{t+\tau}^{\bar{o}} - c) \Big\}.
\end{aligned}
$$

- Guess: Demand price of current owner is:

$$
p_t^o = p^o(\hat{f}_t^o, g_t^o) = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda} + q(g_t^o). \tag{6}
$$

with $q > 0$ and $q' > 0$.

-

$$
q(g_t^o) = \sup_{\tau \geq 0} \mathrm{E}_t^o \left[\left(\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^{\bar{o}}) - c \right) \mathrm{e}^{-r\tau} \right].
$$

The option problem

- $$q(x) \geq \frac{x}{r+\lambda} + q(-x) - c \tag{7}$$
  $$\frac{1}{2}\sigma_g^2 q'' - \rho x q' - rq \leq 0, \tag{8}$$
  with equality in (8) if (7) strict,

- Guess: Stop if $\{x : x \geq k^*\}$, $k^* \geq 0$.

- $$\frac{1}{2}\sigma_g^2 H'' - \rho x H' - rH = 0. \tag{9}$$

- Solutions H to (9) with $H > 0$ and $H' > 0$ in $(-\infty, 0)$ are of the form $\beta_1 h$, h convex, $\lim_{x\to-\infty} h(x) = 0$, $\lim_{x\to-\infty} h'(x) = 0$.
  - An "explicit" formula for h.
  - Kummer functions.

- $$q(x) = \begin{cases} \beta_1 h(x), & \text{for} \quad x < k^* \\ \frac{x}{r+\lambda} + \beta_1 h(-x) - c, & \text{for} \quad x \geq k^*. \end{cases} \tag{10}$$

- "Smooth pasting" $\Rightarrow$

$$\beta_1 = \frac{1}{[h'(k^*) + h'(-k^*)](r+\lambda)}, \tag{11}$$

$$\begin{aligned} &[k^* - c(r+\lambda)][h'(k^*) + h'(-k^*)] \\ &\qquad + h(-k^*) - h(k^*) = 0 \end{aligned} \tag{12}$$

- There exists unique $k^* = k^*(c)$ that solves (12). $k^*(0) = 0$ and $k^*(c) > c(r + \lambda)$ if $c > 0$.
  - Hold stock even though others value it sufficiently more to pay for transaction costs.
  - Variation of model with one group "rational"
- q defined by (10) is an equilibrium option value function.
- The optimal policy consists of exercising immediately if $g^o \geq k^*$, otherwise wait until first time in which $g^o \geq k^*$.
- Bubble: If $x \in (-\infty, k^*)$, $q(x)$ the difference between owner's demand price and his fundamental valuation.
  $b(c) = \frac{h(-k^*(c))}{[h'(k^*(c))+h'(-k^*(c))](r+\lambda)}$
- Mean duration between trades: $\mathrm{E}[\tau(-k^*, k^*)]$

Properties of equilibria with trading costs

- Trading volume
  - If $c = 0$, $k^* = 0$ and $\mathrm{E}[\tau(-k^*, k^*)] = 0$.
  - $\mathrm{E}[\tau(-k^*, k^*)]$ varies continuously with c.
  - $\frac{\partial \mathrm{E}[\tau(-k^*,k^*)]}{\partial c} = \infty$ at $c = 0$.
- Bubble
  - $\frac{\partial b(c)}{\partial c}$ is finite.
  - Bubble increases with overconfidence ϕ, volatility of fundamentals σ_f, decreases with interest rate r.

Greenspan on the possibility of a real estate bubble

"*While stock market turnover is more than 100% annually, the turnover of home ownership is less than 10 per cent annually - scarcely tinder for speculative conflagration.*" (quoted in *Financial Times* of April 22, 2002).

Price and turnover

- Risk-neutrality allows a separate model for each asset.
- Turnover and size of bubble are equilibrium values.
- No causality.
- Cochrane: Cross-sectional regression of market value / book value on share turnover for stocks in NASDAQ 96-00.
- China's A and B shares.

Risk

- Introduce risk averse consumers
- Aggregate consumption is assumed to be a diffusion.
- One equity with dividends = aggregate consumption.
- Groups not symmetric.
- No transaction costs
- Allow short sales

Payoffs and information

- Aggregate dividend process D_t:

$$\frac{dD_t}{D_t} = f_t dt + \sigma_D dZ_t^D, \ \sigma_D > 0$$

- $$df_t = -\lambda(f_t - \bar{f})dt + \sigma_f dZ_t^f, \ \lambda > 0, \sigma_f > 0.$$

- An extra signal:

$$ds_t = -\mu s_t dt + \sigma_s dZ_t^s.$$

- (Z^D, Z^f, Z^s) a three dimensional Brownian.

- Group A agents believe:

$$ds_t = -\mu s_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1-\phi^2} dZ_t^s$$

- Group B agents are rational
- $0 \leq \phi \leq 1$, agents views on correlations of innovations is public information.
- Filtering implies that stationary variance of agent's forecasts are: $\gamma^A = \gamma^A(\phi)$, with γ^A decreasing with ϕ, $\gamma^B = \gamma^A(0)$.

- $$d\hat{f}_t^A = -\lambda(\hat{f}_t^A - \bar{f})dt + \frac{\gamma^A}{\sigma_D^2}(\frac{dD}{D} - \hat{f}^A dt) + \frac{\phi\sigma_f}{\sigma_s}(ds + \mu s)$$

  $$d\hat{f}_t^B = -\lambda(\hat{f}_t^B - \bar{f})dt + \frac{\gamma^B}{\sigma_D^2}(\frac{dD}{D} - \hat{f}^B dt)$$

- $\frac{1}{\sigma_D}(\frac{dD}{D} - \hat{f}^B)$ is a Brownian innovation for B.
- $\frac{1}{\sigma_D}(\frac{dD}{D} - \hat{f}^A)$ is a Brownian innovation for A.
- Difference in beliefs: $\hat{g} = \hat{f}^B - \hat{f}^A$
- $dZ_t^A = dZ_t^B + \frac{\hat{g}_t}{\sigma_D}$

- η change in measure from B's to A's beliefs then by Girsanov:

$$\frac{d\eta_t}{\eta_t} = -\frac{\hat{g}_t}{\sigma_D} dZ_t^B$$

- $d\hat{g}_t = -(\lambda + \frac{\gamma^A}{\sigma_D^2})\hat{g}_t + \frac{\gamma^B - \gamma^A}{\sigma_D} dZ_t^B - \phi\sigma_f dZ_t^s$

- State variables $(D, s, \hat{f}^B, \hat{g}, \eta)$. Two dimensional Brownian (Z^B, Z^s).

Equilibrium

- Agents in each group $C \in \{A, B\}$ have utility function $U(c) = E_0^C \int_0^\infty e^{-bt} \frac{c^\alpha}{\alpha} dt$ and an initial share of rights to output (equity) w^C.
- Recall that S_t is a stochastic discount factor if the date zero price of a payoff Π_t at t is $E_0[S_t \Pi_t]$.
- Form expectations using B probabilities.

- Arrow's procedure (Kreps)
  - Find stochastic discount factor S_t (under B's probabilities) such that when consumers maximize expected utility subject to a single budget constraint:

$$E_0^B[\int_0^\infty S_t c_t dt] \leq w^C E_0^B[\int_0^\infty S_t D_t]dt,$$

$$\text{Supply} = \text{Demand}.$$

  - Show that this equilibrium can be decentralized with long-lived securities.
  - Two Brownians require 3 securities, e.g. riskless bond, equity, consol.

- $S_t = \frac{e^{-bt}(c_t^B)^{\alpha-1}}{(c_0^B)^{\alpha-1}}$
- Since group A shares same utility function and η_t transforms B's beliefs into A's beliefs, $S_t = \frac{\eta_t}{\eta_0}\frac{e^{-bt}(c_t^A)^{\alpha-1}}{(c_0^A)^{\alpha-1}}$
- $c_t^B + c_t^A = D_t$
- $c_t^B = \theta(\eta_t)D_t,\ c_t^A = [1-\theta(\eta_t)]D_t.$
- $$\theta(\eta) = \frac{\left[\frac{1}{\nu^B}\right]^{\frac{1}{1-\alpha}}}{\left[\frac{\eta_t}{\nu^A}\right]^{\frac{1}{1-\alpha}} + \left[\frac{1}{\nu^B}\right]^{\frac{1}{1-\alpha}}},$$

  ν^C the Lagrande multiplier associated with the budget constraint of group C.
- $\log S_t = -b - (\alpha-1)[\log(\theta(\eta_t)) - \log(\theta(\eta_0)) + \log(D_t) - \log(D_0)]$
- Ito's lemma guarantees that $d(\log S_t) = \beta^S dt + \gamma^S dZ_t^B$
- Innovations in Z^B affect S through two channels - changes in output and changes in η (sentiment).

- S_t only depends on Z^S through $\hat{g}$.
- Conditional on state, risk-premia are independent of overconfidence coefficient ϕ.
- Local risk-price of signal (Z^s) risk is zero.
- As $t \to \infty$ positive Martingale η_t concentrates mass at zero.
- $\theta(\eta) \to 1$.
  - Calibrations in Dumas et al. show that nonetheless the share of group A declines very slowly.

Long-Run Risk

Lectures at Collège de France
José A. Scheinkman
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Princeton University
March 2008

In financial economics risk-return tradeoffs show how expected rates of return and consequently asset prices are altered in response to changes in the exposure to the underlying shocks that impinge in the economy. In these lectures we will:

(i) Present some of the recent literature that is concerned with the effect of long run risk on returns and prices.

(ii) Develop an analytical structure that reveals the long-run risk-return relationship in nonlinear continuous time Markov environments. This is done by studying a principal eigenvalue problem for a conveniently chosen family of valuation operators.

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