Unit representation of semiorders
I: Countable sets

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Abstract

This paper proposes a new proof of the existence of constant threshold representations of semiorders on countably infinite sets. The construction treats each indifference-connected component of the semiorder separately. It uses a partition of such an indifference-connected component into indifference classes. Each element in the indifference-connected component is mirrored, using a “ghost” element, into a reference indifference class that is weakly ordered. A numerical representation of this weak order is used as the basis for the construction of the unit representation after an appropriate lifting operation. We apply the procedure to each indifference-connected component and assemble them adequately to obtain an overall unit representation.

Our proof technique has several original features. It uses elementary tools and can be seen as the extension of a technique designed for the finite case, using transfinite induction instead of induction. Moreover, it gives us much control on the representation that is built, so that it is, for exemple, easy to investigate its uniqueness. Finally, we show in a companion paper that our technique can be extended to the general (uncountable) case, almost without changes, through the addition of adequate order-denseness conditions.
Keywords: Semiorder, Numerical Representation, Constant Threshold, Countable sets.

Contents

1 Introduction

Treating two “similar” things as if they were exactly identical has long been recognized as being a source of paradoxes. One of them is the famous sorites paradox: a heap of sand cannot cease to be one if one grain of sand is taken out. Repeating the argument leads to the paradoxical conclusion that one grain of sand is already a heap. Similar examples are well-known, for instance concerning the level of baldness of a man, which is surely not affected by the removal of a single hair. Luce (1956, p. 179) has added to this list of paradoxes with his famous exemple of cups of coffee slightly differently sugared.

Therefore, it is not surprising that the idea of introducing a threshold into preference or perception models has distant origins (see Pirlot and Vincke, 1997 and Fishburn and Monjardet, 1992, for historical accounts of the idea). The formal definition of semiorders is due to Luce (1956). Shortly after, Scott and Suppes (1958) showed that a semiorder defined on a *finite* set always has a numerical representation with positive threshold. This paper is about the existence of such numerical representations of semiorders. Because, it is clear that, if a numerical representation with positive threshold exits, a numerical representation that uses a unit threshold also exists, we will call such representations *unit representations.*

The pioneering work of Scott and Suppes (1958) on finite sets was soon followed by many other alternative proofs using various kinds of arguments. Without aiming at exhaustivity, one can cite Suppes and Zinnes (1963), Scott (1964), Roberts (1971), Rabinovitch (1977), Roberts (1979), Roubens and Vincke (1985), Avery (1992), Roy (1996, Ch. 7), Bogart and West (1999), Balof and Bogart (2003), Troxell (2003), and Isaak (2009).

It is well-known that these results do not extend to the infinite case, even the countably infinite case (see Fishburn, 1985, p. 30). Indeed, the fact that the threshold is constant and positive is not compatible with the existence of infinite (ascending or descending) chains of strict preference that are bounded. This makes semiorders at variance with what happens with many other preference structures (e.g., weak orders, biorders, interval orders, suborders, see Aleskerov et al., 2007, Bridges and Mehta, 1995 and Doignon et al., 1984) for which the finite and the countably infinite cases are identical.

Contrasting with the abundance of results in the finite case, we are only aware of two results in the countably infinite case. In both, an additional condition must be added that forbids the existence of infinite bounded chains of strict preference. The first of these results was given by Manders (1981, Prop. 9, p. 239) in a path-breaking paper. His proof of this result is not easy however. It makes use of the Löwenheim-Skolem theorem (see Narens, 1985, Ch. 6, for an introduction and a discussion of its use in measurement theory) to extend the results dealing with the finite case to the countably infinite one. The result in Beja and Gilboa (1992, Th. 3.8, p. 436) uses more elementary tools. It also leads to possible extensions to the general case (see Bou, Candeal and Induráin, 2010, for a discussion of this extension). However the proof offers little control on the representation that is built. Moreover, it uses at some point expectations w.r.t. some measure (Beja and Gilboa, 1992, p. 444–445).

The aim of this paper is to offer a third proof of the existence of unit representations of semiorders on countably infinite sets. Our aim was at the same time (i) to use only elementary arguments, (ii) to use arguments that would as much as possible be common to the finite and the countably infinite case, (iii) to have good control on the representation that is built and, in particular on its uniqueness.

More precisely, we show that:

- any indifference-connected semiorder (i.e., any semiorder which cannot be decomposed in a series-sum of at least two posets, Schröder, 2003) on a countably infinite set admits a unit representation. We show how to construct such a representation and establish its uniqueness properties,
- under an additional axiom forbidding the existence of infinite bounded chains of strict preference, it is possible to combine the unit representations on all indifference-connected components of the semiorder to build a unit representation of the whole semiorder.

Our construction treats each indifference-connected component separately. Each of these components is partitioned into "indifference classes". Each element in the indifference-connected component is mirrored, using a "ghost" element, into a reference indifference class that is weakly ordered. A numerical representation of this weak order is used as the basis for the construction of the unit representation after an appropriate lifting operation. We apply the procedure to each indifference-connected component and assemble them adequately to lead to the result.

In a companion paper (Bouyssou and Pirlot, 2020b), we show that this proof technique can be extended rather directly, after the addition of order-denseness conditions, to cover the general case. The first complete solution to the problem of the existence of unit representations in the general case was given in Candeal and Induráin (2010) (for earlier partial results see Abrísqueta et al., 2009, Campión et al., 2008, Candeal et al., 2002, Fishburn, 1985, Gensemer, 1987, 1988, Narens, 1994). Candeal and Induráin use the results of Manders (1981) and Beja and Gilboa (1992) for the countably infinite case as a lemma, which gives us a supplementary motivation for presenting the results in the present paper (for more recent results on representations in the general case, we refer to Can, Estevan et al., 2013).

The paper is organized as follows. Section 2 introduces our notation and framework. Section 3 details our construction of the partition of an indifference-connected component into maximum indifference components. Section 4 explains how to build a representation on a single indifference-connected component. Section 5 shows how to assemble the representation built on each indifference-connected component into a single representation. A final section discusses our results and directions for future research.

2 Notation, definitions and preliminary results

2.1 Binary relations

A binary relation R on a set Y is a subset of $Y \times Y$. We often write yRz instead of $(y, z) \in R$. When R is a binary relation on a set Y, we define, for all $x \in Y$, $xR = \{y \in Y : xRy\}$ and $Rx = \{y \in Y : yRx\}$. The asymmetric (resp., symmetric) part of R is the binary relation R^a (resp., R^s) such that yR^az iff $[yRz$ and $Not[zRy]]$ (resp., yR^sz iff $[yRz$ and $zRy]$).

We consider below a binary relation S on a set X. Such a relation can be interpreted as a model for "at least as good" preferences between the objects of X.

From Section 4 on, we assume that X is a denumerable set (finite or countably infinite set). Part of the results, in particular the important construction in Section 3, is valid without restriction on the cardinality of X.

The relation S is a *semiorder* if it is complete (xSy or ySx, for all $x, y \in X$), Ferrers (xSy and zSw imply xSw or zSy, for all $x, y, z, w \in X$) and semi-transitive (xSy and ySz imply xSw or wSz, for all $x, y, z, w \in X$)[1].

In the sequel, we shall often write the semiorder S as a pair (P, I) of relations, where P (resp., I) denotes the asymmetric (resp., symmetric) part of S. The asymmetric part of S is the relation P, interpreted here as a "strict preference" relation. It is a partial order on X, i.e., an asymmetric and transitive relation, which is also Ferrers and semitransitive. The symmetric part of S is the relation I, interpreted as the "indifference" relation. It is reflexive and symmetric but not necessarily transitive. Because S is complete, notice that we could have alternatively defined a semiorder, giving its asymmetric part P, while letting I be the symmetric complement of P (i.e., xIy iff $[Not[xPy]$ and $Not[yPx]]$) and $S = P \cup I$. We refer to Aleskerov et al. (2007), Fishburn (1985), Giarlotta and Watson (2016), Monjardet (1978), Pirlot and Vincke (1997), Roubens and Vincke (1985), Suppes et al. (1989) for detailed studies of various properties of semiorders.

Our vocabulary for binary relations is standard. A *complete preorder* on X is a complete and transitive relation. A *linear order* (or *total order*) on X is a

[1]Note that it is sufficient to impose that S is reflexive instead of complete, since reflexive and Ferrers entail complete.

complete, antisymmetric (i.e., for all $x, y \in X$, xSy and ySx imply $x = y$) and transitive relation. A *strict linear order* is the asymmetric part of a linear order, i.e., a weakly complete, (i.e., for all $x, y \in X$, such that $x \neq y$, xSy or ySx), asymmetric (i.e., for all $x, y \in X$, xSy implies $Not[ySx]$), and transitive relation.

The trace $\succsim_S$ of a semiorder S on X is the relation defined as follows: for all $x, y \in X$, $x \succsim_S y$ if for all $z \in X$, [ySz implies xSz and zSx implies zSy]. In other words, $x \succsim_S y$ if [$yS \subseteq xS$ and $Sx \subseteq Sy$]. The trace $\succsim_S$ is transitive by construction and complete because S is a semiorder. We omit the subscript when there is no ambiguity on the underlying semiorder. It is easy to check that the trace $\succsim_S$ can be equivalently defined using P, i.e., $x \succsim_S y$ if for all $z \in X$, [yPz implies xPz and zPx implies zPy].

We define $\succ$, $\sim$, $\precsim$ and $\prec$ as is usual. We assume w.l.o.g.[2] that X does not contain equivalent pairs of elements, i.e., for all $x, y \in X$, $x \sim y$ entails $x = y$. Hence $\succsim$ is a linear order (i.e., a complete, antisymmetric and transitive relation). This is not restrictive (see Candeal and Induráin, 2010, Lemma 3.2) for the purpose of studying the existence of unit representations.

For all $x, y \in X$ with $x \succsim y$, we define the closed interval $[x, y] = \{z \in X : x \succsim z \succsim y\}$. A *convex* subset of the set X endowed with the complete preorder $\succsim$, is a subset Y containing all the closed intervals determined by pairs of elements in Y. The semi-open $]a, b]$, $[b, a[$ and open $]a, b[$ intervals are defined in the obvious manner.

Finally, we will make use of the following notation: $xI^- = \{y \precsim x : xIy\}$, $xI^+ = \{y \succsim x : xIy\}$.

2.2 Unit representations

The following definition makes precise the type of numerical representation that are sought.

Definition 1 (Unit representation of a semiorder)
A unit representation of the semiorder $S = (P, I)$ on the set X is a function u from X to $\mathbb{R}$ such that, for all $x, y \in X$,

$$\begin{aligned} xPy &\Leftrightarrow u(x) > u(y) + 1, \\ xIy &\Leftrightarrow -1 \leq u(x) - u(y) \leq 1. \end{aligned} \quad (1)$$

⌟

[2]In case X contains equivalent pairs of elements, we consider the quotient of X by the equivalence relation $\sim$, which amounts to identify all the elements in a class of equivalence of $\sim$ to a single representative element. Alternatively, we could work with equivalent elements and consider only numerical representations that assign the same value to all elements in each equivalence class of $\sim$.

The above definition has taken the threshold to be 1. Clearly, if a representation (1) exists, a representation of the same type exists with any threshold $\tau > 0$. We stick to unit thresholds throughout.

A variant of the above definition consists in switching the strict and nonstrict inequalities, i.e., require that

$$\begin{aligned} xPy &\Leftrightarrow u(x) \geq u(y) + 1, \\ xIy &\Leftrightarrow -1 < u(x) - u(y) < 1. \end{aligned} \tag{2}$$

In the paper, we mostly deal with representations of type (1) that we call *strict* representations. Representations of type (2) are called *nonstrict*.

When X is finite, it is well-known that it is always possible to build a representation for which $u(x) - u(y) \neq 1$, for all $x, y \in X$, representations that are at the same time strict and nonstrict exist. Beja and Gilboa (1992, Th. 3.8, p. 436) show that the same is true when X is countably infinite. Our results below will also note this equivalence, which does not carry over to the general case (Bouyssou and Pirlot, 2020a,b).

2.3 Chains

Let R be a relation on the set X. We call an *R-chain*, a sequence x_i of elements of X indexed by a subset of consecutive integers $J \subseteq \mathbb{Z}$ and such that any two consecutive elements of the sequence belong to the relation R (we adopt here the terminology used in the field of ordered sets, see Caspard et al., 2012 or Schröder, 2003. Graph theorists may prefer the term "path"). Formally, the sequence $(x_i, x_i \in X, i \in J)$, where $J \subseteq \mathbb{Z}$ is a subset of consecutive integers and $(x_i, x_{i+1}) \in R$, for all $i, i+1 \in J$ is an R-chain[3]. We shall consider the cases in which $R = P$ and $R = I$ in the sequel, i.e., P-chains and I-chains. Note that an R-chain needs neither have a first nor a last element. In other terms, it can have an infinite number of elements before or after a given element, but not between two given elements.

An R-chain is said to start at $x \in X$, if the set J has a minimum element and x is the element of X indexed by the minimal number in J. In this case, the chain is said to *have a first element*, which is this x. An R-chain is said to terminate at $y \in X$, if the set J has a maximum element and y is the element of X indexed by the maximal number in J. In this case, the chain is said to *have a last element*, which is this y. An R-chain[4] starting at x and terminating at y is finite by definition (i.e. $|J| < \infty$).

[3]Actually, one could consider chains indexed by more general sets of indices. To be precise, one should call the chains defined above "integer-indexed R-chains", but we simply call them R-chains for the sake of conciseness.

[4]We also say, equivalently, "an R-chain from x to y".

A P-chain $(x_i, i \in J)$ has an *upper* (resp., *lower*) *bound* if there exists $a \in X$ (resp., $b \in X$) such that aPx_i (resp., x_iPb) for all $i \in J$. If the chain has both an upper and a lower bound, we say it is *bounded*. Note that the set $\{x_i : i \in J\} \cup \{a, b\}$ is totally ordered by P, but cannot always be indexed by the elements of a subset J' of $\mathbb{Z}$. It cannot be in the case the P chain $(x_i, i \in J)$ has no first or no last element. The elements of a finite subset of X which is totally ordered by P can be indexed by a set J of consecutive integers in order to form a P-chain. If a P-chain $(x_i, i \in J)$ has no last (resp., first) element, then for all $i \in J$, x_{i+k} (resp., x_{i-k}) belongs to the chain, for all $k \in \mathbb{N}$.

2.4 Convex subsets in ordered sets

We start by defining *ordered bipartitions* and establishing simple properties linking ordered bipartitions and convex subsets in an ordered set.

Definition 2
An ordered bipartition of a totally ordered set $(X, \succsim)$ is a partition (A,B) of X, with $x \succ y$, for all $x \in A, y \in B$. ⌟

In the sequel, in the absence of ambiguity, we simply write "bipartition" for "ordered bipartition"[5]. The proof of the following proposition is left to the reader.

Proposition 3
The total order $\succsim$ on X can be extended to a total order on the ordered bipartitions of X, by defining $(A_1, B_1) \succsim (A_2, B_2)$ if $A_1 \subseteq A_2$. Elements of X and ordered bipartitions can also be compared using $\succsim$. Let, for all $x \in X$ and all ordered bipartition (A, B):

$$\begin{array}{lll} x \succsim (A,B) & \text{if} & x \succsim y, \text{ for all } y \in B \\ (A,B) \succsim x & \text{if} & y \succsim x, \text{ for all } y \in A. \end{array}$$

The extension of $\succsim$ to the union of X and the set of ordered bipartitions of X is a complete preorder, satisfying, for all $x \in X$ and all bipartion (A, B), $x \sim (A, B)$ iff x is the least element in A or the greatest element in B.

Definition 4
A convex subset Y of $(X, \succsim)$ is non-terminal if $\exists a, b \in X$ such that $a \succ y \succ b$ for all $y \in Y$. Otherwise, Y is terminal. ⌟

[5]Note that the notion of ordered bipartition corresponds to that of "decomposition" in the terminology of Bridges and Mehta (1995, p. 17). We chose not to adopt the term "decomposition" because we use it below with another meaning.

Proposition 5
A non-terminal non-empty convex subset Y is determined by two bipartitions of $(X, \succsim)$: $(A_1, B_1) \succ (A_2, B_2)$, with $A_1 = \{z \in X : z \succ y, \forall y \in Y\}$, $B_1 = X \setminus A_1$, $B_2 = \{w \in X : y \succ w, \forall y \in Y\}$ and $A_2 = X \setminus B_2$. We have $Y = A_2 \cap B_1$. Conversely, any two such bipartitions determine a non-terminal convex subset $Y = A_2 \cap B_1$.

PROOF
Let Y be a non-terminal non-empty convex subset of X. Hence $\exists a, b \in X$ such that $a \succ y \succ b$ for all $y \in Y$. The sets A_1, B_1, A_2, B_2 are not empty since Y is non-terminal. They are convex. We have $B_1 = Y \cup B_2$ and $A_2 = Y \cup A_1$.

(A_1, B_1) and (A_2, B_2) are bipartitions such that $(A_1, B_1) \succ (A_2, B_2)$. They determine Y in the sense that $Y = A_2 \cap B_1$ and

$$Y = \{y \in X : z \succ y \succ w, \forall z \in A_1, w \in B_2\}. \quad (3)$$

In this sense Y can be viewed as the set comprised between (A_1, B_1) and (A_2, B_2), a sort of interval determined by bipartitions instead of points[6].

Conversely, two bipartitions $(A_1, B_1) \succ (A_2, B_2)$ determine a convex subset Y defined by (3). This set is a convex non empty non-terminal subset. □

Definition 6
We call (A_1, B_1) (resp., (A_2, B_2)) as defined in Proposition 5, the upper (resp., lower) bound of Y. ⌟

Remark 7 (Terminal convex subsets)
If Y is terminal and $Y \neq X$, it is either upper-terminal and has a lower bound (A_2, B_2) or it is lower-terminal and has an upper bound (A_1, B_1). Each of these bipartitions is defined as in Proposition 5. •

There are 4 different cases for a bipartition (A, B) in $(X, \succsim)$:

1. A has a least element a and B has a greatest element b (consider, for instance, $X = \mathbb{Z}$, endowed with its usual order $\geq$ and the bipartition (A, B) with $A = \{x \in \mathbb{Z} : x \geq 0\}$ and $B = \{x \in \mathbb{Z} : x < 0\}$).

2. A has a least element a and B has no greatest element (consider, for instance, $X = \mathbb{Q}$, the set of rational numbers, endowed with its usual order $\geq$ and the bipartition (A, B) with $A = \{x \in \mathbb{Q} : x \geq 0\}$ and $B = \{x \in \mathbb{Q} : x < 0\}$).

[6] A convex subset in an ordered set is the “natural generalization of an interval” (Schröder, 2003, p.225).

3. A has no least element and B has a greatest element b (consider again $X = \mathbb{Q}, \geq$ and, for instance, the bipartition (A, B) with $A = \{x \in \mathbb{Q} : x > 0\}$ and $B = \{x \in \mathbb{Q} : x \leq 0\}$).

4. A has no least element and B has no greatest element (consider again $X = \mathbb{Q}, \geq$ and, for instance, the bipartition (A, B) with $A = \{x \in \mathbb{Q} : x^2 > 2\}$ and $B = \{x \in \mathbb{Q} : x^2 \leq 2\}$).

In the first case, the bipartition (A, B) is usually called a *jump*. In the second and third cases, it is called a *cut* and in the fourth case, a *gap* (see, e.g., Bridges and Mehta, 1995, p. 17).

We apply this categorization to any non-terminal non-empty convex set Y, with bounds $(A_1, B_1) \succ (A_2, B_2)$. If both bounds bipartitions (A_1, B_1), (A_2, B_2) belong to Case 1, let a_i (resp., b_i) denote the least (resp., greatest) element in A_i (resp., B_i), for $i = 1, 2$. Y can be described in 4 manners in terms of intervals: $Y =]a_1, b_2[=]a_1, a_2] = [b_1, b_2[= [b_1, a_2]$. We have, e.g., $Y = [b_1, b_2[= \{y \in X : b_1 \succsim y \succ b_2\} = \{y \in X : z \succ y \succ w, \forall z \in A_1, w \in B_2\}$. The variety of representations is more limited in cases 2 and 3. There is no such representation in Case 4.

Proposition 8 (Case 4)
The bipartition (A,B) is in case 4 iff there is a decreasing sequence $(x_i, i \in \mathbb{N})$ in A, $x_i \succ x_{i+1}, \forall i$, and an increasing sequence $(y_j, j \in \mathbb{N})$ in B, $y_{j+1} \succ y_j, \forall j$, such that

- $x_i \succ y_j, \forall i, j$
- *and* $\nexists z \in X$ *such that* $x_i \succ z \succ y_j$ *for all* i, j.

The proof is left to the reader. The latter formulation, involving z, can be rewritten as follows: for all $z \in X$, $\exists x_i : z \succsim x_i$ or $\exists y_j : y_j \succsim z$. In the case of the bipartition (A, B) of $\mathbb{Q}$, with $A = \{x \in \mathbb{Q} : x^2 > 2\}$ and $B = \{x \in \mathbb{Q} : x^2 \leq 2\}$), a sequence x_i (resp., y_i) could be the sequence of rounded up (resp., down) decimal approximations of $\sqrt{2}$.

2.5 Connected components of the indifference relation

We consider the graph of relation I on X, where I is the indifference relation of a semiorder on X.

Definition 9
The set $\mathcal{D} \subseteq X$ is connected w.r.t. relation I if for all $x, y \in \mathcal{D}$, there is an I-chain joining x and y. ⌟

Abusing notation, we denote by $\succsim$ the restriction to $\mathcal{D} \subseteq X$ of the trace $\succsim$ on X. This abuse of notation is justified by the fact that all elements in an I-connected component compare in the same way to all elements outside this component (see Lemma 13 below). We also call a bipartition of $(\mathcal{D}, \succsim)$ and denote by (A, B) an ordered partition of $\mathcal{D}$ into two subsets A, B such that, for all $x \in A$, $y \in B$, we have $x \succ y$. Such a bipartition is the restriction to $\mathcal{D}$ of a bipartition of $(X, \succsim)$.

Proposition 10
Let $\mathcal{D}$ be a subset of X and (P, I) a semiorder on X. The following properties are equivalent:

1. *$\mathcal{D}$ is connected w.r.t. I,*
2. *for all $x, y \in \mathcal{D}$ with $x \succ y$, there is a decreasing (w.r.t. $\succ$) I-chain joining x and y,*
3. *for all bipartition (A, B) of $(\mathcal{D}, \succsim)$ there is $x \in A, y \in B$ such that $(x, y) \in I$,*
4. *there is no bipartition (A, B) of $(\mathcal{D}, \succsim)$ such that, for all $x \in A, y \in B$, we have $(x, y) \in P$.*

Proof
$1 \Leftrightarrow 2$. It suffices to prove that $(1 \Rightarrow 2)$. Let $x, x_1, \ldots x_i, x_{i+1}, \ldots x_n, y$ be an I-chain from x to y. If some of the vertices in the chain are below y w.r.t. $\succ$, let x_i be the first such vertex. We have $x_{i-1} \succ y \succ x_i$. This implies that $x_{i-1} I y$. Hence $x, x_1, \ldots x_{i-1}, y$ is also an I-chain from x to y.

Assume now w.l.o.g. that $x, x_1, \ldots, x_i, x_{i+1}, \ldots, x_n, y$ is an I-chain of distinct elements from x to y with $x_k \succ y$ for $k = 1, \ldots, n$. Assume that, for some i, we have $x_{i+1} \succ x_i$. If $x_i I y$, we can shorten the I-chain by removing all the x_k's with $k > i$. Else $x_i P y$. For some $k > i + 1$ we must have $x_i \succ x_k$. Let k be the least such index. We have $x_{k-1} \succ x_i \succ x_k$ with $x_{k-1} I x_k$, hence $x_i I x_k$. We may thus drop the sub-chain $x_{i+1}, \ldots, x_{k-1}$. The remaining path is another I-chain from x to y. Repeating this eventually leads to a decreasing (w.r.t. $\succ$) I-chain from x to y.

$3 \Leftrightarrow 4$. This equivalence results immediately from the fact that P and I are exclusive and $P \cup I$ is complete.

$1 \Rightarrow 3$. Let (A, B) be a bipartition of $\mathcal{D}$. Under the hypothesis that all pairs of elements in $\mathcal{D}$ can be joined by an I-chain, we claim that we must have xIy for some $x \in A$ and $y \in B$. Let $z \in A, w \in B$. An I-chain joining z to w cannot be entirely contained in A. Therefore, there is a first arc (x, y), with $x \in A, y \in B$ and xIy, that crosses the cut. This proves the claim.

$4 \Rightarrow 1$. Assume that there is no I-chain between $x, y \in \mathcal{D}$. We suppose, w.l.o.g. that $x \succ y$. Let $A = \{z \in \mathcal{D} : z \succsim x\} \cup \{z \in \mathcal{D} : x \succ z$ and there is an I-chain joining x to $z\}$. Let $B = \mathcal{D} \setminus A$. We have $y \notin A$ and $y \in B$. (A, B) is a bipartition of $\mathcal{D}$ (since $\{z \in \mathcal{D} : z$ and x can be joined by an I-chain $\}$ is convex in $\mathcal{D}$). For all $z \in A, w \in B$, we may not have $(z, w) \in I$, otherwise $w \in A$. Hence we have $(z, w) \in P$, contrary to 4. □

We turn to the study of the connected components of (X, I).

Definition 11
An I-connected component of (X, I) is a maximal connected subset of (X, I). ⌟

Proposition 12
An I-connected component $\mathcal{D}$ of (X, I) is a convex set of $(X, \succsim)$.

Proof
Assume $\mathcal{D}$ is not convex and let $\overline{\mathcal{D}}$ denote the smallest convex set containing $\mathcal{D}$, i.e., the intersection of all convex subsets containing $\mathcal{D}$. Assume that $\overline{\mathcal{D}}$ is not I-connected, i.e., there is a bipartition (A, B) of $\overline{\mathcal{D}}$ such that, for all $x \in A, y \in B$, we have xPy. $(A \cap \mathcal{D}, B \cap \mathcal{D})$ is a bipartition of $\mathcal{D}$ since neither $A \cap \mathcal{D}$ nor $B \cap \mathcal{D}$ is empty (else it would imply that $\mathcal{D}$ is included either in A or in B, which are convex sets, hence $\overline{\mathcal{D}}$ cannot be the smallest convex set containing $\mathcal{D}$). The bipartition $(A \cap \mathcal{D}, B \cap \mathcal{D})$ of $\mathcal{D}$ would be such that for all $x \in A \cap \mathcal{D}, y \in B \cap \mathcal{D}$, we have xPy, a contradiction. □

The following lemma justifies our earlier abuse of notation concerning the trace of S restricted to an I-connected component.

Lemma 13
For all pairs of distinct connected components $\mathcal{D}, \mathcal{E}$ of (X, I), either we have aPb, for all $a \in \mathcal{D}$ and $b \in \mathcal{E}$, or conversely.

Proof
Two elements x, y belonging to different connected components $\mathcal{D}, \mathcal{E}$ of (X, I) cannot be indifferent, by definition. Hence we assume w.l.o.g. that $x \in \mathcal{D}$ and $y \in \mathcal{E}$ satisfy xPy. For any $a \in \mathcal{D}$, we have $a \succ y$ (otherwise y would be between a and x, which is impossible by Proposition 12) and a cannot be indifferent to y (since they belong to different connected components of (X, I)). Hence aPy. For any $b \in \mathcal{E}$, we have $a \succ b$ (since, otherwise, a would be between y and b, which is excluded by Proposition 12 and b cannot be indifferent to a). Consequently, we have aPb. □

Let $\mathfrak{F}$ denote the set of connected components of (X, I). Abusing notation, we denote by P the following relation on $\mathfrak{F}$: for all $\mathcal{D}, \mathcal{E} \in \mathfrak{F}$,

$$\mathcal{D} P \mathcal{E} \quad \text{if} \quad aPb, \text{ for all } a \in \mathcal{D}, b \in \mathcal{E}. \tag{4}$$

Lemma 14
The relation P on $\mathfrak{F}$ is a strict linear order.

PROOF
By Lemma 13, we know that P is a weakly complete binary relation on $\mathfrak{F}$. It inherits its asymmetric and transitive properties from the relation P on X. $\square$

Remark 15 (Linear lexicographic sum decomposition of P)
The latter property of the I-connected components of a semiorder has already been noticed by Manders (1981, p.238). In the language of partially ordered sets, $\mathfrak{F}$ is a *linear lexicographic sum* decomposition of the poset (X, P) (Schröder, 2003, p. 203). In case X is not I-connected, P is said to be *series decomposable* or a *series-sum* poset. On the contrary, if X is I-connected, P is said to be (linearly) *indecomposable*. •

3 Partitioning a connected component into sets of indifferent elements

In this section we consider an I-connected component $\mathcal{D}$ of X. We describe procedures for partitioning $\mathcal{D}$ into sets of indifferent elements. Their construction is recursive. Each set is maximal given the previous ones. These sets are also convex subsets w.r.t to $\succsim$ (we shall use the extension of $\succsim$ to the ordered bipartitions of $\mathcal{D}$ without further notice, see Proposition 3). This construction will play an essential role in the rest of the paper. It is our basic tool to build unit representations. Later, we shall deal separately with each connected component $\mathcal{D}$ of (X, I), build a unit representation of the restriction of the semiorder to each component (Section 4) and then assemble these representations (Section 5).

All results in this section are valid for all semiorders $S = (P, I)$ on a set X and all I-connected components $\mathcal{D}$ of this semiorder. We do not require X (or $\mathcal{D}$) to be denumerable.

We start with introducing the notion of maximal indifference class and describe two procedures for building such a class. Each of these procedures allows to generate any maximal indifference class, as we shall see.

Definition 16
A maximal indifference class of an I-connected component $\mathcal{D}$ of the semiorder $S = (P, I)$ is a subset Y of pairwise indifferent elements such that no element outside Y is indifferent to all elements in Y. ⌟

Proposition 17
A maximal indifference class Y of $\mathcal{D}$ is a convex set in $(\mathcal{D}, \succsim)$.

Proof

Let a, b be two elements in Y, with $a \succ b$. Assume that $c \in X$ is such that $a \succ c \succ b$. Then, aIb implies aIc and cIb. Moreover, c is indifferent to all elements in the set $\{y \in Y : a \succ y\}$ and to all elements in the set $\{y \in Y : y \succ b\}$. The union of these two sets is Y. Consequently, c is indifferent to all elements in Y and therefore, it must belong to Y, by Definition 16. □

As a consequence of Proposition 5, if Y is a non-terminal indifference class of $\mathcal{D}$, it is bounded by two bipartitions, (A_1, B_1), its upper bound, and (A_2, B_2), its lower bound. If Y is terminal but $Y \neq \mathcal{D}$, Y has either an upper or a lower bound bipartition.

3.1 Two procedures for building a maximal indifference class

Consider a bipartition (A, B) in $\mathcal{D}$, a connected component of (X, I). Since $\mathcal{D}$ is I-connected, there are $x \in A$ and $y \in B$ such that xIy (Proposition 10.3). The two procedures are as follows.

Down First Procedure Given: a bipartition (A, B) in $\mathcal{D}$. Do the following:

- select an element y_0 in B, which is indifferent to some element x_0 in A,
- define $I_0^+(y_0) = \{y \in B : y \succsim y_0\}$,
- define $J_0^-(y_0) = \{w \in B : y_0 \succ w \text{ and } wIy, \forall y \in I_0^+(y_0)\}$,
- define $J_0^+(y_0) = \{z \in A : zIw, \forall w \in J_0^-(y_0)\}$,
- let $I^-(A, B) = I_0^+(y_0) \cup J_0^-(y_0) \cup J_0^+(y_0)$.

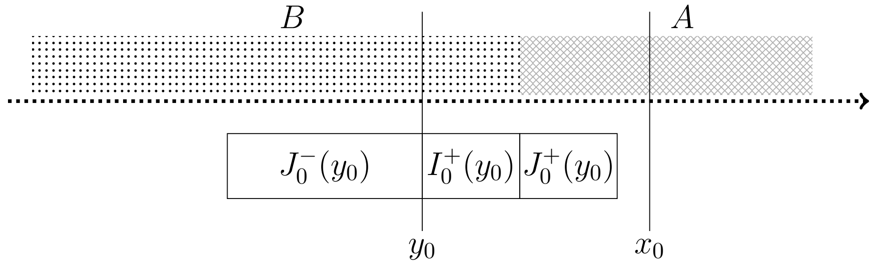
The procedure DFP is illustrated on Figure 1.(a). Note that the resulting set $I^-(A, B)$ does not depend on the particular choice of y_0 and x_0. For all pairs $x_0 \in A, y_0 \in B$, with $x_0 I y_0$, we obtain the same set $I^-(A, B)$).

Proposition 18

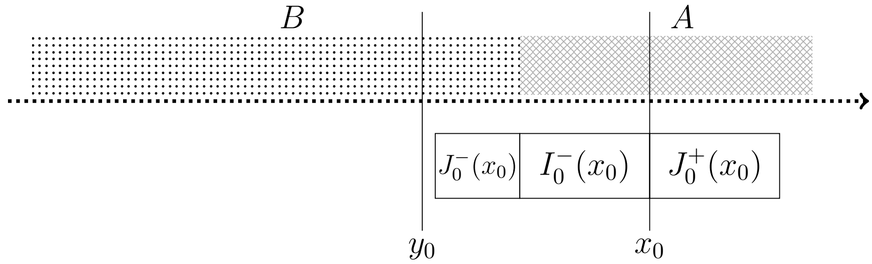
$I^-(A, B)$ is a maximal indifference class.

Proof

$I^-(A, B)$ is an indifference class by construction. It is convex. Indeed, if x, y are two elements of $I^-(A, B)$ and z is such that $x \succ z \succ y$, it is clear that z is indifferent to all elements of $I^-(A, B)$. It remains to prove that $I^-(A, B)$ is maximal.



(a) DFP



(b) UFP

Figure 1: Illustration of procedures Down First (a) and Up First (b). The dotted horizontal axis represents the elements of $\mathcal{D}$ in increasing order w.r.t. $\succ$ (from left to right).

Assume, to the contrary, that $\exists v \notin I^-(A, B)$ such that $vIy, \forall y \in I^-(A, B)$. Then, either $v \prec y, \forall y \in I^-(A, B)$ or $v \succ y, \forall y \in I^-(A, B)$. In the former case, $v \prec y_0$ and $vIy, \forall y \in I_0^+(y_0)$, hence $v \in J_0^-(y_0)$, a contradiction. In the latter case, $v \in A$ and $vIw, \forall w \in J_0^-(y_0)$, hence $v \in J_0^+(y_0)$, a contradiction. $\square$

Proposition 19
If (A_1, B_1) is the upper bound of a maximal indifference class Y, then applying the Down First Procedure starting from $(A, B) = (A_1, B_1)$ yields $I^-(A_1, B_1) = Y$. In such a case, for all $y_0 \in B_1$, $J_0^+(y_0) = \varnothing$, i.e. $Y = I_0^+(y_0) \cup J_0^-(y_0)$.

Proof
Y certainly contains $I_0^+(y_0)$ since y_0 and all elements above it in B_1 are indifferent. We first prove that $Y \subseteq I_0^+(y_0) \cup J_0^-(y_0)$. Suppose $\exists y \in Y$ and $y \notin I_0^+(y_0) \cup J_0^-(y_0)$. Since the latter subset is convex, either $y \succ w, \forall w \in I_0^+(y_0) \cup J_0^-(y_0)$ or $y \prec w, \forall w \in I_0^+(y_0) \cup J_0^-(y_0)$. In the former case, we would have $y \in A_1$, which is excluded. In the latter case, we would have $y \prec y_0$ and $yIw, \forall w \in I_0^+(y_0)$,

which implies $y \in J_0^-(y_0)$, a contradiction. Since Y is included in the indifference class $I_0^+(y_0) \cup J_0^-(y_0)$ and Y is maximal, we must have $Y = I_0^+(y_0) \cup J_0^-(y_0)$. Consequently, $J_0^+(y_0) = \varnothing$. $\square$

Note that this result applies to both non-terminal and lower-terminal maximal indifference classes Y (provided $Y \neq \mathcal{D}$) since the latter all have an upper bound.

Up First Procedure Given: a bipartition (A, B) in $\mathcal{D}$. Do the following:

- select an element x_0 in A, which is indifferent to some element y_0 in B,
- define $I_0^-(x_0) = \{x \in A : x_0 \succsim x\}$,
- define $J_0^+(x_0) = \{z \in A : z \succ x_0; zIx, \forall x \in I_0^-(x_0)\}$,
- define $J_0^-(x_0) = \{w \in B : zIw, \forall z \in J_0^+(x_0)\}$,
- let $I^+(A, B) = I_0^-(x_0) \cup J_0^+(x_0) \cup J_0^-(x_0)$.

The procedure UFP is illustrated on Figure 1.(b). Note that the same set $I^+(A, B)$ is obtained independently of the particular choice of a pair $x_0 \in A, y_0 \in B$ with $x_0 I y_0$.

We have the following two results, whose proofs are similar to those of Propositions 18 and 19 and are therefore omitted.

Proposition 20
$I^+(A, B)$ is a maximal indifference class.

Proposition 21
If (A_2, B_2) is the lower bound of a maximal indifference class Y, then applying the Up First Procedure starting from $(A, B) = (A_2, B_2)$ yields $I^+(A_2, B_2) = Y$. In such a case, for all $x_0 \in A_2$, $J_0^-(x_0) = \varnothing$, i.e. $Y = I_0^-(x_0) \cup J_0^+(x_0)$.

Note that this last result applies to both non-terminal and upper-terminal maximal indifference classes Y (provided $Y \neq \mathcal{D}$) since the latter have a lower bound.

Propositions 19 and 21 imply that by applying the Down First (or the Up First) Procedure to all bipartitions of $\mathcal{D}$ yields all non-terminal maximal indifference classes. If $\mathcal{D}$ has a lower-terminal (resp., an upper-terminal) maximal indifference class, it is also obtained by applying the Down First (resp., Up First) Procedure to all bipartitions of $\mathcal{D}$.

3.2 Coverings by maximal indifference classes and partitions into indifference classes

We construct coverings of a connected component $\mathcal{D}$ by maximal indifference classes. We call UFP (resp., DFP) the Up First (resp., Down First) Procedure described in the previous section.

Procedure for constructing a covering

1. Select a bipartition (A, B) of $\mathcal{D}$

2. Construct an initial maximal indifference class $\mathcal{C}_0 = I_0$ by applying DFP or UFP while starting from the bipartition (A, B). Call (A_1, B_1) (resp., (A_0, B_0)) the upper (resp., lower) bound of I_0 (in case these exist).

3. If there is z in $\mathcal{D}$ above I_0, apply UFP, starting from (A_1, B_1). This yields a maximal indifference class $\mathcal{C}_1 = I^+(A_1, B_1)$ the upper bound of which is (A_2, B_2) (if it exists) and the lower bound is (C_1, D_1). We have $(A_1, B_1) \succsim (C_1, D_1)$. We iterate this process, applying UFP starting from (A_k, B_k), for $k = 2, 3, \ldots$, as long as there is some $z \in \mathcal{D}$ above $I^+(A_{k-1}, B_{k-1})$. The resulting maximal indifference class $\mathcal{C}_k = I^+(A_k, B_k)$ has a lower bound (C_k, D_k) and, possibly, an upper bound denoted by (A_{k+1}, B_{k+1}). We have $(A_k, B_k) \succsim (C_k, D_k)$.

4. If there is w in $\mathcal{D}$ below $\mathcal{C}_0 = I_0$, apply DFP, starting from (A_0, B_0). This yields a maximal indifference class $\mathcal{C}_{-1} = I^-(A_0, B_0)$. We call its upper bound (C_0, D_0) and its lower bound (A_{-1}, B_{-1}) (if the latter exists). We have $(C_0, D_0) \succsim (A_0, B_0)$. We iterate this process, applying DFP starting from (A_{-l}, B_{-l}), for $l = 1, 2, \ldots$, as long as there is some $w \in \mathcal{D}$ below $I^-(A_{-l+1}, B_{-l+1})$. The resulting maximal indifference class $\mathcal{C}_{-l-1} = I^-(A_{-l}, B_{-l})$ has an upper bound (C_k, D_k) and, possibly, a lower bound denoted by (A_{-l-1}, B_{-l-1}). We have $(A_k, B_k) \succsim (C_k, D_k)$.

We shall prove below (see Proposition 24.7) that this procedure ends up, after a finite or, possibly, a countably infinite number of iterations of UFP and DFP, with a covering of the whole connected component $\mathcal{D}$. Before, we associate a family of disjoint indifference classes $(\ldots, I_{-l}, \ldots, I_0, \ldots, I_k, \ldots)$ to the family $(\ldots, \mathcal{C}_{-l}, \ldots, \mathcal{C}_0, \ldots, \mathcal{C}_k, \ldots)$ of maximal indifference classes produced by the above procedure. Note that $\mathcal{C}_0 = I_0$.

Definition 22
For all $k > 0$, we define $I_k = \mathcal{C}_k \setminus \mathcal{C}_{k-1}$. For all $l > 0$, we define $I_{-l} = \mathcal{C}_{-l} \setminus \mathcal{C}_{-l+1}$. ⌟

In the sequel, the index m will take all the values taken by $k \geq 0$ and $-l$, for $l > 0$. This set will be denoted by M. It is a subset of consecutive integers containing 0. The following result is a direct consequence of the procedure for constructing a covering described above. The first case in Proposition 23 is illustrated in Figure 2

Proposition 23
For all $m \in M$ for which there is $w, z \in \mathcal{D}$ such that $z \succ y \succ w$ for all $y \in I_m$, we have:

$$\begin{aligned} I_m &= \{y \in \mathcal{D} : (A_{m+1}, B_{m+1}) \succsim y \succsim (A_m, B_m)\} \\ &= A_m \setminus A_{m+1} \\ &= B_{m+1} \setminus B_m \\ &= B_{m+1} \cap A_m \end{aligned}$$

In case such a w exists but no such z, then

$$\begin{aligned} I_m &= \{y \in \mathcal{D} : y \succsim (A_m, B_m)\} \\ &= A_m \cap \mathcal{D} \\ &= \mathcal{D} \setminus B_m \end{aligned}$$

In case such a z exists but no such w, then

$$\begin{aligned} I_m &= \{y \in \mathcal{D} : (A_{m+1}, B_{m+1}) \succsim y\} \\ &= \mathcal{D} \setminus A_{m+1} \\ &= B_{m+1} \cap \mathcal{D} \end{aligned}$$

If there is neither such a w nor such a z, then $m = 0$ and $I_0 = \mathcal{D}$.

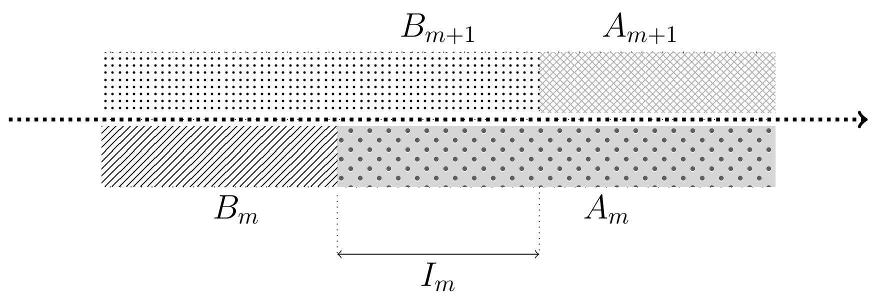


Figure 2: Illustration of the first case in Proposition 23. The dotted horizontal axis represents the elements of $\mathcal{D}$ in increasing order w.r.t. $\succ$ (from left to right).

We collect a number of properties of the sets I_m in the next proposition.

Proposition 24
The sets $I_m, m \in M$ have the following properties:

1. *They are disjoint nonempty convex subsets of $\mathcal{D}$.*

2. *Their elements are pairwise indifferent, i.e., for all $x, y \in I_m$, we have xIy.*

3. *They form an ordered partition w.r.t. $\succ$, i.e. for all $x \in I_{m-1}$ and $z \in I_m$, we have $z \succ x$.*

4. *For all $m \geq 0$ for which I_m and I_{m+1} exist, for all $w \in I_{m+1}$, there is $z \in I_m$ such that we have wPz.*

5. *For all $m < 0$ for which I_m and I_{m+1} exist, for all $v \in I_m$, there is $z \in I_{m+1}$ such that we have zPv.*

6. *For all $m \in M$ for which I_m and I_{m+2} exist, for all $w \in I_{m+2}$, for all $v \in I_m$, we have wPv.*

7. $\mathcal{D} = \cup_{m \in M} \mathcal{C}_m = \cup_{m \in M} I_m$.

PROOF

1. By construction, these subsets are disjoint and nonempty. I_0 is convex since it is a maximal indifference class (Proposition 17). For $m > 0$, if there is z above I_m in $\mathcal{D}$, I_m is determined by the bipartitions (A_m, B_m) and (A_{m+1}, B_{m+1}), i.e., $I_m = A_m \cap B_{m+1}$. By Proposition 5, it is a convex set. If there is no z above I_m, then $I_m = A_{m+1} \cap \mathcal{D}$ hence it is convex. A similar reasoning yields the result for $m < 0$.

2. By construction, I_m is a subset of the maximal indifference class $\mathcal{C}_m$.

3. For $m > 0$, $(A_m, B_m) \succ (A_{m-1}, B_{m-1})$, by construction. For all $x \in I_{m-1}, z \in I_m$, we have $z \succsim (A_m, B_m) \succsim x$. Since $x \sim z$ implies $x = z$ and since $x \neq z$, we have $z \succ x$. The case $m < 0$ is dealt with similarly.

4. For $m > 0$, $\mathcal{C}_m$ was built using UFP. It thus includes all elements indifferent to some element of I_m, which are above this element. Hence there is no $z \in I_{m+1}$ which is indifferent to all $y \in I_m$. This establishes the property. The same argument holds for $m = 0$.

5. For $m < 0$, the sets $\mathcal{C}_m$ were built using DFP. The result is established similarly as for 4.

6. By 4, for all $m \geq -1$, for all $z \in I_{m+2}$, there is $y \in I_{m+1}$ such that zPy. Since for all $v \in I_m$, we have $y \succ v$, hence zPv. By 5, for all $m \leq -2$, for all $y \in I_{m+1}$, there is $z \in I_{m+2}$ such that zPy. Since $y \succ v$ for all $v \in I_m$, we conclude that zPv.

7. Let $x_0 \in I_0$ and $y \in \mathcal{D}$, with $y \succ x$. Since $\mathcal{D}$ is a connected component of I, there is an increasing I-chain joining x_0 to y (by Proposition 10.2). Let $x_0 I x_1 I x_2 I \dots I x_i I \dots I y$ be such an I-chain, with $x_0 \prec x_1 \prec x_2 \dots \prec x_i \prec \dots y$. According to item 6, for all $x_i \in I_m$, we have that $x_{i+1} \in I_m \cup I_{m+1}$ (because all x in I_k for $k \geq m+2$ are such that xPx_i). Hence $y \in \bigcup_{m \geq 0} I_m$. The proof is similar if y is such that $x \succ y$. □

3.3 Remarks about finite coverings

This section deals with the particular case in which an I-connected component of a semiorder can be covered by a finite number of indifference classes. We investigate such coverings that use a minimal number of indifference classes. The reader may want to skip this section without inconvenience since it will not be used in the rest of the paper. The results below indicate one way of building such *minimal coverings* and relate the minimal number of indifference classes in a covering to the maximal length of a chain of P.

Lemma 25
If X contains a P-chain of length K, it cannot be covered using less than $K+1$ indifference classes.

The straightforward proof of this lemma is left to the reader.

Proposition 26
For a semiorder that can be covered by a finite number of indifference classes, the procedure UFP (resp., DFP) started from an element that is indifferent to all elements below (resp., above) it yields a minimal covering by maximal indifference classes.

PROOF
Consider the covering $\{\mathcal{C}_m, m = 0, 1, \dots, K\}$ generated by UFP started from an element indifferent to all elements below it. Let $I_0 = \mathcal{C}_0$ and $I_m = \mathcal{C}_m \setminus \mathcal{C}_{m-1}$, for $m = 1, \dots, K\}$. Pick an element x_K in I_K. Using repeatedly Proposition 24.4, we obtain successively $x_{K-1} \in I_{K-1}, \dots, x_0 \in I_0$ such that $x_m P x_{m-1}$ for all $m = K, \dots, 1$. These elements form a P-chain of length K. By lemma 25, the minimal covering has at least $K+1$ elements and therefore the covering $\{\mathcal{C}_m, m = 0, 1, \dots, K\}$ is minimal. □

Using Lemma 25 and Proposition 26, it is easy to prove the following.

Corollary 27
Let $S = (P, I)$ be a semiorder on X. The maximal length of a P-chain of X is K iff the minimal number of classes in a covering of X by indifference classes is $K + 1$.

3.4 Examples

We illustrate the decomposition in maximal indifference classes by two examples in which X is countably infinite. In the first example X is "discrete" in the sense that each element has an immediate predecessor and an immediate successor w.r.t. the trace $\succsim$. In the second example, X is "dense" in the sense that between two elements of X there is always another element of X.

Example 28
Let $X = \{0, n + \frac{i}{n+1}, n = 1, 2, \ldots \text{ and } i = 1, \ldots, n\} = \{0, 1, 1 + 1/2, 2 + 1/3, 2 + 2/3, 3 + 1/4, 3 + 1/2, 3 + 3/4, 4 + 1/5, \ldots\}$ and let $S = (P, I)$ be the semiorder on X defined by xPy if $x > y + 1$ and xIy if $|x - y| \leq 1$. No pair of distinct elements in X are equivalent, i.e., $x \sim y$ entails $x = y$. The linear order $\succsim$ is thus the same as the natural order on the elements of X. The graph of I on X is connected, hence we have to decompose a single connected component.

The set X has a least element (with respect to $\succsim$), which is 0. Let $B = \{0\}$ and $A = X \setminus \{0\}$. As initial bipartition, we select, for instance, (A, B). Let $x_0 = 1$ and $y_0 = 0$. Applying DFP, starting from $y_0 = 0$, we get $\mathcal{C}_0 = I_0 = \{0, 1\}$. Then, applying UFP as described in section 3 yields:

$$\begin{aligned}
\mathcal{C}_1 &= I_1 = \{1 + 1/2, 2 + 1/3\} \\
\mathcal{C}_2 &= I_2 = \{2 + 2/3, 3 + 1/4, 3 + 1/2\} \\
\mathcal{C}_3 &= I_3 = \{3 + 3/4, 4 + 1/5, 4 + 2/5, 4 + 3/5\} \\
&\ldots
\end{aligned}$$

i.e.,

$$\mathcal{C}_n = I_n = \{n + \frac{n}{n+1}, n + 1 + \frac{1}{n+2}, \ldots, n + 1 + \frac{n}{n+2}\}, \text{ for } n \in \mathbb{N}.$$

Alternatively, we could choose to apply UFP to the bipartition (A, B). Starting from $x_0 = 1$, this yields $\mathcal{C}'_0 = I'_0 = \{1, 1 + 1/2\}$. Applying UFP, successively yields:

$$\begin{aligned}
\mathcal{C}'_1 &= I'_1 = \{2 + 1/3, 2 + 2/3, 3 + 1/4\} \\
\mathcal{C}'_2 &= I'_2 = \{3 + 1/2, 3 + 3/4, 4 + 1/5, 4 + 2/5\} \\
\mathcal{C}'_3 &= I'_3 = \{4 + 3/5, 4 + 4/5, 5 + 1/6, 5 + 2/6, 5 + 3/6\} \\
&\ldots
\end{aligned}$$

i.e.,

$$\mathcal{C}'_n = I'_n = \{n+1+\frac{n}{n+2}, n+1+\frac{n+1}{n+2}, n+2+\frac{1}{n+3}, \dots, n+2+\frac{n+2}{n+3}\}, \text{ for } n \in \mathbb{N}\backslash\{0\}.$$

Since there is an element below $\mathcal{C}'_0$, the decomposition ends up, by using DFP, with:

$$\mathcal{C}'_{-1} = I'_{-1} = \{0\}.$$

Such decompositions depend on the choice of the initial bipartition and the initial choice of the procedure UFP or DFP.

Starting with UFP from bipartition (A'', B''), with $B'' = \{0, 1, 1 + 1/2, 2 + 1/3, 2 + 2/3\}$ and $A'' = X \setminus B''$, yields another decomposition. Indeed, we get $\mathcal{C}''_0 = I''_0 = \{3 + 1/4, 3 + 2/4, 3 + 3/4, 4 + 1/5\}$. The rest of the indifference classes fitting with I''_0 is different from the ones already obtained. ◇

Example 29

Let $X = \mathbb{Q}$, the set of rational numbers endowed with the usual semiorder $S = (P, I)$ defined by xPy if $x > y + 1$ and xIy if $|x - y| \leq 1$. There is no pair of distinct equivalent elements in X. Therefore, the linear order $\succsim$ is the natural order on $X = \mathbb{Q}$. The graph of I on X is connected. We thus have to decompose a single I-connected component.

Consider the initial bipartition (A, B) with $A = \{x \in \mathbb{Q} : x \geq 0\}$ and $B = \{x \in \mathbb{Q} : x < 0\}$. Let $x_0 = 1/2$ and $y_0 = -1/2$. Starting UFP from x_0, yields $\mathcal{C}_0 = I_0 = \{x \in \mathbb{Q} : 0 \leq x \leq 1\} = [0, 1]$. Hence $\mathcal{C}_1 = [1, 2]$ and $I_1 =]1, 2]$. For $k > 0$, we have $\mathcal{C}_k = [k, k + 1]$ and $I_k =]k, k + 1]$. On the other hand, $\mathcal{C}_{-1} = [-1, 0]$ and $I_{-1} = [-1, 0[$. For $l > 0$, we have $\mathcal{C}_{-l} = [-l, -l + 1]$ and $I_{-l} = [-l, -l + 1[$.

With the same initial partition, starting DFP from y_0 would yield $\mathcal{C}'_0 = I'_0 = [-1, 0]$. Hence $\mathcal{C}'_1 = [0, 1]$ and $I'_1 =]0, 1]$. For $k > 0$, we have $\mathcal{C}'_k = [k - 1, k]$ and $I'_k =]k - 1, k]$. On the other hand, $\mathcal{C}'_{-1} = [-2, -1]$ and $I'_{-1} = [-2, -1[$. For $l > 0$, we have $\mathcal{C}'_{-l} = [-l - 1, -l]$ and $I'_{-l} = [-l - 1, -l[$.

In $\mathbb{Q}$, there are bipartitions (A'', B'') where A'' has no least element and B'' has no greatest element (contrary to the bipartition (A, B) defined above). Indeed, let $A'' = \{x \in \mathbb{Q} : x > \sqrt{2}\}$ and $B'' = \{x \in \mathbb{Q} : x < \sqrt{2}\}$. Let $x_0 = 2$ and $y_0 = 1$. Starting UFP from x_0, yields $\mathcal{C}''_0 = I''_0 = \{x \in \mathbb{Q} : \sqrt{2} < x < 1+\sqrt{2}\} =]\sqrt{2}, 1+\sqrt{2}[$. Hence $\mathcal{C}''_1 = I''_1 =]1+\sqrt{2}, 2+\sqrt{2}[$. For $k > 0$, we have $\mathcal{C}''_k = I''_k =]k+\sqrt{2}, k+1+\sqrt{2}[$. On the other hand, $\mathcal{C}''_{-1} = I''_{-1} =]\sqrt{2} - 1, \sqrt{2}[$. For $l > 0$, we have $\mathcal{C}''_{-l} = I''_{-l} = [\sqrt{2} - l, \sqrt{2} - l + 1[$. ◇

Remark 30 (Well-ordered or finite connected components)

In Example 28, the set X is well-ordered by $\succsim$, i.e., every subset of X has a least element w.r.t. $\succsim$. In particular, X itself has a least element, which is 0. In the case $\succsim$ is a well-ordering of a connected component $\mathcal{D}$ of (X, I), the procedure producing partitions into sets of indifferent elements, described in Section 3.2, can be simplified. We can indeed start with the least (w.r.t. $\succsim$) element $y_0 \in \mathcal{D}$. Let (A, B) be the bipartition which defines this element, i.e., $B = \{y_0\}$ and $A = \mathcal{D} \setminus \{y_0\}$. Applying the DFP procedure, we see that $I_0^+(y_0) = \{y_0\}$, $J_0^-(y_0) = \varnothing$, and $J_0^+(y_0) = \{z \neq y_0 : zIy_0\}$. Therefore, I_0 is just the class of elements in $\mathcal{D}$ that are indifferent to y_0. Then we consider the least element y_1 in $\mathcal{D} \setminus I_0$ and we obtain $I_1 = y_1 \cup J_0^+(y_1)$, the class of all elements above y_1 (w.r.t. $\succsim$) and indifferent to y_1. For all $k \geq 1$, I_k is built iteratively by considering the least element y_k which does not belong to $\bigcup_{j=0}^{k-1} I_j$. The set $I_k = \{y_k\} \cup J_0^+(y_k)$ is the class of all elements above y_k (w.r.t. $\succsim$) and indifferent to y_k. Since there are no elements in $\mathcal{D}$ below I_0, the procedure exhausts the elements of $\mathcal{D}$ only by using intervals I_k, with $k \geq 0$. That is exactly what was done in the case of Example 28.

Of course, in case, the reverse order $\precsim$ is a well-ordering, i.e., if every subset of a connected component $\mathcal{D}$ has a *greatest* element w.r.t. $\succsim$, the same construction can be simplified by starting from the greatest (w.r.t. $\succsim$) element in $\mathcal{D}$ and first apply UFP. Only intervals I_{-l}, with $l \geq 0$ will then be used to cover $\mathcal{D}$.

The case in which $\mathcal{D}$ is a finite set is special since every subset of $\mathcal{D}$ has a least and a greatest element. We may thus choose to start from the least element and work upwards or the opposite.

For illustration, consider the semiorder in Example 28 restricted to a finite subset X' of X, say, the elements of X that are smaller than 4. We have $X' = \{0, 1, 1+1/2, 2+1/3, 2+2/3, 3+1/4, 3+1/2, 3+3/4\}$. Starting from $y_0 = 0$ and first applying the DFP procedure, we generate successively the subsets I_0, I_1, I_2 as in Example 28, the last subset I_3 is limited to the singleton $\{3+3/4\}$. In contrast, starting from the greatest element $3 + 3/4$ and applying UFP, would lead to the partition:

$$\begin{aligned} I'_0 &= \{3 + 1/4, 3 + 1/2, 3 + 3/4\} \\ I'_{-1} &= \{2 + 1/3, 2 + 2/3\} \\ I'_{-2} &= \{1, 1 + 1/2\} \\ I'_{-3} &= \{0\}. \end{aligned}$$

•

4 Construction of a unit representation for a denumerable I-connected semiorder

We consider an I-connected component $\mathcal{D}$ of the semiorder $S = (P, I)$ on the set X. We assume that $\mathcal{D}$ is a denumerable set. In this case we show that one can build a unit representation of the semiorder induced on $\mathcal{D}$ without making any additional assumption.

Let $\mathcal{D}$ be a denumerable connected component of (X, I). Choose a bipartition (A_0, B_0) in $\mathcal{D}$ and perform the decomposition in subsets $I_k, k \geq 0$ and $I_{-l}, l > 0$, starting either with DFP or UFP, as described in Section 3. Let M denote the set of consecutive integers, containing 0, for which I_m exists. We have:

$$\mathcal{D} = \bigcup_{m \in M} I_m = (\bigcup_{k \geq 0} I_k) \cup (\bigcup_{l > 0} I_{-l}).$$

The main idea in order to build a unit representation is to create an image of all elements of $\mathcal{D}$ in I_0. These "images", that we call *ghosts* or *dummies*, are sequentially inserted at appropriate positions in the set I_0 augmented with the previously inserted ghosts. In this way, the initial linear order $\succsim$ on I_0 is extended to a complete preorder $\succsim_\varphi$ on I_0 and the set of inserted ghosts at each stage of the process (this set will be denoted by $\tilde{I}_0$ in the sequel). Once this process has come to an end, i.e., when all elements of $\mathcal{D}$ have been associated a representative in the set I_0, at appropriate positions determining the extended preorder, a numerical representation of this ordered set is selected and then "lifted" to yield a unit representation of the semiorder on $\mathcal{D}$. The whole procedure is illustrated by examples in Section 4.3.

4.1 Creating ghosts

We start with the pair of convex subsets I_0, I_1. We create and insert a dummy element in I_0 for each $y \in I_1$. This dummy element is denoted $\varphi_1(y)$ and is referred to as the *ghost* of y. The ghost of y is inserted between the elements of I_0 to which it is preferred and those to which it is indifferent. Since we proceed sequentially, we also have to take the ghosts already inserted into account.

Let us be more precise. Since X is a denumerable set, so is I_1. We order the elements of I_1 in a sequence $\{y_t, t \in T_1\}$, with T_1 a set of consecutive integers starting with 1. To create the ghost $\varphi_1(y_1)$, we define an *ordered bipartition* (A_1^0, B_1^0) in I_0, with

$$\begin{aligned} A_1^0 &= \{x \in I_0 : y_1 I x\}, \\ B_1^0 &= \{x \in I_0 : y_1 P x\}. \end{aligned}$$

By Proposition 24.4, B_1^0 is not empty, while it may happen that A_1^0 be empty.

A dummy element $\varphi_1(y_1)$ is created and inserted in I_0 between A_1^0 and B_1^0 (after all elements of B_1^0 if A_1^0 is empty). We extend the linear order $\succsim$ on I_0 into a complete preorder[7] $\succsim_\varphi$ by setting $a \succ_\varphi \varphi_1(y_1) \succ_\varphi b$ for all $a \in A_1^0$ and $b \in B_1^0$. Since $\succsim_\varphi$ extends $\succsim$, we also have $a \succsim_\varphi b$ for all $a, b \in I_0$ with $a \succsim b$.

Assuming that the ghosts $\varphi_1(y_s)$ of y_s, for $s = 1, \ldots, t-1$, have been created, we now insert the ghost $\varphi_1(y_t)$ of y_t in the set $I_0 \cup \{\varphi_1(y_1), \ldots, \varphi_1(y_{t-1})\}$ ordered by the relation $\succsim_\varphi$ extending $\succsim$. We define the *bipartition* (A_t^0, B_t^0) in this set, letting $A_t^0 = \{x \in I_0 : y_t I x\}$ and $B_t^0 = \{x \in I_0 : y_t P x\}$. By Lemma 32 (see below), for all $a \in C_t^0 = A_t^0 \cup \{\varphi_1(y_s) : y_s \succ y_t, s = 1, \ldots, t-1\}$ and all $b \in D_t^0 = B_t^0 \cup \{\varphi_1(y_s) : y_s \prec y_t, s = 1, \ldots, t-1\}$, we have $a \succ_\varphi b$.

A dummy element $\varphi_1(y_t)$ is inserted between C_t^0 and D_t^0 (or above all elements in D_t^0 in case C_t^0 is empty). The complete preorder $\succsim_\varphi$ is extended by setting $a \succ_\varphi \varphi_1(y_t) \succ_\varphi b$ for all $a \in C_t^0$ and $b \in D_t^0$.

Proceeding sequentially in this way, we finally obtain the set $\tilde{I}_0^1 = I_0 \cup \varphi_1(I_1)$ which is ordered by the (extended) complete preorder $\succsim_\varphi$.

Remark 31
Note that a unit representation of the restriction of the semiorder (P, I) to $I_0 \cup I_1$ can be obtained in the following way:

1. select any representation f of the order $\succsim_\varphi$ on $\tilde{I}_0^1 = I_0 \cup \varphi_1(I_1)$ in the $]0, 1[$ real interval,

2. set
$$u(x) = \begin{cases} f(x) & \text{if } x \in I_0, \\ f(\varphi_1(x)) + 1 & \text{if } x \in I_1. \end{cases}$$

u is clearly a unit representation of the semiorder restricted to $I_0 \cup I_1$. •

Dummies for I_k The generic step for $k > 0$ is as follows. Assume that I_k exists and the ghosts $\varphi_j(I_j)$ of the elements of I_j, for $1 \le j \le k-1$ have previously been inserted. We denote by $\tilde{I_0}^{k-1}$ the current extension of I_0, which contains, in particular, $\varphi_{k-1}(I_{k-1})$. We also assume that the relation $\succsim_\varphi$ has been extended into a complete preorder on $\tilde{I_0}^{k-1}$. We number the elements in I_k as $\{z_t, t \in T_k\}$, with T_k a set of consecutive integers starting with 1. Consider $z_1 \in I_k$. It determines a bipartition (A_1^{k-1}, B_1^{k-1}) in I_{k-1}, with $A_1^{k-1} = \{x \in I_{k-1} : z_1 I x\}$, a possibly empty set, and $B_1^{k-1} = \{x \in I_{k-1} : z_1 P x\}$, a nonempty set (by Proposition 24.4). We insert a ghost $\varphi_k(z_1)$ in $\tilde{I}_0^{k-1}$ between $C_1^{k-1} = \varphi_{k-1}(A_1^{k-1})$ and $D_1^{k-1} = \varphi_{k-1}(B_1^{k-1})$ (in case A_1^{k-1} is empty, we insert $\varphi_k(z_1)$ above all the elements of B_1^{k-1}). There may exist a certain degree of arbitrariness in the precise position of $\varphi_k(z_1)$ w.r.t. the elements of $\tilde{I}_0^{k-1}$ (which was not the case in the initial step). We just choose

[7] In the sequel, we will have to consider that some pairs of elements can be indifferent w.r.t. $\succsim_\varphi$ and, therefore, we may not assume that the extension of $\succ$ is a linear order.

an insertion position that satisfies the constraint w.r.t. the ghosts of I_{k-1} and we extend the preorder $\succsim_\varphi$ accordingly while respecting $a \succ_\varphi \varphi_k(z_1) \succ_\varphi b$ for all $a \in C_1^{k-1}$ and $b \in D_1^{k-1}$.

Proceeding sequentially, assume that $\varphi_k(z_s)$, for $s = 1, \ldots, t-1$, have been created and inserted and the preorder $\succsim_\varphi$ extended to $\tilde{I}_0^{k-1} \cup \{\varphi_k(z_1), \ldots, \varphi_k(z_{t-1})\}$. Consider z_t which determines a bipartition (A_t^{k-1}, B_t^{k-1}) in I_{k-1}, with $A_t^{k-1} = \{x \in I_{k-1} : z_t I x\}$, a possibly empty set, and $B_t^{k-1} = \{x \in I_{k-1} : z_t P x\}$, a nonempty set. By Lemma 32 (see below), for all $a \in C_t^{k-1} = \varphi_{k-1}(A_t^{k-1}) \cup \{\varphi_k(z_s), z_s \succ z_t, s = 1, \ldots, t-1\}$ and all $b \in D_t^{k-1} = \varphi_{k-1}(B_t^{k-1}) \cup \{\varphi_k(z_s), z_s \prec z_t, s = 1, \ldots, t-1\}$, we have $a \succ_\varphi b$.

A ghost $\varphi_k(z_t)$ is inserted in $\tilde{I}_0^{k-1} \cup \{\varphi_k(z_1), \ldots, \varphi_k(z_{t-1})\}$ between C_t^{k-1} and D_t^{k-1} (in case A_t^{k-1} is empty, we insert $\varphi_k(z_t)$ above all the elements of D_t^{k-1}). The precise position of $\varphi_k(z_t)$ w.r.t. the other elements of $\tilde{I}_0^{k-1}$ that lie between C_t^{k-1} and D_t^{k-1} is determined in an arbitrary manner. The complete preorder $\succsim_\varphi$ is extended to $\tilde{I}_0^{k-1} \cup \{\varphi_k(z_1), \ldots, \varphi_k(z_t)\}$ accordingly and satisfies in particular $a \succ_\varphi \varphi_1(z_t) \succ_\varphi b$ for all $a \in C_t^{k-1}$ and $b \in D_t^{k-1}$.

Finally, we obtain the set $\tilde{I}_0^k = \tilde{I}_0^{k-1} \cup \varphi_k(I_k)$, which is ordered by the (extended) complete preorder relation $\succsim_\varphi$.

The following lemma was used repeatedly to prove that it was always possible to insert $\varphi_k(z_t)$ in between C_t^{k-1} and D_t^{k-1}, for $k \geq 1$ and $t \in T_k$.

Lemma 32
Assume that for all j, with $0 < j < k$, φ_j is an injective function from I_j into $\tilde{I}_0^{k-1}$ and assume further that the order $\succ_\varphi$ on the ghosts $\varphi_k(z_s)$, for $s = 1, \ldots, t-1$, reproduces the order $\succ$ on $\{z_s, s = 1, \ldots, t-1\} \subseteq I_k$.
For $k = 1$, we have $a \succ_\varphi b$ for all $a \in C_t^0 = A_t^0 \cup \{\varphi_1(z_s), z_s \succ z_t, s = 1, \ldots, t-1\}$ and all $b \in D_t^0 = B_t^0 \cup \{\varphi_1(z_s), z_s \prec z_t, s = 1, \ldots, t-1\}$.
For $k \geq 2$, we have $a \succ_\varphi b$ for all $a \in C_t^{k-1} = \varphi_{k-1}(A_t^{k-1}) \cup \{\varphi_k(z_s), z_s \succ z_t, s = 1, \ldots, t-1\}$ and all $b \in D_t^{k-1} = \varphi_{k-1}(B_t^{k-1}) \cup \{\varphi_k(z_s), z_s \prec z_t, s = 1, \ldots, t-1\}$.

Proof
We prove the lemma for $k \geq 2$. By hypothesis, we clearly have $a \succ_\varphi b$ for all $a \in \varphi_{k-1}(A_t^{k-1})$ and all $b \in \varphi_{k-1}(B_t^{k-1})$, since $x \succ y$ for all $x \in A_t^{k-1}$ and all $y \in B_t^{k-1}$. Consider now z_s, with $1 \leq s \leq t-1$, such that $z_s \succ z_t$. We have $\varphi_k(z_s) \succ_\varphi \varphi_{k-1}(b)$ for all $b \in B_s^{k-1} = \{x \in I_{k-1} : z_s P x\} \supseteq B_t^{k-1}$, hence $\varphi_k(z_s) \succ_\varphi \varphi_{k-1}(b)$ for all $b \in B_t^{k-1}$. One proves, in a similar way, that for all z_s such that $z_s \prec z_t$, we have $\varphi_k(z_s) \prec_\varphi \varphi_{k-1}(a)$ for all $a \in A_t^{k-1}$. Finally, the insertion procedure guarantees that, for all $z_s, z_{s'}$, with $1 \leq s, s' \leq t-1$, such that $z_s \succ z_t$ and $z_{s'} \prec z_t$, we have $\varphi_k(z_s) \succ_\varphi \varphi_k(z_{s'})$. The proof of the lemma for $k = 1$ is similar. □

Let us now turn to I_{-l} for $l > 0$. We start with $l = 1$.

Dummies for I_{-1} We denote by $\tilde{I}_0$ the current extension of I_0 ordered by the complete preorder $\succsim_\varphi$. We number the elements of I_{-1} as $\{z_t, t \in T_{-1}\}$, with T_{-1} a set of consecutive integers starting with 1. Consider $z_1 \in I_{-1}$. It determines a bipartition (A_1^{-0}, B_1^{-0}) in I_0, with $A_1^{-0} = \{x \in I_0 : xPz_1\}$, a nonempty set (by Proposition 24.5), and $B_1^{-0} = \{x \in I_0 : xIz_1\}$, a possibly empty set. We insert a ghost $\varphi_{-1}(z_1)$ in $\tilde{I}_0$, between A_1^{-0} and B_1^{-0} (in case B_1^{-0} is empty, we insert $\varphi_{-1}(z_1)$ below all elements of A_1^{-0}). There is generally some arbitrariness in the positioning of $\varphi_{-1}(z_1)$ w.r.t. to the elements of $\tilde{I}_0$, namely, the ghosts previously inserted between A_1^{-0} and B_1^{-0}. We just select any insertion for $\varphi_{-1}(z_1)$ that satisfies the constraint and we extend $\succsim_\varphi$ accordingly. In particular this extension satisfies $a \succ_\varphi \varphi_{-1}(z_1) \succ_\varphi b$ for all $a \in A_1^{-0}$ and $b \in B_1^{-0}$.

We then proceed sequentially, assuming that the ghosts $\varphi_{-1}(z_s)$, for $s = 1, \dots, t-1$, have been created and inserted and the complete preorder $\succsim_\varphi$ extended to $\tilde{I}_0 \cup \{\varphi_{-1}(z_1), \dots, \varphi_{-1}(z_{t-1})\}$. Consider z_t, which determines a bipartition (A_t^{-0}, B_t^{-0}) in I_0, with $A_t^{-0} = \{x \in I_0 : xPz_t\}$, a nonempty set, and $B_t^{-0} = \{x \in I_0 : xIz_t\}$, a possibly empty set. By Lemma 33 (see below), for all $a \in C_t^{-0} = A_t^{-0} \cup \{\varphi_{-1}(z_s), z_s \succ z_t, s = 1, \dots, t-1\}$ and all $b \in D_t^{-0} = B_t^{-0} \cup \{\varphi_{-1}(z_s), z_s \prec z_t, s = 1, \dots, t-1\}$, we have $a \succ_\varphi b$.

A dummy element $\varphi_{-1}(z_t)$ is inserted in $\tilde{I}_0 \cup \{\varphi_{-1}(z_1), \dots, \varphi_{-1}(z_{t-1})\}$ between C_t^{-0} and D_t^{-0} (in case D_t^{-0} is empty, $\varphi_{-1}(z_t)$ is inserted below all elements of C_t^{-0}). The precise position of $\varphi_{-1}(z_t)$ w.r.t. the other elements of $\tilde{I}_0$ that lie between C_t^{-0} and D_t^{-0} is determined in an arbitrary manner and the complete preorder $\succsim_\varphi$ is extended accordingly while satisfying $a \succ_\varphi \varphi_{-1}(z_t) \succ_\varphi b$ for all $a \in C_t^{-0}$ and $b \in D_t^{-0}$.

Finally, we obtain the set $\tilde{I}_0^{-1} = \tilde{I}_0 \cup \varphi_{-1}(I_{-1})$, which is ordered by the (extended) complete preorder relation $\succsim_\varphi$.

Dummies for I_{-l} The generic insertion step for subsets I_{-l}, $l = 1, \dots$ is as follows. Let $\tilde{I}_0^{-l+1}$ denote the current extension of I_0. We assume that the ghosts $\varphi_{-j}(z)$ of all $z \in I_{-j}$ for $j = 1, \dots, l-1$ have been inserted in $\tilde{I}_0$ and the complete preorder $\succsim_\varphi$ has been extended to all $\tilde{I}_0^{-l+1}$. Moreover, we assume that the $\succ_\varphi$ order of the ghosts of the elements of I_{-j} reproduces the $\succ$ order in their original subset.

We number the elements in I_{-l} as $\{z_t, t \in T_{-l}\}$, with T_{-l} a set of consecutive integers starting with 1. Consider $z_1 \in I_{-l}$. It determines a bipartition (A_1^{-l+1}, B_1^{-l+1}) in I_{-l+1}, with $A_1^{-l+1} = \{x \in I_{-l+1} : xPz_1\}$, a nonempty set, and $B_1^{-l+1} = \{x \in P_{-l+1} : xPz_1\}$, a possibly empty set. We insert a ghost $\varphi_{-l}(z_1)$ in $\tilde{I}_0^{-l+1}$ between $C_1^{-l+1} = \varphi_{-l+1}(A_1^{-l+1})$ and $D_1^{-l+1} = \varphi_{-l+1}(B_1^{-l+1})$ (in case B_1^{-l+1} is empty, we insert $\tilde{I}_0^{-l+1}$ below all elements of C_1^{-l+1}). The precise position of $\varphi_{-l+1}(z_t)$ w.r.t. the other elements of $\tilde{I}_0^{-l+1}$ that lie between C_1^{-l+1} and D_1^{-l+1} is chosen arbitrarily. The $\succsim_\varphi$ preorder is extended accordingly while ensuring that $a \succ_\varphi \varphi_{-l}(z_1) \succ_\varphi b$ for all $a \in C_1^{-l+1}$ and $b \in D_1^{-l+1}$.

Assuming that $\varphi_{-l}(z_s)$, for $s = 1, \ldots, t-1$, have been created and inserted and the complete preorder $\succsim_\varphi$ extended to $\tilde{I}_0^{-l+1} \cup \{\varphi_{-l}(z_1), \ldots, \varphi_{-l}(z_{t-1})\}$, we consider z_t. This element determines a bipartition (A_t^{-l+1}, B_t^{-l+1}) in I_{-l+1}, with $A_t^{-l+1} = \{x \in I_{-l+1} : xPz_t\}$, a nonempty set, and $B_t^{-l+1} = \{x \in I_{-l+1} : xIz_t\}$, a possibly empty set. By Lemma 33, for all $a \in C_t^{-l+1} = \varphi_{-l+1}(A_t^{-l+1}) \cup \{\varphi_{-l}(z_s), z_s \succ z_t, s = 1, \ldots, t-1\}$ and all $b \in D_t^{-l+1} = \varphi_{-l+1}(B_t^{-l+1}) \cup \{\varphi_{-l}(z_s), z_s \prec z_t, s = 1, \ldots, t-1\}$, we have $a \succ_\varphi b$.

A ghost $\varphi_{-l}(z_t)$ is inserted in $\tilde{I}_0^{-l+1} \cup \{\varphi_{-l}(z_1), \ldots, \varphi_{-l}(z_{t-1})\}$ between C_t^{-l+1} and D_t^{-l+1} (or below all elements of C_t^{-l+1} in case B_t^{-l+1} is empty. The precise position of $\varphi_{-l}(z_t)$ w.r.t. the other elements of $\tilde{I}_0^{-l+1} \cup \{\varphi_{-l}(z_s), s = 1, \ldots, t-1\}$ that lie between C_t^{-l+1} and D_t^{-l+1} is determined in arbitrary manner. The complete preorder $\succsim_\varphi$ is extended accordingly while ensuring that $a \succ_\varphi \varphi_{-l}(z_t) \succ_\varphi b$ for all $a \in C_t^{-l+1}$ and $b \in D_t^{-l+1}$.

Finally, we obtain the set $\tilde{I}_0^{-l} = \tilde{I}_0^{-l+1} \cup \varphi_{-l}(I_{-l})$, which is ordered by the (extended) complete preorder relation $\succsim_\varphi$.

The following lemma proves that the insertion of $\varphi_{-l}(z_t)$ is always possible.

Lemma 33
Assume that for all j, with $1 \geq j < l$, φ_{-j} is an injective function from I_{-j} into $\tilde{I}_0^{-j}$ and assume further that the $\succ_\varphi$ order on the ghosts $\varphi_{-l}(z_s)$, for $s = 1, \ldots, t-1$ reproduces the $\succ_\varphi$ order on $\{z_s, s = 1, \ldots, t-1\} \subseteq I_{-l}$.
For $l = 1$, we have $a \succ_\varphi b$ for all $a \in C_t^{-0} = A_t^{-0} \cup \{\varphi_{-1}(z_s), z_s \succ z_t, s = 1, \ldots, t-1\}$ and all $b \in D_t^{-0} = B_t^{-0} \cup \{\varphi_{-1}(z_s), z_s \prec z_t, s = 1, \ldots, t-1\}$.
For $l \geq 2$, we have $a \succ_\varphi b$ for all $a \in C_t^{-l+1} = \varphi_{-l+1}(A_t^{-l+1}) \cup \{\varphi_{-l}(z_s), z_s \succ z_t, s = 1, \ldots, t-1\}$ and all $b \in D_t^{-l+1} = \varphi_{-l+1}(B_t^{-l+1}) \cup \{\varphi_{-l}(z_s), z_s \prec z_t, s = 1, \ldots, t-1\}$.

The proof of this lemma is similar to that of Lemma 32. It is left to the reader.

At the end of this construction process, involving at most a countably infinite number of steps (each of them involving at most a countably infinite number of insertions), we obtain the set $\tilde{I}_0 = I_0 \cup (\bigcup_{k \geq 0} \varphi_k(I_k)) \cup (\bigcup_{l>0} \varphi_{-l}(I_{-l}) = I_0 \cup (\bigcup_{m \in M} \varphi_m(I_m))$, ordered by $\succsim_\varphi$, which is an extension of the complete preorder $\succsim$ on I_0. The restriction of $\succsim_\varphi$ to $\varphi_m(I_m)$, for $m \in M, m \neq 0$, is an isomorphic image of the $\succsim$ order on I_k.

Remark 34
In the construction described above, we chose to start with building a representation for all I_k, for $k > 0$, in I_0 and then we turned to mapping the subsets I_{-l}, for $l < 0$, into I_0. One could have instead alternated the insertion of I_k and I_{-k}, for all $k > 0$, starting from $k = 1$ and working consecutively. Other schemes can be

considered. The only restriction is that, before inserting I_k (resp., I_{-k}), for $k > 0$, one must make sure that all elements of I_{k-1} (resp., I_{-k+1}) have been inserted. •

Remark 35
In the construction described above, it is not excluded that ghosts of elements from I_k (resp., I_{-l}) are set equivalent, w.r.t. $\succsim_\varphi$, to ghosts of elements from I_m, for $0 \leq m \leq k-2$ (resp., $-l+2 \leq m \leq 0$). That is why we allow the equivalence classes of $\succsim_\varphi$ not to be restricted to pairs (x, x). However, in this section, we do not allow ghosts of elements from I_k (resp., I_{-l}) to be set equivalent to ghosts of elements from I_{k-1} (resp., I_{-l+1}). In Section 4.4, we investigate the precise conditions under which this can be allowed. •

4.2 Construction of a representation

Since $\tilde{I}_0$ is at most countable, there exists a numerical representation of the complete preorder $\succsim_\varphi$ on this set. One way of building a unit representation of the semiorder (P, I) is to select a numerical representation f of $\succsim_\varphi$ in the $]0,1[$ rational or real interval and to define the function u on $\mathcal{D}$ as follows:

$$u(x) = f(\varphi_m(x)) + m \text{ for all } x \in I_m, \tag{5}$$

for all $m \in M$, and interpreting φ_0 as the identity function. We shall refer to equation (5) as to the *lifting equation*.

Enforcing the representation f to range in the $]0,1[$ interval is however a restrictive requirement. Instead we may choose for f any numerical representation of the order $\succsim_\varphi$ on $\tilde{I}_0$ satisfying the following condition: for all $m \in M$,

$$\sup\{|f(\varphi_m(x)) - f(\varphi_m(y))|, x, y \in I_m\} \leq 1. \tag{6}$$

Obviously, this condition is satisfied if the range of f is the $]0,1[$ interval. We shall henceforth refer to condition (6) as to the *unit threshold constraint*.

The following proposition is the first main result of this paper. *It shows that any I-connected component of a semiorder defined on a denumerable set has a unit representation.*

Proposition 36
If f is a numerical representation of $\succsim_\varphi$ on $\tilde{I}_0$ satisfying the unit threshold constraint (6), *then the function u defined by the lifting equation* (5) *is a unit representation of the semiorder $S = (P, I)$ restricted to $\mathcal{D}$, i.e., for all $x, y \in \mathcal{D}$,*

$$xPy \Leftrightarrow u(x) > u(y) + 1,$$
$$xIy \Leftrightarrow -1 \leq u(x) - u(y) \leq 1.$$

Proof

Let x, y be such that xPy. If y belongs to I_k ($k \in M$), we have that x belongs to I_m for $m \geq k+1$ (by Proposition 24). If $x \in I_{k+1}$, we have $u(x) - u(y) = f(\varphi_{k+1}(x)) + k + 1 - f(\varphi_k(y)) - k > 1$, which entails $f(\varphi_{k+1}(x)) > f(\varphi_k(y))$ since $\varphi_{k+1}(x) \succ \varphi_k(y)$ by construction.

If $x \in I_m$, for $m \geq k+2$, we have $u(x) - u(y) = f(\varphi_m(x)) + m - f(\varphi_k(y)) - k > 1$ since $m - k \geq 2$ and $f(\varphi_m(x)) - f(\varphi_k(y)) > -1$. The latter inequality is proved to hold as follows. It is trivially true in case $f(\varphi_m(x)) - f(\varphi_k(y)) \geq 0$. We thus consider the opposite case in which $f(\varphi_m(x)) < f(\varphi_k(y))$. We distinguish three sub-cases.

1. Case $k \geq 0$. Using Proposition 24.4, we know there exists a P-chain $xPz_1P z_2P \ldots Pz_{m-k}$, with $z_i \in I_{m-i}$ for $i = 1, \ldots, m-k$. By construction of the ghosts, we have $f(\varphi_m(x)) > f(\varphi_{m-1}(z_1)) > \ldots > f(\varphi_{k+1}(z_{m-k+1})) > f(\varphi_k(z_{m-k}))$. Therefore $f(\varphi_k(z_{m-k})) < f(\varphi_m(x)) < f(\varphi_k(y))$. Using (6) yields $f(\varphi_m(x)) - f(\varphi_k(y)) > -1$.

2. Case $m \leq 0$. Using Proposition 24.5, we know there exists a P-chain $w_{m-k}Pw_{m-k+1}P \ldots Pw_1Py$, with $w_i \in I_{k+i}$ for $i = 1, \ldots, m-k$. By construction of the ghosts, we have $f(\varphi_m(w_{m-k})) > f(\varphi_{m-1}(w_{m-k+1})) > \ldots > f(\varphi_{k+1}(w_1)) > f(\varphi_k(y))$. Therefore $f(\varphi_m(w_{m-k}) > f(\varphi_k(y)) > f(\varphi_m(x))$. Using (6) yields $f(\varphi_m(x)) - f(\varphi_k(y)) > -1$.

3. Case $m > 0$ and $k < 0$. Using Proposition 24.4, we know there exists a P-chain $xPz_1P \ldots Pz_m$, with $z_i \in I_{m-i}$ for $i = 1, \ldots, m$. By construction of the ghosts, we have $f(\varphi_m(x)) > f(\varphi_0(z_m))$. Using Proposition 24.5, we know there exists a P-chain $w_{-k}Pw_{-k+1}P \ldots Pw_1Py$, with $w_i \in I_{k+i}$ for $i = 1, \ldots, -k$. By construction of the ghosts, we have $f(\varphi_0(w_{-k})) > f(\varphi_{-1}(w_{-k+1})) > \ldots > f(\varphi_{k+1}(w_1)) > f(\varphi_k(y))$. We see that $f(\varphi_0(z_m) \geq f(\varphi_0(w_{-k}))$ is not compatible with $f(\varphi_m(x)) < f(\varphi_k(y))$. Therefore, we have $0 < f(\varphi_k(y)) - f(\varphi_m(x)) < f(\varphi_0(w_{-k})) - f(\varphi_0(z_m) \leq 1$.

Hence, if xPy, then $u(x) > u(y) + 1$.

Consider now a pair $x, y \in \mathcal{D}$ such that xIy. We assume w.l.o.g. that $x \succ y$ and $y \in I_k$ ($k \in M$). By Proposition 24.2, 24.3 and 24.5, we know that $x \in I_k$ or $x \in I_{k+1}$. In the former case, $0 < u(x) - u(y) = f(\varphi_k(x)) + k - f(\varphi_k(y)) - k < 1$, due to condition (6) on f. In the latter case, we have $0 < u(x) - u(y) = f(\varphi_{k+1}(x)) + k + 1 - f(\varphi_k(y)) - k < 1$ because the difference $f(\varphi_{k+1}(x)) - f(\varphi_k(y))$ is negative. To establish this, we consider the following two possible cases:

- $k \geq 0$. By construction of the ghosts and the extension $\succsim_\varphi$ of $\succsim$ for $k \geq 0$, we have $\varphi_k(a) \succ_\varphi \varphi_{k+1}(x) \succ_\varphi \varphi_k(b)$ for all $a \in A = \{z \in I_k : xIz\}$ and all $b \in B = \{z \in I_k : xPz\}$. Since y belongs to A and f represents $\succsim_\varphi$, we have that $f(\varphi_k(y)) > f(\varphi_{k+1}(x))$.

- $k = -l < 0$. By construction of the ghosts and the extension $\succsim_\varphi$ of $\succsim$ for $k = -l < 0$, we have $\varphi_{-l+1}(a) \succ_\varphi \varphi_{-l}(y) \succ_\varphi \varphi_{-l+1}(b)$ for all $a \in A = \{z \in I_{-l+1} : zPy\}$ and all $b \in B = \{z \in I_{-l+1} : zIy\}$. Since x belongs to B and f represents $\succsim_\varphi$, we have that $f(\varphi_{-l}(y)) = f(\varphi_k(y)) > f(\varphi_{-l+1}(x)) = f(\varphi_{k+1}(x))$.

Hence, if xIy, then $|u(x) - u(y)| \leq 1$. □

Remark 37
Note that the hypothesis that X is a denumerable set is used twice in the above construction. First, for constructing a mapping of the whole set $\mathcal{D}$ in the subset I_0, while respecting the order of the elements in their respective original subsets I_m. Second, the existence of a numerical representation of the $\succsim_\varphi$ preorder on $\tilde{I}_0$ rests on this hypothesis. •

Remark 38
Proposition 36 shows that a semiorder on an I-connected set always has a unit representation. It offers an alternative proof of Manders (1981, Prop. 8, p. 237).

Observe also that, if we enforce f to range in the $]0, 1[$ interval, then, in $\tilde{I}_0$, it is impossible that the difference, in terms of f, between two objects is exactly 1. Hence, Proposition 36 also shows that, on an I-connected component, it is always possible to build a representation that is at the same time strict (i.e., it fulfills (1)) and nonstrict (i.e., it fulfills (2)). We will see below that this observation is not limited to the case of an I-connected component. It holds true in the general case of a denumerable semiorder which admits a numerical representation. This also extends Beja and Gilboa (1992, Th. 3.8, p. 436) for the case of an I-connected component of a semiorder. It is not only possible to obtain a strict representation or a nonstrict one but it is also possible to build a representation that is at the same time strict and nonstrict. We will see below (see Remark 63) that this holds true for all semiorders on denumerable sets admitting a numerical representation.

Finally observe that, since X is countable, it is not restrictive to enforce f to range in the $]0, 1[\cap \mathbb{Q}$ interval. This shows that a semiorder on an I-connected component has a unit real representation iff it has a unit *rational* representation. For a semiorder that is I-connected, this gives an alternative proof of Manders (1981, Prop. 7, p. 236). Using the analysis in Section 5 below, it is easy to extend the analysis to a class of semiorders that are not I-connected (see Remark 63). •

4.3 Illustrating the construction of a representation

We build a representation for the I-connected semiorders in Examples 28 and 29.

Example 28 [cont'd]

We consider the decomposition of the semiorder in Example 28 in subsets $I_n, n \in \mathbb{N}$, obtained by starting from bipartition (A, B), with $B = \{0\}$ and $A = X \setminus B$.

- $I_0 = \{0, 1\}$.
- $I_1 = \{1 + 1/2, 2 + 1/3\}$.
  The insertion constraints are: $0 < \varphi_1(1+1/2) < 1 < \varphi_1(2+1/3)$. We insert the elements $\varphi_1(I_1)$ separating them from these in I_0 by a value of $1/2$, i.e.,

  $$\varphi_1(1 + 1/2) = 1/2, \quad \varphi_1(2 + 1/3) = 3/2.$$

  The complete preorder $\succsim_\varphi$ is the natural preorder on the numbers involved in the construction and it will be so in the remaining steps.
- $I_2 = \{2 + 2/3, 3 + 1/4, 3 + 1/2\}$.
  The insertion constraints are: $\varphi_1(1 + 1/2) < \varphi_2(2 + 2/3) < \varphi_2(3 + 1/4) < \varphi_1(2 + 1/3) < \varphi_2(3 + 1/2)$. We insert the elements $\varphi_2(I_2)$ separating them from these in $\varphi_1(I_1) \cup I_0$ by a value of $1/4$; we set, e.g.,

  $$\varphi_2(2 + 2/3) = 3/4, \quad \varphi_2(3 + 1/4) = 5/4, \quad \varphi_2(3 + 1/2) = 7/4.$$

  Note that $\varphi_2(2 + 2/3)$ and $\varphi_2(3 + 1/4)$ could alternatively have been both positioned in the interval $]1/2, 1[$ or both in the interval $]1, 3/2[$.
- $I_3 = \{3 + 3/4, 4 + 1/5, 4 + 2/5, 4 + 3/5\}$.
  The insertion constraints are: $\varphi_2(2 + 2/3) < \varphi_3(3 + 3/4) < \varphi_3(4 + 1/5) < \varphi_2(3 + 1/4) < \varphi_3(4 + 2/5) < \varphi_2(3 + 1/2) < \varphi_3(4 + 3/5)$. We insert the elements $\varphi_3(I_3)$ separating them from these in $\varphi_2(I_2) \cup \varphi_1(I_1) \cup I_0$ by a value of $1/8$, we set, e.g.,

  $$\varphi_3(3{+}3/4) = 7/8,\ \varphi_3(4{+}1/5) = 9/8,\ \varphi_3(4{+}2/5) = 11/8,\ \varphi_3(4{+}3/5) = 15/8.$$

  A number of arbitrary choices have been made in assigning these values. The above ghost insertion choices are represented on Figure 3.

We now indicate how we may define $\varphi_{k+1}(I_{k+1})$ knowing $\varphi_k(I_k)$, for $k \geq 1$. We have:

$$I_k = \{k + \frac{k}{k+1}, k + 1 + \frac{1}{k+2}, \ldots, k + 1 + \frac{k}{k+2}\}, \text{ for } k \geq 1.$$

I_{k+1} has one more element than I_k. One easily verifies that the insertion constraints are : $\varphi_k(k + \frac{k}{k+1}) < \varphi_{k+1}(k + 1 + \frac{k+1}{k+2}) < \varphi_{k+1}(k + 2 + \frac{1}{k+3}) < \varphi_k(k + 1 + \frac{1}{k+2}) <$

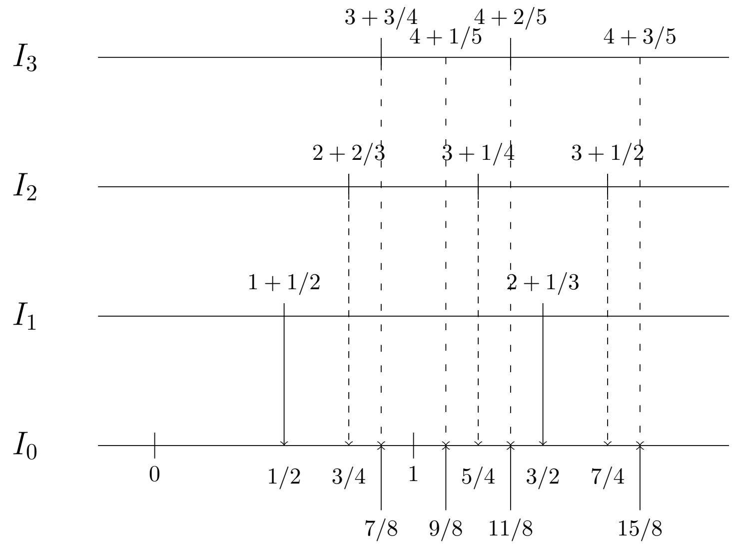


Figure 3: Ghosts insertion illustrated on Example 28

$\varphi_{k+1}(k+2+\frac{2}{k+3}) < \ldots < \varphi_k(k+1+\frac{t}{k+2}) < \varphi_{k+1}(k+2+\frac{t+1}{k+3}) < \ldots < \varphi_k(k+1+\frac{k}{k+2}) < \varphi_{k+1}(k+2+\frac{k+1}{k+3})$, with $2 \leq t \leq k-1$. In words, the ghosts of the two smaller elements in I_{k+1} are positioned between the ghosts of the least and the second least elements in I_k. For the rest, the ghosts positions of both subsets alternate.

We may recursively assign the following values to φ_{k+1}:

$$\varphi_{k+1}(k+1+\frac{k+1}{k+2}) = \varphi_k(k+\frac{k}{k+1}) + \frac{1}{2^{k+1}}$$
$$\varphi_{k+1}(k+2+\frac{1}{k+3}) = \varphi_k(k+\frac{k}{k+1}) + \frac{3}{2^{k+1}}$$
$$\varphi_{k+1}(k+2+\frac{t+1}{k+3}) = \varphi_k(k+1+\frac{t}{k+2}) + \frac{1}{2^{k+1}}, \text{ for } 2 \leq t \leq k.$$

Provided the values of φ_1 and φ_2 are set as indicated above, and the general assignment rules are followed, it is easy to see that the values of φ_i for $i = 1, \ldots, k$ are integer multiples of $\frac{1}{2^k}$. Given the way the values of φ_{k+1} are set, they are odd integer multiples of $\frac{1}{2^{k+1}}$ and hence are distinct from all previously assigned values. The assignment rules ensure that the insertion constraints are satisfied.

The greatest value taken by φ_{k+1} is equal to the greatest value taken by φ_k plus $\frac{1}{2^{k+1}}$. Therefore, $0 < \varphi_{k+1} < 2$ for all $k \geq 0$. A representation f of the $\succsim_\varphi$ preorder, ranging in $]0,1[$, is thus obtained e.g., by setting $f(\varphi_k(x)) = \frac{1}{2}\varphi_k(x)$, for all $x \in I_k$ and all k such that I_k exists. A unit representation of the semiorder is then obtained using the lifting equation (5). $\diamond$

Remark 39
It is worth noticing that the procedure described in Sections 3.1, 3.2, 4.1 and 4.2 unifies the construction of a numerical representation for finite and countable (I-connected) semiorders. Consider for instance the finite semiorder obtained by restricting the set X and the relation in Example 28 to the elements less than or equal to $4+3/5$. A preliminary check for equivalent elements leads to identify the elements $x = 3+3/4$ and $y = 4+1/5$, which have the same predecessors and successors, i.e., $xP = yP$ and $Px = Py = \varnothing$. These elements form an equivalence class, denoted by a. It will be represented by a single ghost and both elements of the class a will be associated the same value in numerical representations (the option of identifying equivalent elements was made in the introductory section). A numerical representation of this finite semiorder can be constructed by the standard procedure. Starting from $x_0 = 0$, we obtain the same intervals I_0, I_1, I_2 while $I_3 = \{a, 4+2/5, 4+3/5\}$. The ghost of a may be set for instance to the value 7/8 while the other ghosts remain unchanged. We get a numerical representation u with unit threshold by setting $f(x) = 1/2\varphi_k(x)$ for $x \in I_k, k = 0, 1, 2, 3$ and applying formula (5). In particular, we have $u(3+3/4) = u(4+1/5) = 3+7/16$. $\bullet$

Example 29 [cont'd]

Let $X = \mathbb{Q}$ and $S = (P, I)$ be the usual semiorder on the rationals (see Example 29). We consider the decomposition obtained by using UFP, starting from the bipartition (A, B), with $A = \{x \in \mathbb{Q} : x \geq 0\}$ and $B = X \setminus A$. We have $I_0 = [0, 1]$, the closed unit rational interval.

For $x \in I_1 =]1, 2]$, $\varphi_1(x)$ is inserted between $A^0_x = \{y \in I_0 : y \geq x - 1\}$ and $B^0_x = \{z \in I_0 : z < x - 1\}$. The element x is the only one in I_1 that has to be inserted between A^0_x and B^0_x. Therefore we create a "dummy" for x, that we label $(x, 1)$ and position (w.r.t. $\succsim_\varphi$) just below $x - 1 \in I_0$. Hence we have: $\varphi_1(x) = (x, 1) \prec_\varphi a$, for all $a \in A^0_x$ and $\varphi_1(x) = (x, 1) \succ_\varphi b$, for all $b \in B^0_x$. Doing this for all $x \in I_1$, we have $(x, 1) \prec_\varphi x - 1$ for all $x \in I_1$. For $x \in I_k, k \geq 1$, we label its ghost $\varphi_k(x) = (x, k)$ and insert it just below $x - k$ and also below (x', j), for all $1 \leq j \leq k$ such that $x' \in I_j$ and $x' - j = x - k$. Hence we have: $\varphi_k(x) = (x, k) \prec_\varphi (x', j) \prec x - k$. In a similar way, for $x \in I_{-l}, l \geq 1$, we create a ghost $\varphi_{-l}(x) = (x, -l)$ and insert it above $x + l$. For $x \in I_{-l}, l \geq 1$, its ghost $\varphi_{-l}(x) = (x, -l)$ is positioned above $x + l$ and also above $(x', -j)$, for all $1 \leq j \leq l$ such that $x' \in I_{-j}$ and $x' + j = x + l$. Eventually, associated to each $x \in I_0 \setminus \{0, 1\}$, we have a family of ghosts $(x + m, m)$, for all $m \in \mathbb{Z}$ (where $(x, 0)$ is interpreted as $x \in I_0$). For the special points $x = 1$ and $x = 0$, the associated ghosts are, respectively, $(1 + k, k), k \geq 0$ and $(-l, l), l \geq 0$. Therefore,

$\tilde{I}_0$ is the set of pairs $(x+m, m), x \in I_0 \setminus \{0,1\}, m \in \mathbb{Z}$ together with the pairs $(1+k, k), k \geq 0$ and $(-l, l), l \geq 0$. This set is ordered by $\succsim_\varphi$ as follows: for all $x, x' \in I_0$ and all $m, m' \in \mathbb{Z}$ such that $(x+m, m), (x'+m', m') \in \tilde{I}_0$, we have $(x+m, m) \succ_\varphi (x'+m', m')$ iff $x > x'$ or $x = x'$ and $m < m'$. The set $\tilde{I}_0$ is countably infinite and $\succsim_\varphi$ is a complete preorder on $\tilde{I}_0$, hence it has a numerical representation f in the rational $]0,1[$ interval[8]. Letting $u(x+m) = f((x+m, m)) + m$, for all $x \in I_0$, yields a numerical representation of the semiorder S on $\mathbb{Q}$. Note that such a representation is at the same type strict and nonstrict (see Remark 38). $\diamond$

4.4 Sparing ghosts

In the construction of a representation described in Sections 4.1 and 4.2, all elements from $I_k, k \geq 1$, are represented by ghosts that are distinct and also distinct from ghosts of elements of I_{k-1} (and similarly for I_{-l} and I_{-l+1}). The complete preorder $\succsim_\varphi$ that is constructed on $\tilde{I}_0$ actually is a linear order since distinct elements in $\tilde{I}_0$ are never equivalent (i.e., the symmetric part $\sim_\varphi$ of $\succsim_\varphi$ reduces to identical pairs). In this section we determine the conditions under which ghosts of elements from I_m and $I_{m'}$, with $m \neq m'$, can be set equivalent (w.r.t. $\succsim_\varphi$). Actually, there is no need for conditions unless $m = k$ and $m' = k-1$ with $k > 0$ or $m = -l$ and $m' = -l+1$ with $l > 0$. We illustrate a variant of the ghost construction on Example 29. While positioning ghosts, we put as many ghosts as possible in the same equivalence class of $\succsim_\varphi$. Although we do not really "spare ghosts" in the true sense, we do not discriminate between as many of them as possible.

We recall the assumption made at the outset that no pair of distinct elements in X are equivalent with respect to $\succsim$, i.e., the equivalence classes of $\sim$ are singletons.

Proposition 40
For $k \geq 1$, let $I_k = \{z_t, t \in T_k\}$, with $T_k \subseteq \mathbb{N}$, a set of consecutive integers starting with 1. In the construction of a representation described in Section 4, the ghost $\varphi_k(z_t)$ can be set equivalent (w.r.t. $\succsim_\varphi$) to the ghost $\varphi_{k-1}(z')$ for some $z' \in I_{k-1}$ if and only if the following two conditions are satisfied:

[8] One way of building such a representation is as follows.

1. Label each element of the denumerable set $I_0 = [0,1] \cap \mathbb{Q}$ by a positive integer, i.e., $I_0 = \{x_n, n \in \mathbb{N}_0\}$; let $n(0)$ (resp. $n(1)$) be such that $x_{n(0)} = 0$ (resp. $x_{n(1)=1}$).
2. Set $g(x_n) = x_n + \sum_{j: x_j < x_n} \frac{1}{2^j} + \frac{1}{2^{n+1}}$.
3. Set $g((x_n+m, m)) = g(x_n) - \mathrm{sign}(m) \frac{1}{2^{n+1}} \sum_{1 \leq j \leq |m|} \frac{1}{2^j}$, whenever $(x_n+m, m) \in \tilde{I}_0$.
4. Set $f((x+m, m)) = \frac{1}{2} g((x_n+m, m)$, whenever $(x+m, m) = (x_n+m, m) \in \tilde{I}_0$.

f is a numerical representation of $\succsim_\varphi$ ranging in the (rational) interval $]0,1[$.

1. $A_t^{k-1} = \{x \in I_{k-1} : z_t I x\}$ *has a least element (w.r.t.* $\succsim$*), which is* z'*,*

2. z_t *is the greatest element in* I_k *that is indifferent to* z'*.*

Similarly, for $l \geq 1$*, let* $I_{-l} = \{z_t, t \in T_{-l}\}$*, with* $T_{-l} \subseteq \mathbb{N}$*, a set of consecutive integers starting with 1. The ghost* $\varphi_{-l}(z_t)$ *can be set equivalent (w.r.t.* $\succsim_\varphi$*) to the ghost* $\varphi_{-l+1}(z')$ *for some* $z' \in I_{-l+1}$ *if and only if the following two conditions are satisfied:*

1. $B_t^{-l+1} = \{x \in I_{-l+1} : z_t I x\}$ *has a greatest element (w.r.t.* $\succsim$*), which is* z'*,*

2. z_t *is the least element in* I_{-l} *that is indifferent to* z'*.*

Proof

Consider the case $k \geq 1$. We first prove that we can set $\varphi_k(z_t)$ equal to $\varphi_{k-1}(z')$ if the conditions are fulfilled. Let $z_s \in I_k$. If $z_s \succ z_t$, we have that $z_s P z'$, by the second hypothesis. Hence we have to set $\varphi_k(z_s) \succ_\varphi \varphi_k(z_t) \sim_\varphi \varphi_{k-1}(z')$, which raises no problem. For all $z \prec z'$, we have $z_t P z$, which is compatible with $\varphi_{k-1}(z) \prec_\varphi \varphi_{k-1}(z') \sim_\varphi \varphi_k(z_t)$.

We now prove the necessity of the two conditions. Suppose that we set $\varphi_k(z_t) \sim_\varphi \varphi_{k-1}(z')$ while there exists z'' with $z' \succ z''$ and $z_t I z''$. Then, using a representation f of the complete preorder on $\tilde{I}_0$, we build the representation of the semiorder according to formula (5), and obtain $u(z_t) = u(z') + 1 > u(z'') + 1$, which contradicts $z_t I z''$.

In a similar way, assuming that we set $\varphi_k(z_t) \sim_\varphi \varphi_{k-1}(z')$ while there exists $z_s \in I_k$ with $z_s \succ z_t$ and $z_s I z'$, and constructing a representation of the semiorder as described in Section 4.2 leads to $u(z_s) > u(z_t) = u(z') + 1$, which contradicts $z_s I z'$.

The case $l \geq 1$ is proved similarly. □

Remark 41

In case z', z_t satisfy the conditions in Proposition 40 for $k \geq 1$ (resp., for $l \geq 1$), the pair (z', z_t) (resp., the pair (z_t, z')) has been called a *hollow* in Pirlot (1990, 1991). This notion, together with the dual notion of *nose* (a minimal strict preference pair), plays a crucial role in the theory of minimal representation of a finite semiorder. It turns out that their role is also important in the characterization of the uncountable semiorders admitting a unit representation (see Bouyssou and Pirlot, 2020a,b). •

Remark 42

In case we "spare ghosts" in the insertion process and apply the representation construction process described in Section 4.2, we obtain a representation that is strict (formula (1)) and not nonstrict (as soon as the semiorder has at least one hollow, i.e., a pair z', z_t satisfying the conditions in Proposition 40). •

Example 29 [cont'd]

Let us reconsider the ghost insertion procedure described in section 4.3 for the case of Example 29, i.e., the usual semiorder $S = (P, I)$ on $\mathbb{Q}$. We insert the ghosts of the elements of $I_1 =]1, 2]$ in $I_0 = [0, 1]$. For $x \in I_1$, $x - 1$ in I_0 is the least rational indifferent to x and x is the greatest indifferent to $x - 1$. Therefore the conditions of Proposition 40 are fulfilled and we may assign the ghost $\varphi_1(x)$ to the indifference class (w.r.t. $\succsim_\varphi$) of $x - 1 \in I_0$. This can be done for all elements in I_1. Each element in I_1 is indifferent to an element in I_0. Only the indifference class of $0 \in I_0$ remains a singleton. We go on with inserting the ghosts of the elements x of I_2. Since x and $x - 1 \in I_1$ fulfill the conditions of Proposition 40, we add $\varphi_2(x)$ to the indifference class of $\varphi_1(x - 1)$ and $x - 2$, for all $x \in I_2$. Going on in the same way for all $I_k, k \geq 1$, and then, similarly, for inserting the ghosts of all the elements of $I_{-l}, l \geq 1$, we eventually come to the following complete preorder $\succsim_\varphi$ on $\tilde{I}_0$. The elements of I_0 are ordered as with $\succsim$. Each element $x \neq 0, 1$ in I_0 is indifferent ($\sim_\varphi$) to the ghosts of $x + m$, for all $m \in \mathbb{Z}$. The cases $x = 0$ and $x = 1$ are particular; $x = 0$ is indifferent to $-l$, for all $l \in \mathbb{N}$; $x = 1$ is indifferent to k, for all $k \in \mathbb{N}$. For obtaining a unit representation of the semiorder S, we may select the canonical representation of the complete preorder $\succsim_\varphi$, i.e., we define $f(x) = x$ for each x in I_0 and all ghosts belonging to its indifference class. Then we apply (5), yielding $u(x) = x - m + m = x$ for all $x \in I_m, m \in \mathbb{Z}$.

Note that we allowed $f(x)$ to range in $[0, 1]$, not in $]0, 1[$. The obtained numerical representation actually returns the semiorder in its initial form. This would not be the case if we restricted the range of $f(x)$ to a subset of $]0, 1[$. Note also that the obtained representation is strict (and not at the same time nonstrict) even if we restrict $f(x)$ to range in $]0, 1[$ (see Remark 42). $\diamond$

4.5 Bounds on the representations

In this section we establish bounds on the values of the unit representations constructed by the procedure described above. The existence of such bounds will be useful for assembling unit representations on the various connected components of (X, I) in order to obtain a unit representation of the whole semiorder (see Section 5).

Proposition 43
Let $\mathcal{D}$ be a connected component of (X, I) and let $\mathcal{D} = \bigcup_{m \in M} I_m = (\bigcup_{k \geq 0} I_k) \cup (\bigcup_{l > 0} I_{-l})$, where the subsets $I_m, m \in M$ are a decomposition of $\mathcal{D}$ as described in Section 3.2. Let f be a representation of the complete preorder $\succsim_\varphi$ on $\tilde{I}_0$ satisfying the unit threshold constraint (6) *and u the representation of the semiorder* (P, I)

on X defined by the lifting equation (5)*. The following inequalities hold: for all $a \in I_0$,*

$$\begin{aligned} k-1 < u(x)-u(a) \leq 2k+1 \qquad & \textit{for all } x \in I_k, k \geq 0, k \in M, \\ -2l-1 \leq u(x)-u(a) < -l+1 \qquad & \textit{for all } x \in I_{-l}, l \geq 0, -l \in M. \end{aligned}$$

There exist semiorders for which the non-strict inequalities above are tight and the strict inequalities cannot be improved.

PROOF

1. For $x \in I_0$, the first double inequality with $k = 0$ holds since xIa and u is a representation of the semiorder.

2. We prove the first double inequality by induction. Assume that $0 < k$ and $k \in M$. Assume that the first double inequality holds for $m = 1, \ldots, k-1$. In particular, for all $y \in I_{k-1}$, we have $k-2 < u(y) - u(a) \leq 2k-1$. By Proposition 24.4, for all $x \in I_k$, there is $y \in I_{k-1}$ such that xPy. Therefore, $u(x) > u(y) + 1 > u(a) + k - 2 + 1$. The strict inequality is thus established. To establish the other inequality, we use the I-connectedness of $\mathcal{D}$, which implies that there are $z \in I_k$ and $y \in I_{k-1}$ such that zIy. By also using xIz, we get : $u(x) \leq u(z) + 1 \leq u(y) + 2 \leq u(a) + 2k - 1 + 2$. The double inequality for $k \geq 0$ thus holds.

3. The proof of the second double inequality is similar. It is left to the reader.

The fact that, for some semiorders, the strict bound cannot be improved and the other bound is tight is shown by Example 44 below. □

Example 44
Let (P, I) be the semiorder on $\mathbb{Z}$ defined for all $x, y \in \mathbb{Z}$ by xPy if $x > y + 1$ and xIy if $|x - y| \leq 1$. Two integers are indifferent if and only if they are consecutive. Let I_0 be $\{0, 1\}$. We have:

$$\begin{aligned} I_k &= \{2k, 2k+1\} \qquad && \text{for } k \geq 0, \\ I_{-l} &= \{-2l, -2l+1\} \qquad && \text{for } l \geq 0. \end{aligned}$$

In case we decide to "spare ghosts", we may set $\varphi_1(2) = 1$, $\varphi_2(4) \sim_\varphi \varphi_1(3) \succ_\varphi \varphi_1(2)$ and, in general, for $k > 0$, $\varphi_k(2k) \sim_\varphi \varphi_{k-1}(2k-1) \succ_\varphi \varphi_{k-1}(2k)$. For I_{-l} and $l > 0$, we may set $\varphi_{-1}(-1) = 0$, $\varphi_{-2}(-3) \sim_\varphi \varphi_{-1}(-2) \prec_\varphi \varphi_{-1}(-1)$ and, in general, $\varphi_{-l}(-2l+1) \sim_\varphi \varphi_{-l+1}(-2l+2) \prec_\varphi \varphi_{-l+1}(-2l+3)$.

We may then build f as follows on $\bigcup_{k \geq 0} \varphi_k(I_k)$: $f(0) = 0$, $f(1) = 1 = f(\varphi_1(2))$, $f(\varphi_1(3)) = f(\varphi_2(4)) = 1 + \varepsilon$ for $0 < \varepsilon \leq 1$, $f(\varphi_{k-1}(2k-1)) = f(\varphi_k(2k)) = 1 + (k-1)\varepsilon$. On $\varphi_{-l}(I_{-l})$, $l \geq 0$, we may set: $f(\varphi_{-1}(-1)) = f(0) = 0$, $f(\varphi_{-1}(-2)) =$

$f(\varphi_{-2}(-3)) = -\varepsilon$, $f(\varphi_{-l+1}(-2l+2)) = f(\varphi_{-l}(-2l+1)) = (-l+1)\varepsilon$. If we set $\varepsilon = 1$ and define u by using (5), we get, for $I_k = \{2k, 2k+1\}$, $u(2k) = f(\varphi_k(2k)) + k = 1 + (k-1) + k = 2k$ and $u(2k+1) = f(\varphi_k(2k+1)) + k = 1 + k + k = 2k+1$. In a similar way, we have $u(x) = x$ for $x \in I_{-l}$ for all $l > 0$. Therefore, $u(x) = x$, for all $x \in \mathbb{Z}$. The inequality $u(x) - u(a) \leq 2k+1$, for all $x \in I_k, k \geq 0$ is satisfied to equality for $x = 2k+1$ and $a = 0$. In a similar way, the inequality $-2l - 1 \leq u(x) - u(a)$ is satisfied to equality for $x = -2l$ and $a = 1$.

Let us build now other ghosts and another representation u that show that the strict inequalities in Proposition 43 cannot be improved. This time, it is decided not to spare ghosts. It is not difficult to see that we may position the ghosts in such a way that the preorder $\succsim_\varphi$ on $\tilde{I}_0$ can be represented by the function f defined as follows. Choose a constant ε such that $0 < \varepsilon \leq 1$. We set $f(0) = 0$ and $f(1) = 1$. For all $k > 0$, we set $f(\varphi_k(2k)) = f(\varphi_{k-1}(2k-2)) + \frac{\varepsilon}{2^k} = \varepsilon(\frac{1}{2} + \ldots \frac{1}{2^k}) = \varepsilon(1 - \frac{1}{2^k})$ and $f(\varphi_k(2k+1)) = f(\varphi_{k-1}(2k-1)) + \frac{\varepsilon}{2^k} = 1 + \varepsilon(1 - \frac{1}{2^k})$. We then define $u(x)$ for $x \in I_k$, $k > 0$, using (5). We have $u(2k) = k + \varepsilon(1 - \frac{1}{2^k})$ and $u(2k+1) = 1 + k + \varepsilon(1 - \frac{1}{2^k})$. Therefore, for $x \in I_k$, $k > 0$, and $a \in I_0$, we have $u(x) - u(a) \geq u(2k) - u(1) = k - 1 + \varepsilon(1 - \frac{1}{2^k}) > k - 1 + \varepsilon$. Since ε may be chosen arbitrarily close to 0, the difference $u(x) - u(a)$ may be arbitrarily close to $k-1$. In a similar way, we may define f on the ghosts of I_{-l} for all $l > 0$ in such a way that $u(-2l) = -l - \varepsilon(1 - \frac{1}{2^l})$ and $u(-2l+1) = 1 - l - \varepsilon(1 - \frac{1}{2^l})$. Therefore, for $x \in I_{-l}$, $l > 0$, and $a \in I_0$, we have $u(x) - u(a) \leq u(-2l+1) - u(0) = -l + 1 - \varepsilon(1 - \frac{1}{2^l})$. Since ε may be chosen arbitrarily close to 0, the difference $u(x) - u(a)$ may be arbitrarily close to $-l+1$. ◇

Remark 45
In case we enforce the representation f of the complete preorder $\succsim_\varphi$ to range in $]0, 1[$ (instead of allowing for the less restrictive unit threshold constraint (6)), we get the following bounds on the representation u: for all $a \in I_0$,

$$\begin{aligned} k - 1 < u(x) - u(a) < k + 1 \qquad & \text{for all } x \in I_k, k \geq 0, k \in M, \\ -l - 1 \leq u(x) - u(a) < -l + 1 \qquad & \text{for all } x \in I_{-l}, l \geq 0, -l \in M. \end{aligned}$$

It is easy to verify that the above inequalities cannot be improved. •

4.6 Uniqueness

The question of the uniqueness of unit representations for semiorders is more delicate than, e.g., for linear orders. In the latter case, it is well-known that the numerical representation of a linear order, when it exists, is unique up to a strictly increasing transformation. The first issue raised by the representation of semiorders is related to the fact that equivalent elements of X are not necessarily represented by the same number. *Regular representations* are precisely these in which equivalent elements are assigned the same number (see Roberts, 1979, p. 60 and p. 252). This issue can be avoided by considering only semiorders without equivalent elements, which we assume in this paper (see Section 2, p. 4).

A second issue relates to the possible existence of several connected components in the graph of (X, I). Basically, the minimal difference between two elements belonging to consecutive connected components has to be larger than 1. The possible ways of assembling numerical representations of the semiorders on different connected components will be discussed in Section 5.

In this section, we focus on the uniqueness of the unit representation of a semiorder restricted to a connected component $\mathcal{D}$ of (X, I). In the construction process of a representation (described in Sections 3.2 and 4), we can point out three types of arbitrary choices, which may result in different representations:

- the choice of the initial bipartition (A, B) and the application of DFP or UFP for generating I_0, in the construction of the partition $\{I_m, m \in M\}$ of $\mathcal{D}$ in Section 3.2,
- in some cases, several positions for inserting ghosts are allowed. Their position is well-determined only with respect to the ghosts in the indifference class I_m inserted at the previous step. It is possible to "spare ghosts" (or not) by "merging" them with some previously inserted ones,
- the choice of a numerical representation of the complete preorder $\succsim_\varphi$ on $\tilde{I}_0$ is arbitrary modulo condition (6), which states that the difference between the values associated to indifferent elements of X must be at most 1.

Actually, the first source of arbitrariness, i.e., the particular partition $\{I_m, m \in M\}$ considered, is irrelevant. Indeed, the procedure for constructing a representation involves three steps: construction of a partition into indifference classes, ghost insertion, choice of a numerical representation of the order on the ghosts and lifting. All unit representations of the semiorder can be obtained by this procedure, whatever the way the initial step is performed. This is shown in Proposition 46 below. The last two steps are the only ones that matter. The process of construction of a representation described in Sections 4.2 and 4.4 can generate *any* unit representation of the semiorder on $\mathcal{D}$ (independently of the chosen partition $\{I_m, m \in M\}$). The decisions made during this process are thus the only degrees of freedom in the construction of a unit representation. In other words, the unit representation of a connected semiorder is unique up to the possible orders on the ghosts and the choice of a representation of the order $\succsim$ on $\tilde{I}_0$ satisfying condition (6). This "uniqueness" result is proved below.

Proposition 46
Let u be any unit representation of the semiorder (P, I) restricted to the connected component $\mathcal{D}$ of (X, I). This representation can be obtained by the construction process described in Section 4.2 and 4.4 by making appropriate feasible choices.

PROOF
Consider any partition $(I_m, m \in M)$ of $\mathcal{D}$ into indifference classes, as described in Section 3. We may define $\tilde{I}_0$ and the complete preorder $\succsim_\varphi$ on the ghosts directly, by using the representation u. Let

$$\tilde{I}_0 = \bigcup_{m \in M} \varphi_m(I_m),$$

setting $\varphi_0(x) = x$ whenever $x \in I_0$. Let $\varphi_m(x)$ (resp., $\varphi_{m'}(y)$) be the ghost of $x \in I_m$ (resp., $y \in I_{m'}$). We define the complete preorder $\succsim_\varphi$ on $\tilde{I}_0$ by:

$$\varphi_m(x) \succsim_\varphi \varphi_{m'}(y) \quad \text{if} \quad u(x) - m \geq u(y) - m'. \tag{7}$$

This order on the ghosts, corresponds to one feasible way of inserting recursively the ghosts as described in Section 4.2 and 4.4. Consider for instance $z \in I_k$, for some $k > 0, k \in M$. The position of $\varphi_k(z)$ in the preorder $\succsim_\varphi$ must satisfy the following requirements:

1. $\varphi_k(z) \succ_\varphi \varphi_k(z')$ for all $z' \in I_k$ with $z \succ z'$. This requirement is satisfied since $z \succ z'$ entails $u(z) - k > u(z') - k$ and therefore $\varphi_k(z) \succ_\varphi \varphi_k(z')$, by definition (7). The requirement for $z' \in I_k$ with $z' \succ z$ is established similarly.

2. assuming $k \geq 1$, we must have $\varphi_k(z) \precsim_\varphi \varphi_{k-1}(z')$ for all $z' \in I_{k-1}$ such that zIz'. We have indeed that $z'Iz$ and $z \succ z'$ entails $0 < u(z) - u(z') \leq 1$. Therefore $u(z) - k \leq u(z') - k + 1$ yields $\varphi_k(z) \precsim_\varphi \varphi_{k-1}(z')$, by definition (7). In the case $z' \in I_{k-1}$ and zPz', we have $u(z) > u(z') + 1$. From $u(z) - k > u(z') - k + 1$ and (7), we deduce $\varphi_k(z) \succ_\varphi \varphi_{k-1}(z')$.

3. the conditions for sparing ghosts, i.e. considering some ghosts as indifferent in the preorder $\succsim_\varphi$ are respected by (7).

The similar conditions relative to the insertion of $\varphi_{-l}(z)$ for $z \in I_{-l}$, for some $l > 0, -l \in M$ follow in an analogous way.

Now let us choose, as representation of the complete preorder $\succsim_\varphi$ on $\tilde{I}_0$, the following function f, defined by:

$$f(\varphi_m(x)) = u(x) - m,$$

for all $x \in I_m$. The function f is a representation of $\succsim_\varphi$ on $\tilde{I}_0$ since $\varphi_m(x) \succsim_\varphi \varphi_{m'}(y)$ iff $f(\varphi_m(x)) = u(x) - m \geq f(\varphi_{m'}(y)) = u(y) - m'$. Function f satisfies (6) since $|u(x) - u(y)| \leq 1$, for all $x, y \in I_m$ and all $m \in M$. Applying (5) restates the unit representation u. □

Remark 47
The construction of a representation for the usual semiorder on the rationals in Section 4.4 (Example 29), illustrates how the canonical representation of this semiorder can be restored. •

5 Building a unit representation for non-connected semiorders

At this stage we know from Proposition 36 that each I-connected component of a semiorder $S = (P, I)$ on a countable set X has a unit representation. When does this imply that the semiorder as a whole has a unit representation? When this is the case, can we build the unit representation of the whole semiorder by assembling the unit representations of its components? Manders (1981) and Beja and Gilboa (1992) have given distinct necessary and sufficient conditions guaranteeing that a semiorder on a countable set admits a unit representation.

The goal of this section is to analyze these conditions and to show how to obtain a representation of the semiorder from those of its I-connected components, provided a representability condition is fulfilled.

5.1 Manders' condition

For the reader's convenience, we recall definitions that allow to formulate the condition used in Manders (1981, p. 238-239).

Let P^* denote the *covering* relation associated to P, i.e., for $x, y \in X$, we have xP^*y if xPy and there is no $z \in X$ such that $xPzPy$. So, xP^*y if y is an immediate successor of x in the partial order P. Manders (1981) defines the relation I^* as follows. For $x, y \in X$, we have xI^*y if xIy or xP^*y or yP^*x. The *transitive closure* of I^* is denoted by $\overline{I^*}$. To illustrate these definitions, let $S = (P, I)$ be the usual semiorder $\mathbb{Q}$ (see Example 29). For all $x, y \in \mathbb{Q}$, we have xP^*y iff $y+1 < x \leq y+2$ and xI^*y iff $|x - y| \leq 2$.

Condition 1 (Manders)
The relation $\overline{I^*}$ is connected.

Manders' condition amounts to say that for all $x, y \in X$, there is an $\overline{I^*}$-chain joining x to y. In other words, it is possible to go from x to y following a path that uses either indifference arcs or jumps between "nearest neighbors" in the P relation.

We give an alternative formulation of Manders'condition and we prove that they are equivalent.

Condition 2
For all $x, y \in X$ with xPy, there is a P^*-chain joining x to y.

Proposition 48
The semiorder $S = (P, I)$ on X satisfies Condition 1 iff it satisfies Condition 2.

PROOF

1. Let $x, y \in X$ with xPy. Assume that $\overline{I^*}$ is connected. Consider an $\overline{I^*}$-chain $(x_i, i = 0, \ldots, n)$ joining $x = x_0$ to $y = x_n$. This chain is composed of pairs of elements belonging to I, P^* or $(P^*)^{-1}$. We may assume w.l.o.g. that $Not[x_i \sim x_{i+1}]$, for all i.

   Assume first that this chain is monotone w.r.t. to the trace $\succsim$, i.e., $x_i \succ x_{i+1}$ for all $i = 0, \ldots, n-1$. This implies that the chain has only pairs belonging to I or P^*. If it has only P^* pairs, there is nothing to prove. Otherwise, for some i, we have (case 1) $x_i I x_{i+1} P^* x_{i+2}$ or (case 2) $x_i P^* x_{i+1} I x_{i+2}$ or else (case 3) the chain has only I pairs.

   Case 1. Let $x_i I x_{i+1} P^* x_{i+2}$. Either we have $x_i P^* x_{i+2}$ or there is $z \in X$ with $x_i P z P x_{i+2}$. In the former case, we may remove the I pair from the chain, going directly from x_i to x_{i+2} using the P^* pair (x_i, x_{i+2}). In the other case, there is $z \in X$ such that $x_i P z P x_{i+2}$. It is easy to prove that $x_i P^* z P^* x_{i+2}$. Indeed, clearly, $x_{i+1} \succ z \succ x_{i+2}$. There is no $w \in X$ such that $x_{i+1} \succ z P w P x_{i+2}$, since this would contradict $x_{i+1} P^* x_{i+2}$. Therefore, $z P^* x_{i+2}$. In a similar way, there is no $w \in X$ such that $x_i P w P z P x_{i+2}$. Otherwise we would have $x_{i+1} \succ w P z P x_{i+2}$ contrary to $x_{i+1} P^* x_{i+2}$. In all cases, we may thus replace the sub-chain $x_i I x_{i+1} P^* x_{i+2}$ by a sub-chain composed of one or two P^* pairs, thus eliminating the I pair.

   Case 2. Let $x_i P^* x_{i+1} I x_{i+2}$, is dealt with similarly. The conclusion is that the I pair can be removed by replacing the initial sub-chain by one or two pairs from P^*.

   Case 3. The chain only has I pairs. We know that xPy. If xP^*y, the result is immediate. Otherwise, there is $z \in X$ with $xPzPy$. For some i, we have $x_i \succsim z \succ x_{i+1}$. If xP^*z, we may replace the initial pairs of the chain (from $x = x_0$ to x_{i+1}) by xP^*zIx_{i+1} which yields another $\overline{I^*}$-chain from x to y. This chain has a configuration that pertains to case 2. We thus apply the procedure for case 2 chains, which results in the elimination of I pairs.

By repeatedly applying the above procedures to case 1 or case 2, we eliminate all I pairs.

To finish, consider an $\overline{I^*}$-chain that is not monotonic w.r.t. $\succsim$. If the chain goes beyond y, let x_j be the first element such that $y \succ x_j$. So, we have $x_{j-1} \succsim y$. Clearly, we have either $x_{j-1}Iy$ or $x_{j-1}P^*y$. In both cases we may replace the initial chain by the shorter $\overline{I^*}$-chain $(x'_i, i = 0, \ldots, j)$ with $x'_i = x_i$, for $i = 1, \ldots, j-1$, and $x'_j = y$. We may thus restrict our attention to $\overline{I^*}$-chains for which $x_i \succsim y$, for all i.

Consider an $\overline{I^*}$-chain $(x_i, i = 0, \ldots, n)$ with $x_0 = x$, $x_n = y$ and $x_i \succsim y$, for all i. Assume that this chain is not monotonic w.r.t. $\succsim$. Let k be the least value of $i, 0 \leq i < n$ such that $x_{k+1} \succ x_k$. Let x_l be the first element in the chain such that $x_k \succ x_l$. Clearly, we have either $x_k I x_l$ or $x_k P^* x_l$. We may thus remove from the chain the pairs $(x_i, x_i + 1)$ for $i = k, \ldots, l-1$, and replace them by the single pair (x_k, x_l) which belongs either to I or P^*. Iterating this procedure, we transform the chain into an $\overline{I^*}$-chain that is monotonic w.r.t. $\succsim$. So the case of non-monotonic $\overline{I^*}$ chain can be reduced to that of monotonic ones. This concludes the first part of the proof.

2. Proving the converse is easy. Assume that Condition 2 is fulfilled. Consider a pair $x, y \in X$. There are three cases. If xIy, the thesis is established. If xPy, Condition 2 entails the existence of a P^*-chain, hence an $\overline{I^*}$-chain from x to y. Finally, if yPx, Condition 2 entails there is a P^*-chain from y to x, hence a $(P^*)^{-1}$-chain from x to y. The latter is also an $\overline{I^*}$-chain. □

5.2 Beja and Gilboa's condition

It reads as follows (Beja and Gilboa, 1992, Axiom 1, p. 435).

Condition 3 (Beja and Gilboa)
For every P-chain $(x_i \in X, i \in J)$, $J \subseteq \mathbb{Z}$, a set of consecutive integers

- *if the P-chain has no last element, for all $z \in X$, there is $n \in J$ such that zPx_n,*
- *if the P-chain has no first element, for all $z \in X$, there is $n \in J$ such that x_nPz.*

In words, this property says that, if we have an infinite P-chain without a last element, there is no object which all elements x_i in the chain are preferred to. In a similar way, for any infinite P-chain without a first element, there is no object which is preferred to all elements in the chain. This property is called *regularity* in Candeal and Induráin (2010).

We give an alternative formulation of Beja and Gilboa's condition and we prove that they are equivalent.

Condition 4 (Bounded *P*-chain condition)
Every bounded P-chain is finite.

This property says that, if $(x_i, i \in J)$ is a P-chain indexed by $J \subseteq \mathbb{Z}$, a set of consecutive integers, and there are $a, b \in X$ such that aPx_iPb for all $i \in J$, then $|J| < \infty$. This condition is clearly necessary for the existence of a unit representation.

Proposition 49
The semiorder $S = (P, I)$ on X satisfies Condition 3 iff it satisfies Condition 4.

Proof

1. Assume that Beja and Gilboa's Condition 3 holds and suppose that Condition 4 is not verified. Hence there are $a, b \in X$ with aPb, and a P-chain $(x_i, i \in J)$ with aPx_iPb for all $i \in J$ and $|J| = \infty$. The chain has either no last element or no first element (or has neither last nor first element). In the former case, we have x_iPb for all $i \in J$, contrary to Condition 3. In the latter case, we have aPx_i for all $i \in J$, which also contradicts Condition 3.

2. Assume that Condition 4 is verified and let us prove that Beja and Gilboa's Condition 3 must hold. Let $(x_i, i \in J)$ be a P-chain without a last element and suppose that Beja and Gilboa's property is violated for that chain, i.e., there is $z \in X$ such that $Not[zPx_i]$ for all $i \in J$. We have $x_{i-2}Px_{i-1}P^{cd}z$ for all $i \in J$, where the co-dual P^{cd} of P is defined by $xP^{cd}y$ iff $Not[yPx]$. A well-known property of semiorders (Pirlot and Vincke, 1997, Th. 3.2, p.53) states that $PPP^{cd} \subseteq P$. Therefore $x_{i-2}Pz$, for all i, which implies x_iPz for all i such that $i-2$ belongs to J.

   Choose an arbitrary $i_0 \in J$ (with $i_0 - 1 \in J$) and consider the truncated P-chain $(x_i, i \in J, i > i_0)$. This is a chain which has no last element and such that $x_{i_0}Px_iPz$, for all $i > i_0$. Applying Condition 4, using the bounds $a = x_{i_0}$ and $b = z$, we conclude that $|\{i \in J, i > i_0\}| < \infty$, which contradicts the initial assumption that the chain has no last element. The proof in case the chain has no first element is similar. □

Remark 50
The Bounded P-chain condition (Condition 4) has the flavor of an Archimedean axiom. It sounds like "Every bounded standard sequence is finite" (Krantz et al.,

1971, p.25). Here the sequence of pairs of objects in P plays a role similar to that of equally spaced preference intervals (see, in particular, the *strong standard sequences* defined in Gonzales, 2003, p. 51). Such properties are required for enabling representations using real numbers. •

In preference to Beja and Gilboa's Condition 3, we shall use the (equivalent) Bounded P-chain condition, which seems more compact. The latter condition actually implies that among all P-chains contained in an interval $[a, b]$, with $a \succsim b$, there is (at least) one having maximal length. Putting it another way, there cannot be P-chains of arbitrary length in a given interval. For proving this result, we need a lemma.

Lemma 51
Let $S = (P, I)$ be a semiorder on X and Z a denumerable subset of X that is totally ordered by P. Then there is a denumerable P-chain that is formed of elements of Z.

PROOF
We distinguish three cases: Z (ordered by P) has no greatest element or it has no least element or it has both a greatest and a least element.

1. If Z has no least element, let us pick any element $y_0 \in Z$. Since y_0 is not a least element in Z, there is $y_1 \in Z$ with $y_0 P y_1$. In turn, y_1 is not a least element, so that there is $y_2 \in Z$ with $y_1 P y_2$. Iterating this, we generate a P-chain $(y_i, i \in \mathbb{N})$, with $y_i P y_{i+1}$ for all $i \in \mathbb{N}$.

2. If Z has no greatest element, a similar process leads to generating a P-chain $(y_{-i}, i \in \mathbb{N})$.

3. If Z has both a least and a greatest element, let w_0 denote its greatest element. We construct w_i recursively, starting with w_0 as follows. Assume that $w_{i-1}, i \geq 1$, has been obtained. If the set $\{z \in Z, w_{i-1} P z\}$ has no greatest element, the construction process stops in w_{i-1}, which is such that $Z_i = \{z \in Z, w_{i-1} P z\} \subseteq Z$ has no greatest element. By applying to Z_i the construction described in item 2, we obtain a P-chain $(y_{-i}, i \in \mathbb{N})$ included in Z_i. Otherwise, if $\{z \in Z, w_{i-1} P z\}$ has a greatest element, we call it w_i and the construction of a P-chain $(w_j, j = 0, \ldots, i)$ continues. If a set Z_i without a greatest element is never met, then a P-chain $(w_i, i \in N)$ is eventually obtained.

In all these cases, a denumerable P-chain can be extracted from the set Z, ordered by P. □

Proposition 52
If P is the asymmetric part of a semiorder, the Bounded P-chain condition is equivalent to the following property: for all $a, b \in X$, with $a \succsim b$, the length of the P-chains $(x_i, i \in J)$ such that $a \succsim x_i \succsim b$ for all $i \in J$ is bounded.

Proof
This property clearly implies the Bounded P-chain condition, so that we only have to prove the direct implication (since bounded length chains are finite). Assume that Condition 4 holds and suppose that, contrary to the thesis, there are P-chains of arbitrary large length contained in interval $[a, b]$. Consider such a chain $(x_1, x_2, \ldots, x_n)$ of finite length n. Since P is the asymmetric part of a semiorder, its trace $\succsim$ defined in Section 2 is a complete preorder and we have $a \succ x_1 \succ x_2 \succ \ldots \succ x_n \succ b$. The chain determines a partition of interval $[a, b]$ into $n+1$ intervals. We first prove that the P-chain can be extended into a P-chain of length $n + 1$.

Since, by hypothesis, there are P-chains of arbitrary length, let us consider a P-chain $(y_j, j \in J')$, with $a \succsim y_j \succsim b$ for all j and $|J'| \geq 2(n + 1) + 1$. Among the intervals $[a, x_1], [x_1, x_2], \ldots, [x_i, x_{i+1}], \ldots, [x_n, b]$ there is at least one containing three successive elements y_{j-1}, y_j, y_{j+1}. These three elements lie in $[a, x_1]$, in $[x_n, b]$ or in one of the intervals $[x_i, x_{i+1}]$. In the latter case, we have: $x_i P y_j P x_{i+1}$ (since $x_i \succsim y_{j-1} P y_j$ implies $x_i P y_j$ and $y_j P y_{j+1} \succsim x_{i+1}$ implies $y_j P x_{i+1}$). We can thus build a P-chain of length $n + 1$ by inserting an additional element y_j between two elements of the chain $(x_i, i = 1, \ldots n)$. The two remaining cases, i.e., $[a, x_1]$ (resp. $[x_n, b]$) contains three consecutive elements y_{j-1}, y_j, y_{j+1}, are dealt with similarly, all leading to prove the existence of a P-chain of length $(n + 1)$.

Under the hypothesis that there are P-chains of arbitrary length in the interval $[a, b]$, we can iterate the previous extension of the chain, inserting, at each step k, at least one additional element z_k. The elements z_k, for $k \in \mathbb{N}_0$ (the set of positive integers), together with the elements $x_1, \ldots, x_n$ of the initial chain, form a countably infinite set Z that is totally ordered by P. It is not necessarily a P-chain but it contains a denumerable P-chain by Lemma 51. All its elements belong to interval $[a, b]$. This contradicts the Bounded P-chain condition. □

An immediate consequence of this property is the following.

Corollary 53
If P is the asymmetric part of a semiorder and satisfies the Bounded P-chain condition, any subset of an interval $[a, b]$ which is totally ordered by P is a finite P-chain. Such subsets have a bounded cardinality which depends on a and b. If such a subset contains a and b, it is a P-chain starting at a and terminating at b.

Proposition 52 states, under the Bounded P-chain condition, that there is a maximal finite length P-chain in each interval $[a, b]$. If a and b belong to the same connected component $\mathcal{D}$ of (X, I), this property is certainly true. Actually, we have the following.

Proposition 54
The semiorder induced by $S = (P, I)$ on a connected component $\mathcal{D}$ of (X, I) satisfies the Bounded P-chain condition.

Proof
Let a and b be two elements of $\mathcal{D}$ and assume w.l.o.g. that $a \succsim b$. By definition of a connected component of (X, I), a and b are linked by a finite chain of I. Therefore, any P-chain in $[a, b]$ must be finite. □

This is no surprise since the Bounded P-chain condition is a necessary condition for the existence of a unit representation of a semiorder and it has been shown in Section 4 that any denumerable connected semiorder admits such a representation. In contrast, semiorders that are not connected may – or not – satisfy the Bounded P-chain condition. Consider, for instance, the set $X = \{(x, z), x \in Y = \mathbb{Z}, z \in \{0, 1\}\}$. The semiorder $S = (P, I)$ on X defined by $(x, 1)P(y, 0)$, for all $x, y \in Y = \mathbb{Z}$, and $(x, z)P(y, z)$, for all $x > y + 1$ and $z \in \{0, 1\}$, does not satisfy the Bounded P-chain condition and has two connected components. If the set Y above is redefined as the integer interval $[-n, n]$, for any fixed $n \in \mathbb{N}, n \geq 1$, then the semiorder also has two connected components but satisfies the Bounded P-chain condition.

5.3 Four equivalent existence conditions

We prove that the four conditions studied in the latter two sections, including Manders' and Beja and Gilboa's, are equivalent.

Proposition 55
For a semiorder $S = (P, I)$ on a set X, Conditions 1, 2, 3 and 4 are equivalent.

Proof
We already proved that the first two and the last two conditions are equivalent (Propositions 48 and 49). We prove below that Conditions 2 and 4 are equivalent.

Assume that Condition 2 holds. Contrary to the thesis, assume that $(x_i, i \in J)$ is a P-chain with aPx_iPb, for all $i \in J$ and $|J| = \infty$. Since aPb, using Condition 2, we know that there is a P^*-chain $(x'_j, j = 0, \ldots, n)$ with $x_0 = a$ and $x_n = b$. Since $|J|$ is infinite, for some j and i, we have $x_j \succsim x_iPx_{i+1}Px_{i+2} \succsim x_{j+1}$. This entails that $x_jPx_{i+1}Px_{j+1}$, contrary to $x_jP^*x_{j+1}$. So, Condition 2 implies Condition 4.

We now assume that Condition 4 holds. Let $x, y \in X$ be such that xPy. If there is no $z_1 \in X$ such that xPz_1Py, then xP^*y and we are done. If there is such a z_1, we have got a P-chain of length 2 joining x to y. Again, there are two cases.

Either, this is a P^*-chain of length 2 (and we are done) or there is z_2 such that xPz_2Pz or zPz_2Py. In the latter case, a P-chain of length 3 can be obtained. By such a process, longer and longer P-chains joining x to y can be obtained recursively. More formally, let $x_i, i = 0, \ldots, n$, with $x_0 = x$ and $x_n = y$ be the P-chain obtained at step n. At step $n + 1$, there are two cases. Either all pairs (x_i, x_{i+1}) belong to P^*, or there is at least one i and $z_n \in X$ such that $x_iPz_nPx_{i+1}$. In the former case, we have constructed a P^*-chain of length n joining x to y and we are done. In the latter case, we have a P-chain of length $n + 1$ joining x to y. The elements of this P-chain are x, y and all elements $z_i, i = 1, \ldots, n$ inserted in the previous and the current step. By renumbering them properly, we have a P-chain $(x'_i, i = 0, \ldots, n + 1)$, with $x'_0 = x$ and $x'_{n+1} = y$. The process stops if a P^*-chain is obtained at some step n. Otherwise, it continues for ever, generating a denumerable set $\{z_n, n \in \mathbb{N}_0\}$, which is totally ordered by P but is not necessarily a P-chain. However, by Lemma 51, there is a denumerable subset of this set that can be ordered in a P-chain. This would contradict Condition 4. Therefore, the algorithm described above stops after a finite number of steps and returns a P^*-chain joining x to y. □

Remark 56
Note that Proposition 55 does not use the hypothesis that X is a countable set. The four conditions are thus equivalent independently of the cardinality of X. •

In the sequel, we shall mainly use the Bounded P-chain condition (Condition 4) as a formulation of the necessary and sufficient condition for the existence of a unit representation of a semiorder on a countable set. In particular, we shall use it to show that such a representation can be obtained by assembling unit representations of the semiorder restricted to its I-connected components.

5.4 Consequences of the Bounded P-chain condition

In Section 2.5, we defined a relation on the set $\mathfrak{F}$ of connected components of the indifference relation. This relation was defined by (4) and denoted by P, abusing notation. Lemma 14 tells us that the relation P on the set $\mathfrak{F}$ of connected components of (X, I) is a strict linear order. The consequences of the Bounded P-chain condition for this order are stated in the following result.

Proposition 57
Under the Bounded P-chain condition, there is an order isomorphism between $\mathfrak{F}$, linearly ordered by P, and a subset $\Gamma \subseteq \mathbb{Z}$ of consecutive integers endowed with the order $>$, i.e., each I-connected component $\mathcal{D} \in \mathfrak{F}$ can be assigned an index i belonging to a subset $\Gamma \subseteq \mathbb{Z}$ of consecutive integers in such a way that for all $i, j \in \Gamma$,

$$i > j \Rightarrow \mathcal{D}_i P \mathcal{D}_j. \tag{8}$$

Proof

With the aim of indexing the elements of $\mathfrak{F}$, we start with an arbitrary connected component $\mathcal{D}$, which we index as $\mathcal{D}_0$. Let us consider an arbitrary other element $\mathcal{E} \in \mathfrak{F}$. Either $\mathcal{D}_0 P \mathcal{E}$ or $\mathcal{E} P \mathcal{D}_0$. We consider the former case only (the latter being dealt with similarly). As a straightforward consequence of Proposition 52, there exists a maximal finite chain $\mathcal{D}_{-i}$, with $i = 1, \ldots, n_1$ and $\mathcal{D}_{-n_1} = \mathcal{E}$, such that $\mathcal{D}_0 P \mathcal{D}_{-1} P \ldots P \mathcal{D}_{-n_1} = \mathcal{E}$. Note that this maximal chain is unique, since it is composed of all elements of $\mathfrak{F}$ that lie between $\mathcal{D}$ and $\mathcal{E}$ w.r.t. P. If there exists $\mathcal{C} \in \mathfrak{F}$ such that $\mathcal{E} P \mathcal{C}$, we iterate the process, indexing by $-n_1 - 1, \ldots, -n_1 - n_2$ the elements of $\mathfrak{F}$ lying between $\mathcal{E}$ and $\mathcal{C}$, with $\mathcal{D}_{-n_1-n_2} = \mathcal{C}$. Such iterations can either stop after a finite number of steps or continue without limit. In the latter case all negative integers will be used but the process will exhaust the elements of $\mathfrak{F}$ to which $\mathcal{D}_0$ is preferred. Indeed, by the Bounded P-chain condition, it is impossible that there is $\mathcal{D} \in \mathfrak{F}$ such that $\mathcal{D}_{-n} P \mathcal{D}$ for all $n \in \mathbb{N}$. We can index in a similar way, using positive integers, the elements of $\mathfrak{F}$ that are preferred to $\mathcal{D}_0$. The set of indices that have been used at the end of this process constitutes the set Γ of consecutive integers. This set can be bounded or unbounded in either direction. The numbering of the connected components fulfills condition (8). □

Remark 58

Proposition 57 shows that, under the Bounded P-chain condition, there is an order isomorphism between the set $\mathfrak{F}$ of the connected components of the graph of the indifference relation on X, ordered by P, and a subset of consecutive integers endowed with the usual order. The Bounded P-chain condition is a sufficient condition for that. It is easy to show that it is not a necessary condition. Let $X = \mathbb{Z} \times \{0, 1\}$. For all $z, w \in \mathbb{Z}$ and $\alpha, \beta \in \{0, 1\}$, we define $(z, \alpha) P (w, \beta)$ if $\alpha > \beta$ or $[\alpha = \beta$ and $z > w + 1]$. The indifference relation I of this semiorder has two connected components, namely $\mathcal{D}_0 = \{(z, 1), z \in \mathbb{Z}\}$ and $\mathcal{D}_1 = \{(z, 0), z \in \mathbb{Z}\}$. We have $\mathcal{D}_0 P \mathcal{D}_1$ although the semiorder does not satisfy the Bounded P-chain condition.

On the other hand it is also easy to construct examples of semiorders that do not satisfy the Bounded P-chain condition and for which $\mathfrak{F}$, ordered by P, is not order-isomorphic with a subset of $\mathbb{Z}, >$. •

A further consequence of the Bounded P-chain condition is the following.

Proposition 59

Let $S = (P, I)$ be a semiorder on the denumerable set X and satisfying the Bounded P-chain condition. Let $\mathcal{D}$ be a connected component of (X, I) and let $\mathcal{D} = \bigcup_{m \in M} I_m$ where the I_m are convex subsets built as described in Section 3.2. If the sequence of subsets $(I_m, m > 0)$ is infinite, then $\mathcal{D}$ is up-terminal, in the sense there is no $y \in X$ with $y \succ z$ for all $z \in \mathcal{D}$. Similarly, if the sequence of subsets

$(I_m, m < 0)$ *is infinite, then* $\mathcal{D}$ *is down-terminal, in the sense there is no* $y \in X$ *with* $z \succ y$ *for all* $z \in \mathcal{D}$.

PROOF
Assume that $(I_m, m = 0, 1, \dots)$ is an infinite sequence of subsets. By Proposition 24.4, there is an infinite sequence of elements $(x_m \in I_m, m = 0, 1, \dots)$ such that $x_{m+1} P x_m$, for all $m > 0$. If there were y such that $y \succ x$ for all $x \in \mathcal{D}$, then, in particular, $y \succ x_{m+1}$ and $x_{m+1} P x_m$, for all $m > 0$, would imply $y P x_m$, for all $m > 0$, contrary to Beja and Gilboa's Condition 3, hence contrary to the Bounded P-chain condition. The other part of the proof is similar (using Proposition 24.5). □

5.5 Assembling unit representations

Let us assume that a unit representation is known for each connected component of (X, I) and that the semiorder $S = (P, I)$ on X satisfies the Bounded P-chain condition. We show how a unit representation of the whole semiorder can be built by assembling the representations on the connected components. This description enables to understand exactly which additional degrees of freedom are available when assembling unit representations. These complete the picture given in Section 4.6, regarding the uniqueness of the representation.

Assuming the Bounded P-chain condition, we know by Proposition 57 that the connected components of (X, I) can be indexed by a subset $\Gamma \subseteq \mathbb{Z}$ of consecutive integers. We assume w.l.o.g. that $0 \in \Gamma$. The following proposition distinguishes four possible cases for a connected component $\mathcal{D}_i$. It provides bounds for the representation on each I-connected component. These bounds will be essential for assembling the representations.

Proposition 60
Let $\mathcal{D}_i \in \mathfrak{F}$ *be a connected component of* (X, I) *and let* u_i *be a representation of the semiorder* (P, I) *restricted to* $\mathcal{D}_i$ *as constructed in Section 4.1. We have the following cases.*

1. *If* $\mathcal{D}_i$ *decomposes in a finite number of subsets* I_k*, for* $k = -n_i, \dots, 0, \dots, m_i$*, then the representation* u_i *is bounded, we have:* $\underline{u}_i < u_i(x) < \overline{u}_i$*, for all* $x \in \mathcal{D}_i$ *and* $m_i + n_i - 2 \le \overline{u}_i - \underline{u}_i \le 2(m_i + n_i) + 2$.

2. *If the sequence of subsets* I_k *is not bounded above but bounded below, i.e., if* I_k *exists for all* $k \in \mathbb{Z}$ *with* $k \ge -n_i$*, then the representation* u_i *has no upper bound but it has a lower bound* $\underline{u}_i$*. In this case, there is no element of* $\mathfrak{F}$ *above* $\mathcal{D}_i$*, i.e.* $i = \max\{j \in \Gamma\}$.

3. *If the sequence of subsets I_k is not bounded below but bounded above, i.e., if I_k exists for all $k \in \mathbb{Z}$ with $k \leq m_i$, then the representation u_i has no lower bound but it has an upper bound $\overline{u}_i$. In this case, there is no element of $\mathfrak{F}$ below $\mathcal{D}_i$, i.e. $i = \min\{j \in \Gamma\}$.*

4. *If the sequence of subsets I_k is not bounded neither above nor below, i.e., if I_k exists for all $k \in \mathbb{Z}$, then the representation u_i has neither an upper nor a lower bound. In this case, $\mathcal{D}_i$ is the only element of $\mathfrak{F}$ ordered by P, i.e. $\{i\} = \Gamma$.*

PROOF

If the sequence I_k is bounded, the boundedness of u_i directly results from Proposition 43. The inequality $m_i + n_i - 2 \leq \overline{u}_i - \underline{u}_i \leq 2(m_i + n_i) + 2$ obtains from Proposition 43 by observing that for any fixed choice of $a \in I_0$ and for all $x \in I_{m_i}$, $y \in I_{-n_i}$, we have

$$m_i - 1 < u_i(x) - u_i(a) \leq 2m_i + 1 \qquad (9)$$

$$n_i - 1 < u_i(a) - u_i(y) \leq 2n_i + 1. \qquad (10)$$

If there are subsets I_k for arbitrary large values of k, Proposition 59 entails that the connected component $\mathcal{D}_i$ is up-terminal, which means that $j \leq i$ for all $j \in \Gamma$. The existence of a lower bound for u_i results from Proposition 43. The case in which there are subsets I_k for arbitrary large negative values of k is similar. □

The following result indicates how to build a unit numerical representation of the semiorder on X. It is the second main result of this paper (the first one being Proposition 36). *It shows that any semiorder on a denumerable set satisfying the Bounded P-chain condition has a unit representation.*

Proposition 61

Let u_i be a representation of the semiorder (P, I) restricted to each connected component $\mathcal{D}_i$, for $i \in \Gamma$. For each bounded u_i, let $\underline{u}_i$ (resp., $\overline{u}_i$) denote its lower (resp., upper) bound. Let ε_i, for $i \in \Gamma, i \neq 0$ be arbitrary nonnegative or positive real numbers. For $i \geq 1$, ε_i must be positive if both bounds $\underline{u}_i$ and $\overline{u}_{i-1}$ are attained, otherwise it can also be set to zero. For $i \leq -1$, ε_i must be positive if both bounds $\underline{u}_{i+1}$ and $\overline{u}_i$ are attained, otherwise it can also be set to zero. The function

$u : X \to \mathbb{R}$ defined by:

$$u(x) = \begin{cases} u_0(x) & \text{for all } x \in \mathcal{D}_0 \\ u_i(x) + \overline{u}_0 - \underline{u}_i + 1 + \varepsilon_i \\ \qquad + \sum_{1 \le j \le i-1} (\overline{u}_j - \underline{u}_j + 1 + \varepsilon_j), & \text{for all } x \in \mathcal{D}_i, i \in \Gamma, i \ge 1 \\ u_i(x) + \underline{u}_0 - \overline{u}_i - 1 - \varepsilon_i \\ \qquad - \sum_{i+1 \le j \le -1} (\overline{u}_j - \underline{u}_j - 1 - \varepsilon_j), & \text{for all } x \in \mathcal{D}_i,\ i \in \Gamma,\ i \le -1 \end{cases}$$

is a unit numerical representation of the semiorder (P, I) on X.

Proof

If $i = 1 \in \Gamma$, then u_0 is bounded above and $u(x) \le \overline{u}_0$, for all $x \in \mathcal{D}_0$. Since $\mathcal{D}_1$ is not down-terminal, u_1 is bounded below by $\underline{u}_1$ and, by definition of u, we have $u(y) = u_1(y) + \overline{u}_0 - \underline{u}_1 + 1 + \varepsilon_1$ for all $y \in \mathcal{D}_1$. In this expression, $\varepsilon_1 \ge 0$. Moreover, $\varepsilon_1 \ne 0$ iff there is $x \in \mathcal{D}_0$ such that $u_0(x) = \overline{u}_0$ and $z \in \mathcal{D}_1$ such that $u_1(z) = \underline{u}_1$. For all $x \in \mathcal{D}_0$ and $y \in \mathcal{D}_1$, we have $u(y) \ge \overline{u}_0 + 1 + \varepsilon_1 \ge u_0(x) + 1 + \varepsilon_1$, which represents correctly the fact that yPx.

If $i + 1 \in \Gamma$ for $i \ge 1$, then $u_i(x) \le \overline{u}_i$, for all $x \in \mathcal{D}_i$. For such an x, $u(x) = u_i(x) + \overline{u}_0 - \underline{u}_i + 1 + \varepsilon_i + \sum_{j=1}^{i-1} (\overline{u}_j - \underline{u}_j + 1 + \varepsilon_j) \le \overline{u}_0 + \sum_{j=1}^{i} (\overline{u}_j - \underline{u}_j + 1 + \varepsilon_j)$. Since $\mathcal{D}_{i+1}$ is not down-terminal, u_{i+1} is bounded below by $\underline{u}_{i+1}$. For all $y \in \mathcal{D}_{i+1}$, we have $u(y) = u_{i+1}(y) + \overline{u}_0 - \underline{u}_{i+1} + 1 + \varepsilon_{i+1} + \sum_{j=1}^{i} (\overline{u}_j - \underline{u}_j + 1 + \varepsilon_j) \ge \overline{u}_0 + 1 + \varepsilon_{i+1} + \sum_{j=1}^{i} (\overline{u}_j - \underline{u}_j + 1 + \varepsilon_j)$. Therefore, $u(y) > u(x) + 1$, for all $y \in \mathcal{D}_{i+1}$ and $x \in \mathcal{D}_i$, which represents correctly the fact that yPx.

On the negative side, if $i = -1 \in \Gamma$, then u_0 is bounded below and $u(x) \ge \underline{u}_0$, for all $x \in \mathcal{D}_0$. Since $\mathcal{D}_{-1}$ is not up-terminal, u_{-1} is bounded above by $\overline{u}_{-1}$ and we have $u(y) = u_{-1}(y) + \underline{u}_0 - \overline{u}_{-1} - 1 - \varepsilon_{-1} \le \underline{u}_0 - 1 - \varepsilon_{-1} \le u_0(x) - 1 - \varepsilon_{-1}$ for all $y \in \mathcal{D}_{-1}, x \in \mathcal{D}_0$. Therefore, $u(x) \ge u(y) + 1 + \varepsilon_{-1}$, for all $x \in \mathcal{D}_0$ and $y \in \mathcal{D}_{-1}$, which represents correctly the fact that xPy.

If $i - 1 \in \Gamma$ for $i \le -1$, then $u_i(x) > \underline{u}_i$, for all $x \in \mathcal{D}_i$. For such an x, $u(x) = u_i(x) + \underline{u}_0 - \overline{u}_i - 1 - \varepsilon_i - \sum_{j=-1}^{i+1} (\overline{u}_j - \underline{u}_j - 1 - \varepsilon_j) \ge \underline{u}_0 - \sum_{j=-1}^{i} (\overline{u}_j - \underline{u}_j - 1 - \varepsilon_j)$. Since $\mathcal{D}_{i-1}$ is not up-terminal, u_{i-1} is bounded above by $\overline{u}_{i-1}$ and we have, for all $y \in \mathcal{D}_{i-1}$, $u(y) = u_{i-1}(y) + \underline{u}_0 - \overline{u}_{i-1} - 1 - \varepsilon_{i-1} - \sum_{j=-1}^{i} (\overline{u}_j - \underline{u}_j - 1 - \varepsilon_j) \le \underline{u}_0 - 1 - \varepsilon_{i-1} - \sum_{j=-1}^{i} (\overline{u}_j - \underline{u}_j - 1 - \varepsilon_j) \le u(x) - 1 - \varepsilon_{i-1}$. Therefore, $u(x) \ge u(y) + 1 + \varepsilon_{i-1}$, for all $x \in \mathcal{D}_{-i}$ and $y \in \mathcal{D}_{i-1}$, which represents correctly the fact that xPy.

The semiorder is correctly represented in each connected component $\mathcal{D}_i$, for $i \in \Gamma$, due to the fact that u_i represents the restriction of the semiorder to $\mathcal{D}_i$ and $u(x)$ obtains by adding the same constant to $u_i(x)$ for all $x \in \mathcal{D}_i$. □

Remark 62 (Uniqueness issue)
The construction of a unit representation of the whole semiorder by assembling unit representations of the semiorders on the connected components described in Proposition 61 is not restrictive. Indeed, it is clear that, starting conversely with a unit representation u of the whole semiorder (P, I), we obtain unit representations u_i by restricting u to $\mathcal{D}_i$ for all $i \in \Gamma$. For all $i-1, i \in \Gamma$, for all $x \in \mathcal{D}_i, y \in \mathcal{D}_{i-1}$, we have $u(x) > u(y)+1$. Denoting by $\overline{u}_{i-1}$ (resp., $\underline{u}_i$) the upper (resp., lower) bound of u on $\mathcal{D}_{i-1}$ (resp., $\mathcal{D}_i$), we define $\varepsilon_i = \underline{u}_i - \overline{u}_{i-1} - 1$. We have that $\varepsilon_i \geq 0$. This number can be 0 only if at least one of the bounds $\underline{u}_i, \overline{u}_{i-1}$ is not attained. These numbers, which are also those used in Proposition 61, determine the minimal difference between the value of elements in consecutive connected components. They are the additional degrees of freedom available in the representation of a semiorder when the latter has several connected components. Putting this result together with the analysis made in Section 4.6 gives a complete picture of the degrees of freedom involved in unit representations of a semiorder. •

Remark 63
As explained in Remark 38, on a single I-connected component, it is always possible to build a representation that is at the same time strict and non strict (see Equations (1) and (2)). It is not difficult to check that the procedure used above to assemble representations of several connected components allows to make the same observation for semiorder that are not restricted to have a single I-connected component. This gives an alternative proof of the observation made in Beja and Gilboa (1992, Th. 3.8, p. 436) (a similar observation was already made by Roberts, 1971, p. 36, footnote), as well as establishing the stronger statement that representations that are at the same time strict and nonstrict always exist on denumerable sets.

Notice also that, if the assembled representations are all rational unit representations, as explained in Remark 38, the above process can always be performed so as to guarantee that the overall representation stays in $\mathbb{Q}$. This gives an alternative proof of Manders (1981, Prop. 7, p. 236). •

Remark 64 (The uncountable case)
The just described process of assembling strict (resp. nonstrict) unit representations of the restrictions of a semiorder to its I-connected components into a strict (resp. nonstrict) unit representation of the whole semiorder does not depend on the assumption that X is denumerable. It only depends on the Bounded P-chain hypothesis and the existence of a strict (resp. nonstrict) unit representation of the restrictions to each I-connected component. This results from the fact that the decompositions of X described in Sections 2.5 and 3 are valid independently of the cardinality of X. Furthermore, if a strict (resp. nonstrict) unit representation exists for each I-connected component of the semiorder and the Bounded P-chain

condition is verified, then the bounds (9) and (10) hold (since they only depend on the number of maximal indifference classes in each I-connected component), which allows assembling the representations, whatever the cardinality of X, in particular, for uncountable X. •

To conclude, we summarize the main results that we proved in this paper as follows.

Theorem 65

1. *Any I-connected semiorder $S = (P, I)$ has a unit representation.*

2. *A semiorder on a denumerable set has a unit representation iff it satisfies the Bounded P-chain condition.*

3. *If a semiorder on a denumerable set has a unit representation,*
   - *it has a representation that is at the same time strict and non-strict;*
   - *it has a representation on $\mathbb{Q}$.*

Statement 1 results from Proposition 36. Statement 2 is proved by Proposition 61. Statement 3 is justified in Remark 63.

6 Discussion

We have offered a new proof of the existence of a unit representation of semiorders on countably infinite sets. Our proof uses only elementary considerations. It is based on the analysis of each I-connected component of the semiorder. On each such component, we build in a recursive way a partition of this component into maximum indifference classes. One such class is taken as a reference set and ghosts representing elements in the other classes are adequately inserted into this reference set. The numerical representation built on the reference set enriched with all ghosts, is then lifted to build the desired unit numerical representation. As announced, it unifies the treatment of the finite and countably infinite cases. Moreover, we feel that our proof is simpler and more direct than the two previous ones in the literature (Beja and Gilboa, 1992, Manders, 1981).

In a companion paper (Bouyssou and Pirlot, 2020b), we show that the same technique can be extended, through the introduction of adequate order-denseness conditions, to cover the general case (see Candeal and Induráin, 2010). Hence, the tools presented in this paper offer a common scheme to build unit numerical representations of semiorders.

Besides the generalization of our results presented in Bouyssou and Pirlot (2020b), the field offers many opportunities for further studies. Let us mention here one of the more intriguing one. It is clear that the function u used in the unit representation of a semiorder can be constrained to take only values in the set of rational numbers $\mathbb{Q}$ (it suffices to do so in Proposition 36, which is always possible since any linearly ordered denumerable set can be embedded into $(\mathbb{Q}, \geq)$). Investigating which semiorders have a representation using a function u taking its values in $\mathbb{Z}$ is an open problem. Its solution would allow to generalize the analysis of minimal integer representations in the finite case proposed in Pirlot (1990, 1991).

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