Study of the Instability of the Sign of the Nonadditivity Index in a Choquet Integral Model

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Abstract. This paper studies the instability of the sign of the nonadditivity index between criteria in a Choquet integral model. Nonadditivity is an essential property of capacities defined on the sets of decision criteria and allows one to flexibly represent the phenomenon of interaction between criteria. In some cases, we show that the sign of the nonadditivity index proposed in the literature depends on arbitrary choice of a numerical representation in the set of all numerical representations compatible with the strict preferential information given by the Decision Maker(DM). This makes its interpretation difficult. We illustrate ours results with examples.

Keywords: Instability, Nonadditivity index, Choquet integral model, Numerical representation, Strict preferential information.

1 Introduction

In Multiple Criteria Decision Making (MCDM), the theory of value functions aims to assign a real number to each alternative, so that the order on the alternatives induced by these reals number does not contradict the preferences of the DM. When the preferences of the DM satisfy preferential independence hypothesis, the value assigned to each alternative can be obtained from an additive model [1]. Since this hypothesis is restrictive [9], the Choquet integral model, more general, was popularized by the work of Michel Grabisch [5, 6]. It is now considered as a central tool in MCDM when one wants to escape the independence hypothesis [8–10].

When a set of preferential information is not compatible with an additive model, it is common to deduce the existence of interaction between criteria. Interaction among multiple decision criteria can be measured by cardinal probabilistic interaction indices, in particular the Shapley interaction index [7]. More details in the literature on axiomatic properties of some cardinal probabilistic interaction indices are given in [8, 3]. In [15], this lack of compatibility with an additive model is simply translated by the notion of nonadditivity index. This article deals with the use of this idea to capture the phenomena of synergy in the framework of nonadditivity. In [12], we used the Shapley interaction index to study the interaction between criteria in a Choquet integral model. In particular, in [12] we show that the positive interaction is always possible for any subset of criteria, but could not transpose this result with the negative interaction. Here, we solve this dual problem using the nonadditivity index.

We show that the sign of the nonadditivity index [15] proposed in the literature is not stable in the set of all capacities compatible with strict preferences of DM. Indeed, we prove that from a null nonadditivity index, we can build a strictly positive and a strictly negative nonadditivity indices while remaining in the set of all capacities compatible with strict preferences of DM.

Within the framework of binary alternatives, we show that it is always possible to represent the strict preferences of the DM with a Choquet integral model inducing the strictly positive nonadditivity indices, then with another inducing the strictly negative nonadditivity indices for all subsets of at least two criteria.

This paper is organized as follows. Section 2 recalls some basic elements on the model of the Choquet integral in MCDM. in Section 3 and 4, we present ours results. We illustrate each result with an example.

2 Notations and definitions

2.1 Framework

Let X be a set of alternatives evaluated on an index set of n criteria $N = \{1, 2, \ldots, n\}$ $(n \geq 2)$. Throughout this paper we use the notation $A \subseteq N_{\geq 2}$ if and only if $A \subseteq N$ and $|A| \geq 2$. The set of all alternatives X is assumed to be a subset of a Cartesian product $X_1 \times X_2 \times \ldots \times X_n$, where X_i is the set of possible levels on criterion $i \in N$. The criteria are recoded numerically using, for all $i \in N$, a function u_i from X_i into $\mathbb{R}$. Using these functions, we assume that the various recoded criteria are commensurate, so we can use the Choquet integral model [11].

2.2 Choquet integral

A generalization of criteria weights consists of assigning weights to subsets of criteria. This can be achieved by a capacity [2] defined as a function μ from the power set 2^N into $[0, 1]$ such that:

- $\mu(\emptyset) = 0$,
- $\mu(N) = 1$,
- $\forall S, T \in 2^N, \big[S \subseteq T \Longrightarrow \mu(S) \leq \mu(T)\big]$ (monotonicity).

For an alternative $x = (x_1, \ldots, x_n) \in X$, the expression of the Choquet integral [7–9] w.r.t. the capacity μ is given by:

$$C_\mu\big(u(x)\big) = \sum_{i=1}^{n} \Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\mu\big(N_{\sigma(i)}\big),$$

where σ is a permutation on N such that: $N_{\sigma(i)} = \{\sigma(i), \ldots, \sigma(n)\}$, $u_{\sigma(0)}(x_{\sigma(0)}) = 0$ and $u_{\sigma(1)}(x_{\sigma(1)}) \leq u_{\sigma(2)}(x_{\sigma(2)}) \leq \ldots \leq u_{\sigma(n)}(x_{\sigma(n)})$.

In the next subsection, we recall the definition of the nonadditivity index.

2.3 Nonadditivity index

We work with the nonadditivity index introduced and studies in [15].

Definition 1. *The nonadditivity index w.r.t. a capacity μ is defined by: for all $A \subseteq N_{\geq 2}$,*

$$\eta_A^\mu = \frac{1}{2^{|A|-1} - 1} \sum_{\substack{(B,\, A \setminus B) \\ \emptyset \subsetneq B \subsetneq A}} \Big(\mu(A) - \mu(B) - \mu(A \setminus B) \Big) \tag{1}$$

For all $A \subseteq N_{\geq 2}$, for each partition $(B, A \setminus B)$ of A with $\emptyset \subsetneq B \subsetneq A$, we compute the difference $\mu(A) - \big(\mu(B) + \mu(A \setminus B)\big)$. Thus η_A^μ corresponds to the arithmetic mean of these differences.

Remark 1. We have $\eta_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j$, therefore the nonadditivity index coincides with the Shapley interaction index I_{ij}^μ, for pairs $\{i, j\} \subseteq N$.

Remark 2 below gives two equivalent expressions of η_A^μ that we find in [15].

Remark 2. Given a capacity μ on N and $A \subseteq N_{\geq 2}$, Equation (1) is equivalent to each of Equations (2) and (3).

$$\eta_A^\mu = \frac{1}{2^{|A|} - 2} \sum_{\emptyset \subsetneq B \subsetneq A} \Big(\mu(A) - \mu(B) - \mu(A \setminus B) \Big) \tag{2}$$

$$\eta_A^\mu = \mu(A) - \frac{1}{2^{|A|-1} - 1} \sum_{\emptyset \subsetneq B \subsetneq A} \mu(B) \tag{3}$$

In the following section, we propose the concept of necessary and possible nonadditivity index similar to that of necessary and possible interaction introduced on [13] in the case of 2-additive Choquet integral model.

3 Necessary and possible nonadditivity index on X

3.1 Some definitions and notations

The DM compares some alternatives only in terms of strict preference P define as follow.

Definition 2. *A strict ordinal preference information P on X is given by:*

$$P = \{(a, b) \in X \times X : \textit{DM strictly prefers } a \textit{ to } b\}$$

We note $a\,P\,b$ or $(a,b) \in P$. The following definition tests if P is representable by a Choquet integral model.

Definition 3. *A strict ordinal preference information P on X is representable by a Choquet integral model if we can find a capacity μ such that: for all $a, b \in X$,*

$$a\,P\,b \Longrightarrow C_\mu\big(u(a)\big) > C_\mu\big(u(b)\big).$$

We denote by C_{Pref} the set of all capacities compatible with P.
The following definition of necessary and possible nonadditivity will be central in the rest of this text. It is inspired from [13] where it was given in the case of 2-additive Choquet integral model.

Definition 4. *Let $A \subseteq N_{\geq 2}$ and P a strict ordinal preference information. We say that:*

1. *There exists a possible positive (resp. null, negative) nonadditivity index between the elements of A if there exists $\mu \in C_{Pref}$ such that $\eta^\mu_A > 0$ (resp. $\eta^\mu_A = 0$, $\eta^\mu_A < 0$),*
2. *There exists a necessary positive (resp. null, negative) nonadditivity index between the elements of A if $\eta^\mu_A > 0$ (resp. $\eta^\mu_A = 0$, $\eta^\mu_A < 0$) for all $\mu \in C_{Pref}$.*

The interpretation of the nonadditivity index is difficult in the case of a possible but not necessary, because it depends on the arbitrary choice of a capacity in C_{Pref}. Indeed, the interpretation of the nonadditivity index really makes sense in the case of the necessary. Our results of the next subsection show that null nonadditivity is not necessary.

3.2 Results on X

Proposition 1 shows that from a null nonadditivity index, we can build a strictly positive nonadditivity index while remaining in C_{Pref}.

Proposition 1. *Let P be a strict ordinal preference information on X and $A \subseteq N_{\geq 2}$. Assume that P is representable by a Choquet integral model using a capacity μ for which $\eta^\mu_A = 0$. Then there exists a capacity $\beta^\mu \in C_{Pref}$ such that $\eta^{\beta^\mu}_A > 0$.*

Proof. Let $A \subseteq N_{\geq 2}$, we suppose that P is representable by a Choquet integral model using a capacity μ such that $\eta^\mu_A = 0$.
Let us define a function β^μ_ε on power set 2^N into $[0,1]$ by:

$$\beta^\mu_\varepsilon(S) = \begin{cases} \dfrac{1}{1+\varepsilon}(\mu(S)+\varepsilon) & \text{if} \quad A \subseteq S \\[2ex] \dfrac{1}{1+\varepsilon}\mu(S) & \text{otherwise.} \end{cases}$$

where ε is a strictly positive real number to be determined as follows.
We show that β^μ_ε is a capacity for all $\varepsilon > 0$.
Let $\varepsilon > 0$. It is obvious that $\beta^\mu_\varepsilon(\emptyset) = 0$ and $\beta^\mu_\varepsilon(N) = 1$.
Let $S, T \subseteq N$ such that $S \subseteq T$.

- If $A \subseteq S$, then $A \subseteq T$ and $\beta_\varepsilon^\mu(T) - \beta_\varepsilon^\mu(S) = \dfrac{1}{1+\varepsilon}(\mu(T) - \mu(S)) \geq 0$ since μ is a capacity and $S \subseteq T$.
- If not$(A \subseteq S)$, then $\beta_\varepsilon^\mu(S) = \dfrac{1}{1+\varepsilon}\mu(S)$. We have $\beta_\varepsilon^\mu(T) = \dfrac{1}{1+\varepsilon}\mu(T)$ (if $A \subseteq T$) or $\beta_\varepsilon^\mu(T) = \dfrac{1}{1+\varepsilon}(\mu(T)+\varepsilon)$, then $\beta_\varepsilon^\mu(T) \geq \dfrac{1}{1+\varepsilon}\mu(T)$ since $\mu(T)+\varepsilon > \mu(T)$. Therefore $\beta_\varepsilon^\mu(T) \geq \dfrac{1}{1+\varepsilon}\mu(T) \geq \dfrac{1}{1+\varepsilon}\mu(S)$ since $S \subseteq T$ and μ is a capacity. Hence, $\beta_\varepsilon^\mu(T) \geq \beta_\varepsilon^\mu(S)$.

In the both cases, we have $\beta_\varepsilon^\mu(T) \geq \beta_\varepsilon^\mu(S)$. Hence β_ε^μ is a capacity for all $\varepsilon > 0$. Besides, we consider the set $\mathit{\Gamma}_{u(x)} = \{i = 1, 2, \cdots, n : A \subseteq N_{\sigma(i)}\}$. We have $1 \in \mathit{\Gamma}_{u(x)}$ for all $x \in X$ since $A \subseteq N = N_{\sigma(1)}$, then $\mathit{\Gamma}_{u(x)} \neq \emptyset$. We then have:

$$\begin{aligned}
C_{\beta_\varepsilon^\mu}\big(u(x)\big) &= \sum_{i=1}^{n}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\beta_\varepsilon^\mu(N_{\sigma(i)}) \\
&= \frac{1}{1+\varepsilon}\sum_{i\notin \mathit{\Gamma}_{u(x)}}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\mu(N_{\sigma(i)}) \\
&+ \frac{1}{1+\varepsilon}\sum_{i\in \mathit{\Gamma}_{u(x)}}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\big(\mu(N_{\sigma(i)})+\varepsilon\big) \\
&= \frac{1}{1+\varepsilon}\sum_{i\notin \mathit{\Gamma}_{u(x)}}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\mu(N_{\sigma(i)}) \\
&+ \frac{1}{1+\varepsilon}\sum_{i\in \mathit{\Gamma}_{u(x)}}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\mu(N_{\sigma(i)}) \\
&+ \frac{1}{1+\varepsilon}\varepsilon\sum_{i\in \mathit{\Gamma}_{u(x)}}\Big(u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big) \\
&= \frac{1}{1+\varepsilon}\Big[C_\mu\big(u(x)\big) + \varepsilon v^\sigma(u(x))\Big] \text{ where } v^\sigma(u(x)) = \sum_{i\in \mathit{\Gamma}_{u(x)}}\Big(u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big).
\end{aligned}$$

We then have $C_{\beta_\varepsilon^\mu}\big(u(a)\big) - C_{\beta_\varepsilon^\mu}\big(u(b)\big) = \dfrac{1}{1+\varepsilon}\Bigg[\bigg(C_\mu\big(u(a)\big) - C_\mu\big(u(b)\big)\bigg) + \varepsilon\bigg(v^{\sigma^a}(u(a)) - v^{\sigma^b}(u(b))\bigg)\Bigg]$ for all $(a,b) \in P$.

ε is such that $C_{\beta_\varepsilon^\mu}\big(u(a)\big) - C_{\beta_\varepsilon^\mu}\big(u(b)\big) > 0$ for all $(a,b) \in P$.
We consider the set $\mathit{\Omega} = \{(a,b) \in P : \; v^{\sigma^a}(u(a)) - v^{\sigma^b}(u(b)) < 0\}$.

- If $\mathit{\Omega} = \emptyset$, then for all $(a,b) \in P$, we have $v^{\sigma^a}(u(a)) - v^{\sigma^b}(u(b)) \geq 0$. Thus for all $(a,b) \in P$, $C_{\beta_\varepsilon^\mu}\big(u(a)\big) - C_{\beta_\varepsilon^\mu}\big(u(b)\big) > 0 \;\; \forall \varepsilon > 0$.
- If $\mathit{\Omega} \neq \emptyset$, we choose ε such that $0 < \varepsilon < \min\limits_{(a,b)\in\mathit{\Omega}}\left(\dfrac{C_\mu\big(u(b)\big) - C_\mu\big(u(a)\big)}{v^{\sigma^a}(u(a)) - v^{\sigma^b}(u(b))}\right)$ in such a way that $C_{\beta_\varepsilon^\mu}\big(u(a)\big) - C_{\beta_\varepsilon^\mu}\big(u(b)\big) > 0$ for all $(a,b) \in P$.

So in both cases we can choose $\varepsilon = \dfrac{1}{2}\min\limits_{(a,b)\in\Omega}\left(\dfrac{C_\mu\big(u(x^j)\big) - C_\mu\big(u(a)\big)}{v^{\sigma^a}(u(a)) - v^{\sigma^b(u(b))}}\right)$ so that $\beta^\mu_\varepsilon \in C_{\text{Pref}}$. Moreover we have:

$$\eta_A^{\beta^\mu_\varepsilon} = \beta^\mu_\varepsilon(A) - \frac{1}{2^{|A|-1}-1}\sum_{\emptyset\neq B\subsetneq A}\beta^\mu_\varepsilon(B)$$
$$= \frac{1}{1+\varepsilon}\left(\varepsilon + \mu(A) - \frac{1}{2^{|A|-1}-1}\sum_{\emptyset\neq B\subsetneq A}\mu(B)\right)$$
$$= \frac{1}{1+\varepsilon}\left(\varepsilon + \eta^\mu_A\right)$$

As $\eta^\mu_A = 0$, we have $\eta_A^{\beta^\mu_\varepsilon} = \dfrac{\varepsilon}{1+\varepsilon} > 0$. Thus there exists a possible positive nonadditivity index for A. Hence there is no null nonadditivity index for A. Therefore the null nonadditivity index is never necessary.

Proposition 2 answers the dual problem. Indeed, it shows that from a null nonadditivity index, we can build a strictly negative nonadditivity index while remaining in C_{Pref}. Note that this dual problem remains open in the case of the necessary and possible interaction with the Shapley interaction index [7].

Proposition 2. *Let P be a strict ordinal preference information on X and $A \subseteq N_{\geq 2}$. Assume that P is representable by a Choquet integral model using a capacity μ for which $\eta^\mu_A = 0$. Then there exists a capacity $\gamma^\mu \in C_{Pref}$ such that $\eta_A^{\gamma^\mu} < 0$.*

Proof. Let $A \subseteq N_{\geq 2}$, we suppose that P is representable by a Choquet integral model using a capacity μ such that $\eta^\mu_A = 0$.
Let us define a function γ^μ_ε on power set 2^N into $[0,1]$ by:

$$\gamma^\mu_\varepsilon(S) = \begin{cases} \dfrac{1}{1+\varepsilon}(\mu(S)+\varepsilon) & \text{if} \quad S \neq \emptyset \\ \\ 0 \quad \text{if } S = \emptyset. \end{cases}$$

where ε is a strictly positive real number to be determined as follows.
We show that γ^μ_ε is a capacity for all $\varepsilon > 0$.
Let $\varepsilon > 0$. It is obvious that $\gamma^\mu_\varepsilon(\emptyset) = 0$ and $\gamma^\mu_\varepsilon(N) = 1$.
Let $S, T \subseteq N$ such that $S \subseteq T$.

- If $S = \emptyset$, then $\gamma^\mu_\varepsilon(S) = 0 \leq \gamma^\mu_\varepsilon(T)$.
- If $S \neq \emptyset$, then $\gamma^\mu_\varepsilon(S) = \dfrac{1}{1+\varepsilon}(\mu(S)+\varepsilon) \leq \dfrac{1}{1+\varepsilon}(\mu(T)+\varepsilon) = \gamma^\mu_\varepsilon(T)$.

In the both cases, we have $\gamma^\mu_\varepsilon(T) \geq \gamma^\mu_\varepsilon(S)$. Hence γ^μ_ε is a capacity for all $\varepsilon > 0$.
Besides, we have $C_{\gamma^\mu_\varepsilon}\big(u(x)\big) = \sum\limits_{i=1}^{n}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\gamma^\mu_\varepsilon(N_{\sigma(i)})$

$$= \frac{1}{1+\varepsilon}\sum_{i=1}^{n}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\big(\mu(N_{\sigma(i)})+\varepsilon\big) \text{ since } N_{\sigma(i)} \neq \emptyset\, \forall\, i \in N$$
$$= \frac{1}{1+\varepsilon}\sum_{i=1}^{n}\Big[u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)})\Big]\mu(N_{\sigma(i)})$$

$$+\frac{\varepsilon}{1+\varepsilon}\sum_{i=1}^{n}\Big[u_{\sigma(i)}\big(x_{\sigma(i)}\big)-u_{\sigma(i-1)}\big(x_{\sigma(i-1)}\big)\Big]$$
$$=\frac{1}{1+\varepsilon}\Big[C_\mu\big(u(x)\big)+\varepsilon v^\sigma\big(u(x)\big)\Big] \text{ with } v^\sigma\big(u(x)\big)=\sum_{i=1}^{n}\Big[u_{\sigma(i)}\big(x_{\sigma(i)}\big)-u_{\sigma(i-1)}\big(x_{\sigma(i-1)}\big)\Big]$$

We then have $C_{\beta^\mu_\varepsilon}\big(u(a)\big)-C_{\beta^\mu_\varepsilon}\big(u(b)\big)=\frac{1}{1+\varepsilon}\Big[\Big(C_\mu\big(u(a)\big)-C_\mu\big(u(b)\big)\Big)+\varepsilon\big(v^{\sigma^a}(u(a))-v^{\sigma^b}(u(b))\big)\Big]$ for all $(a,b)\in P$.

ε is such that $C_{\gamma^\mu_\varepsilon}\big(u(a)\big)-C_{\gamma^\mu_\varepsilon}\big(u(b)\big)>0$ for all $(a,b)\in P$.
We consider the set $\Omega=\{(a,b)\in P: v^{\sigma^a}(u(a))-v^{\sigma^b}(u(b))<0\}$.

- If $\Omega=\emptyset$, then $v^{\sigma^a}(u(a))-v^{\sigma^b}(u(b))\geq 0$ for all $(a,b)\in P$. Thus for all $(a,b)\in P$, $C_{\gamma^\mu_\varepsilon}\big(u(a)\big)-C_{\gamma^\mu_\varepsilon}\big(u(b)\big)>0\ \forall\varepsilon>0$.
- If $\Omega\neq\emptyset$, we choose ε such that $0<\varepsilon<\min\limits_{(a,b)\in\Omega}\left(\frac{C_\mu\big(u(b)\big)-C_\mu\big(u(a)\big)}{v^{\sigma^a}(u(a))-v^{\sigma^b}(u(b))}\right)$ in such a way that $C_{\gamma^\mu_\varepsilon}\big(u(a)\big)-C_{\gamma^\mu_\varepsilon}\big(u(b)\big)>0$ for all $(a,b)\in P$.

So in both cases we can choose $\varepsilon=\frac{1}{2}\min\limits_{(a,b)\in\Omega}\left(\frac{C_\mu\big(u(b)\big)-C_\mu\big(u(a)\big)}{v^{\sigma^a}(u(a))-v^{\sigma^b(u(b))}}\right)$ so that $\gamma^\mu_\varepsilon\in C_{\text{Pref}}$. Moreover we have:

$$\eta_A^{\gamma^\mu_\varepsilon}=\gamma^\mu_\varepsilon(A)-\frac{1}{2^{|A|-1}-1}\sum_{\emptyset\neq B\subsetneq A}\gamma^\mu_\varepsilon(B)$$
$$=\frac{1}{1+\varepsilon}\Big[\varepsilon+\mu(A)-\frac{1}{2^{|A|-1}-1}\sum_{\emptyset\neq B\subsetneq A}(\mu(B)+\varepsilon)\Big]$$
$$=\frac{1}{1+\varepsilon}\Big[\eta^\mu_A+\varepsilon-\frac{1}{2^{|A|-1}-1}\varepsilon(2^{|A|}-2)\Big]$$
$$=\frac{1}{1+\varepsilon}\big(\eta^\mu_A-\varepsilon\big)$$

As $\eta^\mu_A=0$, we have $\eta_A^{\gamma^\mu_\varepsilon}=\frac{-\varepsilon}{1+\varepsilon}<0$. Thus there exists a possible negative nonadditivity index for A. Hence there is no null nonadditivity index for A. Therefore null nonadditivity index is never necessary.

The following example illustrates Propositions 1 and 2

Example 1. $N=\{1,2,3\}$, $X=\{a,\,b,\,c,\,d\}$, $a=(6,11,9)$, $b=(6,13,7)$, $c=(16,11,9), d=(16,13,7)$ and $P=\{(d,c),\,(b,a)\}$.
The strict preference P is representable by the capacity μ (with $\eta^\mu_{23}=0$) given by Table 1 and Choquet integral corresponding is given by Table 2.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(S)$	0.5	0.5	0	1	0.5	0.5	1

Table 1: A capacity $\mu\in C_{\text{Pref}}$ with $\eta^\mu_{23}=0$.

x	d	c	b	a
$C_\mu\big(u(x)\big)$	14.5	13.5	9.5	8.5

Table 2: Choquet integral corresponding at previous capacity μ.

We have $v^{\sigma^d}(d) - v^{\sigma^c}(c) = 7 - 9 = -2 < 0$ and $v^{\sigma^b}(b) - v^{\sigma^a}(a) = 7 - 9 = -2 < 0$ so $\Omega = \{(d,c),\ (b,a)\}$ and we choose $\varepsilon = \dfrac{1}{2}\min\left(\dfrac{8.5-9.5}{7-9}, \dfrac{13.5-14.5}{7-9}\right) = 0.25$. A capacity $\beta_\varepsilon^\mu \in C_{\text{Pref}}$ such that $\eta_{23}^{\beta_\varepsilon} > 0$ and Choquet integral corresponding at β_ε^μ are respectively given by Table 3 and Table 4.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\beta_\varepsilon^\mu(S)$	0.4	0.4	0	0.8	0.4	0.6	1

Table 3: A capacity $\beta_\varepsilon^\mu \in C_{\text{Pref}}$ with $\eta_{23}^{\beta_\varepsilon} > 0$.

x	d	c	b	a
$C_{\beta_\varepsilon^\mu}\big(u(x)\big)$	13	12.6	9	8.6

Table 4: Choquet integral corresponding at previous capacity β_ε^μ.

Indeed, $\eta_{23}^{\beta_\varepsilon^\mu} = \dfrac{\varepsilon}{1+\varepsilon} = \dfrac{0.25}{1+0.25} = 0.2 > 0$.

We have $v^{\sigma^d}(d) - v^{\sigma^c}(c) = 16 - 16 = 0$ and $v^{\sigma^b}(b) - v^{\sigma^a}(a) = 13 - 11 = 2 \geq 0$ so $\Omega = \emptyset$ and we can choose any $\varepsilon > 0$. We take $\varepsilon = 1$. A capacity $\gamma_\varepsilon^\mu \in C_{\text{Pref}}$ such that $\eta_{23}^{\gamma_\varepsilon} < 0$ and Choquet integral corresponding γ_ε^μ are respectively given by Table 5 and Table 6.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\gamma_\varepsilon^\mu(S)$	0.75	0.75	0.5	1	0.75	0.75	1

Table 5: A capacity $\gamma_\varepsilon^\mu \in C_{\text{Pref}}$ with $\eta_{23}^{\gamma_\varepsilon} < 0$.

x	d	c	b	a
$C_{\gamma_\varepsilon^\mu}\big(u(x)\big)$	15.25	14.25	11.25	9.75

Table 6: Choquet integral corresponding at previous capacity γ_ε^μ.

Indeed, $\eta_{23}^{\gamma_\varepsilon^\mu} = \dfrac{-\varepsilon}{1+\varepsilon} = \dfrac{-1}{1+1} = -0.5 < 0$.

In the next section, we define the set of generalized binary alternatives, then we show that on this set, positive and negative nonadditivity index are always possible for all subsets $A \subseteq N_{\geq 2}$.

4 Necessary and possible nonadditivity index with generalized binary alternatives

4.1 Framework of binary alternatives

We assume that the DM can identify two reference levels 0_i and 1_i on each criterion $i \in N$:

- the level 0_i in X_i is considered as a neutral level and we set $u_i(0_i) = 0$,
- the level 1_i in X_i is considered as a good level and we set $u_i(1_i) = 1$.

For all $x = (x_1, \ldots, x_n) \in X$ and $S \subseteq N$, we will sometimes write $u(x)$ as a shorthand for $(u_1(x_1), \ldots, u_n(x_n))$ and we define the alternatives $a_S = (1_S, 0_{-S}) \in X$ such that $a_i = 1_i$ if $i \in S$ and $a_i = 0_i$ otherwise.
We work on the set $\mathcal{B}^g$ which we define as follows.

Definition 5. *We call the set of generalized binary alternatives, the set:*

$$\mathcal{B}^g = \{a_S = (1_S, 0_{-S}) : S \subseteq N\}.$$

We add to the strict preference P a binary relation M modeling the monotonicity relations between generalized binary alternatives, and allowing us to ensure the satisfaction of the monotonicity condition: $\big[S \subseteq T \implies \mu(S) \leq \mu(T)\big]$.

Definition 6. *For all $a_S, a_T \in \mathcal{B}^g$, $a_S \, M \, a_T$ if $\big[not(a_S \, P \, a_T)$ and $S \supseteq T\big]$.*

In the sequel, we need the following basic definition in graph theory [14].

Definition 7. *There exists a strict cycle in $(P \cup M)$ if there exists elements $x_0, x_1, \ldots, x_r$ of $\mathcal{B}^g$ such that $x_0(P \cup M)x_1(P \cup M) \ldots (P \cup M)x_r(P \cup M)x_0$ and for a least one $i \in \{0, \ldots, r-1\}$, $x_i \, P \, x_{i+1}$.*

4.2 Results on binary alternatives

In [12] we find a necessary and sufficient condition for a strict ordinal preference information on $\mathcal{B}^g$ to be representable by a Choquet integral model. Under this condition, Proposition 3 below shows that positive nonadditivity index is always possible for all subsets $A \subseteq N_{\geq 2}$, in a Choquet integral model. In other words, negative and null nonadditivity index are not necessary.

Proposition 3. *Let P be a strict ordinal preference information on $\mathcal{B}^g$ such that $(P \cup M)$ containing no strict cycle. Then there exists a capacity $\mu \in C_{Pref}$ such that $\eta^\mu_A > 0$ for all $A \subseteq N_{\geq 2}$.*

Proof. Assume that $(P \cup M)$ contains no strict cycle, then there exists $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m\}$ a partition of $\mathcal{B}^g$, build by using a suitable topological sorting on $(P \cup M)$ [4].
We construct a partition $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m\}$ as follows:
$\mathcal{B}_i = \{x \in \mathcal{B}^g \backslash (\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{i-1}) : \forall y \in \mathcal{B}^g \backslash (\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{i-1})$, not $[x(P \cup M)y]\}$,
for all $i = 0, 1, 2, \ldots, m$ with $\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{i-1} = \emptyset$ for $i = 0$.

Let us define the mapping $\phi : \mathcal{B}^g \longrightarrow \mathcal{P}(N)$, $f : \mathcal{P}(N) \longrightarrow \mathbb{R}$, $\mu : 2^N \longrightarrow [0,1]$ as follows: $\phi(a_S) = S$, $\mu(S) = \frac{f_S}{f_N}$, where $f_S = f(\phi(a_S))$ for all $S \subseteq N$ and

$$f(\phi(x)) = \begin{cases} 0 & \text{if} \quad \ell = 0, \\ (2n)^\ell & \text{if} \quad \ell \in \{1, 2, \ldots, m\} \end{cases} \quad \forall x \in \mathcal{B}_\ell.$$

We have $\mu \in C_{\text{Pref}}$. Indeed, if $a_S \, P \, a_T$, then $a_S \in \mathcal{B}_q$ and $a_T \in \mathcal{B}_r$ with $q > r$. Therefore $\mu(S) = (2n)^q$ and $\mu(T) = 0$ (if $r = 0$) or $\mu(T) = (2n)^r$ (if $r \geq 1$). But $(2n)^q > \max(0, (2n)^r)$ since $q > r \geq 0$, so $\mu(S) > \mu(T)$.

Let $A \subseteq N_{\geq 2}$ and $\emptyset \subsetneq B \subsetneq A$. There exists $q, r, s \in \{1, 2, \ldots, m\}$ such that $a_A \in \mathcal{B}_q$, $a_B \in \mathcal{B}_r$ and $a_{A \setminus B} \in \mathcal{B}_s$ with $q > r$ and $q > s$.

Hence $\mu(B) = (2n)^r$, $\mu(A \setminus B) = (2n)^s$ and $\mu(A) = (2n)^q = (2n)(2n)^{q-1} > 2(2n)^{q-1} = (2n)^{q-1} + (2n)^{q-1} \geq (2n)^r + (2n)^s$ since $q - 1 \geq r$ and $q - 1 \geq s$.

Therefore $\mu(A) - \mu(B) - \mu(A \setminus B) > 0$ for all B such that $\emptyset \subsetneq B \subsetneq A$. Hence $\sum_{\emptyset \subsetneq B \subsetneq A} \big(\mu(A) - \mu(B) - \mu(A \setminus B)\big) > 0$, so $\eta_A^\mu > 0$.

The following proposition answers the dual question. Indeed, given a strict preference information P on $\mathcal{B}^g$ under the previous conditions, Proposition 4 shows that negative nonadditivity index that is always possible for all subset $A \subseteq N_{\geq 2}$. In other words, positive and null nonadditivity index are not necessary. Note that this dual problem remains open in the case of the necessary and possible interaction with the interaction index [7].

Proposition 4. *Let P be a strict ordinal preference information on $\mathcal{B}^g$ such that $(P \cup M)$ containing no strict cycle. Then there exists a capacity $\mu \in C_{Pref}$ such that $\eta_A^\mu < 0$ for all $A \subseteq N_{\geq 2}$.*

Proof. We construct the partition $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m\}$ and define the mapping ϕ as the proof of the Proposition 3 above.

Now, the functions f and μ are defined as follows: for all $S \subseteq N$, for all $\ell \in \{0, 1, \ldots, m\}$, for all $a_S \in \mathcal{B}_\ell$,

$$\mu(S) = \begin{cases} 0 & \text{if} \quad \ell = 0, \\ \dfrac{\ell+1}{\ell+2} & \text{if} \quad \ell \in \{1, 2, \cdots, m-1\}, \\ 1 & \text{if} \quad \ell = m. \end{cases}$$

Let $a_S, a_T \in \mathcal{B}^g$ such that $a_S \, P \, a_T$. We show that $C_\mu(u(a_S)) > C_\mu(u(a_T))$.

Since $a_S, a_T \in \mathcal{B}^g$ and $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is a partition of $\mathcal{B}^g$, then there exists $r, q \in \{0, 1, \ldots, m\}$ such that $a_S \in \mathcal{B}_r$, $a_T \in \mathcal{B}_q$. As $a_S \, P \, a_T$, then $r > q$. We have $C_\mu(u(a_S)) = \mu(S) = \dfrac{r+1}{r+2}$(if $1 \leq r \leq m-1$) or $\mu(S) = 1$ (if $r = m$), so $C_\mu(u(a_S)) \geq \dfrac{r+1}{r+2}$, since $1 \geq \dfrac{r+1}{r+2}$.

Moreover, $C_\mu(u(a_T)) = \mu(T) = \dfrac{q+1}{q+2}$(if $1 \leq q \leq m-1$) or $\mu(T) = 0$ (if $q = 0$), then $C_\mu(u(a_T)) \leq \dfrac{q+1}{q+2}$, since $0 \leq \dfrac{q+1}{q+2}$. But $r > q$ therefore $\dfrac{r+1}{r+2} > \dfrac{q+1}{q+2}$, since the sequence $(f_n)_{n \in \mathbb{N}}$ is strictly increasing, where $f_n = \dfrac{n+1}{n+2}$ $\forall n \in \mathbb{N}$. Then $C_\mu(u(a_S)) > C_\mu(u(a_T))$. Then $\mu \in C_{\text{Pref}}$.

Let $A \subseteq N_{\geq 2}$ and $\emptyset \subsetneq B \subsetneq A$. Then there exists $q, r, s \in \{1, 2, \ldots, m\}$ such that $a_A \in \mathcal{B}_q$, $a_B \in \mathcal{B}_r$ and $a_{A\setminus B} \in \mathcal{B}_s$ with $q > r$ and $q > s$. Hence $\frac{2}{3} \leq \mu(A) \leq 1$, $\frac{2}{3} \leq \mu(B) \leq 1$ and $\frac{2}{3} \leq \mu(A \setminus B) \leq 1$. Therefore we have, $\mu(B) + \mu(A \setminus B) \geq \frac{2}{3} + \frac{2}{3} = \frac{4}{3} > 1 \geq \mu(A)$, i.e., $\mu(A) - \mu(B) - \mu(A \setminus B) < 0$ for all $\emptyset \subsetneq B \subsetneq A$. Then $\displaystyle\sum_{\emptyset \subsetneq B \subsetneq A} \big(\mu(A) - \mu(B) - \mu(A \setminus B)\big) < 0$, so $\eta^{\mu}_A < 0$.

The following example illustrates Propositions 3 and 4.

Example 2. $N = \{1, 2, 3\}$, $P = \{(a_{23}, a_{12}), (a_2, a_3)\}$.

The binary relation $(P \cup M)$ contains no strict cycle, so P is representable by a Choquet integral model. A suitable topological sorting on $(P \cup M)$ is given by: $\mathcal{B}_0 = \{a_0\}$; $\mathcal{B}_1 = \{a_1, a_3\}$; $\mathcal{B}_2 = \{a_2, a_{13}\}$; $\mathcal{B}_3 = \{a_{12}\}$; $\mathcal{B}_4 = \{a_{23}\}$ and $\mathcal{B}_5 = \{a_{123}\}$. The strict preference P is representable by the following capacities μ and α given by Table 7 and Table 8 respectively.

A	$\{1\}$	$\{3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$6^5 \times \mu(A)$	6	6	6^2	6^2	6^3	6^4	6^5
$6^5 \times \eta^{\mu}_A$				24	174	1254	7244

Table 7: A capacity $\mu \in C_{\text{Pref}}$ and the corresponding nonadditivity index.

A	$\{1\}$	$\{3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\alpha(A)$	2/3	2/3	3/4	3/4	4/5	5/6	6/7
η^{α}_A				−7/12	−37/60	−7/12	−199/315

Table 8: A capacity $\alpha \in C_{\text{Pref}}$ and the corresponding nonadditivity index.

We can see that $\forall A \subseteq N_{\geq 2}$, we have $\eta^{\mu}_A > 0$ and $\eta^{\alpha}_A < 0$.

5 Conclusion

This article studies the nonadditivity index in the Choquet integral model. We make a restriction that the DM only gives strict preference information. The capacity to represent this strict preference information is not unique and the sign of the nonadditivity index can vary depending on the arbitrary choice of a capacity compatible with the strict preference of DM. Therefore we introduce the concept of necessary and possible nonadditivity index similar to that of necessary and possible interaction introduced in [13]. Only necessary nonadditivity index are robust since their sign and, hence, interpretation, does not vary within the set of all representing capacities.

We prove that neither null, nor positive, or negative nonadditivity index is necessary. Thus the sign of nonadditivity index is not stable in the set of all capacities compatible with strict preferences of DM, therefore the interpretation of the nonadditivity index between criteria requires some caution.

In our future research, we will study the case where the ordinal preference information can contain the indifference relation. Moreover, outside the framework of binary alternatives, we will also proposed a linear program allowing to test non stability of the nonadditivity index.

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