Level-strategyproof Belief Aggregation Mechanisms

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Abstract. In the problem of aggregating experts' probabilistic predictions over an ordered set of outcomes, we introduce the axiom of level-strategyproofness (level-SP) and prove that it is a natural notion with several applications. Moreover, it is a robust concept as it implies incentive compatibility in a rich domain of single-peakedness over the space of cumulative distribution functions (CDFs). This contrasts with the literature which assumes single-peaked preferences over the space of probability distributions. Our main results are: (1) a reduction of our problem to the aggregation of CDFs; (2) the axiomatic characterization of level-SP probability aggregation functions with and without the addition of other axioms; (3) impossibility results which provide bounds for our characterization; (4) the axiomatic characterization of two new and practical level-SP methods: the proportional-cumulative method and the middlemost-cumulative method; and (5) the application of proportional-cumulative to extend approval voting, majority rule, and majority judgment methods to situations where voters/experts are uncertain about how to grade the candidates/alternatives to be ranked.[3]

Keywords: Probability Aggregation Functions · ordered Set of Alternatives · Level Strategy-Proofness · Proportional-Cumulative · Middlemost-Cumulative

1 Introduction

Usually, experts are uncertain —their opinions or predictions are probabilistic— and may disagree in their judgments, even if they share a common interest with the regulator [10]. This raises the question: how should one pool their opinions?

Probability aggregation is an important practical problem. With no evidence-based information on the Covid-19 disease, the European Academy of Neurology (EAN) developed an ad-hoc three-round voting method[4] to reach consensus [28]. When a potentially dangerous volcano becomes restless, civil authorities usually turn to scientists for help in anticipating risks. As the science of forecasting eruptions is not exact, volcanologists developed elaborate mathematical models to elicit and aggregate experts' probabilistic opinions [4,11]. Methods that aggregate probabilities are now used in a huge number of applications such as estimating a nuclear risk, weather forecasting, calculating the risks to manned spaceflight due to collision with space debris, or the future of the polar bear population (see [2,29] and chapter 15 in [12]).[5] The Technical University of Delft has developed a software EXCALIBUR (acronym for EXpert CALIBRation). It is based on a method introduced in [12] which combines parametric and quantile inputs for continuous uncertain quantities from weighted experts. It then outputs a probability distribution (or equivalently a CDF).[6]

Aggregating experts' uncertain opinions or probabilistic predictions is a well-studied mathematical/social choice problem [1,31,33,12], sometimes referred to as belief aggregation, or opinion pooling. The formal model

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[4] "In round 1, statements were provided by EAN scientific panels (SPs). In round 2, these statements were circulated to SP members not involved in writing them, asking for agreement/disagreement. Items with agreement > 70% were retained for round 3, in which SP co-chairs rated importance on a five-point Likert scale. Results were graded by importance and reported as consensus statements."

[5] We refer the interested reader to Roger Cooke web page: http://rogermcooke.net which contains useful references, real data, software for expert aggregation, and a wide range of applications.

[6] ANDURYL [23] the successor to EXCALIBUR is open source and scriptable with a graphical user interface.

is the following. Each expert $i \in N$ is asked to provide his prior probability distribution $p_i \in \mathcal{P} = \Delta(\Lambda)$ over a set of outcomes Λ (usually finite or a real interval). The objective of the decision-maker (or the regulator) is to design a PAF = Probability Aggregation Function $\psi : \mathcal{P}^N \to \mathcal{P}$ satisfying some desirable properties.

Most of the literature described so far assumes honest reporting. In practice, judges may have strategic incentives that lead them to misreport their true beliefs. For example, the FDA uses advisory committees since the seventies and follow their recommendations 70% of the time. Each committee is composed of -usually- eleven experts, appointed for four years. Experts have to give individual answers to the FDA and their aggregate vote are used as a collective recommendation. There has been several controversies around conflict of interest in the advisory committees including kickbacks [30]. For example, the pharmaceutical companies GSK, Pfizer or Johnson and Johnson were recently charged several billion dollars for kickbacks.[7] Hence, it is important for the PAF to be incentive-compatible (reporting what you want the outcome to be is an optimal strategy).

When monetary transfers are possible (e.g. the experts can be payed after the realization of the random variable they are predicting), the problem has been well studied since the fifties and several incentive compatible scoring rules have been developed [19,24,35,14]. Our paper deals with situations where monetary transfers are not possible because the realization of the random variable is observed far in the future or the consequences of a mistake are catastrophic (infertility after vaccination, volcanic eruption, global warning).

To the best of our knowledge, incentive compatible belief aggregation without monetary transfers has been studied only as a special case of single-peaked domain restrictions and only when Λ is finite. For strategy-proofness to be stated formally, one needs assumptions about the individual preferences over the set $\mathcal{P}$ of probability distributions over Λ. If the voters' preferences are unrestricted, Gibbard [17] and Satterthwaite's [32] theorems apply and only a dictatorial method is strategyproof. Unfortunately, the Gibbard-Satterthwaite negative conclusion still holds even if one restricts the domain provided that it is "rich" enough (e.g. the class of all convex preferences [36] or the class of generalized single-peaked preferences, including all additively separable convex preferences [27]). Fortunately, under a more severe restriction, the possibility of an anonymous aggregation has been proved recently [18,16] in the domain of the L_1-metric single-peaked preferences on $\mathcal{P}$.[8] Goel et al. [18] proved the existence of a Pareto optimal strategyproof PAF and Freeman et al. [16] identified a large family of strategyproof PAF in the spirit of Moulin's characterization [26].

Suppose the outcome space Λ is a finitely ordered set $\{a_m \succ \cdots \succ a_2 \succ a_1\}$. To define experts' utility functions, we need to compute the cost for expert i if the aggregate probability p is different from its forecast p_i. One natural way is by measuring the distance between the two using some L_q-distance on $\mathcal{P} = \Delta(\Lambda) \subset \mathcal{R}^m$. For example, in [18,16] they measure the cost in terms of the L_1-distance. However, this distance is incoherent with the assumption that our alternatives are ordered. For example, imagine that p_i (the peak of i) is the Dirac mass δ_{a_1} at the smallest alternative a_1 and that p (the output) is the Dirac mass δ_{a_m} at the highest alternative $a_m \in \Lambda$. It is natural to assume that any other probability than p is better from the perspective of expert i, especially the Dirac mass at a_2 which is much closer to a_1 than a_m. However no L_q-distance on $\mathcal{P} = \Delta(\Lambda) \subset \mathcal{R}^m$ will distinguish $\|\delta_{a_1} - \delta_{a_m}\|_{L_q}$ from $\|\delta_{a_1} - \delta_{a_2}\|_{L_q}$ as both are identical (equal to $2^{\frac{1}{q}}$). To capture the intuition that $\text{dist}(\delta_{a_1}, \delta_{a_m})$ must be bigger than $\text{dist}(\delta_{a_1}, \delta_{a_2})$, one needs to measure the distances in the space of cumulative distribution functions (CDFs) $\mathcal{C} = \Sigma(\Lambda)$. This is what our paper will do.

Our main contributions:

- We suppose that the outcome space Λ is a Borel subset of $\mathbf{R}$. For example, Λ can be the set of positive reals (stock market prices), the positive integers (number of infections), a compact interval (temperature in global warning) or a finite set (the Richter magnitude scale for earthquakes, the Volcanic explosivity index).
- We define level-strategyproofness (level-SP) and prove that it implies incentive compatibility for a rich class of single-peaked preferences in the space of CDFs $\mathcal{C}$. Namely if the designer uses a level-SP method

[7] See: https://en.wikipedia.org/wiki/List_of_largest_pharmaceutical_settlements

[8] Single-peaked preferences on $\Delta(\Lambda)$ with Λ finite under the L_1-metric is a very small domain because every peak corresponds exactly to one preference ordering.

then the optimal strategy of an expert will be to vote honestly as soon as his satisfaction is measured by some distance L_q to the peak in the CDF space $\mathcal{C}$ (see section 2.3).
- We prove several possibility theorems characterizing non-dictatorial level-SP aggregation mechanisms in combination or not with other desirable axioms (sections 3, 4 and 5). We also explore the boundaries of our characterizations by establishing several impossibility theorems (sections 2.4 and 5).
- We characterize and compare two new practical solutions to the problem: middlemost-cumulative and proportional-cumulative (section 3).
- We apply proportional-cumulative to extend some voting methods of electing and ranking by allowing experts to express their uncertainties about the quality of the candidates (section 6).

The intuition behind level-SP is simple. Suppose the regulator decision depends on the likelihood of crossing a certain threshold, for example the probability of having a major natural hazard. Then, experts incentives will also depend on the probability of crossing that threshold. If the aggregation rule is level-SP, then whatever the threshold is, no expert, by misrepresenting his truly estimated probability distribution, can obtain that the probability of exceeding the given threshold is closer to what he wanted.[9]

Our main possibility result proves that a probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$, or equivalently its associated cumulative aggregation function $\Psi : \mathcal{C}^N \to \mathcal{C}$, is level-SP iff there is a family of simple voting rules $\{g_a : [0,1]^n \to [0,1]\}_{a\in\Lambda}$ such that each g_a is one-dimensional strategyproof *a la* Moulin [26] and the CDF $P = \Psi(P_1, \ldots, P_n)$ of society at any level $a \in \Lambda$ is computed as $P(a) = g_a(P_1(a), \ldots, P_n(a))$. When we combine this result with the Moulin's formulae [26] we obtain explicit characterizations of level-SP PAF in which the phantoms in Moulin [26] are replaced with monotone right continuous functions.

In addition to level-SP, a natural axiom to satisfy is *certainty preservation*. It says that if all experts agree that some event must happen almost surely or can't happen almost surely, then the aggregation preserves that property.[10] This axiom is classical and is called the *zero preservation property* in [33] or *consensus preservation* in [15]. With level-SP, they imply a form of *independence* (another classical axiom) but on the CDF space, e.g. $g_a = g$ for every $a \in \Lambda$. If in addition, the designer wants to satisfy the *plausibility preservation* axiom[11] then the simple voting rule g associated to Ψ must output one of its inputs. Middlemost-cumulative, (our first main method) is probably the most salient in this class. Its main drawback is its lack of diversity as whenever experts inputs dominate one another, the output is the median expert's input.

Our second main method (proportional-cumulative) will be more diversified as it will be the unique level-SP probability aggregation function satisfying a *proportionality* axiom (which says that if all inputs are the Dirac distributions $\{\delta_{a_i}\}_{i=1,\ldots,n}$, then the output is the average $\frac{1}{n}\sum_{i=1}^{n}\delta_{a_i}$). As application, a recent voting method –majority judgment [5]– is extended to electing and ranking problems where the experts are uncertain about the quality of the alternatives/candidates. For example, in the context of a political election, an expert may think for sure that candidate A will be Good and believe that B may be a Great candidate with probability $\frac{1}{2}$ and a Terrible candidate with probability $\frac{1}{2}$. In the context of Brexit, an expert may think that quitting the EU will benefit the UK with probability $\frac{3}{4}$ and will harm the UK with probability $\frac{1}{4}$.

The paper is organized as follows. Section 2 introduces the model, the new notion of level-strategy-proofness, and its implications together with a quick summary of the fundamental results of Moulin [26]. Section 3 introduces our two main anonymous level-SP methods together with their axiomatic characterizations. Section 4 goes more into details and summarizes all the intermediary characterizations in the anonymous case. Section 5 extends sections 3 and 4 results to the non-anonymous case and explores the boundaries of our characterizations by establishing several impossibilities. Section 6 applies proportional-cumulative to extend majority rule, approval voting and majority judgment voting methods for electing and ranking to situations where experts can express their uncertainties. Section 7 concludes. The appendix contains all the proofs.

[9] The idea of aggregating level events in a binary set-up has been studied by [10] motivated by real applications. Our paper extends in a sense that model and offers a strategically resistant solution.

[10] Formally, for every Borel measurable event A, if $p_i(A) = 0$ for every $i \in N$ then $p(A) = 0$ or equivalently if $p_i(A) = 1$ for every $i \in N$ then $p(A) = 1$.

[11] Whenever all experts agree that some interval has positive probability, so does society.

2 Probability Aggregation Model and the Related Literature

We first recall the classical characterizations of strategyproof aggregation rules when voters have one-dimensional single-peaked preferences (Moulin [26]). Then we describe our probability aggregation model, we introduce the notion of level-strategyproofness (level-SP) and prove it to imply honest reporting for a rich family of single-peaked preferences over the CDF space $\mathcal{C}$. Finally, we show that for an ordered set of outcomes, level-SP is more natural than and incompatible with the classical strategyproof notions, defined with respect to single-peaked preferences over the space of probability distributions.

In the sequel, $N = \{1, \ldots, n\}$ denotes the set of voters or experts, and n their number. For any set Z, we will use the following notations, $\mathbf{z} = (z_1, \ldots, z_n) \in Z^N$ and $\mathbf{z}_{-i}(z'_i)$ to denote the profile which differs from $\mathbf{z}$ only in dimension i which is replaced by z'_i.

2.1 Single-dimensional Strategyproofness: one-SP

We recall Moulin's results [26] and characterizations as they are central to describe our results.

Definition 1 (Single-peaked preference). *A preference order T (represented with $\preccurlyeq$) over the set of alternatives $[0, 1]$ is* ***single-peaked*** *if there exits $x \in [0, 1]$ such that for all $y, z \in [0, 1]$, $z \leq y \leq x \Rightarrow T(z) \preccurlyeq T(y)$ and $x \leq y \leq z \Rightarrow T(z) \preccurlyeq T(y)$. The value x is known as the* ***peak****.*

Definition 2 (Single-dimensional strategyproofness: one-SP). *A voting rule $g : [0, 1]^N \to [0, 1]$ is* ***one-SP*** *iff whenever all experts have single-peaked preferences and all are expected to submit their peaks to be aggregated by g, no expert can obtain a strictly better alternative by reporting a fake peak.*

Remark 1. It can be proved that the previous definition is equivalent to the following property called uncompromisingness in [8]: a voting rule g is one-SP iff for all experts $i \in N$ and for all peak profiles $\mathbf{r} \in [0, 1]^N$:

$$r_i < g(\mathbf{r}) \Rightarrow g(\mathbf{r}) \leq g(\mathbf{r}_{-i}(r'_i))$$

and

$$r_i > g(\mathbf{r}) \Rightarrow g(\mathbf{r}) \geq g(\mathbf{r}_{-i}(r'_i)).$$

We may be interested in rules such that whenever all experts agree on a peak, society chooses that peak. This axiom is usually called unanimity. We will formulate it for a general input space X as it is needed later.

Axiom 1 (Unanimity) *$h : X^N \to X$ is* ***unanimous*** *if for all $x \in X$ we have $h(x, x, \ldots, x) = x$.*

Theorem 1 (Moulin general). *A* ***voting rule*** *$g : [0, 1]^N \to [0, 1]$ is one-SP iff for each coalition of players $S \in 2^N$, there exists a unique "phantom" value $\beta_S \in [0, 1]$ such that: $S \subseteq S'$ implies $\beta_S \leq \beta_{S'}$ and*

$$\forall \boldsymbol{r} = (r_1, \ldots, r_n) \in [0, 1]^N, g(\boldsymbol{r}) = \sup_{S \subseteq N} \min\,(\beta_S, \min_{i \in S} r_i).$$

Moreover the method is unanimous iff $\beta_\emptyset = 0$ and $\beta_N = 1$.

Moulin's formulae is more popular in the anonymous case where experts are treated equally by the rule.

Axiom 2 (Anonymity) *$h : X^N \to X$ is* ***anonymous*** *if $\forall \boldsymbol{x} \in X^N$ and all permutations σ over N:*

$$h(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = h(\boldsymbol{x}).$$

Theorem 2 (Moulin anonymous). *The voting rule $g : [0, 1]^N \to [0, 1]$ is one-SP and anonymous iff there exist $n + 1$ "phantom voters" $\alpha_1 \leq \cdots \leq \alpha_{n+1}$ in [0,1] such that:*

$$\forall \boldsymbol{r} = (r_1, \ldots, r_n) \in [0, 1]^N \in [0, 1]^N, g(\boldsymbol{r}) = \text{median}(r_1, \ldots, r_n, \alpha_1, \ldots, \alpha_{n+1}).$$

Moreover, the method is unanimous if and only if $\alpha_1 = 0$ and $\alpha_{n+1} = 1$.

Remark 2. In Jennings *et al* [21] several equivalent formulae are established, both in the anonymous and the general case. One of the characterizations simplifies the Moulin maxmin formula above by expressing it in a manner resembling the anonymous case. Expressing it as the median of the inputs after determining $n+1$ phantom values that are appropriate considering the input. Thus, the new characterizations in [21] can be used, instead of Moulin's formula, to obtain alternative representations of level-SP probability aggregation functions.

2.2 Probability Aggregation Functions (PAF) and Level StrategyProofness (Level-SP)

It is now time to describe our model. Society (or a decision-maker, or the regulator) wants to estimate a random variable X that ranges over a linearly ordered set of outcomes Λ that we identify with a Borel measurable subset of the real line $\mathbf{R}$. In practice, think of Λ as a finite subset or an interval of $\mathbf{R}$. To construct the estimation, several experts $i \in N = \{1, \ldots, n\}$ are asked to submit their (subjective) probability distribution estimation $p_i \in \mathcal{P}$ where $\mathcal{P}$ denotes the set of Borel probability distributions over Λ. Our objective is to design a Probability Aggregation Function (PAF) $\psi : \mathcal{P}^N \to \mathcal{P}$ which is used to determine society's estimation. As some experts may be strategically behaving, we wish ψ to be incentive compatible. Our paper focuses on problems where society's final decision is based on a threshold, that is to say, the event of outcomes that are below (or above) a certain level. In practice, situations that are represented with tipping points (e.g. climate change, the degree of a volcanic eruption damaging impact) are level-events.

Definition 3 (Level Events). *We define for any outcome $a \in \Lambda$, the* ***level event*** *$\mathcal{E}(a)$ as "the threshold a has not been crossed", e.g.*

$$\mathcal{E}(a) = \{X \leq a\} = \{x \in \Lambda : x \leq a\}$$

The following definition says that a PAF is incentive compatible with respect to level events if, no matter the threshold $a \in \Lambda$ is, no expert can by misreporting obtain that society's probability for the level event $\mathcal{E}(a)$ is closer to the one he/she wishes. Hence, as far as the outcome of the aggregation will be used to make a decision based on the probability of crossing a given threshold, this definition properly describes what it means to be incentive compatible in this context.

Axiom 3 (Level-SP) *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is* ***level-strategyproof***[12] *if for any expert $\forall i \in N$, any input profile $\boldsymbol{p} \in \mathcal{P}^N$, any potential deviation $p'_i \in \mathcal{P}$ of expert i and any threshold $a \in \Lambda$:*

$$p_i(\mathcal{E}(a)) < \psi(\boldsymbol{p})(\mathcal{E}(a)) \Rightarrow \psi(\boldsymbol{p})(\mathcal{E}(a)) \leq \psi(\boldsymbol{p}_{-i}(p'_i))(\mathcal{E}(a))$$

and

$$p_i(\mathcal{E}(a)) > \psi(\boldsymbol{p})(\mathcal{E}(a)) \Rightarrow \psi(\boldsymbol{p})(\mathcal{E}(a)) \geq \psi(\boldsymbol{p}_{-i}(p'_i))(\mathcal{E}(a)).$$

Consequently with this axiom satisfied, if society's final decision (or policy) depends on a level event and expert utility depends on society's final decision then it is a dominant strategy for every expert to vote honestly. Formulated differently, let $\mathcal{C}$ be the set of **cumulative distribution functions** (CDF) over Λ. Let $\pi : \mathcal{P} \to \mathcal{C}$ be the mapping that transforms a probability distribution p over Λ into its CDF P:

$$\forall p \in \mathcal{P}, \forall a \in \Lambda, P(a) = \pi(p)(a) = p(\mathcal{E}(a)) = \int_{x \leq a} \mathrm{d}p(x).$$

To a PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is associated a unique **cumulative aggregation function (CAF)** $\Psi : \mathcal{C}^N \to \mathcal{C}$ that takes the CDFs $\{P_i = \pi(p_i)\}_{i \in N}$ of experts as inputs and returns the CDF of $\psi(\mathbf{p})$ as its output, that is, $\Psi(\pi(p_1), \ldots, \pi(p_n)) = \pi(\psi(p_1, \ldots, p_n))$. With this notation, saying that the PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP is equivalent to saying that the associated CAF $\Psi : \mathcal{C}^N \to \mathcal{C}$ satisfies: for every $a \in \Lambda$, every $\mathbf{P} \in \mathcal{C}^N$, $\mathbf{P'} \in \mathcal{C}^N$ and all $i \in N$:

$$P_i(a) < \Psi(\mathbf{P})(a) \Rightarrow \Psi(\mathbf{P})(a) \leq \Psi(\mathbf{P}_{-i}(P'_i))(a)$$

[12] As our characterizations in next sections show, level-SP methods remain incentive compatible if level sets are defined by strict $\{X < a\}$ or reverse $\{X \geq a\}$, $\{X > a\}$ inequalities.

and

$$P_i(a) > \Psi(\mathbf{P})(a) \Rightarrow \Psi(P)(a) \geq \Psi(\mathbf{P}_{-i}(P'_i))(a).$$

this implies that for every $a \in \Lambda$, every $\mathbf{P} \in \mathcal{C}^N$, $\mathbf{P'} \in \mathcal{C}^N$ and all $i \in N$:

$$|\Psi(\mathbf{P}_{-i}(P'_i))(a) - P_i(a)| \geq |\Psi(\mathbf{P})(a) - P_i(a)| \tag{1}$$

2.3 Robustness of the Level-SP notion

In the following we will show that this definition of incentive compatibility implies a form of robustness with respect to details modeling, that is, level-SP implies the optimality of honest reporting with respect to a large class of single-peaked preferences on the CDF space $\mathcal{C}$, in particular the L_1-distance. This will also allow us to compare our notion with the existing strategyproof concepts in the literature (see the next subsection).

$$\begin{array}{ccc}
p_1, \ldots, p_n & \xrightarrow[\psi]{} & \psi(p_1, \ldots, p_n) \\
\downarrow \pi & & \downarrow \pi \\
P_1, \ldots, P_n & \xrightarrow[\Psi]{} & \Psi(P_1, \ldots, P_n)
\end{array}$$

For each probability measure ν on Λ absolutely continuous w.r.t. the Lebesgue measure and each positive real $r \in \mathbf{R}_+$, we can define an L_r-distance on $\mathcal{C}$ as follows. $\|P - Q\|_{L_r} = [\int_\Lambda |P(a) - Q(a)|^r d\nu(a)]^{1/r}$, then, if ψ verifies level-SP, it also satisfies, $\forall i \in N$, $\forall \mathbf{P} \in \mathcal{C}^N$ any $\forall P'_i \in \mathcal{C}$:

$$\begin{aligned}
\|\Psi(\mathbf{P}_{-i}(P'_i)) - P_i\|_{L_r} &= \left[\int_\Lambda |\Psi(\mathbf{P}_{-i}(P'_i))(a) - P_i(a)|^r \mathrm{d}\nu(a)\right]^{1/r} \\
&\geq \left[\int_\Lambda |\Psi(\mathbf{P})(a) - P_i(a)|^r \mathrm{d}\nu(a)\right]^{1/r} \\
&= \|\Psi(\mathbf{P}) - P_i\|_{L_r}
\end{aligned}$$

Where the inequality follows from (1). Consequently, if the utility of expert i is measured by the use a distance L_r on $\mathcal{C}$ to its peak, e.g.:

$$u_i(p) = -\|\pi(\psi(\mathbf{p})) - \pi(p_i)\|_{L_r} = -\|\Psi(\mathbf{P}) - P_i\|_{L_r}$$

and if level-SP is satisfied, then we obtain $L_r(\mathcal{C})$-strategyproofness for every r, e.g. it is an optimal strategy for an expert to vote honestly as soon as his utility function is measured with some $L_r(\mathcal{C})$ distance to the peak.

2.4 Link with the Literature

An alternative natural way to model the utility of an expert i would be to measure the distance to the peak in the probability space $\mathcal{P}$, e.g.:

$$v_i(p) = -\|\psi(p) - p_i\|_{L_r} = -\left[\int_\Lambda |\psi(p) - p_i|^r \mathrm{d}\nu(a)\right]^{1/r}$$

Call a method $L_r(\mathcal{P})$-SP if honesty is the optimal strategy of an expert with the utility function v_i above. In the special case where $\Lambda = \{a_1, \ldots, a_m\}$ is finite, ν is the uniform distribution and $r = 1$, we obtain the model and strategyproofness concept studied in the budget aggregation problem [16]. The moving phantom mechanisms they defined are proved to be $L_1(\mathcal{P})$-SP. They are not level-SP. In fact, $L_1(\mathcal{P})$-SP and level-SP are incompatible as the following shows.

Theorem 3 (Impossibility 1). *When $\#\Lambda \geq 4$, $n \geq 3$ and the mechanism is certainty preserving*[13]*, level-SP and $L_1(\mathcal{P})$-SP are incompatible unless there is a dictator.*

The question now is, which assumption is more meaningful when outcomes are ordered? Single-peakedness in the probability space $\mathcal{P}$ (as it is usually assumed in the literature) or single-peakedness in the CFD space $\mathcal{C}$ (as we assume)? Suppose $\Lambda = \{a_1, \ldots, a_m\}$ and some expert is single-minded and submit a Dirac mass at $a \in \Lambda$ and that society's aggregation is a Dirac mass at $b \neq a$ then, intuitively, the expert wants b to be as close to a as possible. But, if utilities are measured using some L_r-distance on the probability space $\mathcal{P}$, then the fact that the value of $\|\delta_a - \delta_b\|_{L_r}$ is independent of a and b is not intuitive. In contrast, when utilities are measured with any L_r-distance on the CDF space $\mathcal{C}$ (as in the previous subsection), then the fact that the utility of the expert strictly increases as b goes to a is more intuitive.

To conclude, in a probability aggregation problem where the outcomes are completely independent of one another and there is no underlying alignment or a closeness relation between the outcomes, having a neutral distance (such as the $L_1(\mathcal{P})$) makes complete sense and, in that case, the aggregation problem is somehow solved: we have the existence of interesting strategyproof voting rules in the single-peaked domain induced by the $L_1(\mathcal{P})$-distance as proved in [16] (the moving phantom mechanisms)[14] and we also know that the dictatorial rules are the only strategyproof PAF whenever we slightly enlarge the domain of single-peaked preferences [36,27]. However, when the space of outcomes Λ has a linear structure as in our article, measuring distances in the CDF space is not only natural but also allow the existence of interesting voting rules that are incentive compatible for all $L_r(\mathcal{C})$ single-peaked preferences, $r \in]0, \infty]$. This property is hard to obtain in the probability space even with ordered outcomes. For example the phantom moving mechanisms in [16] are no longer strategyproof if one replaces the distance $L_1(\mathcal{P})$ by $L_2(\mathcal{P})$.

3 Main Anonymous Level-SP Methods

In this section and the rest of the paper, we will interchangeably use the probability input votes $\{p_i\}_{i \in N}$ or the CDF input votes $\{P_i = \pi(p_i)\}_{i \in N}$, depending if we want to establish a property on the PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ or its CAF $\Psi := \pi \circ \psi : \mathcal{C}^N \to \mathcal{C}$. We will abuse notations: when ψ is level-SP, we will also call its corresponding CAF Ψ level-SP.[15]

3.1 Order-Cumulatives

Definition 4 (Order-functions). *The* ***order-function*** $g^k : [0,1]^N \to [0,1]$, $k \in \{1, \ldots, n\}$ *is the voting rule that for a set of n values in $[0,1]$ return the k^{th} smallest value.*

As such $g^1(r_1, \ldots, r_n) = \min(r_1, \ldots, r_n)$, $g^n(r_1, \ldots, r_n) = \max(r_1, \ldots, r_n)$ and when n is odd

$$g^{(n+1)/2}(r_1, \ldots, r_n) = \text{median}(r_1, \ldots, r_n)$$

It is well known and easy to show that the order-functions are one-SP [26,5]. This class plays an important role in majority judgment theory where it has been characterized axiomatically in various ways [5].

Definition 5 (Order-cumulatives). *We denote as the k^{th}* ***order-cumulative*** $\Psi^k : \mathcal{C}^N \to \mathcal{C}$ *the CAF which is defined by applying the k^{th} order function at each level $a \in \Lambda$ in the CDF space:*

$$\forall a \in \Lambda, \Psi^k(\boldsymbol{P})(a) := g^k(P_1(a), \ldots, P_n(a)).$$

We let ψ^k denote the associated probability aggregation function that will also be called an order-cumulative.

[13] The certainty preservation axiom is defined in the next section. It says that whenever all experts believe that some event is certain to happen almost surely, so does society.

[14] It remains an open problem [16] whether the phantom moving mechanisms are the only anonymous continuous $L_1(\mathcal{P})$-SP mechanisms.

[15] One may wonder why we didn't choose to avoid having two notations and just define one model: the aggregation of cumulative distributions problem. The first reason is: except for the level-SP axiom, all the other important axioms are stated on the probability space: certainty preservation, plausibility preservation, or proportionality. The second reason is: level-SP and all the characterizations are better understood on the CDF space.

Consider a safe-to-dangerous scale Λ such as the Richter magnitude scale for earthquakes, the Volcanic explosivity index, or the hurricanes intensity scale. The min order-cumulative Ψ^1 is the most cautious response: since for each threshold, we consider the opinion of the most worried expert. On the other hand, the max order-cumulative Ψ^n is the less paranoid response as its output dominates all experts' CDFs. The middlemost-cumulative method represents the "median" position.

The order-cumulatives are the simplest level-SP methods. They are axiomatically characterized as follows.

Theorem 4 (Order-cumulatives characterization). *The order-cumulatives are the unique level-SP probability aggregation mechanisms that are anonymous, certainty preserving, and plausibility preserving.*

Certainty and plausibility preservation will be formally defined in the next section. Certainty preservation says that if all experts are certain that some Borel measurable event will occur almost surely, so does society. Plausibility preservation says that when an interval has a positive probability for all experts, so does society.[16]

We are particularly interested in the **middlemost-cumulative**. That is to say, the order-cumulative defined by the median order-function when n is odd and when n is even, we have two middlemost-cumulatives: the lower $\Psi^{\frac{n}{2}}$ and the upper $\Psi^{\frac{n}{2}+1}$. It is easy to show that middlemost-cumulatives are **welfare maximizers** if experts' utilities are measured using the $L_1(\mathcal{C})$ distance to the peak.

3.2 Proportional-Cumulative

If there are three experts where two of them believe that $a_1 \in \Lambda$ will occur for sure and the last one believes that a_2 will occur for sure, then any order-cumulative will output one of two Diracs. However, if the experts are equally competent, the "fair" aggregation seems to be $2/3\delta_{a_1} + 1/3\delta_{a_2}$. Such diversity in the output is desirable as it means that all opinions are represented in the aggregation.

Axiom 4 (Proportionality) *If experts are single-minded (every vote p_i is equal to a Dirac mass at $a_i \in \Lambda$, e.g. $p_i = \delta_{a_i}, \forall i \in N$), the aggregation must coincide with the average probability distribution:*

$$\forall (a_1, \dots, a_n) \in \Lambda^N \quad \psi(\delta_{a_1}, \dots, \delta_{a_n}) = \frac{1}{n} \sum_{i \in N} \delta_{a_i}.$$

It is straightforward that the kth-order-cumulative does not verify proportionality as it returns the kth-opinion $\delta_{a_{(k)}}$ after re-ordering the inputs $a_{(1)} \leq \cdots \leq a_{(k)} \leq \cdots \leq a_{(n)}$. Observe that the mean probability aggregation function $\psi(p_1, \dots, p_n) := \frac{1}{n} \sum_{i \in N} p_i$ satisfies proportionality, but it is not level-SP. Surprisingly, there is a unique level-SP proportional method, described now.

Definition 6. *The **proportional-cumulative** $\Psi : \mathcal{C}^N \to \mathcal{C}$ is the aggregation method defined as follows:*

$$\forall \boldsymbol{P} = (P_1, \dots, P_n), \forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \text{median}\left(P_1(a), \dots, P_n(a), \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\right)$$

In the Moulin's model, the one-SP method $\mu : [0,1]^n \to [0,1]$ such that

$$\mu(r_1, \dots, r_n) := \text{median}\left(r_1, \dots, r_n, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\right)$$

is known as the uniform median or linear median. It has been introduced independently by [9] and [22] and it was inspiring in the definition of a proportional moving phantom mechanism called IMM in [16].

Theorem 5 (Proportional-cumulative characterization). *The **proportional-cumulative** is the unique level-SP probability aggregation function that verifies the proportionality axiom.*

[16] One may wonder if a stronger version of plausibility preservation can be satisfied, namely whenever a Borel event has a positive probability for all experts, so does society. We will show that, basically, only the dictatorial rules are "Borel" plausibility preserving.

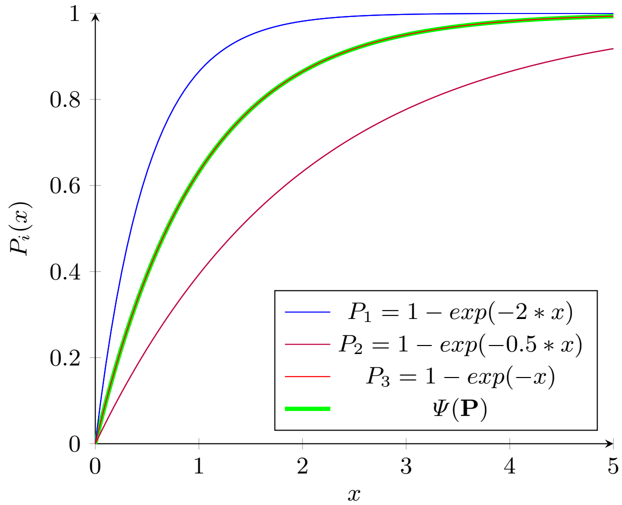

Fig. 1: P_1, P_2 and P_3 are the three cumulative functions of three voters. They stochastically dominate one another: P_1 is more biased to the left, P_3 more to the right. The middlemost-cumulative (in green) coincide with the input of the middlemost voter P_3.

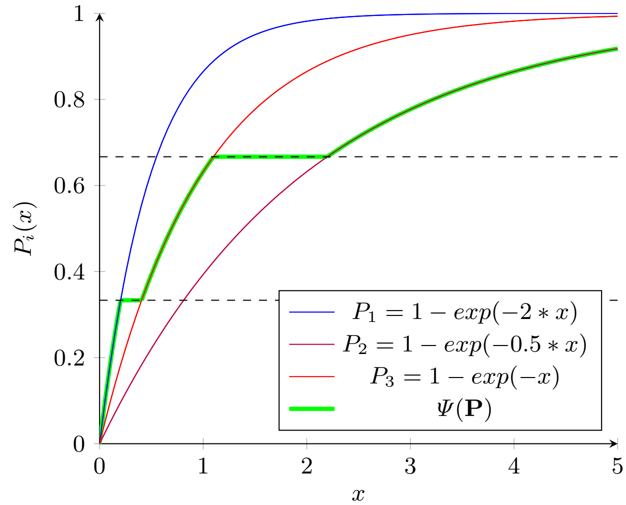

Fig. 2: The same example as in the left figure. The proportional-cumulative (in green) follows at its left the leftist voter P_1 for one third of the mass, follows at its right the rightist voter P_3 for one third of the mass, and similarly for the middle.

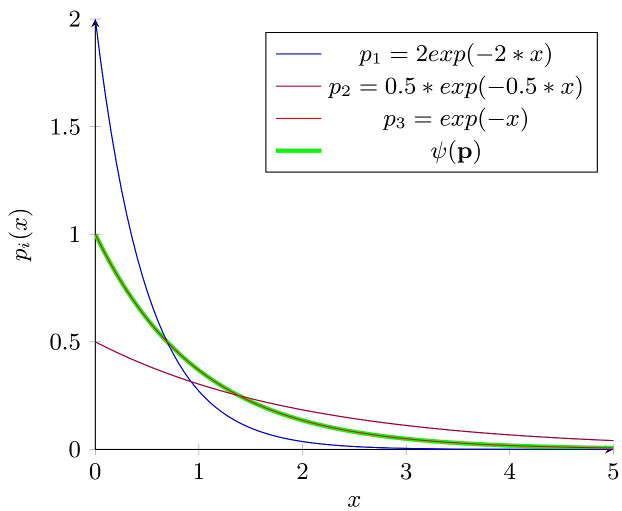

Fig. 3: Densities of the Cumulatives in Figure 1

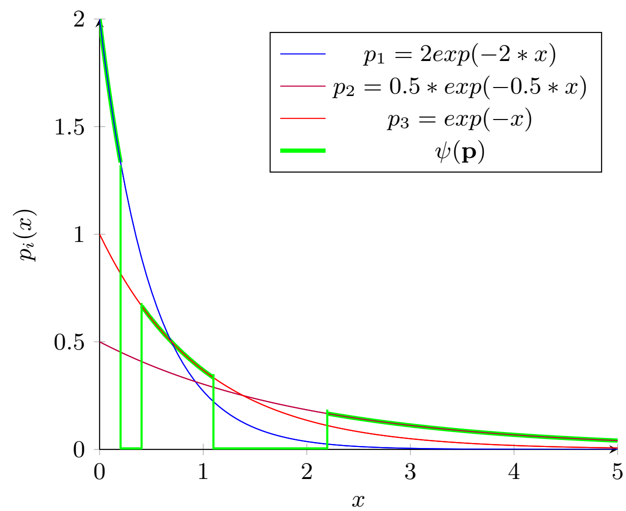

Fig. 4: Densities of the Cumulatives in Figure 2

3.3 Comparing middlemost-cumulative and proportional-cumulative functions

The main drawback of middlemost-cumulative is its lack of diversity. When the various supports S_i of the experts' subjective probabilities p_i are disjoint intervals I_i, or more generally, when the P_i's first-order stochastically dominates one another (as in Fig 1), the outcome of middlemost-cumulative equals the view of the median-expert (see the green curve in Figure 1 (CDFs) and Figure 3 (densities)).

On the contrary, proportional-cumulative takes into account all the experts' views and in a natural way. As Figure 2 shows, the leftmost expert CDF is followed at first by proportional-cumulative for one-third of the probability mass. This is a good property since the fact he is the leftmost expert means that the first third of his probability distribution is what best describes how he differs from the other experts and therefore what he most likely wants to emphasize. Similarly, the rightmost expert is followed by proportional-cumulative for his rightmost third of his opinion, which also best represents how his opinion differs from the group and so what he most likely wants to emphasize. The middle expert is followed for his middle opinion which again is interesting for the same reason. As such, not only do experts contribute for exactly one n-th of the final outcome but proportional-cumulative ensures that the most sticking aspects of their opinion (compared to the group) are conserved.

Given that the order-cumulatives are the unique anonymous level-SP probability aggregation functions that satisfy certainty preservation and plausibility preservation, proportional-cumulative must violate one of the axioms. It is clearly anonymous and is certainty preserving (see next section). Hence, it violates plausibility preservation. That can be seen in Figure 4: the density of proportional-cumulative (in green) for the interval $[1, 2]$ is zero while the experts' densities for this interval are all strictly positive. We believe that the violation of this axiom is not so problematic because proportional-cumulative satisfies a weak version of plausibility: if a level event has a positive probability for all experts, so does society. If the decision-maker's final decision is only based on a level-event, only the plausibility of such an event is of importance.

4 Characterizing Level-SP Methods: Anonymous Case

This section details several characterizations in the anonymous case, the general case is discussed in the next section. We start by identifying all anonymous level-SP methods then we refine this set by adding two classical axioms (certainty and plausibility preservations) until only the order-cumulatives remain.

4.1 Level-SP Anonymous Methods

In the appendix is proved that level-SP implies that for every $a \in \Lambda$ and every input profile $P = (P_1, \ldots, P_n)$, the value of society's CDF $\Psi(P_1, \ldots, P_n)(a) := g_a$ at a depends only on the experts CDF values $(\mathbf{P(a)}) \in [0,1]^N$ in a. This will imply (appendix) that for every a, $g_a : [0,1]^N \to [0,1]$ is one-SP as defined in section 2.1. If in addition we require anonymity (see Axiom 1) then from Moulin's anonymous characterization, we deduce the following result.[17]

Theorem 6 (Anonymous level-SP characterization). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP and anonymous if and only if there exist $n+1$ phantom functions $f_1 \leq \cdots \leq f_{n+1}$ each from $\Lambda \to [0,1]$ increasing and right continuous such that $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated to ψ verifies the formulae:*

$$\forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \text{median}\left(P_1(a), \ldots, P_n(a), f_1(a), \ldots, f_{n+1}(a)\right)$$

where $\lim_{a \to \sup \Lambda} f_{n+1}(a) = 1$. Moreover ψ is unanimous if and only if $f_1 = 0$ and $f_{n+1} = 1$.

In the sequel, **we will call the $\{f_k\}_{k \in \{1,\ldots,n+1\}}$ in the theorem above the phantom functions associated to ψ**. Example of figure 5 corresponds to an anonymous and unanimous PAF (in green on the right side). This PAF does not verify either of the desirable axioms defined in the next two subsections.

[17] Recall that anonymity is defined in Axiom 1 and Unanimity in Axiom 2.

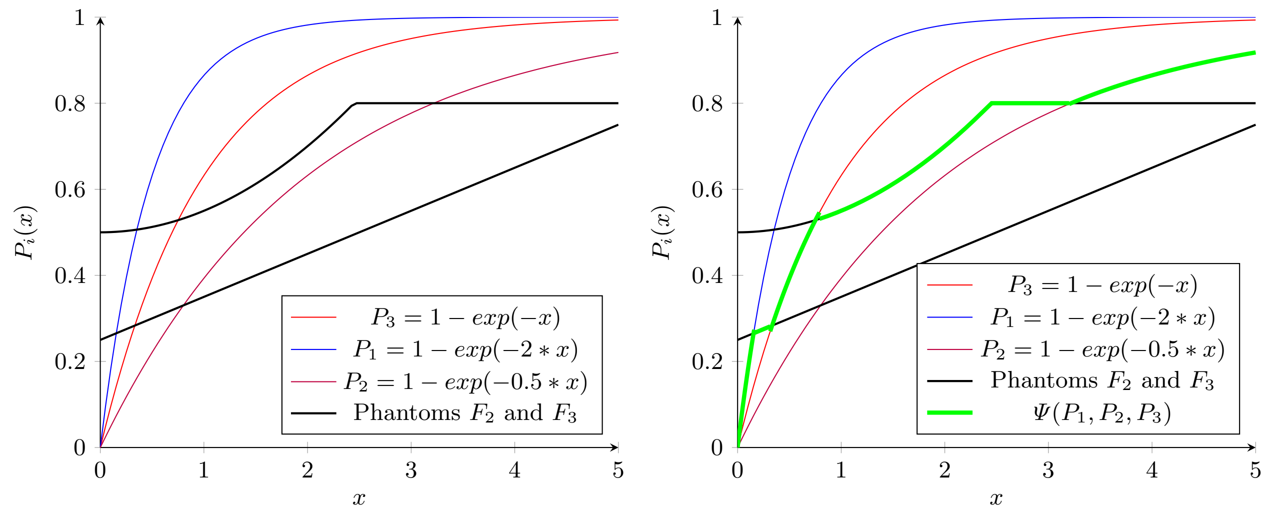

Fig. 5: A Unanimous and Anonymous Level-SP rule. On the left is drawn the cumulative functions P_1, P_2 and P_3 of our 3 voters. In black, is drawn the phantom functions F_2 and F_3 ($F_1 = 0$ and $F4 = 1$ are not drawn). On the right, for each level x in the x-axe, the outcome in green is $median(P_1(x), P_2(x), P_3(x), F_2(x), F_3(x))$.

4.2 Certainty Preservation: Anonymous Case

It is natural to wish that when all experts agree that an event is certain to happen almost surely (or certain not to happen) then the aggregation of their probabilities reflects that too.

Axiom 5 (Certainty preservation) *A PAF ψ **preserves certainty** iff for any probability profiles $\mathbf{p} = (p_1, \ldots, p_n)$ and all events $A \subseteq \Lambda$ Borel measurable:*

$$(p_i(A) = 1 \quad \forall i \in N \Rightarrow \psi(\boldsymbol{p})(A) = 1), \text{ or equivalently, } (p_i(A) = 0 \quad \forall i \in N \Rightarrow \psi(\boldsymbol{p})(A) = 0).$$

In other words, if an event is judged impossible (resp. certain) almost surely by all experts then it is judged impossible (resp. certain) almost surely by the aggregation function. This is a fairly standard axiom in the literature sometimes called the *zero preservation property* or [33] or *consensus preservation* in [15].

Magically, this property alone suffices to imply unanimity and that the phantom functions $a \to f_k(a)$ are constant in a (see the appendix). Combining with the previous theorem, we obtain the following.

Theorem 7 (Certainty preservation characterization (Anonymous)). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP, anonymous and certainty preserving if and only if there are $n-1$ phantom values $f_2 \leq \cdots \leq f_n \in [0, 1]$ such that such that $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated to ψ verifies the following:*

$$\forall a \in \Lambda, \Psi(P_1, \ldots, P_n)(a) = \text{median}(P_1(a), \ldots, P_n(a), f_2, \ldots, f_n).$$

The one-SP anonymous function $g(r_1, \ldots, r_n) = \text{median}(r_1, \ldots, r_n, f_2, \ldots, f_n) : [0, 1]^N \to [0, 1]$ that will be called in the sequel **the one-SP rule associated to ψ or Ψ.** It is interesting to note that proportional-cumulative and the order-cumulatives are, from their definition, certainty preserving.

4.3 Plausibility Preservation: Anonymous Case

We now wish to capture the property according to which: if all experts agree that an outcome may happen with positive probability then the PAF preserves that property.

Axiom 6 (Plausibility preservation) *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ verifies plausibility preservation iff for any input profiles $\boldsymbol{p} \in \mathcal{P}^N$ and any possible interval $I \subseteq \Lambda$:*

$$p_i(I) > 0 \;\; \forall i \in N \implies \psi(\boldsymbol{p})(I) > 0.$$

The next theorem says that the condition for plausibility preservation is that some monotony property must be satisfied by the phantom functions in all intervals where their values are not 0 or 1.

Theorem 8 (Plausibility preservation characterization (Anonymous)). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP, plausibility preserving and anonymous iff each phantom function f_k is strictly increasing on the interval where its value is not in $\{0, 1\}$ and $f_1(a) < 1$ for any $a < \sup \Lambda$ and $f_{n+1}(a) > 0$ for any $a > \inf \Lambda$.*

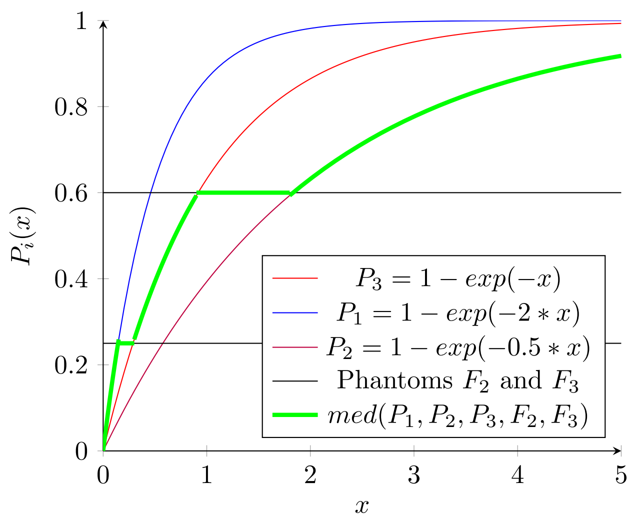

Fig. 6: Certainty Preserving but not Plausibility Preserving, Unanimous and Anonymous Method.

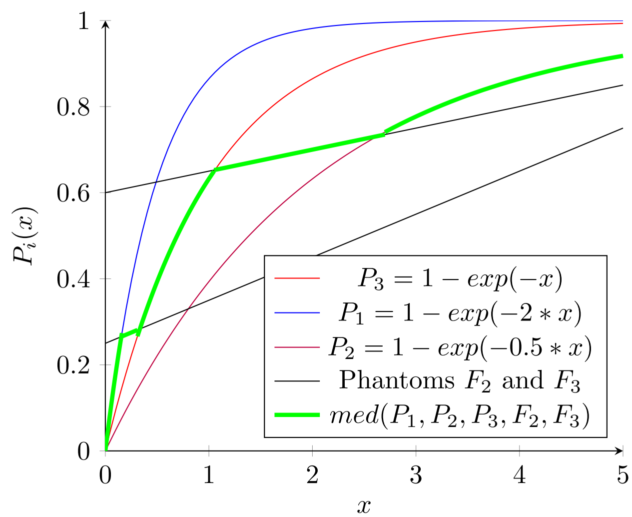

Fig. 7: Plausibility Preserving but not Certainty Preserving, Unanimous and Anonymous Method.

Figure 6 represents a mechanism which is unanimous ($F_1 = 0, F_4 = 1$), anonymous, certainty preserving (phantoms F_2 and F_3 are constant) which is not plausibility preservation (F_2 and F_3 are never equal to 0 or 1 but not strictly increasing). Figure 7 represents a mechanism which is unanimous ($F_1 = 0, F_4 = 1$), anonymous, plausibility preserving (the phantom functions which are not 0 and 1, F_2 and F_3, are strictly increasing) but is not certainty preserving as the phantoms F_2 and F_3 are not constants.

Remark 3. In theorem 4 of section 3, we stated without proof that the unique PAFs that are anonymous, certainty preserving, and plausibility preserving are the order-cumulatives. The proof follows easily from the last theorems. Certainty preservation implies that all the phantoms functions are constants and from plausibility preservation that they must all be equal to 0 or to 1. Hence g, the one-SP associated to ψ, must be an order-function.

As proportional-cumulative is anonymous and certainty preserving, it violates plausibility preservation (as illustrated in figure 4). However, it satisfies the following weaker version.

Axiom 7 (Level plausibility preservation) *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ verifies level plausibility preservation iff for any input profiles $\boldsymbol{p} \in \mathcal{P}^N$ and any level event $\mathcal{E}$:*

$$p_i(\mathcal{E}) > 0 \;\; \forall i \in N \to \psi(\boldsymbol{p})(\mathcal{E}) > 0.$$

Lemma 1. *If a level-SP PAF is unanimous, it is level plausibility preserving.*

As certainty preserving level-SP methods (such as proportional-cumulative) are unanimous, they are level plausibility preserving. In the next section, we explore the world of non necessarily anonymous PAFs, because, in practice, experts are not equally qualified and are usually weighted by the aggregation method.

5 Characterizing Level-SP Methods: General Case

While interesting and more intuitive, the anonymous methods cannot be applied when experts do not have equal expertise. This section characterizes the level-SP methods that are not necessarily anonymous. Thanks to Moulin's maxmin formulae, we can obtain characterizations similar in spirit to the anonymous case. We are also going to explore the limits of our axiomatizations by establishing several impossibilities.

5.1 All Level-SP Probability Aggregation Functions

As in the anonymous case, level-SP implies that we should replace the phantoms in Moulin's characterization in theorem 1 by phantoms functions.

Theorem 9 (Characterization: general case). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP if and only if there exists for every $S \subseteq N$ an increasing right continuous function $f_S : \Lambda \to [0,1]$ such that for all $S \subseteq S'$ we have $f_S \leq f_{S'}$, $\lim_{a \to \sup \Lambda} f_N(a) = 1$ such that $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated to ψ is given by the formulae:*

$$\forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \pi \circ \psi(\boldsymbol{p})(a) = \sup_{S \subseteq N} \min\left(f_S(a), \min_{i \in S} P_i(a)\right).$$

Moreover, ψ is unanimous if and only if $f_\emptyset = 0$ and $f_N = 1$.

As in the anonymous case, **we will call the $\{a \to f_S(a)\}_{S \in 2^N}$ in the theorem above the phantom functions associated to ψ.**

5.2 Certainty Preservation: General Case

The next theorem says that certainty preservation implies level-independence.

Axiom 8 (Level-Independence) *The PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ satisfies **level-independence** if there is voting rule $g : [0,1]^N \to [0,1]$ such that for all $a \in \Lambda$ and all $\boldsymbol{P} \in \mathcal{C}^N$ we have:*

$$\Psi(\boldsymbol{P})(a) = g(P_1(a), \ldots, P_n(a)).$$

Where Ψ is the CDF associated to ψ.

As in the anonymous case, we will refer to g as the one-SP voting rule associated to ψ (or equivalently Ψ).

Theorem 10 (Certainty preservation characterization). *A level-SP PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is certainty preserving if and only it is:*

- *unanimous and level independent.*
- *unanimous and its associated phantom functions $\{a \to f_S(a)\}_{S \in 2^N}$ are constants in a.*

Remark 4. It is unsurprising that certainty preservation implies unanimity since both notions model the idea that if all experts agree on something then the outcome should reflect that agreement.

Let us now explore the boundaries of our result. Level-independence is a weak version of a classical axiom in the literature called *independence*.

Axiom 9 (Independence) *The PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ satisfies **independence** if there is $g : [0,1]^N \to [0,1]$ such that $\psi(p_1, \ldots, p_n)(A) = g(p_1(A), \ldots, p_n(A))$ for every input $\boldsymbol{p} = (p_1, \ldots, p_n)$ and every Borel $A \subset \Lambda$.*

Unfortunately, independence is known to be a very strong axiom as, alone, it implies that ψ must be a weighted average of the expert's inputs [34,25,7,13,15], which is not level-strategyproof unless it is dictatorial.

Theorem 11 (Impossibility 2). *The unique level-SP and independent PAFs are the dictatorials.*

5.3 Plausibility Preservation: General Case

Again, the characterization of plausibility preserving PAFs is essentially the same as in the anonymous case.

Theorem 12 (Plausibility preservation characterization). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP and plausibility preserving if and only if any associated phantom function f_S is strictly increasing on the interval where its value is not in $\{0,1\}$ and $f_\emptyset(a) < 1$ for any $a < \sup \Lambda$ and $f_N(a) > 0$ for any $a > \inf \Lambda$.*

The definition of certainty preservation uses Borel set while the definition of plausibility preservation uses intervals. One might ask what happens if we replace intervals with Borel sets?

Axiom 10 (Borel plausibility preservation) *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ satisfies Borel plausibility preservation iff for any input profiles $\boldsymbol{p} \in \mathcal{P}^N$ and any Borel $A \subset \Lambda$:*

$$p_i(A) > 0 \quad \forall i \in N \implies \psi(\boldsymbol{p})(A) > 0.$$

Borel plausibility preservation is too strong as it results in an impossibility result.

Theorem 13 (Impossibility 3). *When $\Lambda = \mathbf{Z}$, or is a real interval, the unique PAFs that are levels-SP, unanimous, and Borel plausibility preserving are the dictatorials.*

Other impossibilities are proved in the appendix.

5.4 Combining Certainty and Plausibility Preservations: General Case

In the anonymous case combining certainty preservation, and plausibility preservation characterized the order-cumulatives. It is therefore interesting to explore their association without the anonymity axiom.

Theorem 14 (Certainty and plausibility Characterization 1). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP, certainty preserving and plausibility preserving iff it is unanimous and the phantom functions are constants with values in $\{0,1\}$.*

Hence there is a finite number of methods that are level-SP, certainty preserving, and plausibility preserving. Just like with the anonymous case, they all have the property that if the inputs dominate one another, the output is equals to one of the inputs (as in figure 3). Elegantly, this property fully characterizes this class.

Theorem 15 (Certainty and plausibility Characterization 2). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP, certainty preserving and plausibility preserving iff whenever the inputs dominate one another (there is a permutation σ such that $P_{\sigma(1)} \leq \cdots \leq P_{\sigma(n)}$) the output is on of the inputs.*

The next section presents a more diverse level-SP method.

5.5 Weighted proportional-cumulative

This subsection extends proportional-cumulative to problems where experts are not equally competent and have non equal weights. To fully characterize this method, we extend the definition of the proportionality axiom. To motivate this new axiom, imagine a situation where experts have extreme views. In that case, we want the aggregation method to output a diversified solution that takes into account all the views weighted by the expert's degree of expertise. For example, suppose we have two experts, the first represents 95% of the scientific community view and believes global temperature will surely raise by exactly 0.2°C in the next decade. The second represents the remaining 5% of the community and believes that global temperature won't change. A fair compromising PAF would consider the final probability to be a 5% chance that the climate won't change and a 95% chance that it will rise by 0.2°C. The next axiom formalizes this idea.

Axiom 11 (Weighted proportionality) *If experts are single-minded (every vote p_i is equal to the Dirac mass at some $a_i \in \Lambda$, e.g. $p_i = \delta_{a_i}, \forall i \in N$), the aggregation must coincide with the weighted average:*

$$\forall (a_1, \ldots, a_n) \in \Lambda^N \quad \psi(\delta_{a_1}, \ldots, \delta_{a_n}) = \sum_i w_i \delta_{a_i};$$

where $w_i \geq 0$ is the normalized weight attributed to expert $i \in N$, with $\sum_{j \in N} w_j = 1$.

In practice, there is a whole science, regulations, and procedures of how to attribute weights to experts [12]. If the aggregation problem is recurrent and the realized outcome observable (anticipating a volcanic irruption, or predicting the price of an asset or the growth rate), one way to fix weights is by using some non-regret reinforcement-learning algorithm where experts are scored at every time period depending on their realized performance and the weights are changed dynamically proportionally to the accumulated scores [20,12].

The weighted average $\psi(p_1, \ldots, p_n) = \sum_i w_i p_i$ satisfies that axiom but it is not level-strategyproof unless $w_i \neq 0$ for exactly one expert (the dictator). The next theorem shows that there is exactly one PAF function satisfying level-SP and weighted proportionality. This function happens to be certainty preserving and so can be uniquely described by its one-SP voting rule $\mu_{\mathbf{w}} : [0,1]^n \to [0,1]$ defined as follows:

$$\forall \mathbf{r} = (r_1, \ldots, r_n), \mu_{\mathbf{w}}(\mathbf{r}) := \sup \left\{ y \;\middle|\; \sum_{i : r_i \geq y} w_i \geq y \right\}.$$

Before stating this theorem, let us give an equivalent formulation of $\mu_{\mathbf{w}}$ and how it relates to the literature.

Proposition 1. *If all the weights are rationals $w_i = s_i/d$ then:*

$$\forall \boldsymbol{r} = (r_1, \ldots, r_n), \mu_{\boldsymbol{w}}(\boldsymbol{r}) := \text{median}(\overbrace{r_1, \ldots, r_1}^{s_1}, \ldots, \overbrace{r_n, \ldots, r_n}^{s_n}, 0, 1/d, \ldots, 1 - 1/d, 1)$$

Theorem 16 (Weighted proportional-cumulative characterization). *The unique PAF $\psi : \mathcal{P}^n \to \mathcal{P}$ that verifies level-SP and weighted proportionality is the unique certainty preserving one associated to the one-SP rule $\mu_{\boldsymbol{w}}$, that is:*

$$\forall a \in \Lambda, (P_1, \ldots, P_n) \in \mathcal{C}; \Psi(P_1, \ldots, P_n)(a) := \mu_{\boldsymbol{w}}(P_1(a), \ldots, P_n(a))$$

One of the most interesting aspects of the weighted proportional-cumulative is how it responds to dominated inputs. The next theorem shows that, just like what we have seen in the examples with proportional-cumulative, the experts will contribute in the aggregation for exactly their weights and the weighted-cumulative will follow the probability distribution of an expert on the segment that best describes its role in the group. For example, the leftmost expert contributes on his leftmost segment, in other words, exactly the interval where he disagrees most with the other experts.

Proposition 2 (Proportional-cumulative for dominated opinion). *Suppose that Λ is an interval, and that all the P_i are continuous and verify for all $i \in N$, $P_i \geq P_{i+1}$. Then the weighted-proportional level-SP mechanism $\psi : \mathcal{P}^N \to \mathcal{P}$ of weight $\boldsymbol{w}$ can be computed for this input profile as follows:*

$$\psi(\boldsymbol{p})(a) = \begin{cases} p_i(a) & \text{if} \quad \sum_{k \leq i-1} w_k \leq P_i(a) < \sum_{k \leq i} w_k \\ 0 & \text{else} \end{cases}$$

Example in Figure 2 is an illustration of the theorem above.

The following table resumes all our characterizations.

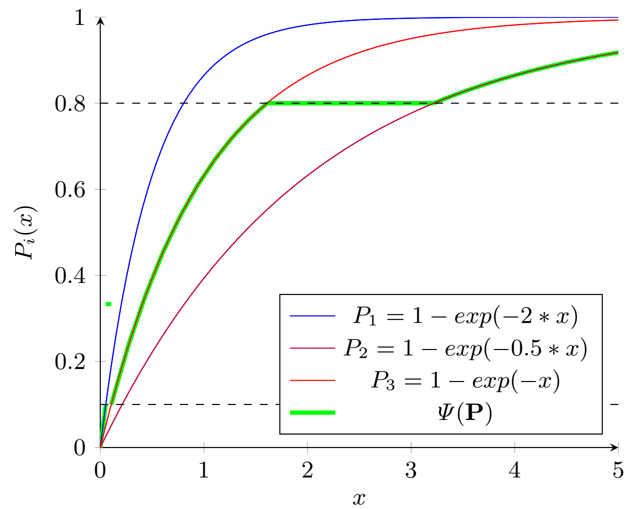

Fig. 8: The Weighted Proportional Cumulative Method with weights 0.1 for P_1, 0.2 for P_2 and 0.7 for P_3

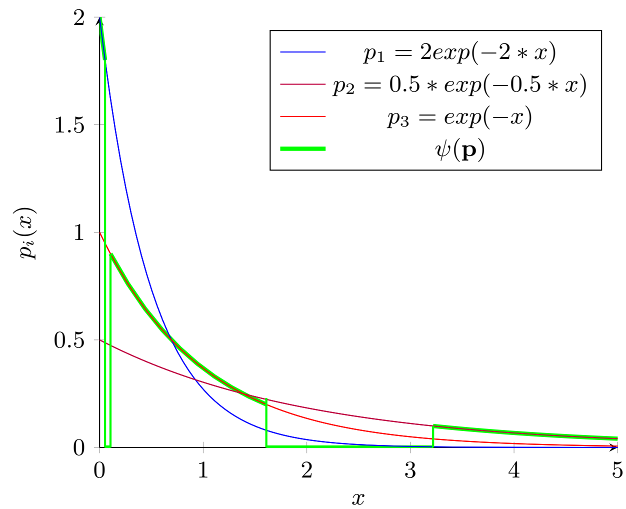

Fig. 9: Probability Density Functions for the Cumulative Distributions in Figure 2

	General case	Anonymous case
Level-SP	$\sup_{S \subseteq N} \min\left(f_S(a), \min_{i \in S}\{P_i(a)\}\right)$ $lim_{a \to \sup \Lambda} f_N(a) = 1$	$median(\mathbf{P(a)}, f_1(a), \ldots, f_{n+1}(a))$ $lim_{a \to \sup \Lambda} f_n(a) = 1$
Certainty Preservation	f_S are constants, $f_N = 1$	f_i are constants, $f_{n+1} = 1$
Plausibility Preservation	f_S is strictly increasing when not equal to 0 or 1	f_i is strictly increasing when not equal to $0, 1$
Plausibility Preservation +Certainty Preservation	$f_S \in \{0, 1\}$	Order Cumulatives
Proportionality	$\mu_{\mathbf{w}}(\mathbf{r}) := \sup\left\{y \mid \sum_{i \mid r_i \geq y} w_i \geq y\right\}$	$median(r_1, \ldots, r_n, 1/n, \ldots, 1 - 1/n)$

6 Application: Electing and Ranking with Uncertain Voters

The objective of this section is to show that proportional-cumulative could be combined with a recent evaluation-based voting method (Majority Judgment [5,6], **MJ**) to construct a strategically robust voting method in situations where voters have uncertainties and doubts about the candidates.

To elect one candidate/alternative or rank several [3], existing voting methods (plurality, Borda, Condorcet, approval voting, etc) implicitly assume that individual voters are certain about their opinions or views. In practice, voters are mostly uncertain. In recruitment committees, members are often hesitating between a good safe candidate vs a risky one. On the day of the Brexit vote, no voter knew with certainty what the final deal between the UK and the EU would be, nor did they know the long-term consequences of such a deal. Similarly, when a reviewer is judging a paper for a conference, they are often uncertain about the quality of some of the papers. To capture those uncertainties, one can use an extension of majority judgment.

Let us first describe how majority judgment (MJ) works for a small number of voters. Suppose we have a numerical scale $\{0, \ldots, 9\}$ and a set of grades ordered from best to worse $A = (9, 7, 6, 5, 2)$ corresponding to the grades given by 5 voters to candidate A. Its majority value is obtained by iterating from the middlemost grade (the median), then down, up, down etc, which gives us the 5 dimensional vector $v(A) = (6, 5, 7, 2, 9)$. Two ordered set of grades A and B are compared in lexicographic order by their majority values. For example if $B = (9, 8, 6, 4, 1)$ then $v(B) = (6, 4, 8, 1, 9)$ and so $v(A) > v(B)$ because $(6, 5, 7, 2, 9) \succcurlyeq (6, 4, 8, 1, 9)$. Majority judgment is an ordinal method as the ranking remains unchanged if the numerical grades are replaced by some qualitative set of grades such as {Great, Good, Average, Poor, Terrible}. Also, the MJ ranking can be extended to a continuum of voters (e.g. a normalized distribution over the set of grades, see [5] chapter 14).

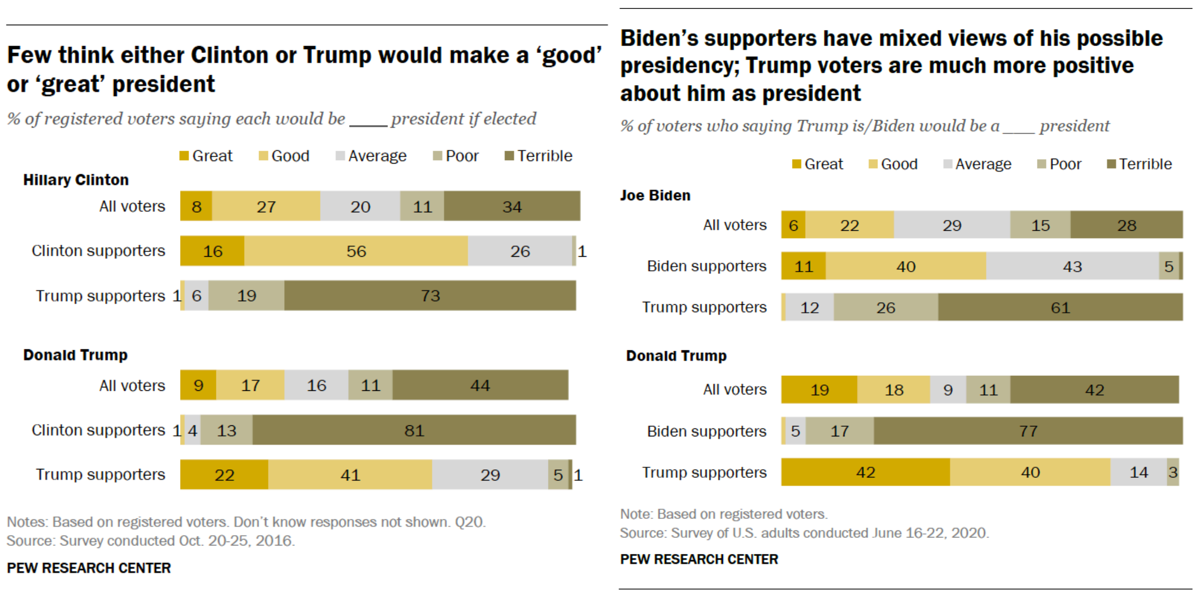

Fig. 10: In majority judgment [6,5], each voter is asked to grade each candidate on a scale Λ such as "Great, Good, Average, Poor, Terrible". The output for each candidate (the merit profile) can be interpreted as a probability distribution over Λ. The winner is the one with the highest median grade (the majority-grade). In case of the same majority-grade, an elaborate iterative rule breaks the tie (see the main text for an example). In the surveys above, Donald Trump's majority grade is "Poor" in both elections and so with majority judgment he loses against both competitors, Clinton and Biden, since their majority grade is "Average". The two figures are extracted from two Pew Research Center survey reports about the 2016 and the 2020 US elections [**?**,**?**].

Imagine N voters who should rank M candidates or alternatives $\mathcal{A} = \{A_1, A_2, A_3, \ldots, A_M\}$. Majority Judgment under Uncertainty **MJU** works as follows:

- *Fixed by the designer:* a scale of grades Λ (such as {Great, Good, Average, Poor, Terrible} and a normalized positive vector $\mathbf{w} = (w_1, \ldots, w_N)$ where w_i is the weight of voter i.
- *Input from voters (the ballots):* each voter $i \in N$ is asked to submit for each candidate $A \in \mathcal{A}$ a probability distribution $p_i^A \in \mathcal{P} = \Delta(\Lambda)$. For example, the voter may think that A_k will be Good for sure, and that A_l will be Great with probability $\frac{2}{3}$ and Terrible with probability $\frac{1}{3}$.
- *Output for voters:* each candidate A is given an aggregate probability distribution p^A **computed using the weighted-proportional-cumulative** that is $P^A(a) = \mu_{\mathbf{w}}(P_1^A(a), \ldots, P_N^A(a))$.
- *Ranking of the alternatives:* classical **majority judgment is applied** to rank the distributions p^{A_1}, $p^{A_2}, \ldots, p^{A_M}$ and consequently the candidates/alternatives in $\mathcal{A}$.

Main properties of MJU:

- *IIA*: adding or dropping a candidate does not change the ranking between the others. This is because MJ is transitive and the ranking between two candidates A and B by MJU depends only on their respective distributions p^A and p^B, which are only a function of the inputs regarding them.
- *Impartiality*: candidates are treated equally by the method. This is because MJ is impartial and MJU applies the same PAF (the weighted-proportional-cumulative) to compute each candidate's A distribution p^A. If we want the method to be *anonymous* (voters are treated equally) just set $w_1 = \cdots = w_N = \frac{1}{N}$.

– *Extends MJ*: if all voters are certain about their choices (i.e. their input are Dirac measures) and have equal weights (anonymity), MJU outputs the empirical distribution of inputs for each candidate, and so the ranking of MJU coincides with the ranking of MJ. Note that proportional-cumulative is the only level-SP method that induces an extension of MJ.
– *Resistance to strategic manipulations*: This is a direct consequence of combining proportional-cumulative (level-strategyproof) and majority judgment (designed to counter strategic manipulations).

The philosophy behind MJ was to allow voters to better express themselves (compared to plurality let say) by submitting a grade on a scale Λ for each candidate. MJU goes further in this philosophy by allowing voters to express their uncertainties about the candidates on the same scale. This can in principle be done without difficulty if the judges are experts (university professors in a recruitment committee), but the task is probably impossible in large elections and some approximation is necessary. For example, one could ask the voters to check, for each candidate, several grades which are then converted to a uniform distribution over the subset of checked grades.

What about the extension of other voting methods to uncertainty? Arrowian voting methods [3] based on ranking-orderings –e.g. voters are asked their full preference ordering over all the candidates (Borda, STV)– cannot easily be extended to uncertainty, at least in a practical way. A possible extension is to submit a probability over the set of permutations over $\mathcal{A}$, e.g. with probability $\frac{1}{2}$ the ranking is $A > B > C > D$ and the probability $\frac{1}{2}$ it is $C > B > A > D$? theoretically, this can be done and several known methods can be consistently extended to uncertainty. In practice, it seems a very difficult task for voters as the interpretation is not clear –how a voter who is uncertain about a few candidates vote?–.

Other methods can be extended. For example, consider binary decision problems such as a referendum (e.g. the Brexit), and extend majority rule —with the Condorcet Jury context in mind— by allowing voters to express their uncertainties (see [10] for a similar approach). More precisely, each voter is asked to report its prior probability that some reform (e.g. the Brexit) is the right decision compared to the status quo. This is represented by a number $p_i \in [0, 1]$ for every voter $i = 1, \ldots, n$. Then aggregate those numbers using proportional-cumulative which coincides with the uniform median rule:

$$p = \mu(p_1, \ldots, p_n) = \text{median}\left(p_1, \ldots, p_n, 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right) = \sup_{q \in [0,1]} \left\{ \frac{\#\{i \in N : p_i \geq q\}}{n} \geq q \right\}.$$

Finally, if $p > 1/2 = \alpha$ then the reform is implemented, otherwise, it is not and we keep the status quo. Observe that when all voters are single-minded and vote extreme values ($p_i \in \{0, 1\}$ for all $i \in N$), then $p = \#\{i \in N : p_i = 1\}/n$, and so the procedure defines an extension of majority rule. But, as the consequences of making a bad decision can be high compared to the status quo (e.g. the Brexit), one may argue that the optimal α must be higher than $\frac{1}{2}$, perhaps $\frac{2}{3}$? In theory, the optimal α can be calculated by the maximization of some vNM expected utility of society. That is, we normalize the society vNM utility to 1 if the reform is a good decision and to 0 if the reform is a bad decision, then the optimal α coincide with the vNM utility of the status quo. In practice, the optimal α can be fixed by some committee (e.g. the house of commons) or it can be determined democratically as the median of society individual risk aversions $\alpha = \text{median}(\alpha_1, \ldots, \alpha_n)$.

7 Conclusion

This paper studies the probability aggregation problem when the set of outcomes Λ is an ordered set. It defines level-strategyproofness and proves it to imply classical strategyproofness for a rich set of single-peaked preferences over the CDF space. Several characterizations, possibility, and impossibility theorems are established when level-strategyproofness is combined with other axioms and two methods are singled out: middlemost-cumulative and proportional-cumulative. Both methods are easy to compute and can be extended to problems where experts are weighted.

The paper gives several arguments supporting the claim that the weighted proportional-cumulative is the best of all level-SP methods. Its unique weakness is the non-satisfaction of plausibility preservation. Despite the fact no unanimous level-SP method satisfies the stronger Borel plausibility preservation, the proportional-cumulative method does satisfy level plausibility preservation. Namely, if all experts agree that the probability of a level set is positive, so does society. Hence, if we are interested in problems where the final decision is level-based, then proportional-cumulative is probably the best incentive compatible method.

In practice, the weighted proportional-cumulative can be used to aggregate experts beliefs in various applications going from nuclear safety, investment banking, volcanology, public health, ecology, engineering to climate change and aeronautics/aerospace. Specific examples include calculating the risks to manned spaceflight due to collision with space debris and quantifying the uncertainty of a groundwater transport model used to predict future contamination with hazardous materials (see Cooke [12], Chapter 15, where those applications and others with real data are described and several aggregation methods are analyzed). Proportional-cumulative can also be combined with some existing voting methods, such as majority rule, approval voting, or majority judgment, to construct methods for electing and ranking where experts can express their uncertainties/doubts about the merit/quality of the candidates/alternatives. Since majority judgment is quite successful in practice,[18] we may expect that its uncertain version will also be of practical interest.

We see several possibilities for extension. A first direction is a definition of level-SP for a multidimensional Λ —or more generally a metric space Λ— and characterizing all the level-SP methods. A second direction is a formal study of all the voting methods where experts are allowed to express their uncertainties and characterizing the remaining methods for some given set of axioms.

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A Implications of level-SP

A.1 Lemmas

Let $\mathcal{C}$ be the set of CDFs on Λ and $M = \sup \Lambda$. For example, if $\Lambda = \mathbf{R}$, then $M = +\infty$. The probability inputs by experts $i = 1, \ldots, n$ are denoted as $p_1, \ldots, p_n$ and their corresponding CDFs are denoted $P_1, \ldots, P_n$. A probability aggregation function (PAF) is denoted ψ and its corresponding cumulative aggregation function (CAF) Ψ. Below we describe the implications of being level-strategyproof.

Lemma 2 (Monotony). *If Ψ is level-SP then for all experts i, for all levels $a \in \Lambda$, and all CDF votes $P_1, \ldots, P_n, P_i'$:*

$$P_i(a) \leq P_i'(a) \Rightarrow \Psi(\boldsymbol{P})(a) \leq \Psi(\boldsymbol{P}_{-i}(P_i'))(a)$$

Proof. We will use a *reductio ad absurdum* to reach our result. Suppose that Ψ verifies level-SP but is not monotonous in all $a \in \Lambda$. Then there is P_i, P_i' and a such that $P_i(a) \leq P_i'(a)$ and $\Psi(\mathbf{P})(a) > \Psi(\mathbf{P}_{-1}(P_i'))(a)$.

If $\Psi(\mathbf{P}_{-1}(P_i'))(a) \geq P_i(a)$. Then:

$$\Psi(\mathbf{P})(a) > \Psi(\mathbf{P}_{-1}(P_i'))(a) \geq P_i(a)$$

This contradicts level-SP.

If $\Psi(\mathbf{P})(a) \leq P_i'(a)$. Then:

$$P_i'(a) \geq \Psi(\mathbf{P})(a) > \Psi(\mathbf{P}_{-1}(P_i'))(a).$$

Then replacing $P_i'(a)$ by $P_i(a)$ improves the output, this contradicts level-SP.

Else:

$$\Psi(\mathbf{P}_{-1}(P_i'))(a) < P_i(a) \leq P_i'(a) < \Psi(\mathbf{P})(a)$$

This contradicts level-SP. Therefore level-SP implies monotony.

Lemma 3 (Level-by-level Independence). *If Ψ is level-SP then for all $a \in \Lambda$ the value of $\Psi(P_1, \ldots, P_n)(a)$ only depends on $P_1(a), \ldots, P_n(a)$.*

Proof. For any $a \in \Lambda$, suppose we have $P_1, \ldots, P_n$ and $P_1', \ldots, P_n'$ such that for all experts i, $P_i(a) = P_i'(a)$. Then according to the monotony lemma $\Psi(P_1, \ldots, P_n)(a) = \Psi(P_1', \ldots, P_n')(a)$. Therefore $\Psi(P_1, \ldots, P_n)(a)$ only depends of $P_1(a), \ldots, P_n(a)$.

The previous lemma implies immediately the existence of a family of one-SP voting rules $\{g_a\}$ that represents Ψ.

Lemma 4. *Let Ψ be a level-SP voting rule. Then, there exists a family of one-SP voting rules $g_a : [0,1]^N \to [0,1]$, $a \in \Lambda$ which represent Ψ. It is uniquely defined so that for every $\boldsymbol{P}$ and a:*

$$\Psi(\boldsymbol{P})(a) = g_a(\boldsymbol{P(a)}).$$

A.2 Proof of the characterizations

Corollary 1 (General case: characterization). *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP if and only if there exists for every $S \subseteq N$ an increasing right continuous function $f_S : \Lambda \to [0,1]$ such that for all $S \subseteq S'$ we have $f_S \leq f_{S'}$, $\lim_{a \to \sup \Lambda} f_N(a) = 1$ such that $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated to ψ is given by the formulae:*

$$\forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \pi \circ \psi(\boldsymbol{p})(a) = \sup_{S \subseteq N} \min\left(f_S(a), \min_{i \in S} P_i(a)\right).$$

Moreover, ψ is unanimous if and only if $f_\emptyset = 0$ and $f_N = 1$.

Proof. $\Rightarrow$: The last lemma gives us the existence of family $\{g_a\}$ of one-SP rules. Therefore, from Moulin, for any $a \in \Lambda$ there is a family of phantom voters $(b_a)_S$ where $S \subseteq \mathcal{N}$ such that:

$$g_a(\mathbf{P}(\mathbf{a})) = \sup_{S \subseteq N} \min\left((b_a)_S, \min_{i \in S} P_i(a)\right)$$

and $(b_a)_S$ is increasing over the subsets of N. It follows that there are 2^n functions $f_S : \Lambda \to [0,1]$ that are increasing over the subsets S of N such that:

$$\forall a \in \Lambda, \Psi(\mathbf{P}(a)) = \pi \circ \psi(\mathbf{p})(a) = g_a(\mathbf{P(a)}) = \sup_{S \subseteq N} \min\left(f_S(a), \min_{i \in S} P_i(a)\right)$$

.

Since $a \to g_a$ must be right continuous and increasing when the inputs are right continuous and increasing (it is necessary if we want the outcome to be a CDF), it follows that so are $a \to f_S(a)$ for every S (take experts reports such that of $i \in S$ then $P_i \geq f_S$ else $P_i \leq f_S$. Then the outcome is always f_S). Since we want the outcome of Ψ to be a CDF we need:

$$\lim_{a \to \sup \Lambda} \Psi(\mathbf{P}(a)) = 1,$$

therefore since $\lim_{a \to \sup \Lambda} P_i(a) = 1$, it follows that $\lim_{a \to \sup \Lambda} g_a(\mathbf{1}) = \lim_{a \to \sup \Lambda} f_N(a) = 1$. Therefore:

$$\lim_{a \to \sup \Lambda} f_N(a) = 1.$$

$\Leftarrow$: Let $g_a(\mathbf{r}) := \sup_{S \subseteq N} \min\left(f_S(a), \min_{i \in S} r_i\right)$. As g_a is one-SP for every a (it respects Moulin's representation), we deduce that Ψ is level-SP. The increasing right continuity of the f_S functions provide that the outcome is right continuous and increasing if the inputs are. Since $\lim_{a \to \sup \Lambda} f_N(a) = 1$, we have (using the formula) that $\lim_{a \to \sup \Lambda}, \min\left(f_N(a), \min_{i \in S} P_i(a)\right) = 1$. As such $\lim_{a \to \sup \Lambda}, \Psi(\mathbf{P})(a) = 1$.

$\circ$: Unanimity is immediate from the conditions for unanimity on a one-SP rule from Moulin's characterization.

Corollary 2 (Moulin-type characterization: anonymous case). *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP and anonymous if and only if there exist $n+1$ phantom functions $f_1 \leq \cdots \leq f_{n+1}$ each from $\Lambda \to [0,1]$ increasing and right continuous in a such that $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated to ψ verifies the formulae:*

$$\forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = median(P_1(a), \ldots, P_n(a), f_1(a), \ldots, f_{n+1}(a))$$

and $\lim_{a \to \sup \Lambda} f_{n+1}(a) = 1$. *Moreover ψ is unanimous if and only if $f_1 = 0$ and $f_{n+1} = 1$.*

Proof. Identical to the above while using the anonymous characterization of one-SP.

A.3 Certainty preservation

Recall that Ψ is level-SP if there is a voting rule $g : [0,1]^n \to [0,1]$ such that for all $\mathbf{P}$ and a, $\Psi(\mathbf{P})(a) = g(P_1(a), \ldots, P_n(a))$.

Theorem 17. *A probability aggregation function ψ is level-SP and is certainty preserving if and only if its associated CAF satisfies level-independence.*

Proof. $\Rightarrow$: Suppose that ψ is level-SP and is certainty preserving.

Let $a < b < M = \sup \Lambda$ and $\mathbf{r}$ be such that $g_a(\mathbf{r}) < g_b(\mathbf{r})$. Then for $\mathbf{P} \in \mathcal{C}^n$ such that for all experts i we have $P_i(a) = P_i(b) = r_i$ one contradicts certainty preservation. Therefore there is g such that for all $a < M$, $\Psi(\mathbf{P})(a) = g(P_1(a), \ldots, P_n(a))$.

For $a < M$, if all experts agree $P_i(a) = 1$ then by certainty preservation $\Psi(\mathbf{P})(a) = \Psi(\mathbf{P})(M) = 1$. Therefore $\Psi(\mathbf{P})(M) = g(P_1(M), \ldots, P_N(M))$.

For $a < M$, if all experts agree $P_i(a) = 0$ then by certainty preservation $\Psi(\mathbf{P})(a) = 0$ therefore $\{0,1\}$ are possible outcome of g. Therefore by Moulin's characterization of g, g is surjective.

In conclusion, we have shown that ψ is level-independent.

$\Leftarrow$: Suppose that ψ is level-independent and level-SP. Denote by g the associated (necessarily one-SP) voting rule. Let A be an internal in Λ. Let us suppose that all experts agree that A is impossible, e.g. $p_i(A) = 0$ for every $i \in N$. Then $p_i(A) = P_i(\sup A) - lim_{a < \inf A, a \to \inf A} P_i(a) = 0$. Therefore

$$\Psi(\mathbf{P})(\sup A) - lim_{a < \inf A, a \to \inf A} \Psi(\mathbf{P})(a)$$

$$= g(P_1(\sup A), \ldots, P_n(\sup A)) - lim_{a < \inf A, a \to \inf A} g(P_1(a), .., P_n(a)) = 0,$$

the first equality is a consequence of level independence and the second from the continuity g (that can be deduced from the Moulin's characterization). Hence $\psi(\mathbf{p})(A) = 0$ and by sigma-additivity it follows that for any A Borel measurable we also have that if all experts agree that A is impossible, this is preserved by the aggregation function. Therefore ψ is certainty preserving.

Corollary 3. *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP and certainty preserving if and only if the associated phantom functions $f_S : \Lambda \to [0,1]$ are constant over $\Lambda / sup\Lambda$. Therefore they can be chosen without loss of generality as constants over Λ.*

Proof. $\Rightarrow$: We have shown that Ψ is level-independent, therefore $\Psi(\mathbf{P})(a) = g(P_1(a), \dots, P_N(a))$. As such g is one-SP and is defined by constant phantom values. It follows that the phantom functions are constants.

$\Leftarrow$: Suppose the phantom functions f_S are all constants. Then for all $i = 1, .., n$, and for any a, b such that $P_i(a) = P_i(b)$ we have $\Psi(\mathbf{P})(a) = \Psi(\mathbf{P})(b)$. Therefore ψ is level-independent (and level-SP) and so satisfies certainty preservation (as shown in the previous proof).

Corollary 4. *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP, anonymous, and certainty preserving if and only if there is n phantom values $f_1 \leq \dots \leq f_n \in [0,1]$ such that such that $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated to ψ verifies the following:*

$$\forall a \in \Lambda, \Psi(P_1, \dots, P_n)(a) = median(P_1(a), \dots, P_n(a), f_1, \dots, f_n, 1).$$

Proof. Identical to the above while using the Moulin anonymous characterization.

A.4 Plausibility Preservation

Theorem 18. *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP and plausibility preserving if and only if the associated One-SP family $\{g_a\}_{a \in \Lambda}$ satisfies:*

$$\forall \boldsymbol{r} \in \{0,1\}^N, (a < b \wedge g_a(\boldsymbol{r}) \neq 1 \wedge g_b(\boldsymbol{r}) \neq 0) \Rightarrow g_a(\boldsymbol{r}) < g_b(\boldsymbol{r}).$$

and g_a is not a constant equal to 0 or 1 for any $a \neq \sup \Lambda$.

Proof. $\Rightarrow$: Suppose that ψ verifies plausibility preservation. Let us suppose that there is $a < b$ and $\mathbf{r} \in \{0,1\}^N$ such that $g_a(\mathbf{r}) = g_b(\mathbf{r})$. Then for experts such that $P_i(c) = 1 - (b-c)\epsilon$ if $r_i = 1$ and $P_i(c) = c\epsilon - a\epsilon$ if $r_i = 0$ for an ϵ ¿ 0 small enough, we have by plausibility preservation that $\Psi(\mathbf{P})(a) < g_b(\mathbf{P})(b)$. This is true no matter how small $\epsilon > 0$ is. Since for $\epsilon = 0$ we have $\Psi(\mathbf{P})(a) = g_b(\mathbf{P}(b))$, we can conclude by level-SP that $\Psi(\mathbf{P})(a) = g_b(\mathbf{P}(b)) \in \{0,1\}$.

As such g_a is not strictly increasing it is worth 0 or 1.

$\Leftarrow$: Suppose we have $a < b$, and all experts agree that $P_i(b) - P_i(a) > 0$. If $\Psi(\mathbf{P})(a) = 1$ then since $P_i(a) < 1$ for all i we have g_a constant equal to 1 (by level-SP), which contradicts the assumption. Therefore $\Psi(\mathbf{P})(a) \neq 1$. Similarly, $\Psi(\mathbf{P})(b) \neq 0$. Therefore $\Psi(\mathbf{P})(a) < \Psi(\mathbf{P})(b)$. Consequently, ψ verifies plausibility preservation.

Corollary 5. *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP and plausibility preserving if and only if any associated phantom functions f_S is strictly increasing on the interval where its value is not in $\{0,1\}$ and $f_\emptyset(a) < 1$ for any $a < \sup \Lambda$ and $f_N(a) > 0$ for any $a > \inf \Lambda$.*

Proof. $\Rightarrow$: Since we have:

$$\forall \mathbf{r} \in \{0,1\}^N, (a < b \wedge g_a(\mathbf{r}) \neq 1 \wedge g_b(\mathbf{r}) \neq 0) \Rightarrow g_a(\mathbf{r}) < g_b(\mathbf{r}).$$

We have that the f_S functions verify:

$$(a < b \wedge f_S(a) \neq 1 \wedge f_S(b) \neq 0) \Rightarrow f_S(a) < f_S(b).$$

Also since g_a is not constant to 0 or 1 we have $f_\emptyset(a) \neq 1$ and $f_N(a) \neq 0$.

$\Leftarrow$: Suppose that the f_S are strictly increasing when not in $\{0,1\}$. Suppose that for all i, $p_i(a) > 0$. For any $b < a$,we have $f_S(b) \leq f_S(a)$ and $P_i(b) < P_i(a)$. As such:

$$\min(f_S(b), \min_{i \in S} P_i(b)) \leq \min(f_S(a), \min_{i \in S} P_i(a))$$

with equality iff $f_S(a) = 0$.

$$\sup_{S \subseteq N} \min(f_S(b), \min_{i \in S} P_i(b)) \leq \sup_{S \subseteq N} \min(f_S(a), \min_{i \in S} P_i(a))$$

with equality iff $f_\emptyset(b) = 1$ or $f_N(a) = 0$. Therefore as long as the conditions for equality do not happen, we verify plausibility preservation.

Corollary 6. *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP, plausibility preserving and anonymous if and only if each phantom function f_k is strictly increasing on the interval where its value is not in $\{0,1\}$ and $f_1(a) < 1$ for any $a < \sup \Lambda$ and $f_{n+1}(a) > 0$ for any $a > \inf \Lambda$.*

Proof. Identical to the above while using the anonymous characterization.

A.5 Combining level-SP, certainty and plausibility preservations

Theorem 19. *A probability aggregation function ψ is level-SP, certainty preserving and plausibility preserving if and only if the output of its associated one-SP aggregation rule g always output one of its inputs (e.g. $g(r_1, \ldots, r_n) \subset \{r_1, \ldots, r_n\}$ for every $(r_1, \ldots, r_n) \in [0,1]^N$).*

Proof. $\Rightarrow$:Suppose that for any $a < \sup \Lambda$, $g(s_1, \ldots, s_n) \notin \{s_1, \ldots, s_n\}$ then by one-SP there is $\mathbf{r} \in \{0,1\}^N$ such that $g(s_1, \ldots, s_n) = g(r_1, \ldots, r_n)$ (if $s_i < g(s_1, \ldots, s_n)$, define $r_i = 0$ and if $s_i > g(s_1, \ldots, s_n)$, define $r_i = 1$). Since ψ is plausibility preserving for any $\mathbf{r} \in \{0,1\}^N$, we have that the function $a \to g_a(\mathbf{r})$ is strictly increasing when not equal to 0 or 1. Since ψ is level-independent $a \to g_a(\mathbf{r})$ must be a constant. It follows that it is worth 0 or 1. As such $g(s_1, \ldots, s_n) \in \{0,1\}$.

Suppose $g(s_1, \ldots, s_n) = 0$, then by one-SP $0 = g(s_1, \ldots, s_n) = g(1, \ldots, 1) = 1$. This is impossible. Similarly if $g(s_1, \ldots, s_n) = 1$, then by one-SP $g = g(s_1, \ldots, s_n) = g(0, \ldots, 0) = 0$ (certainty preservation). This is impossible. Therefore, we reach a contradiction and conclude that $g(s_1, \ldots, s_n) \in \{s_1, \ldots, s_n\}$.

$\Leftarrow$: Suppose that there is a g such that for all a, for all $\mathbf{r}$, $\Psi(\mathbf{P})(a) = g(\mathbf{P(a)}) \subset \{P_1(a), \ldots, P_n(a)\}$ with g One-SP and onto. Then ψ is certainty preserving and level-SP. We also have $\forall \mathbf{r} \in \{0,1\}^N$, $f(\mathbf{r}) \in \{0,1\}$. Therefore ψ is plausibility preserving.

Corollary 7. *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP, certainty preserving and plausibility preserving if and only if the phantom functions can be chosen as constants with values in $\{0,1\}$ with $f_N = 1$ and $f_\emptyset = 0$.*

Proof. $\Rightarrow$: Let g be the rule given by level-independence, for all $\mathbf{r}$ we have $g(\mathbf{r}) \in \{0,1\}$ therefore all f_S are in $\{0,1\}$.

$\Leftarrow$: Since phantoms functions are constants, ψ is certainty preserving and since for all $\mathbf{r}$ we have $g(\mathbf{r}) \in \{0,1\}$ and that $f_\emptyset = 0$ and $f_N = 1$, ψ is plausibility preserving.

Corollary 8 (Order Cumulatives). *A probability aggregation function is anonymous, level-SP, certainty and plausibility preserving if and only if its associated one-SP aggregation rule is an order function.*

Proof. $\Rightarrow$: As shown above, all f_S are in $\{0,1\}$ and since $g(\mathbf{r}) = median(r_1, \ldots, r_n, f_1, \ldots, f_{n+1})$ for some $f_1 \leq \cdots \leq f_{n+1}$, there is k such that $0 = f_1 = f_2 = \cdots = f_{k-1} < f_k = f_{k+1} = \cdots = 1$. Thus, we deduce that $g(\mathbf{r})$ is always the k-th greatest element of $\mathbf{r}$ and so g is an order function.

$\Leftarrow$: Since an order function can be written as $median(r_1, \ldots, r_n, f_1, \ldots, f_{n+1})$ where all the phantoms are worth 0 or 1, the corresponding ψ is anonymous, certainty preserving, plausibility preserving and level-SP.

Theorem 20. *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP, certainty preserving and plausibility preserving if and only if when all the experts are dominating each other the outcome is one of the inputs.*

Proof. Suppose that ψ is level-SP plausibility and certainty preserving and that the P_i are strictly dominating each other. Without loss of generality let us choose the P_i with the i in increasing order. For any a, let $\Psi(\mathbf{P})(a) = P_i(a)$. There is such an i since all phantoms are constants in $\{0,1\}$.

We therefore have that $f_{S-i} = 0$ and $f_S = 1$. It follows that $\Psi(\mathbf{P}) = P_i$. By continuity we have that this is true even if the domination is not strict.

$\Leftarrow$: Let us suppose level-SP and that whenever the input is a set of dominated functions, the outcome is one of the inputs. For any given $a < \sup \Lambda$ let us consider the one-SP rule g_a associated to a. Since the outcome is always one of the inputs, we deduce that the phantom voters for g_a are all equal to 0 or 1 (if not we could create a situation where one of the phantoms is chosen at some point by selecting an appropriate set of dominated expertise). As such the outcome of g_a only depends on the order of the inputs. Since for all a the order of the inputs when considering dominating P_i are the same and since the outcome always corresponds to the same expert, it follows that all g_a are equal. Therefore the phantom functions f_S are constants in $\{0,1\}$. Therefore we are certainty preserving and plausibility preserving.

B Characterization of the weighted proportional-cumulative

Recall that the weighted proportionality axiom says the following: If experts are single-minded (every i'vote p_i equals the Dirac mass at $a_i \in \Lambda$, e.g. $p_i = \delta_{a_i}, \forall i \in N$), the aggregation must coincide with the weighted average:

$$\forall (a_1, \ldots, a_n) \in \Lambda^N \quad \psi(\delta_{a_1}, \ldots, \delta_{a_n}) = \sum_i w_i \delta_{a_i};$$

where $w_i \geq 0$ is the normalized weight attributed to expert $i \in N$, with $\sum_{j \in N} w_j = 1$.

Theorem 21 (Weighted Proportional-Cumulative). *The unique probability aggregation function $\psi : \mathcal{P}^n \to \mathcal{P}$ that satisfy the level-SP and weighted proportional axioms is the unique level-independent one associated to the SP-Moulin aggregation rule $\mu_{\boldsymbol{w}}$ that is:*

$$\forall a \in \Lambda, (P_1, \dots, P_n) \in \mathcal{C}; \Psi(P_1, \dots, P_n)(a) = \mu_{\boldsymbol{w}}(P_1(a), \dots, P_n(a))$$

where

$$\forall \boldsymbol{r} = (r_1, \dots, r_n), \mu_{\boldsymbol{w}}(\boldsymbol{r}) := \sup \left\{ y : \sum_{i | r_i \geq y} w_i \geq y \right\}.$$

Proof. $\Rightarrow$: The weighted proportionality axiom can be rewritten in terms of CDF as follows:

$$\forall (a_1, \dots, a_n) \in \Lambda^N, \Psi(\delta_{a \geq a_1}, \dots, \delta_{a \geq a_n})(a) = \sum_{i : a_i \leq a} w_i.$$

Let ψ be level-SP and weighted proportional. Let Ψ be the CAF associated to ψ and f_S be its phantom functions. Let us first show that (w.l.o.g):

$$\forall S \subseteq N, \forall a \in \Lambda : f_S(a) = \sum_{i \in S} w_i.$$

Let us take any $S \subseteq N$ and $a < \sup \Lambda$. Suppose that all experts $i \in S$ are single-minded in a and all other experts are single-minded in any $b > a$. By weighted proportional we therefore have $g_a(\mathbf{P(a)}) = \sum_{i \in S} w_i$. By level-SP we have $g_a(\mathbf{P(a)}) = f_S(a)$. We conclude that for all alternatives $a < \sup \Lambda$ we have $f_S(a) = \sum_{i \in S} w_i$ and we can assume without loss of generality that this also holds for $a = \sup \Lambda$. Hence we conclude that the phantoms associated to ψ are the constant phantom functions $f_S = \sum_{i \in S} w_i$. Consequently, g_a is independent on a (the phantoms as constant), and we denote by g that common function (which is the one-SP rule associated to ψ). We want to show that $g = \mu_w$, that is for all $a \in \Lambda$ and for all $\mathbf{r} \in [0,1]^N$:

$$g_a(\mathbf{r}) = g(\mathbf{r}) = \mu_w(\mathbf{r}).$$

- Suppose that there is j such that $g(\mathbf{r}) = r_j$, then there is S and $S' = S - \{k | r_j = r_k\}$ such that for all $k \in S$, we have $r_j = \min(f_S, \min_{k \in S} r_k)$ and $r_j \leq f_{S'}$ (Moulin's characterization).
  - For any $y > r_j$, we have $\sum_{i : r_i \geq y} w_i \leq \sum_{i : r_i > r_j} w_i = f_{S'} \leq r_j < y$. Therefore by definition of μ_w we have $\mu_w(\mathbf{r}) \leq r_j$.
  - For any $y \leq r_j$, we have $\sum_{i : r_i \geq y} w_i \geq \sum_{i : r_i \geq r_j} = f_S \geq r_j \geq y$. Therefore by definition of μ_w, $\mu_w(\mathbf{r}) \geq r_j$.

  By combining the above we obtain that if there is j such that $g(\mathbf{r}) = r_j$, then $\mu_w(\mathbf{r}) = r_j$.
- Suppose that there is no i such that $g(\mathbf{r}) = r_i$, then by Moulin's characterization there is an S such that $g(\mathbf{r}) = f_S$. According to Moulin's characterization $f_S = \min(f_S, \min_{i \in S} r_i)$ therefore if $k \in S$ then $r_k > f_S$. Suppose there is an i such that if $k \notin S$ and $r_k > f_S$ then for $S' = S \cup r_k$ since $f'_S \geq f_S$ we have $\min(f_{S'}, \min_{i \in S'} r_i) \geq f_S$. Therefore according to Moulin's characterization $f_S = f_{S'}$. It follows that we can choose S such that for all $k \notin S$, $r_k < f_S$.
  - For any $y > f_S$ we have $\sum_{i : r_i \geq y} w_i \leq \sum_{i : r_i \geq f_S} w_i = f_S < y$, therefore $\mu_w(\mathbf{r}) \leq r_j$.
  - For $y \leq r_j$, we have $\sum_{i : r_i \geq y} w_i \geq \sum_{i : r_i \geq f_S} w_i = f_S \geq y$ therefore $\mu_w(\mathbf{r}) \geq f_S$. Therefore $\mu_w(\mathbf{r}) = f_S$.

  It follows that $g = \mu_w$.

$\Leftarrow$: If Ψ is the level independent associated to μ_w then for $\mathbf{r} \in \{0,1\}^N$. We have $\mu_w(\mathbf{r}) = \sum_{i : r_i = 1} w_i$. As such we verify weighted proportionality.

Proposition 3. *Assuming all weights are rationals $w_i = s_i / d$ then we have:*

$$\forall \boldsymbol{r} = (r_1, \dots, r_n), \mu_{\boldsymbol{w}}(\boldsymbol{r}) = median(\overbrace{r_1, \dots, r_1}^{s_1}, \dots, \overbrace{r_n, \dots, r_n}^{s_n}, 0, 1/d, \dots, 1 - 1/d, 1)$$

Proof. First let us show that if we have d experts and that all the voters have the same weight $\frac{1}{d}$, then:

$$\forall \mathbf{r'} = (r'_1, \dots, r'_d), \mu_d(\mathbf{r'}) := \mu_{\mathbf{w} = (\frac{1}{d}, \dots, \frac{1}{d})}(\mathbf{r'}) := median(r'_1, \dots, r'_d, 0, 1/d, \dots, 1 - 1/d, 1).$$

Since all the experts have the same weight, we are anonymous. We can therefore use Moulin's anonymous characterization. By considering $\mathbf{r} \in \{0,1\}^d$ we find, by definition of a proportional method, $\mu_d = (\sum_i r_i)/d$. It follows that the phantom value f_i in Moulin median characterization is $f_i = i/d$.

Now that we have established the characterization of μ_d, we wish to establish that if we have n experts such that each expert has weight s_i/d where $s_i \in \mathbf{N}$, then:

$$\forall \mathbf{r}, \mu_w(\mathbf{r}) = \mu_d(\overbrace{r_1, \ldots, r_1}^{s_1}, \ldots, \overbrace{r_n, \ldots, r_n}^{s_n}) = median(\overbrace{r_1, \ldots, r_1}^{s_1}, \ldots, \overbrace{r_n, \ldots, r_n}^{s_n}, 1/d, \ldots, 1-1/d)$$

We can simply consider that for each expert i, if expert i has weight $w_i = s_i/d$ then he is duplicated so as to appear s_i times in the anonymous case with d experts.

C Domination property

Proposition 4 (Proportional-cumulative for dominated opinion). *Suppose that Λ is an interval, and that all the P_i are continuous and verify for all $i \in N$, $P_i \geq P_{i+1}$. Then the weighted-proportional level-SP mechanism $\psi : \mathcal{P}^N \to \mathcal{P}$ of weight $\boldsymbol{w}$ can be computed for this input profile as follows:*

$$\psi(\boldsymbol{p})(a) = \begin{cases} p_i(a) \text{ if } & \sum_{k \leq i-1} w_k \leq P_i(a) < \sum_{k \leq i} w_k \\ 0 & \text{else} \end{cases}$$

Proof. Suppose for all $i \in N$, $P_i \geq P_{i+1}$. Then $\mu_w(\mathbf{P(a)}) = \sup \left\{ y : \sum_{i:P_i(a) \geq y} w_i \geq y \right\}$.

- Suppose that $\sum_{k \leq j-1} w_k \leq P_j(a) < \sum_{k \leq j} w_k$. Let us determine $\mu_w(\mathbf{P(a)})$.
  - For $y > P_i(a)$ we have $\{k : P_k(a) \geq y\} \subseteq \{k < j\}$. Therefore $\sum_{k:P_k(a) \geq y} w_k \leq \sum_{k<j} w_k \leq P_i(a) < y$. Therefore $\mu_w(\mathbf{P(a)}) \leq P_i(a)$.
  - For $y \leq P_i(a)$ we have $\{k \leq j\} \subseteq \{k : P_k(a) \geq y\}$. As such $\sum_{k:P_k(a) \geq y} w_k \geq \sum_{k \leq j} w_k \geq P_i(a) \geq y$.

  We have therefore shown that $\mu_w(\mathbf{P(a)}) = P_j(a)$.
  By continuity and monotony of the P_i's there is $a_1 \leq a$ such that $P_j(a_1) = \sum_{k \leq j-1} w_k$ and $a_2 > a$ such that $P_j(a_1) = \sum_{k \leq j} w_k$. Therefore on the interval $[a_1, a_2]$, $\Psi(\mathbf{P}) = P_j$. It follows that $\psi(\mathbf{p})(a) = p_j(a)$.
- Suppose that there is no j such that $\sum_{k \leq j-1} w_k \leq P_j(a) < \sum_{k \leq j} w_k$. Then there is a j such that $\Psi(\mathbf{P})(a) = \sum_{k \leq j} w_k$, $P_{j+1}(a) < \sum_{k \leq j} w_k < P_j(a)$. By continuity of the P_i's there is $a_1 < a$ such that $\Psi(\mathbf{P})(a_1) = P_j(a_1) = \sum_{k \leq j} w_k$ and $a_2 > a_1$ such that $\Psi(\mathbf{P})(a_2) = P_{j+1}(a_2) = \sum_{k \leq j} w_k$.
  Therefore $\Psi(\mathbf{P})(a_1) = \Psi(\mathbf{P})(a_2)$. It follows that $\psi(\mathbf{p})(a) = 0$.

D Impossibilities

D.1 With $L_1(\mathcal{P})$-SP

Proposition 5 (Impossibility 1). *When $\#\Lambda \geq 4$, there are at least 3 experts and the mechanism is certainty preserving, level-SP and $L_1(\mathcal{P})$-SP are incompatible.*

Proof. Let us consider a certainty preserving level-SP method and 4 alternatives $a_1 \prec a_2 \prec a_3 \prec a_4$. For more than 3 players, we simply consider duplicates of the same player, the proof, therefore, remains the same. Therefore, we assume w.l.o.g. 3 players.

Let $P_1(a_1) = P_3(a_3) = 0$ and $P_1(a_2) = 1$. Let us suppose that $f_{\{1,2\}} < 1$, let $\epsilon > 0$ be small enough so that $f_{\{1,2\}} + \epsilon < 1$. For $p_2 = (f_{\{2\}} + \epsilon, 0, f_{\{1,2\}} - f_{\{2\}}, 1 - f_{\{1,2\}} - \epsilon)$. We have:

$$\begin{aligned}
&p_2 = (f_{\{2\}} + \epsilon, 0, f_{\{1,2\}} - f_{\{2\}}, 1 - f_{\{1,2\}} - \epsilon) \\
&P_2 = (f_{\{2\}} + \epsilon, f_{\{2\}} + A, f_{\{1,2\}} + \epsilon, 1) \\
&\Psi(\mathbf{P}) = (f_{\{2\}}, f_{\{2\}} + \epsilon, f_{\{1,2\}}, 1) \\
&\psi(\mathbf{p}) = (f_{\{2\}}, \epsilon, f_{\{1,2\}} - f_{\{2\}} - \epsilon, 1 - f_{\{1,2\}} - 2\epsilon) \\
&\|\psi(\mathbf{p}) - p_2\|_1 = 4\epsilon.
\end{aligned}$$

However for $p_2' = (f_{\{2\}}, 0, f_{\{1,2\}} - f_{\{2\}}, 1 - f_{\{1,2\}})$ we obtain $\psi = (f_{\{2\}}, 0, f_{\{1,2\}} - f_{\{2\}}, 1 - f_{\{1,2\}})$. Therefore:

$$\|\psi(\mathbf{p}_{-1}(p_1')) - p_2\|_1 = 2\epsilon.$$

Therefore, we have contradicted $L_1(\mathcal{P})$-SP.

As such assuming a $L_1(\mathcal{P})$-SP solution exists, we must have $f_{\{1,2\}} = 1$, by a symmetrical proof we can show that must also have $f_1 = 0$. By switching the players around we, therefore, determine that the only possible solution is middlemost-cumulative.

For $p_1 = (1/2, 0, 1/2, 0)$ $p_2 = (0, 0, 0, 1)$, $p_3(0, 1/2, 0, 1/2)$. Then $\Psi(\mathbf{P}) = (0, 1/2, 1/2, 1)$. Therefore $\psi(\mathbf{p}) = (0, 1/2, 0, 1/2)$. As such $\|\psi(\mathbf{p}) - p_2\|_1 = 2$.

Whereas, if player 1 changes to $p_1'(0, 0, 1/2, 1/2)$, then $\Psi(\mathbf{P}_{-i}(P_1')) = (0, 0, 1/2, 1)$. Therefore, we have that $\psi(\mathbf{p}_{-i}(p_1')) = (0, 0, 1/2, 1/2)$. As such $\|\psi(\mathbf{p}_{-1}(p_1')) - p_2\|_1 = 2\epsilon$. We have contradicted $L_1(\mathcal{P})$-SP.

D.2 With Borel plausibility

This proof needs some reworking.

Theorem 22 (Impossibility 3). *When $\Lambda = \mathbf{Z}$, or is an interval the unique probability aggregation functions that are level-SP, unanimous, and Borel plausibility preserving are the dictatorials.*

Proof. Let ψ be a probability aggregation function that is level-SP, unanimous, and Borel plausibility preserving on $\mathbf{Z}$.

Suppose we can choose $\mathbf{p}$ and experts $i \neq j$ such that there is $a < b$ that verify $\Psi(\mathbf{P})(a) = P_i(a) = P_i(b) = \Psi(\mathbf{P})(b)$ and for all experts $k \neq i$ we have $P_k(a) \neq \Psi(\mathbf{P})(a)$ and $p_i([a, b]) > 0$. Suppose we can also have $[c, d[$ disjoint from $[a, b]$ such that $\Psi(\mathbf{P})(c) = P_j(c) = P_j(d) = \Psi(\mathbf{P})(d)$ and $P_i(c) \neq \Psi(\mathbf{P})(c)$ and $p_i([c, d[) > 0$.

For such an example, we contradict Borel preservation for $A = [a, b] \cup [c, d[$. As such no such example exists. We wil call this the "two intervals lemma".

Let us show that their exists an expert i and an interval a, b such that we can find $\mathbf{p}$ that verifies $\Psi(\mathbf{P})(a) = P_i(a) = P_i(b)\Psi(\mathbf{P})(b)$ and for all $k \neq i$, $P_k(a) \neq \Psi(\mathbf{P})(a)$.

By unanimity, we have that $f_\emptyset = 0$ and $f_N = 1$. As such for all $a \in \mathbf{Z}$ there is S and i such that $f_S = 0$ and $f_{S \cup \{i\}} > 0$.

Since N is finite and $\mathbf{Z}$ isn't by pigeonhole principle we can find two elements $a < b$ an i and a S such that $f_S(a) = f_S(b) = 0$ and $f_(a)S \cup \{i\} > 0$. We have found our interval $[a, b]$ and our player i.

It follows from our "two interval lemma" that there is only one i that verifies the previous (or we could create two intervals as desired). It follows that for all S such that $i \notin S$ we have $f_S = 0$.

A similar proof (were we start from $f_N = 1$ instead of $f_\emptyset = 0$) gives that for all S such that $i \in S$ we have $f_S = 1$.

We, therefore, have that i is a dictator.

E Diversity

Axiom 12 (Diversity) *For every Diracs inputs $(p_1, \ldots, p_n) = (\delta_{a_1}, \ldots, \delta_{a_n})$, there are positive weights $w_1 > 0, \ldots, w_n > 0$ such that $\psi(\delta_{a_1}, \ldots, \delta_{a_n}) = \sum_{i=1}^n w_i \delta_{a_i}$.*

Proposition 6. *Diversity is equivalent to certainty preservation and f_S strictly increasing with S.*

Proof. $\Rightarrow$: Suppose that this is true then for all $a < M$, if all experts are single-minded.

If no expert wanted a then $\Psi(a) = \Psi(b)$ where $b < a$ and for all $b < c \leq a$ we have no experts want c. It follows that for all S we have $f_S(a) = f_S(b)$. Therefore we are level-independent. Since whenever an expert chooses an alternative the weight of that alternative must be felt ($w_i > 0$) then for all $S \neq \emptyset$ for all $i \in S$, $f_S > f_{S-i}$.

If we are level-independent and the f_S are strictly increasing, suppose that all experts are single-minded then for any set of single-minded experts.

$$\Psi(\mathbf{P})(a) = f_{\{i : a_i \leq a\}}(a)$$

Therefore for all a, $\psi(\delta_{a_1}, \ldots, \delta_{a_n})(a) = f_{\{i : a_i \leq a\}} - f_{\{i : a_i < a\}}$.

$$\psi(\delta_{a_1}, \ldots, \delta_{a_n}) = \sum_{i=1}^n (f_{\{i : a_j \leq a_i\}} - f_{\{i : a_j < a_i\}}) \delta_{a_i}$$

The next result implies that not only is proportionality incompatible with level plausibility but this incompatibility holds for all diverse methods.

Theorem 23 (Impossibility 4). *For 3 or more experts, there are no level-SP probability aggregation functions that satisfy diversity and preserve plausibility.*

Proof. Suppose that $\psi : \mathcal{P}^N \to \mathcal{P}$ is a level-SP probability aggregation function that satisfies diversity and preserves plausibility.

Then since diversity implies certainty preservation and that we require plausibility preservation, we have all the $f_S \in \{0, 1\}^N$.

Since diversity also implies that f_S is strictly increasing with S. By pigeonhole principle, we reach a contradiction.

An interesting question now is: can we guarantee more diversity?

Theorem 24 (Impossibility 5). *When there are at least* 3 *alternatives and* 2 *players there are no methods such that the outcomes support is always the union of the input support.*

Proof. It is immediate that the property "outcomes support is always the union of the input support" implies certainty preservation. Let g be the associated One-SP voting rule. If one of the phantoms is not $\{0, 1\}$ then we can move the players around such that that phantom is selected on an interval where at least one player is increasing. This is a contradiction. Therefore we are plausibility preserving. By taking a set of dominated inputs we can create a situation where on an interval $[a, b]$ the outcome does not change but one of the players input is increasing.