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Ordinal Power Indices

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Introduction

The design of procedures aimed at ranking individuals according to how they behave in various groups is of great importance in many practical situations. The problem occurs in a variety of scenarios coming from social choice theory, cooperative game theory or multiattribute decision theory, and examples include: comparing researchers in a scientific department by taking into account their impact across different teams [Papapetrou et al., 2011]; finding the most influential political parties in a parliament based on past alliances within alternative majority coalitions [Marošević and Soldo, 2018]; rating attributes according to their influence in a multiattribute decision context, where independence of attributes is not verified because of mutual interactions (see [Bouyssou and Marchant, 2007] for a discussion on winning coalitions of criteria, [Boutilier et al., 2004] for CP-nets concerned by qualitative conditional dependence and independence of preference statements under a *ceteris paribus* interpretation); and quantifying individual's productivity in the presence of teamwork taking into account that contribution of an individual to a team may also depend on the individual's productivity, since individuals who are more productive bring more expertise, team-building skills, and visibility, and they contribute more on average [Flores-Szwagrzak and Treibich, 2020].

In many real world applications, a precise evaluation of coalitions' "power" may be hard or even impossible due to a bunch of unknown factors: existence of uncertain data, complexity of the analysis, missing information or difficulties in the update, etc. In such situations, measuring the importance of individuals using classical power indices is not always straightforward. In this case, it may be interesting to consider only ordinal information concerning binary comparisons between coalitions. For instance, suppose director of a department wants to evaluate performance of professors based on their contribution in scientific groups. Also, assume the only information provided to the director is that one group performs better than another one or that the two groups have the same level of performance. This is a valid assumption since it is not possible to evaluate performance of scientific groups by numbers; the performance of a scientific group depends on a combination of factors like the number of publication made by the group, the importance of the subjects to the department, the number of citations, the quality of their papers, and many other factors that may be hard to quantify.

Some scientists, modeled the lack of information in such situations with probabilistic methods [Suijs et al., 1999], or with estimating the value of coalitions using intervals [Branzei et al., 2010]. However, these methods are not always applicable due to various types of uncertainty. In this thesis, we follow the same approach as in [Moretti and Öztürk, 2017] and [Bernardi et al., 2017], and model the worth of coalitions in an ordinal way using a binary relation which is defined over the set of coalitions. Therefore, throughout the thesis we provide answers to the general question of how to obtain a ranking over a finite set N of individuals (called a *social ranking*), given a ranking over the elements of a power set 2^N (called a *power relation*, normally denoted by   or).

In the problem of professors' evaluation, suppose given a set of five professors $N = \{1, 2, 3, 4, 5\}$, the director of department wants to rank them. Also, assume the information provided to the director is the power relation $\{2, 4, 5\} \quad \{1, 3\} \quad \{1, 2\} \quad \{2, 3\} \quad \{2, 4, 5\} \quad \{3, 5\} \quad \{2, 4\} \quad \{2, 5\} \quad \{1, 4\}$ that indicates the relative performance of different scientific groups. For instance, $\{2, 4, 5\} \quad \{1, 3\}$ means a group consisting of three professors 2, 4, and 5 performs strictly better than a group of professors 1 and 3, and $\{2, 4, 5\} \quad \{3, 5\}$ means the performance of the corresponding groups are the same.

Our goal is not only to define a social ranking over a set of individuals, but the majority of our research concerns a set of properties (axioms) that social ranking rules should satisfy. To the best of our knowledge, the issue was first introduced by [Marichal and Roubens, 1998], but it was formally studied by [Moretti, 2015] and [Moretti and Öztürk, 2017], where social ranking solutions were analysed following a property-driven approach. They evaluate the effect of basic properties in the combination of social ranking, and show that the pairwise combination of these natural properties yields either to impossibility (i.e., no social ranking exists), or to flattening (i.e., all the individuals are equally ranked), or to dictatorship. Within the same framework, [Bernardi et al., 2017] axiomatically characterized a social ranking solution based on the idea that the most influential individuals are those appearing more frequently in the highest positions in the ranking of coalitions. A more practical approach to this problem have been studied in [Fayard and Escoffier, 2018] in which the authors implement a social ranking rule proposed in Chapter 2 in order to find an approximation of the minimum number of coalitions to be removed in order to satisfy the transitivity. To explore new methods to rank individuals given an ordinal ranking over their coalitions, in this thesis, we use different notions from classical social choice theory and cooperative game theory.

Along with the aim of this thesis, in Chapter 1 we do a literature review on the contexts related to the thesis. We discuss axiomatic study and its components. We also describe the type of results expected from the study and their importance. We review the axiomatic studies that have been done in the contexts of voting theory, cooperative game theory, and ranking sets of objects. We finally study the recent advancement in our ranking problem.

Chapter 2 introduces our first approach to address the ranking problem. In this chapter, we investigate the use of a *ceteris paribus* majority principle as a social ranking solution. According to this ranking method, two individuals are ranked using information from *ceteris paribus* (i.e., everything else being equal) comparisons over all possible coalitions. This suggests an interpretation of our problem along the lines of a virtual election, with groups of individuals (coalitions) playing the role of voters. Unfortunately, the *ceteris patibus* majority solution can lead to a *Condorcet-like paradox*. Therefore, a domain restriction over the family of ranking of coalitions is proposed to guarantee the transitivity of the ranking over the individuals. The chapter concludes by a discussion on possible interpretations of incompleteness of the power relation. We propose a new social ranking rule to take into consideration a specific interpretation of incompleteness.

Chapter 3 presents another method to rank individuals. The new solution is defined by the extension of the notions of marginal contribution and of the Banzhaf index in classical cooperative games, and is called *ordinal Banzhaf solution*. We restrict our attention to power relations as linear orders, and we characterize the resulting *ordinal Banzhaf solution* by means of a set of axioms inspired from those introduced in Chapter 2. The similarity between the axiomatic study of the two solutions motivates us to explore the similarities and differences of the solutions in more detail.

Chapter 4 of the thesis is dedicated to axiomatic study of families of *weighted ceteris paribus* majority rules. The social ranking rules in this chapter are weighted extension of the *ceteris paribus* majority rule to rank more than two individuals. Based on the interpretation of the ranking problems, the weights assigned to coalitions (voters) can be a function of the coalitions, the set of individuals getting compared by the coalition, the combination of them, their sizes, and so on. Since the weight functions can be defined in infinitely many ways, each interpretation results in a specific family of weighted social ranking rules which is a sub-family of other families. The inclusion relation between the families forms a tree whose edges show the corresponding inclusion relations between families, and the main goal of the chapter is to analyse each family of solutions as a subset of another family of solutions by axiomatic study of their properties.

The contributions of the thesis are published in the proceedings of international conferences, namely IJCAI 2018 [Haret et al., 2018] and IJCAI 2019 [Khani et al., 2019].

1 Literature Review

1.1 Social Choice Theory

Social choice theory is a theoretical framework to investigate and analyze how to combine individuals' preferences to reach a collective decision or *social welfare* in some sense. The increasing interest to social choice comes from its strong connection to other fields of computer science, as well as a vast exchange between them. Precisely speaking, one reason of this importance stems from imported notions from computer science to the context of social choice in order to solve problems originated in social choice, like computing the complexity of methods in social ranking, or, for instance, specifying voting methods where manipulation is hardly plausible. On the other hand, developed techniques in social choice theory can be used in order to solve problems in the context of computer science and artificial intelligence [Brandt et al., 2016b]. An example is the applications of social choice in order to develop page ranking systems in Internet search engines [Chevaleyre et al., 2007], [Tennenholtz, 2004]. As we will see, in the vast part of the thesis we benefit from one of the original ideas in social choice theory in order to find solutions for a complex ranking problem.

The topics covered in (computational) social choice can be classified based on two distinct lines: the nature of the social choice problem we deal with, and the type of formal or computational techniques studied [Chevaleyre et al., 2007]. In this thesis, we mainly focus on three of them: *voting theory*, *coalition formation*, and *ranking systems*.

- **Voting theory**. Given a set of voters and a set of candidates, the question of interest in voting theory is how to aggregate the opinion of voters (represented as their ballots over a set of candidates) to find a ranking over candidates, or to find the best candidate. This question arises in many areas like business, social organisations, or politics. The root of this question comes back to Romain times, when Pliny and other senators in the Senate had to decide on the fate of a number of prisoners: acquittal (A), banishment (B), or condemnation to death (C). Although option A, favored by Pliny, had the largest number of supporters, it did not have an absolute majority. One of the proponents of harsh punishment then strategically moved to withdraw proposal C, leaving its former supporters to back option B, which is obviously the winner of majority contest between A and B. If senators have voted on all three options using plurality rule, then option A would have won. This example illustrates several interesting features of voting rules. For example it may be interpreted as demonstrating a lack of fairness of the plurality rule: even though a majority of voters believes A to be inferior to one of the other options (namely, B), A still wins [Brandt et al., 2016b]. In fact, when there are just two candidates choosing the best candidate is straightforward. It goes the way of the *majority*. However, when there are more than two candidates there is no one obvious way of choosing the best candidate. Different methods are proposed that each one takes care of some specific sense of fairness. Another clue on the use of voting rule in the Middle Ages is the writings of the Catalan philosopher, poet, and missionary Ramon Llull (1232-1316) about voting rules. He supported the idea that election outcomes should be based on direct majority contests between pairs of candidates. The rule that he referred to seems to be the one nowadays known as Copeland rule [Copeland, 1951], under which the candidate who wins the largest number of pairwise majority contests is elected.

Another attempts in the field of voting rules are the works of the French engineer Jean-Charles de Borda (1733-1799) and the French philosopher and mathematician Marquis de Condorcet (1743-1794). The discussion between these two scientists motivated Borda to propose a method of voting, today known as the Borda rule. According to this rule, the winner of an election is chosen by giving each candidate, a number of points corresponding to the number of candidates ranked lower [Brandt et al., 2016b]. He argued the superiority of his method over plurality rule by an example indicating the Borda rule is a *single-winner* voting rule.

Although early works on the context of collective decision making and voting theory was completely limited to design various voting rules and comparing their pros and cons, this manner had changed due to the seminal work of Kenneth Arrow in 1963, in which he followed a broader view and highlighted some common properties in all the proposed voting rules. Arrow explained the philosophical and economical motivations to define different voting rules in mathematical terms as axioms [Arrow, 1963].

- **Coalition Formation**. Since coalition formation is considered as a choice process involving more than two individuals [van Deemen, 1991], social choice theory is important in the context: the process of forming coalitions is considered as a preference aggregation problem in which each player has a preference over possible coalitions, and the question is how to aggregate such preferences in order to form the coalitions. Another important question arises after forming coalitions: how to distribute the sharing benefits or costs of a coalition among its members. To do so, a kind of ranking over the set of players based on their performance in the coalition is needed. These problems are studied in the field of cooperative game theory [van Deemen, 1991]. Cooperative game theory analyzes how coalitions of individuals can form, and how they should distribute the sharing benefits or costs of their cooperation. The notion of cooperative games was first introduced in the essays of Von Neumann Morgensten [von Neumann and Morgenstern, 2007] as an attempt to distinguish two approaches of cooperative and non-cooperative games.

Simple games are cooperative games in which the coalitions are partitioned in two sets, the set of winning coalitions and the other coalitions. Simple games are used as a model for binary voting situations: in the case of having two candidates, the agents in favor of the most appealing one form a winning coalition and the others will be the losing coalition. The decisions of the winners concern the whole set of players and the losers are obliged to take these decisions for granted, whether the effects of the winners' decisions are favourable for them or not [van Deemen, 1991]. An example is the majority voting game. In this game only a majority coalition of voters can win, i.e. determine a winning alternative. The Benefit of such abstraction is that simple games can be studied without referring to specific rules like majority, plurality and so on. Power indices like Shapley-Shubik [Shapley, 1953], and Banzhaf index [Banzhaf III, 1964] are introduced in order to measure the power of agents in each coalition. A real situation where power indices are used is the problem that some Nordic countries were once faced to join or not to join the European Union. The problem arose since Nordic countries traditionally give high priorities (compared to European Union) to Environmental protections. Hence, the question was how the new members of the European Union could affect the environmental standards of European Union, e.g., to adjust them to their own standards. The Shapley value provides an appropriate solution by evaluating the power of each member when they join the EU [Holzinger, 1995].

- **Ranking sets of objects**. Imagine a given set of statements, each one of them with a degree of plausibility, and suppose the goal is to choose a set of statements that are more plausible than the others. At the first glance, the answer to this problem seems to be simple: just order the statements based on their degree of plausibility, and then select the statements at the top of the ranking as the set of statements which is highly plausible [Packard, 1981]. However, note that in many cases a combination of two plausible statements is not necessarily plausible because combination of two plausible statements may form an inconsistent set of statements. The consideration of having inconsistency when two plausible items are combined was the beginning of a field study whose aim is to rank sets of objects when a ranking over the the set of objects is provided. In [Barberà et al., 2004], the authors categorize the ranking problem into three classes of *complete uncertainty*, *opportunity sets*, and *sets as final outcomes*. These categories are defined based on what is the goal of dealing with sets of objects. Different answers are provided for the problem according to the category that the ranking problem falls into.

1.1.1 Solutions to Social Choice Problems

As we have seen in the previous section, different problems in the context of social choice can be defined. Given a society of individuals or agents, the problem, in general, deals with a collective decision making process that best reflects the opinion of the members in the society. The main parts in all these problems is a set of individuals or agents and a set of candidates. In social choice theory, the decision making problems are mainly indicated by specifying the data pertaining to a set of individuals and the data related to a set of alternatives (which is the opinion of members in the society about the candidates). For instance, consider a voting scenario in which some colleagues want to decide on choosing a date for a special event (such as Christmas dinner or a movie screening). The data related to the voters (colleagues) and candidates (different dates) as well as the preferences of the colleagues about the dates can be expressed in a Doodle.

In cooperative game theory, consider the example of a football team who wins a match and suppose there are some bonus to be given to players according to their contribution in the team. In this case, the level of players' contribution in the team as well as the ways that the bonus can be divided among them represents the data related to the cooperative game problem. Note that it is possible to observe this problem as an aggregation problem in which individuals are represented by players in the team and alternatives are different possible ways to distribute the bonus.

In the context of ranking sets of individuals, the goal is to lift ranking over individuals to ranking(s) over sets of individuals. Considering all possible rankings over sets of individuals as a set of alternatives, the ranking problem can be defined as selecting alternative(s) that best fits to the ranking over individuals and possible positive or negative synergy among them.

A solution to social choice problems is, in general, single-valued or multi-valued based on the context in which the problem is defined and its interpretation. In some contexts, the interest is on solutions which result in just one outcome, while in the others the interest is to find out more than one outcome by a solution. For instance, in the context of voting theory, specially in most of political electoral systems, an admissible solution is to have just one winner candidate. On the other hand, multi-valued solutions are appropriate, for instance, in the problem of ranking sets of statements, in which indifference between sets of statements is a possibility.

Note that for a given problem there may exist more than one solution resulting in different outcomes, and defining such solutions is not a hard task to do. For example, for the voting problem it is possible to design a voting system that follows the opinion of just one of the voters to choose the best candidate (called dictatorship), or the system that goes by the concept of majority. However, what is essential in defining the solutions is to see which one is more reasonable in the sense of satisfying a set of intuitive and naturally appealing properties. To see the importance of verifying solutions from property driven approach, consider the voting problem. In the case of two candidates the most logical way to select the candidates is to go with majority. However, by having more than two candidates, majority may not be considered as an appealing approach in some applications, because sometimes it brings about cycles in the final outcome.

[Arrow, 1963] studied the solutions of voting theory from the property driven approach, and he argued that any acceptable method of aggregation (resulting a pre-order over the set of candidates) should satisfy at least the following three axioms:

1. If every individual ranks x above y, then so should society.
2. It should be possible to determine the relative social ranking of x and y by considering only the relative ranking of x and y supplied by each of the individuals.
3. The voting system should always return exactly one clear final ranking.

For example, notice that the majority rule does satisfy two of these requirements, but it may generate a preference order with cycles, when there are more than two individuals. As we will see in the next section, analysing properties of solutions in social choice theory is beneficial from many facets.

In the next section, we discuss axiomatic study and its components. We also investigate the type of results expected from axiomatic study and their importance.

1.1.2 Axiomatization

Several aspects of computational social choice have been investigated in order to qualify the solutions for a given social choice problem [Brandt et al., 2016b]. Optimality of a solution in sense of computation burden is one of the important issues in order to solve concrete problems. Evaluating solutions according to their complexity in order to solve a problem is widely studied and investigated in the context of computational social choice. However, the classical approach to study solutions in social choice theory is to analyze sets of properties or axioms that they satisfy. In the terminology of social choice, such kind of studies are called *axiomatic study* [Thomson, 2001]. Until recently, axiomatic study had been the primary method of investigation in a few branches of economics and game theory, such as social choice and utility theory. However, in the last years this method is expanded specially in two domains of bargaining theory and coalition formation. Such expansions shed new light to axiomatic techniques and nowadays, they are used more purposefully in order to compare different solutions or even finding new solutions that satisfy remarkable pleasant properties [Thomson, 2001].

Axiomatic study is urged by the need to differentiate between solutions that are plausible for a social choice problem. As mentioned in the previous section, for a specific domain of problems several intuitively appealing solutions may exist. Axiomatic study allows us to validate the existence of solutions according to simple and natural interpretations. Furthermore, when there is no solution with intuitive natural interpretation, axiomatic study can help us to find one. Given a domain of problems, axiomatic study starts with a list of desirable properties for the domain and ends up by describing the solutions to the problem as precisely as possible using the given properties. It also allows to investigate the logical relations between axioms, and to see how changes in the domain of problems can affect the axioms. Normally, axiomatic study of solutions leads to a characterization theorem which is a description of the solution according to given properties, although the ultimate goal of axiomatic study goes further than that. In fact, the goal of axiomatic study should be to understand and describe the implications of the list of properties as precisely as possible [Thomson, 2001].

There are two reasons that motivate analyzing solutions from the axiomatic perspective. The first reason is that although defining solutions is not a cumbersome task per se, focusing just on defining solutions prevents us from exploring the whole space of solutions for a given problem. There might be other solutions that satisfy much more appealing properties than currently defined ones, and we cannot achieve them without defining the properties and combining them. Thus, axiomatic study helps us to have a broader view over the space of all feasible solutions, and to lighten all the corners of the solutions' space in order to find nice kind of solutions. It is also worth mentioning that sometimes axiomatic study frees us from effortless searching for solutions

[Thomson, 2001]. For instance, the most intuitive voting method in the case of two candidates is the majority rule that satisfies a set of benign properties, yet moving to the domains with three candidates or more, majority may result in cyclic ranking over candidates, which is not appealing in many practical scenarios. Accordingly, one may look for other solutions that meet the mentioned requirements while avoid cycles in the ranking results. However, [Arrow, 1963] proved an impossibility result which expresses that such kind of solutions are not available!

The second reason for axiomatic study is that sometimes by intuition a solution may be recognized to return right answers, however other solutions may exist, in particular situations, that are equally successful for these examples. These solutions can be reached by axiomatic evaluation of the properties.

The other important note about axiomatic study is that in many of the cases the characterization is done for a specific solution. For instance, solutions like majority rule and Borda are widely studied and applied in the literature of social choice theory. One may ask what properties are satisfied by these solutions that make them so practical? Answer to this question is somehow clarified by the characterization done by [May, 1952]. This kind of characterizations is legitimate when the solution is widely used in practice or in theoretical literature, like majority rule in voting theory or Shapley value in cooperative game theory. Axiomatic study of these solutions reveals the reason of their prevalent use in the literature.

As we will see in the next chapters, most of the solutions introduced to solve the ranking problem in this thesis are inspired from well-known and widely used solutions in other contexts like voting theory and cooperative game theory. Therefore, figuring out how the properties of solutions change when they adjusted to other frameworks is worthy, and it gives better insight to define new solutions specific to different frameworks.

What we have mentioned about the importance of axiomatic study was about merits of characterizing single solutions, however, characterizing families of solutions merits more. In fact, by considering a big family of solutions (that have a common structure) we can define some small sets of axioms, even one axiom, and see which members of the family satisfy them. Such axioms result in theorems that analyze inclusion relation between two families of solutions. The axiomatic study of the inclusion relation can be continued until reaching a single member whose belonging to a family is characterized with a set of axioms. Defining axioms again and again for members of hierarchical families of solutions will branch off in several directions, and each branch leads to just one single solution. To clarify, consider the family of all scoring methods in voting theory. Such scoring methods score candidates by some specific scoring systems. Based on the context that the scoring methods are applied to, there are infinitely many scoring system in order to score candidates [Chebotarev and Shamis, 1998]. A sub-family of such scoring rules contains voting methods that score candidates based on their position in voters' preferences. This sub-family has infinitely many scoring systems as well. For instance, in the original Borda's system the scoring of the candidates depends on the number of candidates stand in the voting process [Young, 1974]: if there are five candidates, the highly ranked candidate in a voter's preference gets the score of five, the candidate in the second position gets the score of four, and so on until the last positioned candidate gets the score of one. Other variant of the scoring rule can start scoring from zero instead of one (in the example of five candidates, assigning the score zero to the last positioned candidate and score of four to the highly ranked one). Another scoring rule can score candidates proportional to their position in a given preference (the highly ranked one get the score of $\frac{1}{1}$, the second one gets the score of $\frac{1}{2}$ and so on) [Fraenkel and Grofman, 2014]. From this sub-family other sub-families are plausible, and the hierarchical structure can be continued to reach only one member (For instance, Borda rule). The axiomatic study in this context, is to analyze the inclusion relation between different families, like the family of general scoring rules and those with scoring systems related to the positioning of candidates, or, for instance, characterizing a specific scoring method like Borda, as a member of the sub-family of scoring rules with scoring system based on the positioning of candidates.

The relation between axioms that characterize a solution is the essential part of axiomatic study that needs more attention. The characterization theorem of a specific solution (rule), denoted by *solution* is of form "A solution satisfies a list of properties **if and only if** it is *solution* ". It is relatively easy to prove that *solution* satisfies a set of axioms, since we can deal with axioms one by one and the interaction between them is not important. However, proving the other part of the theorem, saying "if a solution satisfies a certain list of properties then it is the particular solution" needs to consider also the interaction among the axioms. More precisely, we need to look at all the axioms in the assumption at the same time in order to prove the theorem. This part of the characterization theorem is called *uniqueness part*. The importance of independence of the axioms can be seen in this part of the theorem: independence means deleting any one of the axioms from the list of axioms, other solutions (different from *solution*) fulfill the remaining list of axioms. Taking care of the independence of axioms in the characterization theorem simply guarantees that the results provided in the theorem are in the more general form: if one of the axioms in the theorem is redundant we widen the scope of the results by deleting it [Thomson, 2001].

Taking all the considerations into account, in the following sections we review the axiomatic study of the voting theory introduced by [Arrow, 1963] and [May, 1952]. Also, Section 1.3 is devoted to axiomatic study in the context of cooperative game theory.

1.2 Voting Theory

In this section, we review some of the literature in the context of voting theory. We study the problem of aggregating individuals' preferences, and analyze the famous aggregation methods (solutions) from property driven approach. Kenneth Arrow is one of the pioneering scientists who studied the problem of finding aggregation methods as a collective decision making process given a set of alternatives, among which there is a choice to be made. He introduced the class of solutions for the aggregation problem, called *social welfare functions*.

Referring to it as a voting theory, the nature of voters and alternatives depends on the settings of the problem. For instance, in an electoral system the voters are individuals in a society and the alternatives stand as the candidates in an election. Because voters and candidates may vary by the definition of voting problem, conventionally, they indicate voters by natural numbers and the candidates by lower case letters. Also, since the problem of finding an aggregation method arises in a very general context, the aggregation methods are normally designed before being used in an aggregation process. Thus, the question of interest forms more precisely as: Given a set of voters and also a set of alternatives, which procedure yields a social ordering of the alternatives, no matter what voters' preferences are over the alternatives.

To answer this question, Arrow defined a framework in which voters are illustrated as a set $N = \{1, \ldots, n\}$ ($n \geq 2$), which is a finite set of individuals or voters. Also, he showed the set of alternatives as $\mathcal{X} = \{x, y, z, \ldots\}$, which is mainly assumed to be finite. In this framework, Arrow assumed that each voter i in N is endowed with a preference over the alternatives in $\mathcal{X}$, which is merely ordinal. The preference of voter i on the set of alternatives is indicated by a binary relation R_i, which ranks the alternatives from worst to best according to voter i's view point. Since the rationality of voters are crucial in Arrow's framework, he presumed that the preferences made by voters are complete and transitive. The preferences made by voter i means that given any $x, y \quad \mathcal{X}$, $x\ R_i\ y$ refers to "individual i weakly prefers x to y". The main idea for this assumption is that, in reality, alternatives are not "interpersonally comparable": in practice, there is no saying that how much more strongly one voter prefers one alternative to others while the other voter's preference on the set of alternatives is other way around. We write $x\ P_i\ y$ if $x\ R_i\ y$ and not $y\ R_i\ x$ (read it as "individual i strictly prefers x to y"), and $x\ I_i\ y$ if $x\ R_i\ y$ and $y\ R_i\ x$ (read it as "individual i is indifferent between x and y"). Denoting the set of all binary relations over the set $\mathcal{X}$ as $\beta(\mathcal{X})$, we write $R_i \quad \beta(\mathcal{X})$ for all $i \quad N$.

A preference profile in this framework is an n-tuple $\mathcal{R} = (R_1, ..., R_n) \quad \beta(\mathcal{X})^n$ which is a vector of binary relations of n voters over the set of alternatives. As an example, a framework containing three voters and three alternatives is illustrated as a set $N = \{1, 2, 3\}$ of voter and the set of alternatives $\mathcal{X} = \{a, b, c\}$. Also each voter has a preference over the set $\mathcal{X}$. For instance, the preferences for each voter can be as follows:

$$a\ R_1\ b\ R_1\ c$$

$$b\ R_2\ a\ R_2\ c$$

$$b\ R_3\ c\ R_3\ a.$$

Now a social welfare function,F, is defined as a function that assigns to each profile $(R_1, R_2, \ldots, R_n)$ (in some domain of admissible profiles) a social preference relation $R = F(R_1, R_2, \ldots, R_n)$ on $\mathcal{X}$. Everywhere which is clear from the context instead of using F we refer to the collective ranking over the set of individuals corresponds to $(R_1, R_2, \ldots, R_n)$ as R [Arrow, 1963]. For instance, in the example above if majority results in a social ordering over the set of alternatives, then it returns $b\ R\ a\ R\ c$ as the result of aggregation process. Note that, in general, for any two alternatives $x, y \quad \mathcal{X}$ the relation $x\ R\ y$ refers to the collective weak preference of x to y, and $x\ I\ y$ indicates the collective indifference between the two individuals ($x\ R\ y$ and $y\ R\ x$).

One of the paradigmatic examples of social welfare functions that fits to the model, in most cases, is the *pair-wised majority rule*. This social welfare function is widely discussed by [De Condorcet, 1785]. Based on this solution, for any two individuals $x, y \quad \mathcal{X}$, $x\ R\ y$ if the majority of voters prefers x to y. From its definition, it is easy to verify that this type of social welfare does not guarantee transitivity of the final ranking over candidates. In fact, there are situations where applying majority rule in order to select among more than two alternatives yields a cycle in the collective ranking of alternatives, which prevents selection of the best alternative(s). This case is called *Condorcet paradox*, named after the Marquis de Condorcet (1743-1794). The following example explains one of the situations that *Condorcet paradox* happens.

Example 1. *Consider five individuals declare their preferences by providing a ranking of the elements of a set of alternatives* $\mathcal{X} = \{x, y, z\}$*, as table 1.1. If we accept to use majority*

Individual 1	$x\ R_1\ y\ R_1\ z$
Individual 2	$x\ R_2\ y\ R_2\ z$
Individual 3	$y\ R_3\ z\ R_3\ x$
Individual 4	$z\ R_4\ y\ R_4\ x$
Individual 5	$z\ R_5\ x\ R_5\ y$

Table 1.1: Preferences of individuals on $\{x, y, z\}$.

rule:rank x above y if and only if a majority of the individuals do, and similarly for all other pairs of alternatives, then we must rank x above y (as three out of five individuals do) and y above z (as, again, three out of five individuals do). This suggests that the collective preference order should be x R y R z. But this solution is in conflict with the fact that three out of five individuals rank z above x [Endriss, 2011].

[Arrow, 1963] analyzed this kind of situations by abstracting the pairwise majority rule, which results in extracting a set of axioms. Investigating the relations between the axioms reveals cons and pros of the majority rule. Note that, unlike the axiomatic studies in geometry or logic, where the axioms are appropriate for any case, the axioms provided by Arrow are not sure truth; some axioms that are appealing in some situations may not be appropriate for others [Thomson, 2001]. These axioms just arrange a model to judge different social ranking rules from the sense of fairness or other interesting criteria in contexts of economy and social choice. The axioms proposed by Arrow are listed here:

- Universal domain: the domain of F is the set of all logically possible profiles of complete and transitive individual preference orderings.

  This property means that the solution should accept all different types of preferences which are rational.

- Ordering: for any profile $(R_1, R_2, ..., R_n)$ in the domain of F, the social preference relation R is weak ordering.

  This property refers to the point that the result of collective decision should be rational. In other words, it must avoid cycles in the final ranking. This is very important since the goal of many voting problems is to select the uniquely best alternative between a group of alternatives, and cycles are in conflict with the goal.

  Note that *ordering* is the property that already rules out the pairwise majority rule when it serves to rank at least three alternatives.

- Weak Pareto principle: for any profile $(R_1, R_2, ..., R_n)$ in the domain of F, if for all $i \quad N$ it holds $x\ P_i\ y$, then $x\ P\ \ y$.

  This axiom refers to the idea that the collective decision should respect the opinion of all voters. In the extreme case, if all the voters prefer one candidate than the other one, so does society. According to the concept of majority, pair-wise majority rule satisfies *Weak Pareto principle*.

- Independence of irrelevant alternatives: for any two profiles $(R_1, R_2, \dots, R_n)$ and $(R_1, R_2, \dots, R_n)$ in the domain of F and any $x, y \quad \mathcal{X}$, if for all $i \quad N\ R_i$'s ranking between x and y coincides with R_i 's ranking between x and y, then $x\ R\ y$ if and only if $x\ R\ \ y$.

  This axiom notes that the ranking of any two alternatives like x and y just depends on the preferences of voters over the two alternatives and not the others. As an example, consider ranking of three alternatives $\mathcal{X} = \{x, y, z\}$ by preferences of three individuals $N = \{1, 2, 3\}$. Suppose the preference profile R is given by:

  $$x\ R_1\ y\ R_1\ z$$

  $$y\ R_2\ x\ R_2\ z$$

  $$y\ R_3\ z\ R_3\ x.$$

  The axiom *independence of irrelevant alternatives* affirms that if the profile be restricted to any of two individuals, here let's say x, y:

  $$x\ R_1\ y$$

  $$y\ R_2\ x$$

  $$y\ R_3\ x$$

  then the ranking of x and y should not change. For instance, pairwise majority rule satisfies the property, and in both cases it ranks y higher than x.

- Non-dictatorship: there does not exist an individual $i \qquad N$ such that, for all $(R_1, R_2, \dots, R_n)$ in the domain of F and all $x, y \quad \mathcal{X}$, $x\ P_i\ y$ implies $x\ P\ \ y$.

  Finally, non-dictatorship emphasizes that the ranking solution should not be in favor of one voter and lets it to dictate its preferences to other voters. Since pair-wise majority rule respects the opinion of all individuals, it satisfies non-dictatorship

Arrow proved that the five conditions are incompatible. This result called *impossibility theorem* is an important turning point in the axiomatic study of voting rules. It very well pictures the paradox between the above mentioned axioms when it turns to rank more than two candidates:

Theorem 1.2.1. *If $|\mathcal{X}| > 2$, there exists no preference aggregation rule satisfying universal domain, ordering, the weak Pareto principle, independence of irrelevant alternatives, and non-dictatorship.*

During the years, different versions of impossibility theorem have been stated, and different proofs have been provided [Geanakoplos, 2005].

As we mentioned earlier, the axiom *universal domain* assumes non-restricted domain of preference profiles for a social ranking solution. For example, consider three alternatives $\mathcal{X} = \{x, y, z\}$ and three individuals $N = \{1, 2, 3\}$. Universal domain forces the social ranking solution to map any preference profile formed by the individuals to a collective weak order over the set of alternatives. For instance, in this case each individual can have 13 different preferences over the three alternatives[1] and therefore there are 13^3 preference profiles to be mapped to collective weak orders over the set of candidates. In fact, *universal domain* is a very strong axiom demanding a lot about social ranking rules (to map each one of 13^3 preference profiles to a specific collective ranking). However, sometimes nature of alternatives and preferences of individuals are so determined that not all individuals' preferences can arise. When studying such a case within Arrow's framework, there is no need for a social welfare function to be able to handle each and every tuple of individual orderings; some but not all profiles are admissible, and the domain is said to be restricted. In fortunate cases, it is, then, possible to find a social welfare function that, apart from the axiom universal domain, meets all assumptions and conditions of Arrow's theorem. Such domains are said to be *Arrow consistent*.

As an example, imagine a situation where the domain of preference profiles are restricted to those in which majority of individuals have the same preference over the set of candidates. In the case of three individuals and three alternatives, one preference profile can be as:

$$a\, R_1\, b\, R_1\, c$$
$$a\, R_2\, b\, R_2\, c$$
$$c\, R_3\, b\, R_3\, a.$$

Reckoning the aggregation rule by pair-wise majority rule, it is easy to see that the result is the majority preference: $a\ R\ b\ R\ c$, which is a weak order (satisfaction of ordering axiom). Considering all such preference profiles, it is easy to see that pair-wise majority rule satisfies all the other axioms. It meets weak Pareto property because if all individuals rank the same, the collective ranking would be the preferences of individuals. It is non-dictatorial since it considers preference of the majority of individuals, and finally it satisfies independence of irrelevant alternatives because the aggregation process is done by pair-wise comparing of individuals. Hence, the domain of preference profiles in which the majority of individuals has the same preferences is an example of what is called *"arrow consistent"* domain.

Another well known domain restriction that provides an arrow consistent domain is called *single peakedness*, which is introduced by [Black et al., 1958]. *Single-peakedness* requires the assumption that there exists a one dimensional scale that orders candidates from left to right (there is an ordering L over the candidates). In this case, the preference of an individual is *single-peaked* when the individual prefers x to y if x is between y and her most preferred alternative

[1] The preferences are $a\ P\ b\ P\ c$,$a\ P\ c\ P\ b$, $b\ P\ a\ P\ c$, $b\ P\ c\ P\ a$, $c\ P\ a\ P\ b$,$c\ P\ b\ P\ a$, $a\ I\ b\ P\ c$, $a\ P\ b\ I\ c$, $a\ I\ c\ P\ b$, $b\ P\ c\ I\ a$, $b\ P\ c\ P\ a$, $c\ P\ b\ I\ a$, $a\ I\ b\ I\ c$

based on the ordering L.
As an example, suppose the alternatives in the set $\mathcal{X} = \{x, y, z\}$ can be ordered on a line from left to right based on a property, like their political tendencies when it comes to political voting or their experience when it comes to academic voting. Let's assume, based on this ordering, x is the extreme left alternative, y is the middle one, and z is the extreme right alternative. In this case, a preference profile of a voter i is single peaked if the voter ranks alternatives in any way except as $x \, R_i \, z \, R_i \, y$ or $z \, R_i \, x \, R_i \, y$. The domain restriction by *single peakedness* can be justified by the evidence that people have some ideal points and they do not prefer alternatives that are far from the ideal point to those that are closer. Therefore, this domain restriction is common in many real world voting situations.

If the preferences of all the voters are single-peaked then the majority rule guarantees the transitivity of ranking over candidates when there are more than two candidates. For instance, by accepting the above mentioned ordering over alternatives x, y, z, if we want to modify the preference profiles of individuals in Example 1 to avoid cycles (restricting it to single peakedness), we can change the preference profile of individual 5 in a way that alternative y is not the worst alternative.

Until now we have reviewed one of the main results in the context of voting theory called *impossibility theory*, and analyzed one of the possibilities to avoid such kind of impossibility results. In the rest of the section, we review another important result in voting theory regarding the characterization of majority rule which was first introduced by [May, 1952].

Since the pattern of collective ranking may be build up by knowing the ranking over any two pairs of alternatives, May considered the base case of having only two alternatives. In fact, he assumed there exists a set of n individuals who have preferences over two individuals x and y. For any two individual x and y there are three possible ways to rank them: to prefer x to y, to prefer y to x, or to be indifferent between x and y. Each individual in this setting is associated with a variable D_i that takes the values 1, 0, and -1 respectively if xP_iy, xI_iy, and yP_ix. In the same way, variable D illustrates the decision made by the whole group of voters, and is valued by 1, 0, and -1 respectively if xPy, xIy, and yPx (P and I are counterparts of P_i and I_i for collective decisions).

He denoted the social welfare function by $D = F(D_1, D_2, ..., D_n)$. He referred to this function as *group decision function* which maps the n-fold Cartesian product $U \times U \times \cdots \times U$ onto U where $U = \{-1, 0, 1\}$.

As we mentioned before, the most familiar form of this social welfare function is *simple majority*. It counts the number of times x is preferred to y ($N_{xy}(1)$) and compares it to the number of times that alternative y is preferred to alternative x ($N_{xy}(-1)$). By the simple majority rule, $D = -1, 0, 1$ respectively if $N_{xy}(1) - N_{xy}(-1)$ is negative, zero, or positive. Due to the importance of majority rule in many practical voting scenarios May made four weak conditions that provide sufficient and necessary conditions for a group decision function to be majority rule. These properties are listed below:

- The group decision function is a single valued function defined over the elements of $U \times U \times ... \times U$. This property is called *decisiveness*. It comes from the need to select only one best candidate in many cases. Based on decisiveness, a social welfare function is a one-valued function for every unique vector as $(D_1, ..., D_n)$.

- The group decision function is a symmetric function of its arguments. Referring to this property as *symmetry*, it means that the value of the group decision function is determined only by the preferences, no matter to which individuals those preferences belong. This axiom is sometimes called *anonymity*. The idea of the axiom is appealing since it imposes that the group decision function should be fair in the sense that it does not privilege one voter over the others.

- The third property is called *neutrality*. It means that if each individual reverses her preference over the two alternatives, then the value of the group decision function should be reversed as well. On the other words, the social welfare function should not have tendency toward one candidate than the others. More formally: $F(-D_1, -D_2, \dots, -D_n) = -F(D_1, D_2, \dots, D_n)$

- Finally, the forth property which is called *positive responsiveness* claims that the group decision function responds to changes in voters' preferences in a "positive" way: if $D = F(D_1, D_2, ..., D_n) = 0$ or 1 and $D_i = D_i$ for all $i = j$ and $D_j > D_i$, then $D = F(D_1, D_2, ..., D_n) = 1$.

May characterized the simple majority rule using the four axioms in the following theorem [May, 1952].

Theorem 1.2.2. *Given a set of voters and two alternatives, a social welfare function is simple majority rule if and only if it always satisfies decisiveness, anonymity, neutrality, and positively responsiveness.*

Another formulation of the axiomatic study for majority rule is proposed by [Merlin, 2003].

Suppose a set $N = \{1, 2, \dots, n\}$ of individuals and a set $\mathcal{X} = \{a_1, ..., a_m\}$ of alternatives $(m \geq 2)$ are given. Also, assume the preference of individual i is represented by a complete preorder R_i over the set $\mathcal{X}$ of alternatives. In this setting, a *social welfare function* is a function f that maps a n-fold Cartesian product $\mathcal{R}(N)^n$ onto $\mathcal{R}(N)$ ($\mathcal{R}(N)$ indicates the set of complete preorders over the set N).

In this framework, [Merlin, 2003] introduced three conditions that are considered to be the basic requirements of democracy, and then concluded that simple majority rule is the social welfare function that satisfies all the three conditions. Here, we briefly explain these conditions or axioms.

The first axiom called *anonymity*. It emphasizes that a democratic social welfare function should give the same power to individuals or voters who participate in the voting.

Definition 1.2.3 (Anonymity). *Suppose V and V are two sets of voters of the same size ($|V| = |V|$). Also assume $\Gamma(V, V)$ refers to the set of all permutations $\gamma : V \rightarrow V$. For a given profile $(R_i)_{i\ V}$, $\gamma((R_i)_{i}\ N)$ is equal to $(R_j)_{j\ V}$ such that $R_i = R_j$ if and only if $j = \gamma(i)$. A social welfare function f satisfies anonymity if and only if for any $\gamma\ \Gamma(V, V)$ it holds that $f(\gamma((R_i)_{i\ V})) = f((R_j)_{j\ V})$.*

This axiom is equivalent to the axiom symmetry of May, extended to more than two alternatives. The second axiom takes care of fairness among alternatives in the sense that the social welfare function should not be biased in favor of one alternative. Reversing the name of alternatives should also reverse the result of collective ranking. This axiom is called *neutrality*. Note that since indifference does not play a role in ranking individuals, we assume the preferences of individuals are linear orders, i.e., complete, transitive, and antisymmetric (the preference of individual i is indicated by L_i).

Definition 1.2.4 (Neutrality)**.** *Consider a set V of individuals, and suppose $\Sigma(A)$ is the set of all permutations over the set of alternatives $A = \{a_1, a_2, ..., a_m\}$. For a binary relation R on the set A and $\sigma \quad \Sigma(A)$, we define $\sigma(R)$ by $a_{\sigma(i)}\ \sigma(R)\ a_{\sigma(j)}$ if and only if $a_i\ R\ a_j$. For any profile of linear orders $(L_i)_{i\ V}$ we define $\sigma((L_i)_{i\ V})$ as $(\sigma(L_i))_{i\ V}$. A social welfare function f satisfies neutrality if and only if for any permutation $\sigma \quad \Sigma(A)$ it holds that $f(\sigma((L_i)_{i\ V})) = \sigma(f(L_i)_{i\ V})$.*

The third axiom affirms the positive respond of social welfare functions to any changes in preferences of individuals. We refer to it as *monotonicity*.

Definition 1.2.5 (Monotonicity)**.** *Consider a set of individuals V, such that for a preference profile $(R_i)_{i\ V}$ the social welfare function f results that $x\ P\ y$ or $x\ I\ y$. Now lets form another preference profile $(R_i)_{i\ V}$ such that there exists $j \quad V$ with ($x\ I_j\ y$ and $x\ P_j\ y$) or ($y\ P_j\ x$ and $x\ I_j\ y$) and for all $i = j$ it holds that $R_i = R_i$. Then f is monotone if and only if applying it on $(R_i)_{i\ V}$ results that $x\ P\ y$.*

Finally, a characterization of the majority rule is given by [Merlin, 2003].

Theorem 1.2.6. *Let $A = \{x, y\}$, a social welfare function satisfies anonymity, neutrality, and monotonicity if and only if it satisfies simple majority rule.*

In the next section, we review cooperative game theory. We recall the solutions related to distribute the share of coalitions among players and we analyse them from property-driven approach.

1.3 Cooperative Game Theory

One of the approaches to address the ranking problem, in the thesis, is inspired from solutions in the context of cooperative game theory. In this section, we review some of the basic notions in cooperative game theory which are used in Chapter 3.

In general, game theory concerns mathematical models of interaction among rational decision makers called *players*. Rationality in the context of game theory reflects the point that players always make decisions to increase their utility. Problems of game theory are generally divided into two branches of cooperative games and non-cooperative games. Non-cooperative game is a game with competition between individual players. Such kind of games provide a rich language to analyze the interaction among players in detail, and to predict the impact of individuals' decisions on the final outcome.

Cooperative game theory, on the other hand, assumes that groups of players, called coalitions, are the actors since players in a coalition bind agreements to cooperate and share utility (when the utility is quantifiable). The basic framework for cooperative games, which is known as *characteristic function*, was introduced by [Morgenstern and Von Neumann, 1953]. In this model, the characteristic function indicates the worth or value of each coalition (not to each individual in the coalition). Examples of cooperative games can be seen in many real group activities in which a group of individuals pursue a common goal. For instance, a football team whose goal is to score and win a match, or a group of researchers who cooperate to achieve new results or technologies. In all such activities, a binding agreement is conducted between members before starting to act.

A key concern in cooperative game theory is to find out the rational outcome of a game. Outcome means how worth of a coalition should be divided among its members. More formally, in cooperative game theory the problem is how to share the benefits or costs of the grand coalition (specified by characteristic function) among members in the coalition. Different possibilities for this sharing are summarized in various vectors called *payoff vector*s, which indicate value to each individual in the coalition. In this case, solution to the problem is called *solution concept* which for every game identifies some subsets of the possible payoff vectors. Note that, from a set of all outcomes, only some of them may be desirable. For instance, if the contribution of all agents in forming a coalition be the same, an outcome that allocates the entire value of coalition to just one agent is less appealing to one that divide the value among agents equally.

So far, we have talked about sharing the worth of coalitions, but in some cases there is not really worth to share. Forming coalitions puts individuals in a special state of the world and so they experience a corresponding satisfaction. From this point of view, games are divided into two classes of *transferable utility games* (TU-games) and *non-transferable utility games* (NTU-games). Since in this thesis we use some ideas about solution concepts related to TU-games it is worth studying them. Solution concepts for these games can be evaluated according to two sets of criteria: *fairness*, i.e., how well each agent's payoff reflects his contribution, and *stability*, i.e., what are the incentives for the agents to stay in the coalition structure.

Study of solution concepts in a game can be done from two points of view: normative and descriptive. In the descriptive point of view, the solution concept characterizes what players actually do. In fact, one may want to predict the likely outcome of the interactions among players, and the resulting payoff is understood as a natural consequence of the interactions. In the normative approach, a solution concept characterizes what a rational player should do: one can set up a number of normative goals, typically illustrated by axioms, and try to derive their logical implications [Chalkiadakis et al., 2011].

In the next section, we study transferable utility games in more detail. We also review a type of TU-games which is widely used as a model of voting systems, and we explore the axiomatic study of solution concepts for these games.

1.3.1 Transferable Utility Games

Suppose in a cooperative game, each agent has a "utility function" that is expressed in currency units. In this case, a common currency enables agents compare and share their alternative outcomes. Such kind of games are called *TU games*. TU-games involve a set of players and a characteristic function which represents the value that each coalition can achieve. To formally define TU-games we need some notations.

We consider a set $N = \{1, 2, ..., n\}$ of agents, a coalition is a subset of N. The set N is also called the grand coalition. We indicate the set of all coalitions formed by the members of N as 2^N. During the thesis, we indicate each coalition by upper case Latin alphabet (A,B,...). TU-games are formally defined as below:

Definition 1.3.1. *A transferable utility game (TU-game) is defined as a pair* (N, v)*, where* N *is a set of agents and* $v : 2^N \rightarrow \mathbb{R}$ *is a characteristic function.*

Whenever set of players is clear from the text, we simply refer to games by their characteristic functions. Note that based on the definition, the worth of each coalition does not depend on the worth of other coalitions. In general, for TU-games, it is assumed that the worth of empty coalition ($\emptyset$) is zero.
It is worthy here to mention some of the properties of TU-games:

- **Additive.** When a TU-game is additive it means that the worth of each coalition is the same whether its members cooperate or not. In such games, there is no positive or negative synergy among the agents in a coalition. Formally $\forall S, U \in 2^N, v(S \cup U) = v(S) + v(U)$.

- **Monotone.** When a TU-game is monotone it refers to the idea that as the size of coalition increases it will be worth more. More precisely, $\forall S, U \in 2^N$ such that $S \subseteq U$ it holds that $v(U) \geq v(S)$. Note that, many of the TU-games are not monotonic because as the members added to a coalition, the overhead caused by the cost of communication or maintenance increases, and as a result the worth of coalition reduces.

- **Superaditive.** In a TU-game which is superadditive, coalitions can best off by merging together. Formally, $\forall S, U \in 2^N, S \cap U = \emptyset, v(S \cup U) \geq v(S) + v(U)$. In such games, players have incentive to form the grand coalition.

A family of TU-games, called *simple games*, is important from the point that they can be used as a model for many voting situations [Banzhaf III, 1964, Dubey et al., 1981]. The formal definition of simple games is as follows.

Definition 1.3.2 (Simple game)**.** *A game* (N, v) *is a simple if* $v : 2^N \rightarrow \{0, 1\}$*,* $v(N) = 1$*, and* (N, v) *be monotonic.*

Simple games can be used as a model for voting situations like *majority game*. For a set of n agents, assume they have to decide to accept or reject an alternative using majority vote. Majority game is defined as below.

Definition 1.3.3 (majority game). *A simple game (N, v) is majority game when the characteristic function $v : 2^N \to \{0, 1\}$ is defined as:*

$$v(S) = \begin{cases} 1 & \text{if } |S| > \frac{n}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

In the same way, simple games can be considered as a model of weighted majority rule.

Definition 1.3.4 (weighted majority game). *A simple game (N, v) is a weighted majority game if there is a number q, called quota, and a vector $w = (w_1, \ldots, w_n)$ of weights such that*

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

In general, this game is identified by $[q; w_1, w_2, ..., w_n]$.

Example 2. *Consider a voting scenario with 10 voters who have the same power. In order to legislate a rule, more than half of the votes should be cast in favor of the rule. This scenario can be modelled by a weighted majority game as* $[6; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$.

Example 3. *Let's assume a voting scenario with* 5 *voters $N = \{1, 2, 3, 4, 5\}$ is given. Also suppose the weight of the voters are defined as $w_1 = 3, w_2 = 5, w_3 = w_4 = w_5 = 1$. In order to pass a rule, at least* 4 *votes should be cast. In this case, the voting scenario is modelled as* $[4; 3, 5, 1, 1, 1]$.

Considering simple games as a model for voting situations allows us to study the solution concepts for simple games which are beneficial to measure the power of each individual in the process of decision making. These kinds of solution concepts, also called *power indices*, are reviewed with more details in the next section.

1.3.2 Solution Concepts

A solution (or one-point solution) for TU-games is a mapping that assigns a vector of payoffs $(x_1, x_2, ..., x_n)$ (x_1 indicates the payoff assigned to individual 1, x_2 the payoff assigned to individual 2 and so on) to each characteristic function game (N, v).

Focusing on semi-values (a class of solution concepts whose members satisfy a set of standard axioms) [Carreras et al., 2003], [Dubey et al., 1981], two solution concepts the *Shapley value* [Shapley, 1953] and the *Banzhaf index* [Banzhaf III, 1964] are important specially in voting situations where we want to evaluate the influence of players. In the context of voting theory, these two solution concepts are referred to as power indices.

In order to define these power indices, we first need to introduce the notion of *marginal contribution*.

Definition 1.3.5 (marginal contribution). *For* (N, v) *be a TU-game, the marginal contribution of individual* $i \in N$ *w.r.t coalition* $S \in 2^N \setminus \{i\}$, *is* $mc_i(S, v) = v(S \cup \{i\}) - v(S)$.

According to Definition 1.3.5, the marginal contribution of individual i to coalition $S \cup \{i\}$ indicates the value that is added or lost from coalition S when individual i joins the coalition. [Penrose, 1946] and [Banzhaf III, 1964] are the two scientists who independently introduced the Banzhaf index. This index was first applied to measure the power of players in a voting committee, thus, it was initially defined only for simple games. By the use of the notion of marginal contribution we are able to define the Banzhaf index.

Definition 1.3.6 (Banzhaf index). *For* (N, v) *be a TU-game, the Banzhaf index* $\beta(v)$ *of* v *is the* n*-vector* $\beta(v) = (\beta_1(v), \beta_2(v), \ldots, \beta_n(v))$, *such that for each* $i \in N$:

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \in 2^{N \setminus \{i\}}} mc_i(S, v). \tag{1.3}$$

Example 4. *Imagine a voting scenario with three voters* a, b, c, *and suppose it is modelled by a simple game* $[6; 4, 3, 2]$. *In this scenario the winning coalitions are*

$$ab, ac, abc.$$

By Banzhaf index the power of each voter is specified as below:

$$\beta_a(v) = \frac{3}{4}, \beta_b(v) = \frac{2}{4}, \beta_c(v) = \frac{2}{4}.$$

To clarify the process of calculating the Banzhaf index, let's explain the Banzhaf index for individual a. *In this case, we need to compute the marginal contribution of* a *when it joins all other coalitions, which are* 4 *coalitions as indicated in table 1.2.*

$\rightarrow a$	$mc_a(a, v) = 0$
$b \rightarrow ab$	$mc_a(ab, v) = 1$
$c \rightarrow ac$	$mc_a(ac, v) = 1$
$bc \rightarrow abc$	$mc_a(abc, v) = 1$

Table 1.2: Marginal contributions of individual a entering to coalitions formed by $\{b, c\}$.

Since there are three marginal contributions with value one and the others with value zero, its average implies the Banzhaf index for individual i, *which is* $\frac{3}{4}$.

Another important solution concept is introduced by [Shapley, 1953] for simple games. Shapley value assumes the formation of the grand coalition, and evaluates the average performance of individuals based on different ways they join grand coalition.

Definition 1.3.7 (Shapley value)**.** *For* (N, v) *be a TU-game, the Shapley value* $\varphi(v)$ *of* v *is the* n*-vector* $\varphi(v) = (\varphi_1(v), \varphi_2(v), \ldots, \varphi_n(v))$ *such that for each* $i \quad N$*:*

$$\varphi_i(v) = \Sigma_{S \quad 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} mc_i(S, v) \tag{1.4}$$

where s *refers to size of coalition* S*.*

Example 5. *Consider a simple game as* $[6; 4, 3, 2]$ *where there are three voters* a, b, c *with the weights* $w_a = 4, w_b = 3, w_c = 2$*. The Shapley value for individual* a *is computed using marginal contributions in table 1.3.*

abc	$mc_a(a, v) = 0$
acb	$mc_a(a, v) = 0$
bac	$mc_a(ba, v) = 1$
cab	$mc_a(ca, v) = 1$
bca	$mc_a(bca, v) = 1$
cba	$mc_a(cba, v) = 1$

Table 1.3: Marginal contributions of individual *a* by different ways of entering to grand coalition

Considering all the ways that individual a can enter into the grand coalition, its marginal contribution is two times zero and four times one. Therefore, by averaging on the values, we get the value $\frac{4}{6}$ *as the Shapley value of individual a.*

An axiomatic characterization of Shapley value is given in [Shapley, 1953]. In this paper, Shapley formulated a number of properties that a one-point solution should (or might) have, and then he showed that the Shapley value is the only solution with these properties. A slightly modified version of the axioms are reviewed in the rest of this section [Dubey, 1975], [Dubey and Shapley, 1979].

- The first axiom called *efficiency*, and it means that the values assigned to the individuals in the grand coalition must be equal to the value of the grand coalition ($\Sigma_{i=1}^{n} \varphi_i(v) = v(N)$).

- The next axiom is *null player*. Let's refer to a player who does not change the worth of a coalition as a "null player" (for all $S \quad 2^N$, $v(S) - v(S \setminus i) = 0$). The axiom *Null player* asserts that the value assigned to a Null player, by the solution concept, should be zero ($\varphi_i(v) = 0$ for all player i which is a null player).

- The third axiom which is called *symmetry* means that if two players have the same performance when they join a coalition, then they should get the same value. More precisely, $\varphi_i(v) = \varphi_j(v)$ for any players i, j such that $v(S \quad \{i\}) = v(S \quad \{j\})$, for every $S \quad N \setminus \{i, j\}$.

- The last axiom *additivity* points out that if two independent games v and v are combined, their value must be added player by player, e.g., for any player i ($\varphi_i(v + v) = \varphi_i(v) + \varphi_i(v)$).

The following theorem indicates the characterization of the Shapley value [Shapley, 1953].

Theorem 1.3.8. *There is a unique function φ, which satisfies the axioms Efficiency, Null player, Symmetry, and Additivity. Moreover, this φ is just the Shapley value.*

In addition to the axiomatic study of Shapley value, an axiomatic study is provided to characterize the Banzhaf index [Dubey and Shapley, 1979]. The authors analyse the Banzhaf index using the four axioms.

The characterization theorem is as follows [Dubey and Shapley, 1979]. In the following theorem $\eta_i(v)$ refers to the total number of *swings* of player i, and $\overline{\eta}(v) = \Sigma_{i \quad N}\eta_i(v)$ (player $i \quad N$ said to swing outside S if $v(S) = 0$ while $v(S \quad \{i\}) = 1$).

Theorem 1.3.9. *For the class of all simple games, there is a unique function φ that satisfies the following properties:*

A1. *If i is a null player in v then $\varphi_i(v) = 0$.*

A2. $\Sigma_{i=1}^{n}\varphi_i(v) = \overline{\eta}(v)$

A3. *Symmetry.*

A4. *For any two simple games v and w it holds that $\varphi(v \quad w) + \varphi(v \quad w) = \varphi(v) + \varphi(w)$. $((v \quad w)(S) = max\{v(S), w(S)\}$ and $(v \quad w)(S) = min\{v(S), w(S)\}$ for any coalition $S \quad 2^N$).*

Moreover, φ is the Banzhaf index.

In this section we have explored the axiomatic study of solution concepts in cooperative game theory. In the next section, we investigate the axiomatic study in the context of ranking sets of individuals.

1.4 Ranking Sets of Individuals

One of the problems that is inspired to many individual and collective decision making problems is the problem of ranking sets of individuals when the ranking over individuals are given. As an example, suppose you are given a finite set of statements, each of which is plausible to one degree or another, and asked to pick a most plausible consistent subset. Therefore, what is needed is to define a ranking over the set of statements based on their degree of plausibility. A natural way to proceed would be first to order the plausibility of the consistent subsets and then to randomly choose a most plausible one. The problem reduces, then, to ordering the consistent subsets.

Formally, suppose a decision maker is provided by a ranking R over the set $\mathcal{X} = \{x, y, z\}$ of objects or individuals. The symmetric and strict parts of the ranking R are respectively indicated by I and P. The structure of ranking R depends on the context of the ranking problem. In general, we assume that it is a binary relation. The decision maker, then, faces the problem of finding a binary relation $\quad 2^{\mathcal{X}} \times 2^{\mathcal{X}}$, which forms an ordinal ranking over the subsets of $2^{\mathcal{X}}$.

This problem is extensively studied in recent years due to its importance in many practical situations. These applications are, for instance, comparing the stability of coalitions in cooperative game theory or evaluating the likelihood of a set of events to occur [Barberà et al., 1984, Barberà et al., 2004]. This ranking problem is the inverse of the ranking problem that we study in this thesis. Therefore, it is worthy to briefly study this ranking problem.

Based on the objective of decision maker to rank subsets of objects, the ranking over individuals can be extended to rank over subsets of individuals under three different interpretations.

Complete uncertainty. Suppose the decision maker is endowed with a preference over objects, and assume the decision making is a two-step process. In the first step, the decision maker could rank subsets of objects, while in the second step the nature selects one of the alternatives from the highest ranked subset (random selection). In this case, the decision maker should rank subsets with most preferred objects higher. By doing this, the decision maker increases the probability of favored objects to be selected in the second step, by nature.

Opportunity sets. Suppose the decision maker faces the process of selecting one object from a set of objects. And since the decision maker is not sure about her preference in the future, she does the decision making in two steps. In the first step, she selects a set of objects that provides her a bigger opportunity to choose an object in the second step, after being sure about her preferences.

Sets as final outcomes. Each set contains objects that are assumed to materialize simultaneously. In this kind of problems, in order to rank subsets of objects we need to consider the possible interaction among the objects as well as their individual ranking. Examples of this kind of scenarios are ranking coalitions based on how stable they are, or election of members to join an organization.

Note that in the context of complete uncertainty or the one of opportunity sets, where objects are mutually exclusive and precisely one object materializes at a later stage, it is quite reasonable to discard the possibility that complementarity or incompatibility may affect ranking sets.

In order to clarify the idea behind these assumptions, consider a set of two alternatives $\mathcal{X} = \{x, y\}$. Suppose alternative x is preferred to alternative y, and the goal is to rank the three subsets $\{x\}$, $\{y\}$, and $\{x, y\}$. Under the interpretation of *complete uncertainty*, it is reasonable to rank $\{x\}$ over $\{x, y\}$ since it is possible that y be chosen when $\{x, y\}$ is selected. The same way, the set $\{x, y\}$ is expected to be ranked higher than $\{y\}$ because selecting $\{x, y\}$ makes it possible to choose x at the final step.

As an example, imagine a voting scenario in which the aggregation is done by a social choice correspondence which maps the preference profiles of voters to a subset of candidates (some candidates can be equally ranked according to the voting rule). In such scenario, voters tend to rank different subsets of candidates that can be realized in the outcome. This ranking should be done based on the fact that some other voters may tend to manipulate their votes in order to make one of the candidates winner and the others loser. This kind of ranking sets of candidates is categorized in the context of complete uncertainty, since candidates are mutually exclusive and just one of them will be materialized as the outcome.

Under the interpretation of *opportunity sets*, the two sets $\{x, y\}$ and $\{x\}$ could be indifferent, and $\{y\}$ could be ranked below them. Example of ranking sets as opportunity sets is selecting a restaurant to eat based on their menus. For example, suppose someone is eager to eat in French restaurant and she considers a restaurant as french if they have goose liver in their menu. Now imagine a restaurant with two types of food in its menu as food x and food y (which are not goose liver), and another restaurant offers a menu as $\{x, y, w\}$ where w in the goose liver. Although the individual is not particularly interested in goose liver, she selects the second set of options because it assures that he will eat in a French restaurant [Baharad and Nitzan, 2000].

Finally, by the interpretation of *sets as final outcomes*, if objects are goods, $\{x, y\}$ might be ranked higher than the others. However, it completely depends on the nature of the objects x and y and the kind of incompatibility or complementary effects between them [Barberà et al., 2004]. An example of this ranking problem is the ranking of sets on statements by their level of consistency when statements are ranked based on their degree of plausibility. This problem is studied by scientists according to the above mentioned interpretations (see, for instance, [Bossert, 1995, Kreps, 1979, Roth, 1985]).

In the following sections, based on the interpretation of the ranking problem, we explore the properties of methods to lift ranking over objects (individuals) to ranking over subsets of them. Note that during these sections whenever which is needed we interchangeably use the words object and individual.

1.4.1 Complete Uncertainty

In this section, we consider the problem of ranking sets of objects when the decision maker's goal to rank the subsets is to increase the chance of selecting the highly preferred object, by nature, in the second round. A familiar example of ranking sets categorized in the context of complete uncertainty is to decide to go to the beach regarding uncertain information about the weather (if it will be sunny or not). Suppose g indicates your choice "to go to beach" and $-g$ denotes your choice "not to go to beach", and s refers to the sunny weather while $-s$ not sunny weather. If the decision maker chooses to go to the beach, she implicitly selects the set $\{(g, s), (g, -s)\}$ and when she chooses not to go she takes the set $\{(-g, s), (-g, -s)\}$. In this case, the decision making problem reduces to the problem of ranking sets regarding the fact that nature chooses one of the members in the selected set and the decision maker is uncertain about it [Packard, 1979]. In this example, the decision maker faces to rank some subsets of objects (g, s), $(g, -s)$, $(-g, s)$, and $(-g, -s)$, by knowing that nature selects one of the objects from the most preferred subset of the decision maker.

Different methods have been proposed in order to solve the problem. For instance, because the decision maker is not the one who makes the final decision, it seems logical that she makes decision pessimistically. Therefore, one way to solve this kind of problem can be the maxi-min method.
Given a set A, object x is the minimum of A ($x = min(A)$) if for all other objects $y \quad A$, the decision maker prefers y to x ($y\ P\ x$). In the same way, object x is called the maximum of set A ($x = max(A)$) if for all the other objects $y \quad A$, the decision maker prefers x to y ($x\ P\ y$). The relation $\quad 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ is maxi-min when for any two sets A and B, $A \quad B \Leftrightarrow min(A) \geq min(B)$.

As far as we know, the axiomatic analysis of the ranking methods was first done by [Fishburn, 1992] with the aim to answer the question of how a voter can compare possible outcomes of the voting (possible sets of alternatives). His work is based on the interpretation of complete uncertainty, and the idea is that each voter wants to increase the probability of its most preferred alternative to be selected in the outcome, with regard to the fact that the other voters can manipulate their votes. From the same interpretation, more recent approaches to this problem is done by [Kannai and Peleg, 1984] and [Packard, 1981]. They investigate the possible logical properties for the solutions in the context of complete uncertainty which somehow justify another method of ranking sets called *mixed min-max* method.

One of the properties they inspect is called *dominance*, also known as Gärdenfors principle. This property means that adding an object which is better than all the objects in a set will improve the set. On the other hand, it expresses that adding an object which is worst than the other objects in the subset will decline the subset, simply because it reduces the chance of selecting the best object from the subset by nature. Note that this axiom is completely ignorant about the possible interaction between objects, which is a valid assumption for the context of complete uncertainty.

Definition 1.4.1 (Dominance)**.** *A binary relation on* $2^{\mathcal{X}}$ *satisfies dominance iff for all* A $2^{\mathcal{X}}$ *and for all* x $\mathcal{X}$*,*

- $[x\ P\ y,\ \ y\ \ A]\quad A\quad \{x\}\quad A$
- $[y\ P\ x,\ \ y\ \ A]\quad A\quad A\quad \{x\}.$

Another benign property called *independence*. It emphasizes that if one set is preferred to another one by a decision maker, then adding the same object to both of the sets should not reverse the preference of the decision maker over the sets.

Definition 1.4.2 (Independence)**.** *A binary relation on* $2^{\mathcal{X}}$ *satisfies independence iff for all* A, B $2^{\mathcal{X}}$, *for all* x $\mathcal{X} \setminus (A\quad B)$,
$A\quad B\quad (A\quad \{x\})\quad (B\quad \{x\}).$

Along with the definition of these two properties, the following Lemma plays an important role in ranking sets of objects [Kannai and Peleg, 1984].

Lemma 1.4.3. *Assume that the ranking over sets is transitive, and* A *is a nonempty subset of objects. If satisfies dominance and independence then* $A\quad \{min(A), max(A)\}$.

As indicated in the Lemma, if dominance and independence are satisfied, any set A $\mathcal{X}$ must be indifferent to a set consisting of the best and the worst elements of the set A. As a result having access to a restricted ranking over singletons and two-element sets is sufficient to recover the whole ranking [Barberà et al., 2004].

[Packard, 1979] introduced different ranking methods in order to lift the ranking from individuals to the set of coalitions along which a set of axioms that characterize some of the ranking methods. Also [Kannai and Peleg, 1984] proved an impossibility theorem regarding the axioms *Dominance* and *Independence* which is worth mentioning here.

Theorem 1.4.4 (Impossibility). *Consider a set $\mathcal{X}$, containing more than six objects ($|\mathcal{X}| > 6$). If the ranking over the objects is a linear order, then there is no complete preorder over $2^{\mathcal{X}}$ satisfying both Independence and Dominance.*

Other results containing both possibility and impossibility results are discussed in [Barberà et al., 2004] and [Bouveret et al., 2009]. Also many other extensions have been proposed under complete uncertainty [Barberà et al., 2004], particularly *lexi-min* and *lexi-max* extensions [Bossert, 1995] and [Pattanaik and Peleg, 1984].

1.4.2 Opportunity Sets

For this family of problems the sets are interpreted as opportunity sets from which the decision maker chooses one object. An example is the set of candidates available to a voter in an election. Also, the budget set of standard consumer theory, which consists of all consumption bundles that a consumer can afford, given her wealth and the prevailing prices, is an immediate example of an opportunity set [Gravel, 2009].

As mentioned before, one plausible interpretation is that the decision maker faces a two step decision making process that selecting a set of objects in the first step constraints the opportunity of choosing one feasible object in the second step. The problem of choosing a restaurant is one of this kind. Suppose the decision maker knows about the menus of different restaurants and consider all the restaurants to be the same except in their menus. Also, assume she wants to eat in a French restaurant and the sign of a French restaurant is to have a goose liver in the menu. Thus, she ranks menus with goose liver higher than the others, so that in the future she can select a food in a French restaurant according to her taste. This explanation entails some ranking methods as follows:

Suppose $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$ are two sets, then it holds that $A \quad B$ if and only if for all $x \quad B$ there exists a member $y \quad A$ such that $y\ R\ x$ (indifference allowed).

It is also easy to verify that ranking methods of this kind satisfy the following property called *Extension Robustness*, with the meaning that adding a set A to another set B, which is as good as A, determines a set that is indifferent to A.

Definition 1.4.5 (Extension Robustness). *A binary relation on $2^{\mathcal{X}}$ satisfies extension robustness iff for all $A, B \quad 2^{\mathcal{X}}$ it holds that $A \quad B \Leftrightarrow A \quad A \quad B$*

It is possible to prove that every mentioned ranking method can be characterized by the extension robustness.

Another interpretation of the opportunity sets comes from the notion of *flexibility*. According to this notion, decision maker prefers a set of objects that provides her a greater opportunity in the future to select one object. For instance, look again at the example of restaurants. Suppose the decision maker prefers a menu containing only steak to a menu containing only chicken. But she prefers the menu containing both steak and chicken because it engenders more flexibility to choose between them. This kind of interpretation is discussed in [Kreps, 1979].
The validation of this method comes from the possible uncertainty of decision maker about her

preferences over objects in the future. For instance, a decision maker who is uncertain about her future taste prefers to choose a menu with wider range of option. The proposed ranking methods obviously do not satisfy *extension robustness* because, according to it, the set $A \quad B$ provides more flexibility, to choose in the future, than the set A.

One of the properties of such ranking rules is a kind of monotonicity that implies sets with bigger cardinality get higher rank by the ranking method.

Definition 1.4.6 (Desire of flexibility)**.** *The binary relation satisfies "desire of flexibility" iff for any two sets A and B, $B \quad A$ implies that $A \quad B$.*

Mentioning flexibility of choice as a kind of freedom, [Pattanaik and Xu, 1990] analyzed three well-defined properties that each ranking method should satisfy.

Indifference between no choice situations. Consider a binary ranking defined over $2^{\mathcal{X}}$. For all $x, y \quad \mathcal{X}$ it holds that $\{x\} \quad \{y\}$.
This axiom explains that all sets that offer only one object to the decision maker have the same ranking no matter what is the proposed object in the sets.

Simple extension monotonicity. Consider a binary relation over $2^{\mathcal{X}}$, for all $x, y \quad \mathcal{X}$, $\{x, y\} \quad \{y\}$.
This axiom is simply indicates that the ranking method should look at the size of coalitions, not to the objects inside the sets.

Strong independence. Consider a binary relation over $2^{\mathcal{X}}$, then for all $A, B \quad 2^{\mathcal{X}}$ and for each $x \quad \mathcal{X} \setminus A \quad B$ it holds that $A \quad B \Leftrightarrow A \quad \{x\} \quad B \quad \{x\}$.
Strong independence reflects that when one set is preferred to another one, increasing the size of them by adding one arbitrary object will not affect the ranking between the sets.

[Pattanaik and Xu, 1990] defined a ranking method based on the cardinality of the sets.

Definition 1.4.7 (cardinality-based ordering)**.** *The binary relation over the power set of objects $2^{\mathcal{X}}$ is called cardinality-based ordering iff:*

$$A, B \quad 2^{\mathcal{X}}, A \quad B \Leftrightarrow |A| \geq |B|$$

Finally, he proved that the three axioms *indifference between no choice situations*, *simple extension monotonicity*, and *strong independence* characterize cardinality-based ordering [Pattanaik and Xu, 1990].

Theorem 1.4.8. *The binary relation $\quad 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ is the cardinality-based ordering iff satisfies indifference between no choice situations, simple extension monotonicity, and strong independence.*

In the next section, we explore another interpretation of ranking sets of objects when the elements in sets are not mutually exclusive.

1.4.3 Sets as Final Outcomes

In this section, we review the literature on the problem of ranking sets of objects, when they materialize simultaneously. Examples of this family of problems are formation of coalitions, election of new members to join an organisation, and college admissions problem. Different approaches are proposed to solve this family of problems.

The first approach that we review is proposed by [Roth, 1985] to solve the college admissions problem. The approach is called *fixed cardinality ranking* which assumes the sets of a fixed cardinality can be formed. In the college admissions problem, colleges are assumed to have a fixed quota q N, indicating the maximum number of students they can admit. Therefore, having an initial ranking over students, colleges look for ranking over sets of students of size q. In order to analyse problems in this approach, [Roth, 1985] introduced a property called *responsiveness*. A ranking over sets of objects satisfies *responsiveness* if for any two sets that differ in only one object, the ranking prefers the set containing the more preferred objects.

To formally define the property we need a further notation. Given a set of objects $\mathcal{X}$, let's indicate the set of subsets of size q with $\mathcal{A}_q = \{S \quad 2^{\mathcal{X}} s.t\ |S| = q\}$. The responsiveness formally defined as follows [Roth, 1985].

Definition 1.4.9 (Responsiveness)**.** *Let R be a binary relation on the set $\mathcal{X}$. A binary relation on the set $\mathcal{A}_q$ satisfies responsiveness iff for all $S \quad \mathcal{A}_q$, for all $a \quad \mathcal{X}$, and for all $b \quad \mathcal{X} \setminus S$ we have that* $[[S \quad (S \setminus \{a\}) \quad \{b\}] \Leftrightarrow aRb]$ *and* $[[(S \setminus \{a\}) \quad \{b\} \quad S] \Leftrightarrow bRa]$

To solve the problem of ranking subsets of fixed size q, [Bossert, 1995] proposed a lexicographical method called *lexicographic rank-ordered rule*. He mainly considered existence of a linear order over the set of individuals, and also assumed the objects in sets are numbered in decreasing order. This assumption enables the decision maker to lexicographically compare any two sets of objects. A characterization of the ranking method is given by [Bossert, 1995] using the Responsiveness and the well-known axiom *neutrality* which ensures that the names of objects are irrelevant in establishing the ranking. The neutrality axiom in this context is defined as below.

Definition 1.4.10. *For any two sets $A, B \quad \mathcal{A}_q$ (q N), the set of individuals $\mathcal{X}$ and any one to one mapping $f : A \quad B \rightarrow \mathcal{X}$, a linear order $\quad 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ satisfies neutrality iff* $[xRy \Leftrightarrow f(x)\, R\, f(y) \quad x \quad A,\ y \quad B] \quad [A \quad B \Leftrightarrow f(A) \quad f(B)]$.

In this definition $f(A)$ and $f(B)$ refer to sets whose members are obtained by respectively mapping members in A and B using f.

By considering the domain of the ranking problem to be the set of linear orders, it is possible to characterize the *lexicographic rank-ordered rule* using the two axioms neutrality and responsiveness [Bossert, 1995].

Lemma 1.4.11. *Given a set of n objects, a quota $2 \leq q < n$, and a linear order R over the set $\mathcal{X}$, ranking $\quad 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ is obtained by lexicographic rank-ordered rule if and only if it satisfies neutrality and responsiveness.*

A still different approach to solve the problem is proposed by [Fishburn, 1992]. In his approach Fishburn utilized four types of information that their combination establish an extension of ranking over objects. These information are not just the primitive information, but information on complements of alternatives. Three types of these information come from the necessity that the established ranking over singleton sets must be congruent with the ranking of corresponding objects. In fact, partitioning the set of objects based on decision maker's preferences into two classes of approved objects and disapproved objects helps to induce more rankings over subsets.

The forth type of information is a *signed ordering* on the complements of the singleton sets. Considering the set of objects as potential candidates to form a committee of specific size, signed ordering allows for the consideration of comparisons like "it is more important to prevent a candidate *a* from being in the committee than having candidate *b* in the committee", or "leaving candidate *a* off the committee is preferred to leaving *b* off the committee", etc [Moretti et al., 2016]. The properties of the signed order as well as possible extensions are available in [Fishburn, 1992].

1.5 Social Ranking Solutions

The problem of ranking individuals given a ranking over coalitions formed by them, is first studied by [Moretti and Öztürk, 2017], from property driven approach. The authors in the paper axiomatically study the problem and explore solutions satisfying a set of meaningful properties.

Given a set N of individuals, they denote the ranking over coalitions formed by the individuals as , which is a total preorder (complete and transitive) over the power set of N (2^N). This ranking is called a power relation. The authors indicate set of all total preorders over the set of individuals by $\mathcal{T}^N$, and the set of total preorders over the set of coalitions by $\mathcal{T}^{2^N}$. The ranking problem is to find a total preorder over the set N of individual (social ranking) when a power relation on subsets of N is given. More formally, a social ranking rule is defined as a function $\rho : \mathcal{T}^{2^N} \to \mathcal{T}^N$ which maps a total preorder on a set of coalitions to a total preorder over the set of individuals. In this case, for any two individuals i and j the notation $i\rho(\ )j$ refers to the weak preference of i to j. The authors introduce two axioms of the ranking methods, and they analyse the effect of the axioms on defining ranking methods. These properties are listed as follows.

Dominance. A social ranking rule $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \to \mathcal{T}^N$ satisfies *dominance* on $\mathcal{C}^{2^N} \quad \mathcal{T}^{2^N}$ if and only if for all $\quad \mathcal{T}^{2^N}$, and for any two individuals $i, j \quad N$, if i dominates j in , then $i\rho(\ )j$ (and not $j\rho(\ )i$ if i strictly dominates j in).

This axiom states that if each coalition containing specific individual like i always be ranked higher than coalition S when i is substituted by another individual j, then i should be ranked higher than j.

The second axiom is *symmetry*. This axiom rules out the ranking methods that rank individuals based on their name and not their performance.

Symmetry. A social ranking rule $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \to \mathcal{T}^N$ satisfies *symmetry* if and only if

$$i\rho(\ )j \Leftrightarrow p\rho(\ )q$$

for all $i, j, p, q \quad N$ and $\quad \mathcal{C}^{2^N}$ such that $|D^k_{ij}(\quad)| = |D^k_{pq}(\quad)|$ and $|D^k_{ji}(\quad)| = |D^k_{qp}(\quad)|$ for any $k = 0, 1, ..., n-2$ ($D^k_{ij}(\quad) = \{S \quad 2^{N\setminus\{i,j\}}, |S| = k, S \quad \{i\} \quad S \quad \{j\}\}$).

They examine if the two intuitive ranking methods, *primitive* and *complement primitive*, satisfy *Dominance* and *Symmetry*. They proved that, for a set of three individuals primitive and complete primitive social rankings satisfy axioms dominance and symmetry. Given a power relation $\quad \mathcal{T}^{2^N}$, a social ranking $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \to \mathcal{T}^N$ is called primitive iff for any individuals $i, j \quad N$ it holds that $i\rho(\quad)j \Leftrightarrow \{i\} \quad \{j\}$. Also if for a coalition S, the complement of S is defined as $S \quad = N \setminus S$, then the social ranking $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \to \mathcal{T}^N$ is called *complete primitive* iff for any two individuals $i, j \quad N$ we have $i\rho(\quad)j \Leftrightarrow \{j\} \quad \{i\}$.

One of the important results in the paper is a theorem that illustrates the incompatibility of the axioms symmetry and dominance when there are more than three individuals to be ranked. They show that combination of these natural properties yields either to impossibility (i.e., no social ranking exists), or to flattening (i.e., all the individuals are equally ranked), or to dictatorship (i.e., the social ranking is imposed by the relative comparison of coalitions of a given size) [Moretti and Öztürk, 2017].

Theorem 1.5.1. *Let $|N| > 3$. There is no social ranking solution $\rho : \mathcal{T}^{2^N} \to \mathcal{T}^N$ which satisfies dominance and symmetry on $\mathcal{T}^{2^N}$.*

Within the same framework, [Bernardi et al., 2017] axiomatically characterize a social ranking solution based on the idea that the most influential individuals are those appearing more frequently in the highest positions in the ranking of coalitions. Following this reasoning, they fix a set of properties that a solution should meet, and explore the possible ranking methods satisfying them.

Neutrality. A social ranking rule ρ satisfies neutrality if and only if for any two individuals and a power relation $\quad \mathcal{T}^{2^N}$, it holds that

$$i\rho(\quad)j \Leftrightarrow \sigma(i)\rho(\quad_\sigma)\sigma(j)$$

where σ is a bijection on N such that for any power relation $\quad \mathcal{T}^{2^N}, \quad_\sigma$ is defined as below

$$\sigma(i) \quad_\sigma \sigma(j) \Leftrightarrow i \quad j.$$

The neutrality axiom is based on the idea that a solution should preserve the ranking of individuals in a society over permutations of the individuals' names.

Coalitional Anonymity. A social ranking rule ρ satisfies *coalitional anonymity* if and only if for any two power relations $\quad , \quad \mathcal{T}^{2^N}$, any two individulas $i, j \quad N$, and bijection π on $2^{N\setminus\{i,j\}}$ it holds that $i\rho(\quad)j \Leftrightarrow i\rho(\quad)j$ when for all $S, T \quad 2^{N\setminus\{i,j\}}$ we have

$$S \quad \{i\} \quad T \quad \{j\} \Leftrightarrow \pi(S) \quad \{i\} \quad T \quad \{j\}.$$

The axiom essentially states that the ranking between any two elements i, j should be independent of the other elements.

Monotonicity. We say that a solution ρ is monotone if for any power relation $\mathcal{T}^{2^N}$, every individuals i, j N such that $i\rho(\ )j$ and $j\rho(\ )i$, and any power relation $\mathcal{T}^{2^N}$ which is obtained by just strictly improving the ranking of a subset containing i but not j, applying the social ranking solution on ranks i strictly better than j ($i\rho(\ )j$ and not $j\rho(\ )i$).

Independence from the worst set. Consider a power relation $\mathcal{T}^{2^N}$ is given as

$$S_1 \quad S_2 \quad \dots \quad S_{2^n-1},$$

in which $S_1, S_2, \dots, S_{2^N-1}$ 2^N, and let's Σ_1 Σ_2 $\dots$ Σ is an order in which subsets S_t are grouped in equivalence class Σ_k. We say a social ranking rule ρ satisfies *Independence from the worst set* if for any power relation with the associated ordering Σ_1 Σ_2 $\dots$ Σ (> 2), and i, j N such that i is strictly better that j ($i\rho(\ )j$ and not $j\rho(\ )i$) we should have i is strictly better than j in the power relation ($i\rho(\ )j$ and not $j\rho(\ )i$) when $\mathcal{T}^{2^N}$ is obtained from by partitioning Σ to $T_1, T_2, \dots, T_m$:

$$\Sigma_1 \text{ A } \Sigma_2 \text{ A } \dots \text{ A } \Sigma_{-1} \text{ A } T_1 \text{ A } \dots \text{ A } T_m$$

This axiom enlightens a method to rank individuals given a power relation. The axiom gives more importance to subsets ranked higher in the power relation. A possible interpretation of the property is the evaluation of professors based on their scientific collaboration in different groups. Once that a total order between two professors is established on the basis of their scientific productivity over all groups, the possible use of a secondary criterion for groups' evaluation (e.g., the educational offer of a team) affecting only coalitions with the lowest scientific productivity, may not impact a total order defined according to the most important evaluation's criteria [Bernardi et al., 2017].

They have defined a social ranking rule called *lexicographic excellence solution* which follows the notion of lexicographical ordering over the equivalence classes of subsets in a given power relation. Based on this solution, to rank individuals i and j given a power relation , starting from the highly ranked class of subsets, we count the number of times that each one of i and j shows up in the subsets of the class. Finding difference between the numbers of presence for the two individuals terminates the process, and ranks the one shows up more upper than the other one. If we face indifference between the numbers of presence for the two individuals, the process continues for other equivalence classes. If indifference occurs for all the equivalence classes then the two individuals are considered to be indifference.

They have proved a theorem that characterizes the lexicographic excellence solution using the four mentioned axioms [Bernardi et al., 2017].

Theorem 1.5.2. *The lexicographic excellence solution is the unique solution fulfilling axioms neutrality, coalitional anonymity, monotonicity, and independence from the worst set.*

The majority of work on the problem of ranking individuals when ranking over coalitions formed by them are given is done based on axiomatic study of different solutions. A more practical approach in this context is given in [Fayard and Escoffier, 2018], which is based on a solution introduced in Chapter 2. Since the introduced solution does not guarantee transitivity of ranking over individuals, In Their paper, the authors implement a social ranking rule to find an approximation of the where a minimum number of coalitions are removed in order to satisfy the transitivity. They called the solution *CP-majority with maximum coalitions*. Still another empirical approach is provided in [Allouche et al., 2020]. In this paper, the authors investigate the manipulability for social ranking rules when each individual prefers to improve its position in social ranking. In order to be coherent during the thesis let us fix the notations that we use in the chapters.

1.6 Conclusion

In this chapter, we have reviewed the literature on the main concepts relating to the ranking problem in the thesis. Specially, we have got familiar with axiomatic study and its importance in social choice theory. We have investigated problems in voting theory. We have studied the corresponding solutions and analyse the properties that they satisfy. Also we have explored the notions of cooperative game theory. We have inspected the solutions in order to solve the problem of distributing the sharing of coalitions among their members, and we have studied them from property driven approach.

We have also considered recent advancement in the inverse of our ranking problem, which is lifting the ranking over individuals to a ranking over their coalitions (subsets). We have explored different interpretations of the problem, and for each one we have reviewed the axiomatic study of the ranking methods. Finally, the chapter is concluded by studying the recent works on the problem of ranking individuals when ranking over their coalitions are given.

In the next chapter we design a social ranking rule based on the concept of majority in the classical social choice theory, and we analyse the properties of the ranking method. Also, since majority rule does not guarantee transitivity of collective ranking, we inspect some restrictions on structure of power relations to avoid cycles in the final ranking result.
Chapter 3 establishes a ranking rule based on classical solution concepts in cooperative game theory. Specially, by extending the notion of marginal contribution to ordinal framework we define a solution and study it from property driven approach.

Finally, Chapter 4 is devoted to explore possible extensions of the *ceteris paribus* majority rules to weighted versions. More precisely, considering the ranking rule introduced in Chapter 2, we categorize possible extensions of the solution to different families of weighted solutions and investigate the relation between the families from property driven approach.

2

Ceteris Paribus majority rule

Abstract

In this chapter, we study the problem of finding a social ranking over individuals or objects given a ranking over coalitions formed by them. We investigate the use of a *ceteris paribus* majority principle as a social ranking solution. We study the problem from property driven approach. Particularly, we follow an axiomatic study inspired from classical axioms of social choice theory. Faced with a Condorcet-like paradox, we analyze the consequences of restricting the domain according to an adapted version of single-peakedness. We conclude with a discussion on different interpretations of incompleteness of the ranking over coalitions and its exploitation for defining new social rankings, providing a new rule as an example.

Contents

2.1 Introduction

In this chapter, in order to solve the ranking problem, introduced in 1.5, we utilize some of the well-known concepts in the theory of voting.

The problem of ranking individuals given an ordinal ranking over coalitions formed by them is of great relevance in the context of decision theory, social choice theory and game theory.

Consider, for instance, the problem of estimating the "power" of countries in the process of collective decision making in an international parliament, or evaluating the influence of a belief on making belief bases consistent or inconsistent [Hunter and Konieczny, 2010]. However, in such situations, as in many others, the worth of each group (or coalition) is, in general, hardly quantifiable, with the only available information about the relative strength of groups being purely ordinal. Thus, we assume given a set of individuals, as an input we have an ordinal ranking over subsets (coalitions) of individuals, and as an output we are looking for a ranking over the set of individuals. Since the aim of the ranking procedure is to compare any two individuals, we assume the ranking over individuals to be a complete preorder (transitive and reflexive).

More precisely, given a set N of individuals, we consider a power relation ( ) as a binary relation over the set of coalitions, which indicates their relative performance ($2^N \times 2^N$). Given a power relation, we are looking for a mapping that maps the power relation to an ordinal ranking over the set of individuals. Notice that, during this chapter, we do not impose any property over the binary relations in the domain of a solution (the set of power relations).

Suppose there is a set N of five researchers in the department $N = \{1, 2, 3, 4, 5\}$, and assume a power relation ( ) demonstrates the relative performance of teams: 2345 245 1234 13 12 23 145 35 24 14 34 [1]. The power relation illustrates that, for instance, team 2345 performs strictly better that team 245, or teams 1234 and 13 are performing in the same level of satisfaction. Therefore, we are looking for a map that assigns a total preorder to a given power relation. We call this mapping a *social ranking solution*.

The introduced social ranking rule, in this chapter, owes a great deal to the simple majority rule in classical voting theory. The ubiquitous use of majority in many of practical voting procedures prompts us to extend it to our coalitional setting and to study it from property driven approach. More specifically, we propose a social ranking rule based on a *ceteris paribus* majority principle, provide an axiomatic characterization of it, and analyze conditions under which the social ranking is transitive.
The simple intuition behind this social ranking rule makes it worthy to study: individuals i and j are compared using only information from the power relation that ranks them under a *ceteris paribus* (i.e., everything else being equal) interpretation. More precisely, if S is a coalition containing neither i nor j, we only look at the relation between S $\{i\}$ and S $\{j\}$ in order to infer some information about the relative strength of i and j, as it is shown in the following example.

Example 6. *Consider the power relation in the example of researchers' evaluation:* 2345 245 1234 13 12 23 145 35 24 14 34*. Suppose researchers* 1 *and* 2 *are up for a promotion. The ceteris paribus ranking of candidates* 1 *and* 2 *implies that only three comparisons from can be used:* 245 145, 24 14 *and* 13 23*. These comparisons are interpreted as saying, e.g., that keeping* 45 *equal, the team containing* 2 *(i.e.,* 245*) performs better than the team containing* 1 *(i.e.,* 145*). The (ceteris parisbus) majority principle states that* 2 *should be rewarded, since candidate* 2 *wins against* 1 *in two comparisons (i.e.,* 245 145 *and* 24 14*) whereas* 1 *wins against* 2 *only in one (i.e,* 13 23*).*

[1] Throughout the thesis, we often write teams without commas and parentheses, e.g., we write 245 instead of $\{2, 4, 5\}$.

Unfortunately, if we do not assume any restriction over the domain of power relations, it is easy to show that the *ceteris paribus* majority solution can lead to a *Condorcet-like paradox* in the social ranking. In order to mitigate this issue, we were able to identify a restriction on the power relation domain that is analogous to the classical single-peakedness property from social choice theory [Black et al., 1958], and prevents the Condorcet paradox.

Applying the *ceteris paribus* principle on the problem of ranking individuals, it is (virtually) converted to an electoral system which is different from classical voting scenarios from two points:

1. The voters are coalitions and may include more than one individual: in example 6 one voter is coalition 45 containing individuals 4 and 5, and the other voters are singleton coalitions 3, and 4.

2. One individual can be a candidate and also a part of a voter: consider comparison 12 23 in the power relation in example (6), the coalition containing only 2 acts as a voter, while in the comparison 245 145, 2 is a candidate.

Another distinguishing feature of this chapter relates to the incompleteness of power relations. In both papers [Moretti and Öztürk, 2017] and [Bernardi et al., 2017] it is assumed that one has access to a ranking over all possible coalitions, i.e., that the power relation is a *total or complete preorder* over the elements of 2^N. However, in many situations this assumption might not be satisfied, e.g., due to missing data, incomparability of certain coalitions or their impossibility to be formed, etc. Therefore, we also consider power relations that are not necessarily complete. In Section 2.3 incompleteness does not provide any complementary information for the social ranking rule, though in other cases the lack of comparisons may be a source of information and can be exploited in the definition of a social ranking. In this light, in Section 2.5 we briefly discuss different interpretations of incompleteness and conclude with an example of a social ranking rule based on the idea of *information level for coalitions*.

The contributions of this chapter are published in the proceeding of international conference IJCAI-18 [Haret et al., 2018].

The chapter is organized as follows: In section 2.2 we fix all the notations we are going to use; Section 2.3 presents the characterization of a social ranking solution based on the *ceteris paribus majority* principle; Section 2.4 deals with the analysis of single-peakedness in our framework; Section 2.5 discusses incompleteness of the power relation, and Section 2.6 concludes the chapter.

2.2 Preliminaries and Notations

Before going into details, let's unify the notations that we are going to use in different chapters of the thesis:

Let $N = \{1, \dots, n\}$ be a finite set of elements or individuals and let $R \quad N \times N$ be a *binary relation* on N (xRy meaning that x is in relation R with y, for $x, y \quad N$). A binary relation R on N is said to be: *reflexive*, if for each $i \quad N$, iRi; *transitive*, if for each $i, j, z \quad N$, (iRj

and $jRk)$ iRk; *total*, if for each i, j N, $i = j$ iRj or jRi; *antisymmetric*, if for each i, j N, iRj and jRi $i = j$. A *preorder* is a reflexive and transitive binary relation. A preorder that is total is called *total preorder*. An antysimmetric total preorder is called *linear order* (each equivalence class is a singleton). We denote by $\mathcal{T}(N)$ the set of all total preorders on N, and by $\mathcal{L}(2^N)$ the set of all linear orders on 2^N. A *power relation* is a binary relation $\mathcal{B}(2^N)$ where $\mathcal{B}(2^N)$ is the family of all subsets of $2^N \times 2^N$. For all S, T 2^N, S T means that (S, T) and (T, S) / and S T means that (S, T) and (T, S) . The problem that we study is to rank individuals in a set N given a power relation ( ) defined over the power set of individuals(2^N). We refer to ranking over individuals as a *social ranking solution* or *solution*. Given a set of individuals, throughout the thesis, we study the social ranking solutions that applying them on a power relation results in a total preorder over the set of individuals. Formally, a *social ranking solution* on A N is a function $R_A : \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ associating to each power relation $\mathcal{B}(2^N)$ a total preorder $R_A($ $)$ (or R_A) over the elements of A. By this definition, the notion $iR_A j$ means that applying the social ranking solution to the power relation gives the result that i is ranked higher than or equal to j. When R_A is a total preorder, we denote by I_A its symmetric part, and by P_A its asymmetric part. In Chapter 4 a social ranking solution is defined as a function that maps each power relation to a set of linear orders over the set of individuals($2^{\mathcal{L}(N)}$).

2.3 Social Ranking Solutions

Starting from the classical approaches to the voting procedure, in this section we expand the simple majority rule to the domain of coalitional voting systems and reformulate the properties introduced by [May, 1952] in our coalitional setting. Investigating how the properties of simple majority rule will change when we apply them to the domain of our problem helps us to better understand the advantages and disadvantages of the proposed social ranking rule and, probably, find out the applications that such method best fits into.

The first property discussed in this section says that each coalition should influence the social ranking of two alternatives i and j equally, so we can interchange the relation involving coalitions S $\{i\}$ and S $\{j\}$ with the one T $\{i\}$ and T $\{j\}$ involving another coalition T different from S but having the same kind of relation, and without changing the final social ranking over i and j. In the following, recall that, given two power relations and , the notations and denote indifference in and in , respectively.

Definition 2.3.1 (Equality of Coalitions)**.** *Let A N. A solution $R_A : \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ satisfies the property of Equality of Coalitions (EC) if and only if for all power relations , $\mathcal{B}(2^N)$, i, j A and bijection $\pi : 2^{N\setminus\{i,j\}} \to 2^{N\setminus\{i,j\}}$ such that S $\{i\}$ S $\{j\} \Leftrightarrow \pi(S)$ $\{i\}$ $\pi(S)$ $\{j\}$ for all S $2^{N\setminus\{i,j\}}$, it holds that $iR_A j \Leftrightarrow iR_A j$.*

Differently stated, the social ranking of two individuals i and j should only depend on the ranking expressed by coalitions over i and j, regardless of the number and the identity of coalitions' members. In particular, a coalition of one member has the same influence as a coalition with many members.

Example 7. *Consider the power relation in example (6), to evaluate researchers' performance in a department with five researchers. To compare individuals* 1 *and* 2*, a social ranking rule R, which follows a ceteris paribus majority principle, evaluates their performance in teams* 45, 4, *and* 3*, by referring to ceteris paribus comparisons* 245 145, 24 14 *and* 13 23*. Suppose the social ranking rule R satisfies equality of coalitions. Let's form another power relation in which the coalitions are permuted in the relative ceteris paribus comparisons. Suppose in a new power relation , the ceteris paribus comparisons to compare individuals* 1 *and* 2 *are* 145 A 245, 24 A 14*, and* 23 A 13 *(to form power relation , coalitions* 45 *and* 3 *are permuted). The social ranking rule R ranks* 1 *and* 2 *in the same way as it ranks them in power relation . In this example we have* $1R_A2 \Leftrightarrow 1R_A2$*.*

The next condition states that a solution should not favor any candidate in A N: if the name of two individuals in A is reversed, the social ranking remains in favor of the individual who performs better.

Definition 2.3.2 (Neutrality)**.** *Let* A N*. A solution* $R_A : \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ *satisfies the property of Neutrality (N) if and only if for all power relations* , $\mathcal{B}(2^N)$ *and* i, j A *such that* S $\{i\}$ S $\{j\} \Leftrightarrow S$ $\{j\}$ S $\{i\}$ *for all* S $2^{N\setminus\{i,j\}}$*, it holds that* $iR_Aj \Leftrightarrow jR_Ai$*.*

This property points out a very natural demand in many social ranking situations. But satisfaction of neutrality property is problematic in some ranking problems. Suppose, given a power relation, the aim of the ranking problem is to extract the only top-ranked individual. In many of such situations tie breaking rules are beneficial. However, tie breaking rules refer to the idea that the electoral system has specific mechanisms (like randomness or lexicographical methods) to select one candidate out of the others, if more than one individual are highly ranked. Such mechanisms obviously violate the neutrality axiom.

Example 8. *Consider the power relation in Example 6, and suppose the aim is to compare researchers* $A = \{1, 2\}$*. A social ranking, which follows ceteris paribus principle, ranks individuals by referring to ceteris paribus comparisons* 245 145, 24 14*, and* 13 23*. Assume that the social ranking rule R satisfies neutrality, if the name of individuals* 1 *and* 2 *is reversed by forming another power relation with ceteris paribus comparisons* 145 A 245, 14 A 24*, and* 23 A 13*, then the social ranking rule reverses the ranking of the individuals as well:* $1R_A2 \Leftrightarrow 2R_A1$*.*

The next property states that a solution should be coherent with changes of the power relation of coalitions. More precisely: if, on a power relation, a social ranking solution is indifferent or in favor of i with respect to j, and if the power relation of all coalitions remains the same except that a single coalition becomes favorable to i, then the social ranking becomes strictly favorable to i.

Definition 2.3.3 (Positive Responsiveness)**.** *Let* A N*. A solution* $R_A : \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ *satisfies the property of Positive Responsiveness (PR) if and only if for all power relations* , $\mathcal{B}(2^N)$*,* i, j A *with* iR_Aj *and such that for some* T $2^{N\setminus\{i,j\}}$*,* $[T$ $\{i\}$ T $\{j\}$ *and* T $\{i\}$ A T $\{j\}]$*, or,* $[T$ $\{j\}$ T $\{i\}$ *and* T $\{i\}$ T $\{j\}]$ *and* S $\{i\}$ S $\{j\} \Leftrightarrow$ S $\{i\}$ S $\{j\}$ *for all* S $2^{N\setminus\{i,j\}}$ *with* $S = T$*, it holds that* iP_Aj*, but not* jP_Ai*.*

The following example clarifies the idea.

Example 9. *Consider the power relation of Example 6, and suppose a social ranking rule R satisfies positive responsiveness, and the goal is to compare researchers* 1 *and* 2 *($A = \{1, 2\}$). Referring to ceteris paribus comparisons in the power relation (*245 145, 24 14, *and* 13 23*), let's assume that the social ranking rule ranks individual* 2 *at least as high as individual* 1 *in the power relation ($2R_A 1$). Now, suppose individual* 2 *starts to perform better when joining coalition* 3 *by forming a power relation , with ceteris paribus comparisons* 245 A 145, 24 A 14, 13 23*. Based on the property of positive responsiveness, the social ranking R responds positively to the improvement of individual* 2*, and ranks it strictly higher than individual* 1*. In another way, suppose, given the power relation , the social ranking rule ranks* 1 *at least as high as* 2*. Now if individual* 2 *starts to perform poorly when it joins coalition* 45*, and has the same performance as* 1*, by forming power relation with* 245 145, 24 A 14, 23 13*, then the social ranking rule ranks* 1 *strictly higher than* 2 *($1P_A 2$).*

In the rest of this section, we introduce the extension of majority rule to our coalitional framework, and characterize it using the three axioms.
Before formally defining the ranking rule, we need to introduce some further notations. Given a power relation $\mathcal{B}(2^N)$ and two elements i, j N we define two sets: $D_{ij}( ) = \{S$ $2^{N\setminus\{i,j\}} : S \{i\} S \{j\}\}$ and $E_{ij}( ) = \{S$ $2^{N\setminus\{i,j\}} : S \{i\} S \{j\}\}$. The union of these two sets represents the *informative part* of the power relation for ranking individuals i and j. We denote the cardinalities of $D_{ij}( )$ and $E_{ij}( )$ by d_{ij} and e_{ij}, respectively. *Ceteris paribus* majority rule ranks one individuals higher than the other one if it performs better in majority of cooperations:

Definition 2.3.4 (Ceteris Paribus Majority)**.** *Let $\mathcal{B}(2^N)$. The ceteris paribus majority relation (CP-majority) is the binary relation R $N \times N$ such that for all i, j N:*

$$iR\ j \Leftrightarrow d_{ij}( ) \geq d_{ji}( ).$$

Example 10. *Applying ceteris paribus majority rule on the power relation of example (6)*

2345 245 1234 13 12 23 145 35 24 14 34

ranks individual 2 *higher than individual* 1*, since individual* 2 *performs two times better than individual* 1 *in coalitions* 4 *and* 45*, while* 1 *just performs one time better than* 2 *by joining coalition* 3 *(Table 2.1).*

Coalitions	Comparisons
3	13 23
4	24 14
45	245 145

Table 2.1: CP-comparisons to compare individuals 1, 2 in power relation of Example 6.

The following result characterizes CP-majority rule using the properties introduced above.

Theorem 2.3.5. *Let $A = \{i, j\} \quad N$ be a set with only two alternatives. A solution $R_A : \mathcal{B}(2^N) \longrightarrow \mathcal{T}(A)$ associates to each $\quad \mathcal{B}(2^N)$ the corresponding CP-majority relation $R \quad \cap A \times A$ if and only if it satisfies axioms EC, N and PR.*

Proof. ( ) (The existence part)A map assigning to each $\quad 2^N \times 2^N$ the corresponding CP-majority relation $R \quad \cap A \times A$ is a solution since the two alternatives in A can always be compared (it also is obviously transitive and reflexive). Hereby, we denote such a map the CP-majority solution (on only two alternatives). EC is satisfied since the CP-majority relation only depends on the numbers $d_{ij}(\ )$ and $d_{ji}(\ )$, and not on which coalitions are in favor of one or the other alternative. The Neutrality property is clearly also satisfied, since the interchange of the alternatives does not affect the definition of the CP-majority relation. Also notice that the CP-majority satisfies the PR property: if $d_{ij}(\ ) = d_{ji}(\ )$, the change of one single indifference $S \quad \{i\} \quad S \quad \{j\}$ with $S \quad E_{ij}(\ )$ in favor of i or j breaks the tie.
( ) (Uniqueness part) Suppose R_A satisfies EC, N and PR. Let $\ ,\ \quad \mathcal{B}(2^N)$ be such that $d_{ij}(\ ) = d_{ij}(\ )$ and $d_{ji}(\ ) = d_{ji}(\ )$. Define a permutation π of the elements in $2^{N \setminus \{i,j\}}$ such that the elements of $D_{ij}(\ )$ are in a one-to-one correspondence with the elements in $D_{ij}(\ )$, the elements of $D_{ji}(\ )$ are in a one-to-one correspondence with the elements in $D_{ji}(\ )$ and the elements of $E_{ij}(\ )$ are in a one-to-one correspondence with the elements in $E_{ij}(\ )$. Then, property EC implies that:

$$iR_A j \Leftrightarrow iR_A j. \tag{2.1}$$

Now, suppose now that $d_{ij}(\ ) = d_{ji}(\ )$, and $d_{ij}(\ ) = d_{ji}(\ )$. Define another coalitional relation $< \quad B$ such that $D_{ij}(<) = D_{ji}(\ )$, $D_{ji}(<) = D_{ij}(\ )$ and $E_{ij}(<) = E_{ji}(\ )$. By the property N, we have that $xR^<_A y \Leftrightarrow yR_A x$ Moreover, since $d_{ij}(\ ) = d_{ij}(<)$ and $d_{ji}(\ ) = d_{ji}(<)$, for the previous arguments, by EC we have that $iR^<_A j \Leftrightarrow iR_A j$. So, $jR_A i \Leftrightarrow iR_A j$, and together with relation (2.1), we have that $iR_A j \Leftrightarrow jR_A i$. Since R_A must be total, we have then proved that:

$$d_{ij}(\ ) = d_{ji}(\ ) \qquad iI_A j. \tag{2.2}$$

Now, take a power relation $\quad \mathcal{B}(2^N)$ such that $d_{ij}(\ ) > d_{ji}(\ )$. Take $X \quad D_{ij}(\ )$ with $|X| = d_{ij}(\ ) - d_{ji}(\ )$. Define another power relation $\quad \mathcal{B}(2^N)$ such that $S \quad \{i\}$ A $S \quad \{j\}$ for all $S \quad D_{ij}(\ ) \setminus X$, $S \quad \{i\} \quad S \quad \{j\}$ for all $S \quad E_{ij}(\ ) \quad X$ and $S \quad \{j\}$ A $S \quad \{i\}$ for all $S \quad D_{ji}(\ )$. Clearly, $d_{ij}(\ ) = d_{ji}(\ )$ and therefore, by relation (2.2), $iI_A j$. Break a tie for precisely one element $S \quad X$ such that now $S \quad \{i\}$ A $S \quad \{j\}$: by property PR, we have that now $iP_A j$. By induction, using this result and the PR property, we have that breaking $0 < m \leq |X|$ ties in favour of i for m elements of E, always gives $iP_A j$. So, if now $S \quad \{i\}$ A $S \quad \{j\}$ for all $S \quad X$, we have that $d_{ij}(\ ) = d_{ij}(\ )$ and $d_{ji}(\ ) = d_{ji}(\ )$, and by EC we obtain that $iP_A j$. More precisely, we have proved that:

$$d_{ij}(\ ) > d_{ji}(\ ) \qquad iP_A j, \tag{2.3}$$

and by the N property it immediately follows that:

$$d_{ij}(\ ) < d_{ji}(\ ) \qquad jP_A i. \tag{2.4}$$

Relations (2.2), (2.3) and (2.4) are the definition of the CP-majority relation, which concludes the proof. □

Axioms EC, N and PR are independent.

Proof. In order to establish the independence of the three axioms, we show that for each pair of axioms there are social ranking rules that satisfies them but not the remaining one:

- Consider a set N of individuals and a power relation . To compare individuals $\{i, j\} = A$, let's assume d_{ij}^{k} indicates number of times $S \quad \{i\} \quad S \quad \{j\}$ for $S \quad 2^{N \setminus \{i,j\}}$ and $|S| = k$. If a social ranking solution defined as $iR_A j \Leftrightarrow \Sigma_k k \times d_{ij}^{k}(\quad) \geq \Sigma_k k \times d_{ji}^{k}(\quad)$, then it is easy to check that it satisfies Neutrality and Positive responsiveness but not Equality of Coalitions.

- Consider a set N of individuals and a power relation . Imagine social ranking rules R such that to compare $\{i, j\} = A$ we have $iR_A j \Leftrightarrow d_{ij}(\quad) \geq k \times d_{ji}(\quad)$ and $jR_A i \Leftrightarrow k \times d_{ji}(\quad) \geq d_{ij}(\quad)$ when $k = 1$. In this case, when k is greater than one, the ranking system is in favor of individual j, and when k is less than one, the ranking system is in favor of individual i. These social ranking rules obviously satisfy Equality of coalitions, positive responsiveness but not Neutrality.

- Consider a set N of individuals and a power relation . To compare individuals $\{i, j\} = A$ let's define social ranking rules that rank one individual higher than the other one if it gets less votes, e.g., $iR_A j \Leftrightarrow d_{ji}(\quad) - d_{ij}(\quad) \geq 0$. Such social ranking rules satisfy neutrality, and anonymity but not positive responsiveness.

□

The *ceteris paribus* simple majority solution is grounded in intuitive and appealing principles. However, it turns out that strict Condorcet-like cycles are possible for more than two candidates, similarly to classical voting theory.

Example 11. *Let's recall the power relation of Example 6.*

$$2345 \quad 245 \quad 1234 \quad 13 \quad 12 \quad 23 \quad 145 \quad 35 \quad 24 \quad 14 \quad 34.$$

If we use the ceteris paribus majority rule to compare the three individuals 1, 2, *and* 3 *we get the result that:*
$3R \; 2$, *since* $13 \quad 12$, $2R \; 1$, *since* $245 \quad 145$, $24 \quad 14$ *and* $13 \quad 23$, *but* $1R \; 3$, *since* $12 \quad 23$, *which obviously produce a cycle.*

The question raised by Example 11 is whether there are reasonable assumptions about the power relation under which strict Condorcet-like majority cycles can be avoided. Section 2.4 introduces a domain restriction which acts as a sufficient condition for avoiding cycles in the majority solution. Notice that *monotonic* power relations (each individual has a positive effect when joining a coalition) belong to another type of domain restriction. Even if such an assumption seems natural in some contexts (for instance, in multi-attribute decision making), it might be violated in others (for instance, in the context of our Example 6, the performance of a researcher may decrease joining a larger team). Moreover, the ranking generated by the CP-majority is only affected by comparisons between coalitions of the type $S \quad \{i\}$ and $S \quad \{j\}$, where the monotonicity condition does not apply.

2.4 Single-peakedness of the Power Relation

It is an important insight from the classical voting literature that certain restrictions on the preferences of the voters are sufficient to guarantee a feasible majority solution. From this perspective, as mentioned in the last chapter, an interesting restriction is what we will call here *individual single-peakedness* [Black et al., 1958]. In a classical voting scenario, a basic assumption is that there exists a linear order on candidates; then, supposing voters rank candidates linearly (i.e., no ties), it goes on to say that a voter V's preference $_V$ over candidates is *individually single-peaked* if, for any candidates i, j and k such that $i \quad j \quad k$, it is not the case that both $i \quad _V \; j$ and $k \quad _V \; j$. [Sen, 1966] proved that if voters' preference $_V$ is single-peaked for considering any three candidates in the set of all candidates, then the $_V$ is single peaked over the whole set of candidates. We will formalize a similar property for the power relation , which we will, for simplicity, assume to be a linear order.

Definition 2.4.1 (Social single peakedness). *The (linear) power relation is socially single-peaked if there exists a linear order on the set of items N such that for any $i, j, k \quad N$ for which $i \quad j \quad k$ and any $S \quad 2^{N\setminus\{i,j,k\}}$, none of the following conditions holds:*

(sp$_1$) $S \quad \{i\} \quad S \quad \{j\}$ *and* $S \quad \{k\} \quad S \quad \{j\}$,

(sp$_2$) $S \quad \{i,k\} \quad S \quad \{i,j\}$ *and* $S \quad \{i,k\} \quad S \quad \{j,k\}$.

Intuitively, the linear order stands for a dimension along which items in N can be ranked. We then assume that the power relation orders coalitions in a manner consistent with , as follows: a coalition S containing neither i, j nor k is interpreted as a voter with the power to rank i, j and k. Then, according to sp$_1$, S does not rank the median candidate j as the least preferred of the lot; according to sp$_2$, the extreme candidates i and k are not the most preferred among the combinations $\{i,j\}$, $\{i,k\}$ and $\{j,k\}$. Thus, though clearly not identical with it, single-peakedness evokes the similarly named condition from voting theory.

We further exploit this (by now familiar) tactic of interpreting coalitions as voters in order to generate an individually single-peaked voting profile out of a socially single-peaked power relation .

Definition 2.4.2 (Revealed preferences). *Given a power relation on N, a linear order on N, items $i, j, k \quad N$ such that $i \quad j \quad k$, and a coalition $S \quad 2^{N\setminus\{i,j,k\}}$, the S-revealed (respectively, $ijkS$-revealed) preference relations $>_S$ (respectively, $>_{ijkS}$) based on , are defined as follows:*

- $i >_S j$ *iff* $S \quad \{i\} \quad S \quad \{j\}$,
- $i >_{ijkS} j$ *iff* $(S \quad \{i,j,k\}) \setminus \{j\} \quad (S \quad \{i,j,k\}) \setminus \{i\}$.

1 vs. 2		2 vs. 4		1 vs. 4		revealed orders
1	2	2	4	1	4	$1 > 2 > 4$
13	23	23	34	13	34	$2 >_3 4 >_3 1$
14	24	12	14	12	24	$2 >_{124} 1 >_{124} 4$
134	234	123	134	123	234	$2 >_{1234} 1 >_{1234} 4$

Table 2.2: Power relation and revealed orders

Intuitively, $>_s$ and $>_{ijkS}$ stand in for the preferences of S and S $\{i, j, k\}$ over candidates i and j. Relation $>_S$ encodes the fact that coalition S prefers i to j, since adding i to S leads to better performance than adding j; $>_{ijkS}$ encodes the fact that coalition S $\{i, j, k\}$ prefers i to j, since losing i from S $\{i, j, k\}$ leaves the coalition in a worse position than losing j. Stated differently, $>_{ijkS}$ expresses the fact that i is more valuable to S $\{i, j, k\}$ than j.

We can show now that social single-peakedness of (as introduced in Definition 2.4.1) implies individual single-peakedness of the revealed preference relations.

Lemma 2.4.3. *If is a power relation on N, i, j, k N and S $2^{N \setminus \{i,j,k\}}$, then is socially single-peaked iff $>_S$ and $_{ijk}$ are individually single-peaked.*

Proof. Assume, first, that the power relation is socially single-peaked but that $>_S$ is not individually single-peaked, for some coalition S. Then there are i, j, k / S such that i j k and $i >_S j$, $k >_S j$, which implies that S $\{i\}$ S $\{j\}$ and S $\{k\}$ S $\{j\}$. But this contradicts condition sp_1, and hence the social single-peakedness of . Similarly, if $>_{ijkS}$ is not individually single-peaked, it follows that S $\{i, k\}$ S $\{i, j\}$ and S $\{i, k\}$ S $\{j, k\}$, contradicting condition sp_2. The proof that individual single-peakedness of $>_S$ and $>_{ijkS}$ implies social single-peakedness of is analogous. □

The revealed preference relations allow us to interpret the *ceteris paribus* majority solution over items i, j, k as the result of an election over i, j, k where the voters are coalitions S and S $\{i, j, k\}$, for S $2^{N \setminus \{i,j,k\}}$. The following example illustrates this.

Example 12. *Suppose we want to rank items 1, 2 and 4 from a set $N = \{1, 2, 3, 4\}$ of items and we are given a power relation which generates the revealed relations showed in Table 2.2. We have orders $>$ and $>_3$, corresponding to the revealed preferences of coalitions S $2^{N \setminus \{1,2,4\}}$, as well as $>_{124}$ and $>_{1234}$, corresponding to revealed preferences of coalitions S $\{1, 2, 4\}$. The majority relation in the election over the revealed preferences is $2 > 1 > 4$, which corresponds to the* ceteris paribus *majority solution over . Notice that the revealed preference orders are individually single-peaked (with linear order 1 2 3 4 over items), and that the social ranking is socially single peaked (with the same linear order over items).*

Now we can state our main result of this section.

Theorem 2.4.4. *If the power relation is socially single-peaked, then for any items i, j, k N, it does not hold that iR jR kR i (i.e., the* ceteris paribus *majority solution does produce any non-transitive cycles).*

Proof. For every coalition $S \in 2^{N\setminus\{i,j,k\}}$, construct a profile of votes over i, j and k from the revealed preference relations $>_S$ and $>_{ijkS}$. We have that $iR_{\succeq} j$ iff i is a majority winner over j in this profile. Since, by Lemma 2.4.3, the relations $>_S$ and $>_{ijkS}$ are individually single-peaked, we get that there is no majority cycle between i, j and k in the final result, which implies that there is no cycle between i, j and k in the *ceteris paribus* majority solution. □

As an illustration of how a socially single-peaked power relation can be obtained, consider the fact that a linear order $\succeq$ over the elements of 2^N can be numerically represented by a *characteristic function* $v\colon 2^N \to \mathbb{R}$ such that $S \succeq T$ iff $v(S) > v(T)$ for all $S, T \in 2^N$. Suppose now that the marginal contribution $v(S \cup \{i\}) - v(S)$ of player $i \in N \setminus S$ is somehow (inversely) related to the distance, on a policy scale, of the ideal position of player i from the jointly preferred position of coalition $S \in 2^N$ (the lower the distance from the joint position, the higher the marginal contribution). More precisely, suppose that:

- agents in N have a preferred ideal position $x_i \in [0, +\infty)$, where the line $[0, +\infty)$ represents the policy scale, and
- each coalition $S \in 2^N$ is also characterized by a jointly preferred position $p_S \in [0, +\infty)$ on the same policy scale, e.g., resulting from an aggregation process over the individual positions of players in S, or provided by an external actor (see, e.g., the model of coalition formation in [Bilal et al., 2001]).

For every $S \in 2^N$ and $i, j \in N \setminus S$, we assume that the following monotonicity relation exists between the distance $d_{iS} = |p_S - x_i|$ and the marginal contribution of i:

$$d_{iS} < d_{jS} \Leftrightarrow v(S \cup \{i\}) - v(S) > v(S \cup \{j\}) - v(S). \tag{2.5}$$

In addition, we assume that the jointly preferred position of a coalition monotonically increases over the policy scale with the positions of its members, that is:

$$x_i < x_j \Rightarrow p_{S \cup \{i\}} \leq p_{S \cup \{j\}}, \tag{2.6}$$

for every $S \in 2^N$ and $i, j \in N \setminus S$. This is the case, for instance, when the jointly preferred position x_S is computed as the median of the individual positions x_i in S. The following proposition shows that the power relation $\succeq$ is single-peaked according to Definition 2.4.1.

Proposition 2.4.5. *Let $\succeq$ be a linear order on 2^N and let $v\colon 2^N \to \mathbb{R}$ be such that $S \succeq T$ iff $v(S) > v(T)$ for all $S, T \in 2^N$. Consider the vectors $x \in \mathbb{R}^N_+$ and $p \in \mathbb{R}^{2^N}_+$ satisfying conditions (2.5) and (2.6) for all $S \in 2^N$ and $i, j \in N \setminus S$. Then, $\succeq$ is socially single peaked.*

Proof. Take $i, j, k \in N$ with $x_i < x_j < x_k$. Notice that since $\succeq$ is a linear order over 2^N, then by Definition 2.4.2, relations $\succeq_S$ and $\succeq_{ijkS}$, for each $S \in 2^{N\setminus\{i,j,k\}}$, are linear orders over N. To prove that $\succeq_S$ and $\succeq_{ijkS}$ are single-peaked (with respect the ordering (i, j, k)), it remains to show that if $i \succeq_S j$ and $i \succeq_{ijkS} j$, then $j \succeq_S k$ and $j \succeq_{ijkS} k$.

Let $S \in 2^{N\setminus\{i,j,k\}}$. First, suppose that $i \succeq_S j$ or, equivalently, $S \cup \{i\} \succeq S \cup \{j\}$. Then, $v(S \cup \{i\}) - v(S) > v(S \cup \{j\}) - v(S)$ and by relation (2.5) $d_{iS} < d_{jS}$. Consequently,

$p_S < \frac{x_i+x_j}{2}$, and since $x_j < x_k$ we have that $d_{jS} < d_{kS}$. Then, by relation (2.5), we have $v(S \quad \{j\}) - v(S) > v(S \quad \{k\}) - v(S)$ and, by the definition of v as a numerical representation of , it follows that $S \quad \{j\} \quad S \quad \{k\}$, implying that $j \quad _S \; k$.

Now, suppose that $i \quad _{ijkS} \; j$ or, equivalently, $S \quad \{i,k\} \quad S \quad \{j,k\}$. Then, $v(S \quad \{i,k\}) - v(S \quad \{k\}) > v(S \quad \{j,k\}) - v(S \quad \{k\})$ and by relation (2.5), we have $d_{iS \quad \{k\}} < d_{jS \quad \{k\}}$. Consequently, $p_{S \quad \{k\}} < \frac{x_i+x_j}{2}$. Moreover, by relation (2.6), since $x_i < x_k$ we have that $p_{S \quad \{i\}} \leq p_{S \quad \{k\}}$. So, $p_{S \quad \{i\}} < \frac{x_i+x_j}{2}$, then $d_{jS \quad \{i\}} < d_{kS \quad \{i\}}$, and again by relation (2.5), $v(S \quad \{i,j\}) - v(S \quad \{i\}) > v(S \quad \{i,k\}) - v(S \quad \{i\})$. By the definition of v as numerical representation of , we obtain $S \quad \{i,j\} \quad S \quad \{i,k\}$, implying that $j \quad _{ijkS} \; k$. □

Example 13. *Consider a set $N = \{1,2,3,4\}$ of four agents, with individual preferred position $x_i = i$ for each $i \quad N$, the linear power relation in Table 2.2 that can be numerically represented by a characteristic function v satisfying condition (2.5), and jointly preferred positions $p_S = median([x_i]_{i \quad S}) -$ for each $S \quad 2^N$.* [2] *To be more specific, we have that the jointly preferred positions are $p = 0$ (by convention), $p_{\{i\}} = x_i$, $p_{\{i,j\}} = \frac{x_i+x_j}{2} -$, $p_{\{i,j,k\}} = x_j$ if $x_i < x_j < x_k$. The single-peaked linear orders $_S$ and $_{124S}$ on 1, 2, 4, with $S \quad \{ \; , \{3\}\}$, correspond to the revealed orders in Table 2.2.*

A different, though related way to obtain socially single-peaked power relations starts off assuming that there is a valuation $v\colon N \to \mathbb{R}$ on the items themselves such that $i < j$ iff $v(i) < v(j)$, and that $v(S) = \sum_{i \quad S} v(i)$. In other words, coalitions are ranked according to the sum of the values of their members. This also leads to a socially single-peaked power relation , which we will be denoted as $_\Sigma$.

Proposition 2.4.6. *Power relation $_\Sigma$ is socially single-peaked.*

Proof. Take the linear order on items of N to be given by the valuation v, i.e., $i \quad j$ iff $v(i) < v(j)$. We obtain that $v(S \quad \{i\}) = v(S) + v(i)$ and $v(S \quad \{i,j\}) = v(S) + v(i) + v(j)$ and it is straightforward to check that conditions sp_1 and sp_2 are satisfied. □

Finally, note that (as per Theorem 2.4.4) social single-peakedness provides only a sufficient condition under which supports application of the *ceteris paribus* majority rule. As we have shown in this section, some natural interpretations of the power relation turn out to satisfy it, but nonetheless social single-peakedness should not be thought of as exhaustive of the cases favorable to the *ceteris paribus* majority rule. Consider, for instance, a (total and transitive) power relation $_M$ such that for all non-empty coalitions $S, T \quad 2^N$

$$S \quad _M \; T \Leftrightarrow \{b(S)\} \quad _M \; \{b(T)\}, \tag{2.7}$$

where, for each $S \quad 2^N$, $S = \;$, $b(S)$ is a best element of S, i.e., such that $\{b(S)\} \quad _M \; \{i\}$ for each $i \quad S$. Even if $_M$ is not socially single peaked (some ties may occur in $_M$), we show now that the CP-majority relation $R^{\quad M}$ is transitive. The CP-majority relation $R^{\quad M}$ is transitive.

[2]Since is a linear order, the factor $(0, \frac{1}{2})$ is used to break ties $d_{iS} = d_{jS}$ in favor of the element with the lowest individual position $\min\{i,j\}$.

Proof. First, note that for each $x, y \in N$, if $\{x\} \succsim_M \{y\}$, then there is no $S \in 2^{N \setminus \{x,y\}}$ such that $S \cup \{y\} \succ_M S \cup \{x\}$. So, $d_{yx}^M = 0$. Now, assume that $\{x\} \succ_M \{y\}$. Then, $d_{xy}^M \geq 1 > 0 = d_{yx}^M$. On the other hand, if $\{x\} \sim_M \{y\}$, we have that $S \cup \{y\} \sim_M S \cup \{x\}$ for each $S \in 2^{N \setminus \{x,y\}}$, implying $d_{xy}^M = d_{yx}^M = 0$. We have then shown that $xR^{M}y$ if and only if $\{x\} \succsim_M \{y\}$, and the transitivity of R^{M} follows from the definition of $\succsim_M$. □

2.5 When Incompleteness is a Source of Information to Rank Individuals

In the previous section, we have defined a social ranking rule which ranks individuals according to their performance when joining different coalitions. However, as we mentioned in the introduction, in such a context the relation between individuals in a coalition or (and) the incompleteness of power relation may bring some hidden information that helps to rank individuals more precisely. By precisely it means ranking individuals in a way to match better to the context in which a power relation is defined. More clearly, given a power relation $\succsim$ the ranking of individuals i and j can differ by considering different possibilities cause the power relation to form. For instance, the incompleteness of power relation may relate to the absence of some coalitions (let's refer to it as case 1), or it may reflect the fact that we are not able to compare some of the coalitions. There are some situations where coalitions have different quality competences and it is not possible to compare them (Let's refer to it as case 2). In case 2, the incompleteness may be related to the heterogeneity of teams and reflects an incomparability. In Example 6, the absence of coalition 234 may be interpreted as the case that all the coalitions in the power relation are qualifies in terms of their research activities like the papers they have published, while coalition 234 is a coalition qualified in terms of teaching skills, and, hence, this is the reason it is not showed up in the power relation.

Moreover, coalitions may not form for other various reasons. For instance, coalitions may not form because the structure of problem is defined in a way that forming some coalitions is forbidden. Imagine in a company each project must be done by a team of experts. It seems reasonable that the teams cannot contain employees of the same expertise (case 1a). On the other hand, think about a scenario where a coalition forms to permit individuals in the coalition to evaluate each other. In this case, some coalitions might not form because some members prefer to focus on small number of individuals to evaluate, or some coalitions form because the members prefer to explore more and evaluate more number of individuals by cooperating with them (case 1b). The other possibility is that individuals are free to choose their teammates according to the level of friendship. In this case, some coalitions do not form because some individuals are not friend with others and prefer not to cooperate with them (case 1c).

Each one of these scenarios may somehow affect the ranking of individuals in a power relation. For instance, in case 1c, the absence of a coalition is a negative factor for the individuals forming it, while it is not the case in 1a. Hence, the exploitation of incompleteness to define a social ranking rule must depend on the reasons of incompleteness, and different solution must be considered for alternative contexts.

Example 14. *Consider a set of individual* $N = \{1, 2, 3, 4, 5\}$ *and a power relation given as* 435 235 15 25 135 14 24 45 12*. By CP-majority rule we have that individual* 1 *is ranked higher than individual* 2*, and also individuals* 2 *and* 5 *are considered to be indifferent. However, it is possible to extract some information about incompleteness of the power relation. For instance, coalition* 23 *is not in the power relation while* 235 *is in the power relation. By the interpretation of case 1c, absence of* 23 *and presence of* 235 *could be related to the situation where "*2 *and* 3 *are not friend and they do not want to cooperate, however, if* 5 *plays the role of a third party between them then the coalition* 235 *can form". By this interpretation, individual* 5 *can be assumed to have more importance than individual* 2*. On the other hand, according to interpretation of case 1b, coalitions* 35*,* 4*, and* 5 *cooperate with the other individuals to evaluate them. Some coalitions tend to explore more, like coalition* 35 *that cooperates with three different individuals (not belonging to coalition* 35*), and some coalitions cooperate less like* 4 *or* 5 *since they can collaborate with four other individuals (not belonging to coalitions* 4 *and* 5*), yet they only cooperate with three of them. So here the question is that shall we keep the ranking of individuals* 1 *and* 2 *as is given by CP-majority rule? Or shall we give different weights (more weight or less weight) to the coalitions that explore more?*

In the following, we present a new social ranking solution as an example of ranking rules that can be defined in case 1b. The general idea of this solution is to classify the coalitions in terms of their information level and to compare two candidates using the information inferred from the "most" informed coalitions.

Following the interpretation in case 1b, it means that coalitions that involve in more comparisons has more experience and a higher level of information. More formally, let $S \quad 2^N$ and $\overline{S} = N \setminus S$ be its complement. We define the set of comparisons in which S is involved in the power relation as the set $S = \{x \quad \overline{S} : y \quad \overline{S} \setminus \{x\}$ s.t. $S \quad \{x\} \quad S \quad \{y\}$ or $S \quad \{y\} \quad S \quad \{x\}\}$. Inversely the number of comparisons that a coalition does not participate in shows how the coalition is *ignorant*. Consequently, the set of elements that cannot be compared by means of a coalition S is given by $\overline{S} \setminus S$, and we call its cardinality $|\overline{S} \setminus S \ |$ the *ignorance* of S.

Example 15. *Consider the power relation in Example 14:* 435 235 15 25 135 14 24 45 12*, and suppose the goal is to compare individuals* 1 *and* 2*. By considering the case 1b the ignorance levels of coalitions* 35*,* 4*, and* 5 *respectively are* $|3 \setminus 3| = 0$*,* $|4 \setminus 3| = 1$*, and* $|4 \setminus 3| = 1$*.*

We denote by c_t, $t \quad \{0, \ldots, n\}$, the set of all coalitions S such that $|\overline{S} \setminus S \ | = t$, and call it the ignorant class t. So, the class c_0 (i.e., the class of most aware coalitions having the lowest ignorance), contains all coalitions that are involved in the comparison among all the elements in their complement set (e.g., c_0 coincides with the power set 2^N if is total); the class c_1 (i.e., the class with the second lowest ignorance) contains all coalitions involved in the comparison of all elements, except one, in their complement set, etc. Obviously, some classes c_t may be empty, and each coalition S may belong to at most one class c_t. The family of all $n + 1$ classes defined over a power relation is denoted by $\mathcal{C} = \{c_0, \ldots, c_n\}$.

For instance, in Example (15) coalition 35 belongs to the class c_0 since it is the most aware coalition in the power relation, while 4 and 5 belong to the class c_1 since they ignore doing some of the comparisons.

Given two elements i, j N, with a slight abuse of notation we denote by

$$D_{ij}(\mathcal{C} \ ) = (D_{ij}(c_0, \ ), \ldots, D_{ij}(c_n, \ ))$$

the restriction of $D_{ij}(\ )$ on $\mathcal{C}$, where $D_{ij}(c_t, \ )$, for each t $\{0, \ldots, n\}$, is the (possibly empty) set of coalitions S c_t such that S $\{i\}$ is strictly stronger than S $\{j\}$, and by

$$d_{ij}(\mathcal{C} \ ) = (d_{ij}(c_0, \ ), \ldots, d_{ij}(c_n, \ ))$$

the vector of cardinalities $d_{ij}(c_t, \ ) = |d_{xy}(c_t, \ )|$.

In the following, we introduce a generalized version of the CP-majority relation aimed at giving more weight to coalitions with a lower ignorance. First, consider the notion of lexicographic order among vectors $\mathrm{v} = (v_0, \ldots, v_n)$ and $\mathrm{w} = (w_0, \ldots, w_n)$: $\mathrm{v} \geq_L \mathrm{w}$ if either $\mathrm{v} = \mathrm{w}$ or $k : v_t = w_t, t = 1, \ldots, k-1$ $v_k > w_k$.

Definition 2.5.1 (Informative CP-Majority). *Let* $\mathcal{B}(2^N)$*. The Informative CP-majority relation (ICP-majority) is the binary relation* $R^{\mathcal{C}}$ $N \times N$ *such that for all* i, j N*:*

$$iR^{\mathcal{C}} \ j \Leftrightarrow d_{ij}(\mathcal{C} \ ) \geq_L d_{ji}(\mathcal{C} \ ).$$

Example 16. *Consider again the power relation of Example 15. The CP-majority ranks individual* 1 *higher than individual* 2 *since the ceteris paribus comparisons are* 235 135, 15 25*, and* 14 24*. Coalitions can be classified by their information levels:* 35 *belongs to* c_0 *and coalitions* 4 *and* 5 *belong to* c_1*. Using information level,* $D_{12}(c_0, \ ) =$ *,* $D_{21}(c_0, \ ) = \{35\}$*. Moreover, we have that* $D_{12}(c_1, \ ) = \{4, 5\}$ *and* $D_{21}(c_1, \ ) =$ *, whereas all the other sets* $D_{12}(c_t, \ )$ *and* $D_{21}(c_t, \ )$ *(*t $\{2, \ldots, n\}$*) are empty. So,* $d_{21}(\mathcal{C} \ ) = (1, 0, 0, 0, 0, 0) \geq_L d_{15}(\mathcal{C} \ ) = (0, 1, 1, 0, 0, 0)$*, implying that, according to the ICP-majority,* 2 *is ranked better than* 1*.*

A deeper discussion of the meaning of incompleteness in our context and the information that can be derived from an incomplete power relation leads to alternative definitions of social ranking solutions. Such an analysis also concerns the axiomatic characterization of solutions, as well as their computational aspects.

2.6 Conclusion and Future Works

In this chapter, we have studied a social ranking solution based on a *ceteris paribus* majority principle; we have provided an axiomatic characterization (on only two alternatives) and we have studied a domain restriction which guarantees the transitivity of the generated rankings. Some hints on how to exploit the incompleteness of a power relation are also provided.

A direction for future research is to further investigate the necessary and sufficient conditions over the domain of power relations which guarantee the transitivity of the ranking induced by the CP-majority. Another open issue is the definition of social ranking solutions that benefit from a larger amount of information in the power relation (and not only focusing on the information coming from the comparison of *ceteris paribus* coalitions). Moreover, alternative criteria could be used to generate the classes of coalitions used for the ICP-majority rule. For instance, one could argue that the weight of a class is related to the overall probability that the coalition in that class form. Another interesting problem is the analysis of the robustness of our social ranking solutions to “small” changes in the power relation. In this perspective, a related issue deals with an application to multi-criteria decision making (MCDM) where, given the relative strength of coalitions of criteria (represented by a power relation), a social ranking solution can be used as an alternative method to compare the importance of criteria, independently from the (arbitrary to some extent) weight assigned to coalitions by a capacity [Grabisch and Labreuche, 2010].

3

Ordinal Banzhaf Solution

Abstract

In this chapter, we introduce a new method to rank single elements given an order over their sets. For this purpose, we extend the game theoretic notion of marginal contribution and of Banzhaf index to our ordinal framework. Furthermore, we characterize the resulting *ordinal Banzhaf solution* by means of a set of properties inspired from those used to axiomatically characterize another solution from the literature: the *ceteris paribus* majority. Finally, we show that the computational procedure for these two social ranking solutions boils down to a weighted combination of comparisons over the same subsets of elements.

Contents

3.1 Introduction

In decision making and social choice theory, a number of studies are devoted to ranking individuals based on the performance of coalitions formed by them. As we have studied in Chapter 1, *power indices* like the Banzhaf index [Banzhaf III, 1964] and the Shapley value [Shapley, 1953] are described from the need to measure individual's *a priori* power in certain cooperative games (simple games). These power indices are based on the role that each individual may get when they join a coalition, which is codified with the notion of *marginal contribution*. Such methods can be used in a variety of applications, such as, finding the most “valuable" items, when the preferences of a user are defined over their combinations; or comparing the influence of different countries inside an international council (for instance, the European Union Council).

In cooperative game theory, some assumptions are made conventionally. For instance, it is assumed that the coalitions are quantifiable, and also their values are monotonic, in the sense that if one coalition be part of another coalition then the value of the first coalition would be less than or equal to the value of the second one. However, in many practical situations, it is not possible to compute the worth of coalitions quantitatively, or even monotonicity may not necessarily hold. For example, the value of a coalition may decrease by joining new members when there is an overhead caused by the cost of communication and cooperation, or when some individuals in the coalitions are not friend and have some negative synergy in between. These possibilities intrigue us to assume the existence of a binary relation over sets of coalitions.

In this chapter, as in chapter 2, we assume a binary relation over subsets of individuals is given, and we are looking for a mapping to transform the ranking over subsets of individuals to a ranking over the set of individuals, which is a complete preorder. Following the main concept of majority, in this section, we utilize a part of comparisons in the power relation that somehow indicates the ordinal version of the classical marginal contributions of individuals [Banzhaf III, 1964]. We refer to the social ranking rule in this chapter as *ordinal Banzhaf relation*. For this solution, we provide an axiomatic characterization which is mostly inspired from the axiomatic study in Chapter 2 for the *ceteris paribus* majority solution on a set of two individuals.

Both the CP-majority rule and the ordinal Banzhaf solution suggest an interpretation of our social ranking problem along the lines of a virtual election, with groups of individuals (coalitions) playing the role of voters: according to the CP-majority solution, a coalition S prefers individual i to individual j if $S \quad \{i\} \quad S \quad \{j\}$, i.e. coalition $S \quad \{i\}$ is “stronger” than coalition $S \quad \{j\}$; according to the ordinal Banzhaf solution, coalition S approves an individual i if $S \quad \{i\} \quad S$, i.e. the marginal contribution of i to $S \quad \{i\}$ is positive. Under this interpretation, we propose a new family of relations on the elements of N that we call *weighted majority* relations. We investigate some members of the family and we show that the CP-majority and the ordinal Banzhaf solution are special cases of this family, when the power relation is a linear order over coalitions.

The contributions of this chapter are published in the proceeding of the international conference IJCAI-19 [Khani et al., 2019].

The remaining of the chapter is organized as follows. Section 3.2 is devoted to the definition of the ordinal Banzhaf relation and its main features as a social ranking solution. Section 3.3 is devoted to the discussion of an axiomatic characterization of the ordinal Banzhaf solution and to its comparison with the CP-majority. Section 3.4 introduces the family of weighted majority rules, and study some of its members. Section 3.5 concludes the chapter.

3.2 Ordinal Banzhaf Index

In this section, we fix the settings of our ranking method. As mentioned in the Introduction, classical power indices like the Shapley value and the Banzhaf index are used in games with the

assumption that the characteristic functions of the cooperative games follow a kind of monotonic behavior. In this section, we extend the notion of *marginal contribution*, which is commonly applied to define frequently used power indices, to our ordinal framework where there is no assumption on the ranking over coalitions to be directly proportional to their size ($|S| \geq |T| \Leftrightarrow S \succeq T$). To motivate our work, we start by showing that the Banzhaf value is very sensitive to small changes in valuation of coalitions. But before that let's recall some important notions of cooperative game theory:
A *Transferable Utility* (TU)-game is a pair (N, v) where v is a function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The *Banzhaf value* $\beta(v)$ of v is the n-vector $\beta(v) = (\beta_1(v), \beta_2(v), \ldots, \beta_n(v))$, such that for each $i \in N$:

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \in 2^{N \setminus \{i\}}} \left(v(S \cup \{i\}) - v(S) \right). \tag{3.1}$$

For example, consider a set $N = \{1, 2, 3\}$ of individuals, and suppose the following power relation is given over the set of coalitions (power set of N) :

$$123 \succeq 12 \succeq 1 \succeq 23 \succeq 2 \succeq 13 \succeq 3 \succeq \emptyset.$$

If a real-valued function is available representing the "strength" of each coalition on a numerical scale such that $S \succeq T \Leftrightarrow v(S) \geq v(T)$, it would be possible to compare the social ranking (power) of individuals 1 and 2 using the Banzhaf value of v. It is easy to check that the difference of Banzhaf values $\beta_i(v) - \beta_j(v)$ for each $i, j, k \in \{1, 2, 3\}$ can be written as follows:

$$\beta_i(v) - \beta_j(v) = \frac{1}{2}(v(i) - v(j)) + \frac{1}{2}(v(ik) - v(jk)). \tag{3.2}$$

One can verify that the difference $\beta_1(v) - \beta_2(v)$ can be made positive or negative with a suitable choice of v compatible with the constraint $v(1) > v(23) > v(2) > v(13)$. For instance, consider the functions v_ϵ and v'_ϵ such that $v_\epsilon(1) = 4$, $v_\epsilon(23) = 3$, $v_\epsilon(2) = 2$, $v_\epsilon(13) = 1 + \epsilon$ and $v'_\epsilon(1) = 4$, $v'_\epsilon(23) = 3$, $v'_\epsilon(2) = 2$, $v'_\epsilon(13) = 1 - \epsilon$, with $1 > \epsilon > 0$. Both v_ϵ and v'_ϵ satisfy the aforementioned constraints, but according to equation 3.2, $\beta_1(v_\epsilon) > \beta_2(v_\epsilon)$ and $\beta_2(v'_\epsilon) > \beta_1(v'_\epsilon)$, even for very small ϵ. In order to get more robust results to evaluate individuals, our goal is to introduce a social ranking solution inspired from the classical notion of Banzhaf value.

As the first step, the extension of the classical marginal contribution to our ordinal framework is provided.

Definition 3.2.1 (Ordinal marginal contribution)**.** *Let $\succeq \in \mathcal{B}(2^N)$. The ordinal marginal contribution $m_i^S(\succeq)$ of individual i w.r.t. coalition S, $i \notin S$, in power relation $\succeq$ is defined as:*

$$m_i^S(\succeq) = \begin{cases} 1 & \text{if } S \cup \{i\} \succ S, \\ -1 & \text{if } S \succ S \cup \{i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Note that in Definition 3.2.1 we assume that the marginal contribution can be negative due to the nature of our power relation, which is not necessarily monotonic.

Example 17. *Consider the power relation such that:*

$$\begin{array}{l} 1 \quad 2 \quad 12 \quad 3 \quad 13 \quad 23 \quad 234 \quad 34 \quad 14(\ldots)\,^{1} \\ (\ldots) \quad 4 \quad 24 \quad 124 \quad 1234 \quad 134 \qquad 123. \end{array} \tag{3.4}$$

In , the ordinal marginal contribution of individual 2 *w.r.t. coalition* 134*,* $m_2^{134}(\ )$*, is equal to* 1 *since* 1234 134 *holds. However, the ordinal marginal contribution of individual* 2 *w.r.t. coalition* 4*,* $m_2^{4}(\ )$ *, is* −1 *due to* 4 24*.*

The possible negative values for an *ordinal marginal contribution* can be validated by the assumption that the cooperation between individuals may be forced by an external factor. For example, in a company, teams may be arranged by external authorities to perform tasks. In this case, cooperation may sometimes cause increase in the performance or, on the other hand, may result in a decrease. Also, note that the empty set ( ) in the power relation can be seen as a benchmark to discriminate between performance of coalitions, as performing positively or negatively.

The intuitive ranking method, when ordinal marginal contributions are computed, is to follow the concept of majority. Putting away the cases that the ordinal marginal contribution of an individual is zero, the number of times that the individual performs more positively and less negatively indicates the power of individual in cooperation. More precisely, let's assume U_i refers to the set of coalitions not containing individual i, $U_i = \{S \quad 2^N ; i \ / \ S\}$, (in the same way we define $U_{ij} = \{S \quad 2^N : i, j \ / \ S\}$) and suppose $u_i^{+,}$ ($u_i^{-,}$) shows the number of coalitions $S \quad U_i$ such that $m_i^S(\ ) = 1$ ($m_i^S(\ ) = -1$). The difference $s_i = u_i^{+,} - u_i^{-,}$, which is called *ordinal Banzhaf score* of individual i in , represents the number of times one individual performs positively than negatively. The social ranking rule that corresponds to the ordinal Banzhaf score is called *ordinal Banzhaf relation*, and it ranks one individual higher than the other one if the individual has a higher ordinal Banzhaf score. The social ranking rule is formally defined as below:

Definition 3.2.2 (Ordinal Banzhaf relation)**.** *Let* $\mathcal{B}(2^N)$ *and* $A \quad N$*. The ordinal Banzhaf relation is the binary relation* $\hat{R}_A \quad A \times A$ *such that for all* $i, j \quad A$*:*

$$i\hat{R}_A j \Leftrightarrow s_i \geq s_j .$$

Remark 1. *From the definition of the ordinal Banzhaf score, it immediately follows that the relation* $\hat{R}_A$ *on* $A \quad N$ *is transitive and total. So,* $\hat{R}_A$ *is a social ranking solution.*

Example 18. *Consider the power relation of Example 17 and let* $A = \{1, 2\}$*. To compare individuals in* A*, we refer to the comparisons that provide the marginalistic information about the individuals (marginalistic comparisons). The sets* U_1 *and* U_2 *specify sets of coalitions that each one of the individuals can join them:*

$$U_1 = \{\ , 2, 3, 4, 23, 24, 34, 234\}$$

[1] dots "..." are not part of the power relation, we use it to split the line and avoid line overflow.

and

$$U_2 = \{\ , 1, 3, 4, 13, 14, 34, 134\}.$$

The ordinal marginal contributions and the ordinal Banzhaf scores of individuals 1 *and* 2 *are reported in Table 3.1. Since* $s_2 = -2 > -4 = s_1$, *it follows* $2\hat{P}_A 1$.

S U_1	$m_1^S(\ )$	S U_2	$m_2^S(\ )$
	1		1
2	−1	1	−1
3	−1	3	−1
4	1	4	−1
23	−1	13	−1
24	−1	14	−1
34	−1	34	1
234	−1	134	1
	$s_1 = -4$		$s_2 = -2$

Table 3.1: Ordinal marginal contributions of individuals 1 and 2 for the power relation (26).

The next example highlights the situation in which individuals are ranked the same

Example 19. *Consider* 123 12 1 23 2 13 3 . *Let* $A = \{1, 2\}$ *be the set of elements to be ranked. We have that* $m_1 = m_1^2 = m_1^{23} = 1$, $m_2 = m_2^3 = m_2^{13} = 1$, *and* $m_1^3 = m_2^1 = 0$. *As a result,* $s_1 = s_2 = 4$. *By the ordinal Banzhaf relation,* 1 *and* 2 *are indifferent, i.e.* $1\hat{I}_A 2$.

As shown in Examples 18 and 19, given a power relation as a linear order on 2^N, the social ranking provided by the ordinal Banzhaf relation does not depend on the choice of a compatible cardinal function v, and therefore it answers to the initial question of this section concerning robustness. Another natural question is whether it always exists a cardinal evaluation v compatible with , such that the ranking provided by the classical Banzhaf value on v coincides with the ranking provided by the ordinal Banzhaf relation on . A negative answer to this question follows from Example 20.

Example 20. *Consider the power relation such that* 123 12 1 23 3 13 2 . *Let* $A = \{1, 2\}$ *be the set of elements to be ranked. Consider every compatible cardinal function* v *such that* $v(S) \geq v(T) \Leftrightarrow S \quad T$ *for each* $S, T \quad 2^N$. *By relation (3.2) we have that*

$$\beta_1(v) - \beta_2(v) = \frac{1}{2}(v(1) - v(2)) + \frac{1}{2}(v(13) - v(23)).$$

Since $v(1) - v(2) > v(23) - v(13)$, *we have that* $\beta_1(v) > \beta_2(v)$ *(independently from the choice of* v*). On the other hand* $m_1 = m_1^2 = m_1^{23} = 1$ *and* $m_1^3 = -1$, *whereas* $m_2 = m_2^1 = m_2^3 = m_2^{13} = 1$. *So,* $s_1 = 2$ *and* $s_2 = 4$. *Then, according to the ordinal Banzhaf relation,* 2 *is strictly better than*

1*, i.e.* $2 \hat{P}_A 1$*, yielding an opposite conclusion with respect to the classical Banzhaf value for every compatible function v.*

In the next section, we analyse the ordinal Banzhaf rule from property driven approach and investigate its similarities and differences to the *ceteris paribus* majority rule.

3.3 Axiomatic Analysis

In this section, we introduce a set of axioms which are inspired from those in classical social choice theory [May, 1952] and in the axiomatic approach presented in [Haret et al., 2018]. The axioms in this section follow the same spirit as the axioms in Chapter 2 (the spirit of majority), however, they are applied to a different informative part of power relations.

The first property states that to rank any two individuals, a social ranking solution should not care about the name of coalitions (and as a result their size and members) that they join. What is important is how they perform by joining the coalitions. According to this property, permuting coalitions in a way that preserves the number of positive and negative ordinal marginal contributions of individuals should not affect ranking of the individuals. So, a positive (negative) ordinal marginal contribution to distinct coalitions S and T should carry the same weight.

Definition 3.3.1. *(Coalitional Anonymity, CA) Let* $A \subseteq N$*. A solution* $R_A : \mathcal{C} \subseteq \mathcal{B}(2^N) \to \mathcal{T}(A)$ *satisfies the coalitional anonymity axiom on* $\mathcal{C}$ *if and only if for all power relations* $\succeq, \succeq' \in \mathcal{C}$*, for all individuals* $i, j \in A$ *and bijections* $\pi^i : U_i \to U_i$ *and* $\pi^j : U_j \to U_j$ *such that* $S \cup \{i\} \succeq S \Leftrightarrow \pi^i(S) \cup \{i\} \succeq' \pi^i(S)$ *for all* $S \in U_i$ *and* $S \cup \{j\} \succeq S \Leftrightarrow \pi^j(S) \cup \{j\} \succeq' \pi^j(S)$ *for all* $S \in U_j$*, then it holds that* $i R_A j \Leftrightarrow i R'_A j$*.*

The following example clarifies the notion.

Example 21. *Consider the power relation of Example 17, and suppose we want to compare individuals* 1 *and* 2 *given a social ranking rule R that satisfies coalitional anonymity. To compare individuals* 1 *and* 2*, we look at their ordinal marginal contributions reported in Table (3.1). Now, imagine Table 3.2 illustrates the ordinal marginal contributions of individuals* 1 *and* 2 *in the power relation below.*

$$3 \succ \emptyset \succ 2 \succ 12 \succ 1 \succ 23 \succ 123 \succ 13 \succ 24 (\ldots)$$
$$(\ldots) \succ 124 \succ 4 \succ 14 \succ 34 \succ 134 \succ 1234 \succ 234.$$

$S \quad U_1$	$m_1^S(\ \ )$	$S \quad U_2$	$m_2^S(\ \ )$
	-1		-1
2	1	1	-1
3	-1	3	-1
4	-1	4	1
23	-1	13	1
24	-1	14	1
34	-1	34	-1
234	1	134	-1

Table 3.2: Ordinal marginal contributions of individuals 1 and 2 for the power relation .

It is easy to verify that bijections $\pi^1 : U_1 \to U_1$ and $\pi^2 : U_2 \to U_2$ transform ordinal marginal contributions in Table 3.1 to their counterparts in Table 3.2, when they map every coalition to themselves, except that:

$$\pi^1(\ \ ) = 2, \pi^1(2) =$$

$$\pi^1(4) = 234, \pi^1(234) = 4$$

$$\pi^2(\ \ ) = 4, \pi^2(4) =$$

$$\pi^2(34) = 13, \pi^2(13) = 34$$

$$\pi^2(134) = 14, \pi^2(14) = 134.$$

Thus, by the axiom coalitional anonymity the social ranking solution ranks individuals 1 *and* 2 *in the power relation as they are ranked in the power relation ,* $1R_A 2 \Leftrightarrow 1R_A 2$.

The second axiom is a classical neutrality axiom, and it states that a social ranking solution should not be biased in favor of one alternative. So, if the names of individuals i and j are reversed (if i and j exchange their performances), the ranking of individuals i and j must also be reversed. Before introducing its definition, we need some further notation. Let $\sigma : N \to N$ be a bijection. For a set $S = \{i, j, k, ..., t\} \quad N$, we denote the image of S through σ, $\sigma(S) = \{\sigma(i), \sigma(j), \sigma(k), ..., \sigma(t)\}$.

Definition 3.3.2. *(Neutrality, N) Let $A \quad N$. A solution $R_A : C \quad \mathcal{B}(2^N) \to \mathcal{T}(A)$ satisfies the neutrality axiom on C if and only if for all power relations , C and each bijection $\sigma : N \to N$ such that $\sigma(A) = A$ and $S \quad T \Leftrightarrow \sigma(S) \quad \sigma(T)$ for all $S, T \quad 2^N$, then it holds that $iR_A j \Leftrightarrow \sigma(i) R_A \sigma(j)$ for every $i, j \quad A$.*

The following example illustrates what is expected from a social ranking rule that satisfies neutrality.

Example 22. *Referring to the power relation in Example 17, suppose the power relation is defined as*

$$2 \ A \ 1 \ A \ 12 \ A \ 3 \ A \ 23 \ A \ 13 \ A \ 134 \ A \ 34 \ A \ 24(...)$$
$$(...) \ A \ 4 \ A \ 14 \ A \ 124 \ A \ 1234 \ A \ 234 \ A \quad A \ 123.$$

Applying a social ranking R, which satisfies neutrality, on the power relation should result a ranking over individuals 1 *and* 2 *which is the inverse of the ranking obtained from* ($1R_A2 \Leftrightarrow 2R_A1$)*. This is because everywhere that individual* 1 (2) *performs positively (negatively) in power relation individual* 2 (1) *performs positively (negatively) in . This can be verified by comparing Table 3.1 with Table 3.3, which indicates the marginal contributions of individuals* 1 *and* 2 *is the power relation .*

$S \quad U_1$	$m_1^S(\ )$	$S \quad U_2$	$m_2^S(\ )$
	1		1
2	-1	1	-1
3	-1	3	-1
4	-1	4	1
23	-1	13	-1
24	-1	14	-1
34	1	34	-1
234	1	134	-1

Table 3.3: Ordinal marginal contributions of individuals 1 and 2 for the power relation .

The next axiom says that an appealing social ranking rule needs to be coherent with the modifications on the performance of different coalitions. Therefore, suppose that in a given power relation, the social ranking rule ranks individual i higher than or indifferent to j. If the power relation remains the same for all coalitions except the one that becomes in favor of i, then the social ranking rule must rank individual i strictly better than j.

Definition 3.3.3. *(Monotonicity, M) Let $A \quad N$. A solution $R_A : C \quad \mathcal{B}(2^N) \to \mathcal{T}(A)$ satisfies the monotonicity axiom on C if and only if for all power relations , C and $i, j \quad A$ such that:*

- *there exists a coalition $S \quad U_i$ such that $S \quad S \quad i$ and $S \quad i$ A S, and*
- *$T \quad i \quad T \Leftrightarrow T \quad i$ A T and $T \quad j \quad T \Leftrightarrow T \quad j$ A T for all the other coalitions $T \quad 2^N, T = S$,*

then it holds that $iR_Aj \quad iP_Aj$.

The name *monotonicity* reflects the idea that increases in the number of positive performances of one individual enhances its ranking, as long as the number of negative performances does not change.

Example 23. *Consider the power relation in Example 17, and suppose enhancing performance of individual* 2 *results in a power relation :*

$$1 \text{ A } 2 \text{ A } 12 \text{ A } 23 \text{ A } 3 \text{ A } 13 \text{ A } 234 \text{ A } 34 \text{ A } 14(...)$$
$$(...) \text{ A } 4 \text{ A } 24 \text{ A } 124 \text{ A } 1234 \text{ A } 134 \text{ A } \quad \text{ A } 123.$$

Table 3.4 shows the ordinal marginal contributions of individuals 1 *and* 2 *in the power relation* .

S U_1	$m_1^S($ $)$	S U_2	$m_2^S($ $)$
	1		1
2	−1	1	−1
3	−1	3	1
4	1	4	−1
23	−1	13	−1
24	−1	14	−1
34	−1	34	1
234	−1	134	1

Table 3.4: Ordinal marginal contributions of individuals 1 and 2 for the power relation .

The ordinal marginal contributions of individuals 1 *and* 2 *in Tables 3.1 and 3.4 are the same except for individual* 2 *when it joins coalition* 3*. Suppose social ranking R satisfies Monotonicity. If it ranks* 2 *at least as good as* 1 *in the power relation* $(2R_A1)$*, then we expect that applying it to the power relation* *ranks* 2 *strictly better than* 1 $(2P_A1)$*.*

The main result of this section is presented as follows. This theorem characterizes the ordinal Banzhaf solution with the three axioms *Coalitional Anonymity*, *Neutrality*, and *Monotonicity* when power relations belong to set of all linear orders $\mathcal{L}(2^N)$.

Theorem 3.3.4. *Let* A N*. A solution* $R_A : \mathcal{L}(2^N) \to \mathcal{T}(A)$ *is the ordinal Banzhaf solution if and only if it satisfies the three axioms CA, N and M on* $\mathcal{L}(2^N)$*.*

Proof. ( ) (The existence part.) First, we prove that the ordinal Banzhaf solution $\hat{R}_A$, satisfies the three axioms N, CA and M on $\mathcal{L}(2^N)$. Consider two power relations , $\mathcal{L}(2^N)$ such that for all individuals i, j A the following conditions hold:

i) There exists a bijection $\pi^i : U_i \to U_i$ with S i $S \Leftrightarrow \pi^i(S)$ i A $\pi^i(S)$ for all S U_i;

ii) there exists a bijection $\pi^j : U_j \to U_j$ with S j $S \Leftrightarrow \pi^j(S)$ j A $\pi^j(S)$ for all S U_j.

We first show that it holds $i\hat{R}_A j \Leftrightarrow i\hat{R}_A j$. Since condition (i) holds it means that there is a bijection from the set of coalitions S U_i with $m_i^S(\ ) = 1$ $(m_i^S(\ ) = -1)$ to the set of all S U_i with $m_i^S(\ ) = 1$ $(m_i^S(\ ) = -1)$. Moreover, from condition (ii) it also follows that there exists a bijection from the set of S U_j with $m_j^S(\ ) = 1$ $(m_j^S(\ ) = -1)$ to the set of all S U_j with $m_j^S(\ ) = 1$ $(m_j^S(\ ) = -1)$. Then we have that

$$s_i = u_i^{+,} - u_i^{-,} = u_i^{+,} - u_i^{-,} = s_i$$

and

$$s_j = u_j^{+,} - u_j^{-,} = u_j^{+,} - u_j^{-,} = s_j ,$$

that directly imply

$$i\hat{R}_A j \Leftrightarrow i\hat{R}_A j. \tag{3.5}$$

By conditions (i) and (ii) and relation (3.5) it follows that $\hat{R}_A$ satisfies the property of coalitional anonymity (CA).

Consider two power relations , $\mathcal{L}(2^N)$, two individuals i, j A and a bijection σ : $N \to N$ with $\sigma(i)$ A for each i A such that S $T \Leftrightarrow \sigma(S)$ A $\sigma(T)$ for all S, T 2^N. We now show that for these power relations $i\hat{R}_A j \Leftrightarrow \sigma(i)\hat{R}_A\sigma(j)$. First, notice that for all S U_i S i $S \Leftrightarrow \sigma(S)$ $\sigma(i)$ A $\sigma(S)$, as well as for all S U_j S j $S \Leftrightarrow \sigma(S)$ $\sigma(j)$ A $\sigma(S)$. More precisely, for each S U_i with $m_i^S(\ ) = 1$ ($m_i^S(\ ) = -1$), we have that $\sigma(S)$ $U_{\sigma(i)}$ and $m_{\sigma(i)}^{\sigma(S)}(\ ) = 1$ ($m_{\sigma(i)}^{\sigma(S)}(\ ) = -1$), and for each S U_j with $m_j^S(\ ) = 1$ ($m_j^S(\ ) = -1$), we have that $\sigma(S)$ U_i and $m_{\sigma(j)}^{\sigma(S)}(\ ) = 1$ ($m_{\sigma(j)}^{\sigma(S)}(\ ) = -1$). From this it follows that

$$s_i = u_i^{+,} - u_i^{-,} = u_{\sigma(i)}^{+,} - u_{\sigma(i)}^{-,} = s_{\sigma(i)}$$

and

$$s_j = u_j^{+,} - u_j^{-,} = u_{\sigma(j)}^{+,} - u_{\sigma(j)}^{-,} = s_{\sigma(j)}$$

implying that: $i\hat{R}_A j \Leftrightarrow \sigma(i)\hat{R}_A\sigma(j)$, as it is required by the neutrality property (N).

Finally, consider two power relations , $\mathcal{L}(2^N)$ and suppose that for any two individuals i, j A the following conditions hold:

iii) there exists a coalition S U_i such that S S i and S i A S

iv) T i $T \Leftrightarrow T$ i A T and V j $V \Leftrightarrow V$ j A V for all the other coalitions T U_i, $T = S$, and V U_j.

We want to prove that $i\hat{R}_A j$ $i\hat{P}_A j$. According to condition (iii) and (iv), we have that

$$s_i = u_i^{+,} - u_i^{-,} > u_i^{+,} - u_i^{-,} = s_i \tag{3.6}$$

and

$$s_j = u_j^{+,} - u_j^{-,} = u_j^{+,} - u_j^{-,} = s_j . \tag{3.7}$$

Moreover, if $i\hat{R}_A j$, by definition of ordinal Banzhaf score, we have that

$$s_i = u_i^{+,} - u_i^{-,} \geq u_j^{+,} - u_j^{-,} = s_j \tag{3.8}$$

Then, by relations (3.6), (3.7) and (3.8) it immediately follows that

$$s_i = u_i^{+,} - u_i^{-,} > u_j^{+,} - u_j^{-,} = s_j , \tag{3.9}$$

which means that $i\hat{P}_A j$.

( ) (The uniqueness part.) We have to prove that if a solution R_A satisfies axioms CA, N and M on $\mathcal{L}(2^N)$ then it is the ordinal Banzhaf solution $\hat{R}_A$, i.e. $iR_A j \Leftrightarrow s_i \geq s_j$ for all $\mathcal{L}(2^N)$ and i, j A.

We start showing that if R_A satisfies axioms CA and N on $\mathcal{L}(2^N)$, then for all $\mathcal{L}(2^N)$ and i, j A such that $s_i = s_j$, we have that $iI_A j$.

Consider a power relation $\mathcal{L}(2^N)$ with $s_i = s_j$, for some i, j A. By Remark 1 and by the fact that there are no indifferences in the power relation, we also have that

$$u_i^{+,} = u_j^{+,} \text{ and } u_i^{-,} = u_j^{-,} . \tag{3.10}$$

Now, consider another power relation such that for all S, T 2^N,

$$S \quad T \Leftrightarrow \sigma(S) \quad \sigma(T),$$

where $\sigma : N \to N$ is a bijection with $\sigma(i) = j, \sigma(j) = i$ and $\sigma(k) = k$ for all k $A, k = i$ and $k = j$. By axiom N it holds that

$$iR_A j \Leftrightarrow jR_A i. \tag{3.11}$$

Moreover, by construction of , it holds that

$$u_i^{+,} = u_i^{+,} , u_i^{-,} = u_i^{-,} , u_j^{+,} = u_j^{+,} , u_j^{-,} = u_j^{-,} . \tag{3.12}$$

Then it is easy to define a bijection $\pi^i : U_i \to U_i$ such that $S \quad i \quad S \Leftrightarrow \pi^i(S) \quad i$ A $\pi^i(S)$ for all S U_i (defining a one-to-one correspondence between elements S U_i with $m_i^S( ) = 1$ and those with $m_i^S( ) = 1$, and a one-to-one correspondence between S U_i with $m_i^S( ) = -1$, and those with $m_i^S( ) = -1$) and, in a similar way, another bijection $\pi^j : U_j \to U_j$ such that $S \quad j \quad S \Leftrightarrow \pi^j(S) \quad j$ A $\pi^j(S)$ for all S U_j. Therefore, from the CA axiom, we have that

$$iR_A j \Leftrightarrow iR_A j. \tag{3.13}$$

From relation (3.11) and (3.13), and since R_A is total, it immediately follows that

$$iI_A\, j.$$

Now, consider a power relation $\mathcal{L}(2^N)$ such that $q = u_i^{+,} > u_j^{+,} = p$ for some integer numbers p and q $\{0, 1, \ldots, 2^{n-1}\}$. One can opportunely rearrange the relation within each set $\{S \quad ij, S \quad i, S \quad j, S\}$ for all S U_{ij} to obtain a new power relation $\mathcal{L}(2^N)$ such that $u_i^{+,} = u_j^{+,} = p$ (for instance, just taking $q - p$ coalitions S U_{ij}, with $S \quad ij \quad S \quad i$ or $S \quad j \quad S$ and inverting the relation). Then, since R_A satisfies both N and CA, we have that $iI_A\, j$. Using a similar argument, and restoring precisely one of the previously changed comparison to move from to , we can now form another power relation with $u_i^{+,} = p + 1$ and $u_j^{+,} = p$. By the M axiom of R_A we have now that $iP_A\, j$. By applying this procedure a sufficient number of times, it is then possible to reconstruct the power relation from in $q - p$ steps, and by the application of the M axiom of R_A at each step, we can conclude that $iP_A j$. □

Remark 2. *In the claim of Theorem 3.3.4, it is possible to substitute the domain of linear orders $\mathcal{L}(2^N)$ with the larger domain of power relations $C \quad \mathcal{B}(2^N)$ such that for each $\quad C$ and all $S \quad U_{ij}$ the following two conditions hold: c.1) $\quad$ is transitive and total on $\{S \quad ij, S \quad i, S \quad j, S\}$; c.2) only strict comparisons hold, i.e. for all $A, B \quad \{S \quad ij, S \quad i, S \quad j, S\}$, $A = B$, we never have $A \quad B$.*

Note that the *ordinal Banzhaf relation* is based on the *ordinal Banzhaf score*, and the ordinal Banzhaf score does not consider indifferences in its definition, then it is easy to verify that the existence part of Theorem 3.3.4 holds when the power relations belong to the set of all binary relations (the *ordinal Banzhaf relation* still satisfies the three axioms). However, the uniqueness part of the theorem holds only if, in addition to the three axioms, the domain be transitive and complete with no indifferences.

We devote the rest of this section to compare some fundamental features of the *ceteris paribus* majority relation introduced in Chapter 2 and the ordinal Banzhaf solution. To do that, we first recall the definition of the *ceteris paribus* majority rule as well as some notations from Chapter 2. The set of all coalitions $S \quad U_{ij}$ for which $S \quad i \quad S \quad j$ (*ceteris paribus* comparison) is denoted by $D_{ij}(\ )$. In addition, the cardinality of the set $D_{ij}(\ )$ is denoted by $d_{ij}(\ )$. The *ceteris paribus majority* relation is then defined as follows:

Definition 3.3.5 (Ceteris Paribus (CP-)Majority relation)**.** *Let $\_ \quad \mathcal{B}(2^N)$ and $A \quad N$. The Ceteris Paribus (CP-) majority relation is the binary relation $\bar{R}_A \quad N \times N$ such that for all $i, j \quad N$:*

$$i\bar{R}_A j \Leftrightarrow d_{ij}(\ ) \geq d_{ji}(\ ).$$

The following example reveals how the CP-majority rule and the ordinal Banzhaf index act differently to rank two individuals.

Example 24. *Consider the power relation defined in Example 17. Table 3.5 illustrates the CP-comparisons between* 1 *and* 2*. If $\bar{R}$ is ceteris paribus majority solution, then it holds that $1\bar{P}_A 2$ ($d_{12}(\ ) = 3$ and $d_{21}(\ ) = 1$), whereas for the ordinal Banzhaf solution $\hat{R}$ we have $2\hat{P}_A 1$, according to the Table 3.1.*

$S \quad U_{12}$	$S \quad 1$ vs. $S \quad 2$	
	1	2
3	13	23
4	14	24
34	134	234

Table 3.5: CP-comparisons on $\quad$ of Example 1.

By comparing the axioms *equality of coalitions* and *positive responsiveness* as well as the corresponding characterization theorem in Section 2.3 of Chapter 2 to *coalitional anonymity* and *monotonicity* presented in this chapter and Theorem 3.3.4, we find out two important differences between the *CP-majority rule* and the *ordinal Banzhaf solution*:

i) As extensively discussed in chapter 2, the CP-majority relation, is not necessarily transitive, if $|A| > 2$, whereas the ordinal Banzhaf solution yields a transitive relation over the elements of A, for any $A \subseteq N$.

ii) The axiomatic characterization for the CP-majority solution holds true over the domain of all binary relations $\mathcal{B}(2^N)$, while the one for the ordinal Banzhaf solution applies to the restricted domain of linear orders $\mathcal{L}(2^N)$ (or the larger one mentioned in Remark 2).

Even if the CP-majority solution and the ordinal Banzhaf solution may rank individuals in a very different manner (see, for instance, Example 24), they share some fundamental similarities, at least over sets with only two elements.

First, for both of the theorems to characterize the CP-majority relation and the ordinal Banzhaf rule, the same neutrality axiom is used. Actually, the axiom of neutrality as introduced in this chapter implies, on the same domain $C \subseteq \mathcal{B}(2^N)$, the axiom of neutrality used in the context of CP-majority relation, that only considers the particular bijection $\sigma : N \to N$ such that for $i, j \in N$, $\sigma(i) = j$ and $\sigma(j) = i$, and $\sigma(k) = k$ for all $k \in N \setminus \{i, j\}$.

In addition, for the ordinal Banzhaf solution, the coalitional anonymity axiom plays a role similar to the one played by the equality of coalitions for the CP-majority rule: the social ranking must be invariant with respect to particular permutations of coalitions. However, how coalitions are permuted is different in the two axioms, focusing on permutations preserving the number of CP-comparisons in the axiom equality of coalitions, and on permutations preserving the number of positive and negative ordinal marginal contributions in the coalitional anonymity.

Finally, the positive responsiveness axiom for the CP-majority solution and the monotonicity axiom for the ordinal Banzhaf solution follow a similar principle for breaking ties in favour of individuals that improve their position. However, there are two main differences here: first, in the CP-majority, we consider improvements on CP-comparisons, while for the ordinal Banzhaf solution we consider improvements on ordinal marginal contributions; second, due to the domain restriction on $\mathcal{L}(2^N)$, the possibility to have indifference is not considered for the characterization of the ordinal Banzhaf solution. We used different names for the same idea of breaking ties since in the *ceteris paribus* majority rule increasing the performance of one individual means decrease in the performance of the other one, while for the ordinal Banzhad rule increase in the performance of one individual does not mean decrease in the performance of the other one, thereby it does not necessarily positively respond to increase of performance in one individual, the positive respond depends on the other individual to keep the same performances as before.

A further similarity between the two solutions is discussed in the next section, where the ordinal Banzhaf solution and the CP-majority are presented as two special members of a new family of the *weighted majority relations*.

3.4 Weighted Majority Relations

In this section, we define a family of social ranking rules. The members of this family are extensions of *ceteris paribus* majority rule to weighted versions. Each member of the family weighs the *ceteris paribus* comparisons according to particular features in the power relation, for instance, the size of the coalitions in CP-comparisons or the type of members inside the coalitions. We define a member of this family which coincides with the previously mentioned ordinal Banzhaf solution on the domain of power relations as linear orders.

We start by rewriting the definition of the CP-majority rule as follows.
Let $A \quad N$ and $i, j \quad A$. Then,

$$i\bar{R}_A j \iff |D_{ij}(\ )| \geq |D_{ji}(\ )| \iff \sum_{S \ U_{ij}} \bar{d}_{ij}^S(\ ) \geq 0,$$

where

$$\bar{d}_{ij}^S(\ ) = \begin{cases} 1 & \text{if } S \quad i \quad S \quad j, \\ -1 & \text{if } S \quad j \quad S \quad i, \\ 0 & \text{otherwise,} \end{cases} \tag{3.14}$$

for all $S \ \_ U_{ij}$. We can generalize this definition, to any non-negative linear combination of the terms $\bar{d}_{ij}^S$, for all $S \quad U_{ij}$, by multiplying each $\bar{d}_{ij}^S$ with a weight value. Since the weight values can be any non-negative value, this generalization results in a family of social ranking rules called *the family of weighted majority rules*.

Definition 3.4.1 (Weighted majority relation)**.** *Let* $\quad \mathcal{B}(2^N)$, $A \quad N$ *and let* $\mathrm{w} = [w_{ij}^S]_{i,j \ A,S \ U_{ij}}$ *be a weight scheme such that* $w_{ij}^S \geq 0$ *for all* $i, j \quad A$ *and* $S \quad U_{ij}$*. The weighted majority relation associated to* w *is the binary relation* $R_A^{\ ,\mathrm{w}} \quad A \times A$ *such that for all* $i, j \quad A \quad N$*:*

$$iR_A^{\ ,\mathrm{w}} j \iff \sum_{S \ U_{ij}} w_{ij}^S \bar{d}_{ij}^S(\ ) \geq 0.$$

Obviously, if $w_{ij}^S = 1$ for all $i, j \quad A$ and $S \quad U_{ij}$, then we get the CP-majority, i.e. $R_A^{\ ,\mathrm{w}} = \bar{R}_A$.

Example 25. *Let's recall the power relation in Example 17:*

$$\begin{array}{c} 1 \quad 2 \quad 12 \quad 3 \quad 13 \quad 23 \quad 234 \quad 34 \quad 14(\dots) \\ (\dots) \quad 4 \quad 24 \quad 124 \quad 1234 \quad 134 \qquad 123. \end{array} \tag{3.15}$$

Suppose we want to compare individuals 1 *and* 2*. The corresponding ceteris paribus comparisons are* 1 $\quad$ 2, 13 $\quad$ 23, 14 $\quad$ 24*, and* 234 $\quad$ 134*. Let's assume that the weight scheme related to CP-comparisons indicates the sizes of the relevant coalitions:*

$$w_{12} = 0, w_{12}^{\{3\}} = 1, w_{12}^{\{4\}} = 1, w_{12}^{\{34\}} = 2.$$

In this case, the weighted CP-majority rule ranks 1 *and* 2 *the same: first note that* $d_{12} = 1, d_{12}^{\{3\}} = 1, d_{12}^{\{4\}} = 1, d_{12}^{\{34\}} = 0$*, and therefore*

$$0 \times d_{12} + 1 \times d_{12}^{\{3\}} + 1 \times d_{12}^{\{4\}} + 2 \times d_{12}^{\{34\}} \geq 0 \iff 1R_A 2.$$

Symmetrically it holds that $d_{21} = 0, d_{21}^{\{3\}} = 0, d_{21}^{\{4\}} = 0, d_{21}^{\{34\}} = 1$ *and, as a result,*

$$0 \times d_{21} + 1 \times d_{21}^{\{3\}} + 1 \times d_{21}^{\{4\}} + 2 \times d_{21}^{\{34\}} \geq 0 \iff 2R_A 1.$$

In the following we define a weight scheme corresponding to a weighted CP-majority rule that coincides with the *ordinal Banzhaf solution*. The weight scheme is called *Banzhaf distance* which is defined as follows.

Definition 3.4.2 (Banzhaf (Bz-) distance). *Let* $\mathcal{L}(2^N)$*, i, j N and let S U_{ij}. The Banzhaf (Bz-) distance between i and j with respect to S $2^{N\setminus\{i,j\}}$ is denoted by* $\hat{\mathrm{w}} = [\hat{w}^S_{ij}(\ )]_{i,j\ \ A,S\ \ U_{ij}}$ *and is defined as the cardinality of an intersection as follows:*

$$|\{S, S \quad ij\} \cap \{T : S \quad i \quad T \quad S \quad j \text{ or } S \quad j \quad T \quad S \quad i\}|.$$

Note that $\hat{w}^S_{ij}(\ )$ is just the number of S and S ij between S i and S j in the power relation . For instance, if we have S 1 S 12 S 2 S, then $\hat{w}^S_{12}(\ ) = 1$ and if we have S 3 S S 34 S 4, then $\hat{w}^S_{34}(\ ) = 2$.

The *Banzhaf distance* of two individuals i and j w.r.t coalition S in a power relation partially illustrates how individuals i and j are able to manipulate the performance of coalition S. For instance, regarding to S 1 S 12 S 2 S, the *Banzhaf distance* of 1 and 2 w.r.t coalition S is one, and it indicates that substituting 2 with 1 in coalition S 2 improves the performance of coalition even more than what it would be when both individuals i and j are presented in coalition S.

Remark 3. *Notice that the Bz-distance $\hat{w}^S_{ij}(\ )$ is a well-defined metric: it can only take values* 0,1*, or* 2 *(non-negativity);* $\hat{w}^S_{ii}(\ ) = 0$ *(identity);* $\hat{w}^S_{ij}(\ ) = \hat{w}^S_{ji}(\ )$ *(symmetry);* $\hat{w}^S_{ik}(\ ) \leq \hat{w}^S_{ij}(\ ) + \hat{w}^S_{jk}(\ )$ *for all S 2^N with i, j, k S (triangle inequality).*

We refer to the member of the family of all weighted majority rules whose weights are the *Banzhaf distances* of CP-comparisons as the *Banzhaf-distance based CP-majority* rule.
The following theorem refers to one of the important results in this section. It shows that the ordinal Banzhaf index and the *Banzhaf-distance based CP-majority rule* coincide in the domain of all linear orders.

Theorem 3.4.3. *Let* $\mathcal{L}(2^N)$ *and A N. We have that*

$$R_A^{\ ,\hat{\mathrm{w}}} = \hat{R}_A.$$

Proof. In order to prove the theorem we need to indicate the the Banzhaf-distance based CP-majority rule follows ranking method of the ordinal Banzhaf solution. Precisely we need to prove that for all i, j A $iR_A^{\ ,\hat{\mathrm{w}}}j \Leftrightarrow s_i \geq s_j$ or

$$iR_A^{\ ,\hat{\mathrm{w}}}j \Leftrightarrow s_i - s_j \geq 0.$$

First note that we can rewrite the difference of ordinal Banzhaf scores $s_i - s_j$ as follows

$$s_i - s_j = \sum_{S\ \ U_i} m^S_i(\ ) - \sum_{S\ \ U_j} m^S_j(\ ) =$$

$$= \sum_{S\ \ U_{ij}} (m^S_i(\ ) + m^{S\ \ j}_i(\ )) - (m^S_j(\ ) + m^{S\ \ i}_j(\ )).$$

Consider all coalitions $S \quad U_{ij}$ such that $\bar{d}^S_{ij}(\ ) = 1$ as reported in Table 3.6 (the case $\bar{d}^S_{ij}(\ ) = -1$ is very similar). According to the values of $\hat{w}^S_{ij}$ and $s_i - s_j$ in the power relation it follows that

$$\sum_{S \quad U_{ij}} \hat{w}^S_{ij} \bar{d}^S_{ij}(\ )$$

$$= \frac{1}{2} \left(\sum_{S \quad U_{ij}} (m^S_i(\ ) + m^{S \quad j}_i(\ )) - (m^S_j(\ ) + m^{S \quad i}_j(\ )) \right).$$

Therefore, we have that $\sum_{S \quad U_{ij}} \hat{w}^S_{ij} \bar{d}^S_{ij}(\ ) \geq 0$ iff $s_i \geq s_j$, which concludes the proof.

$S \quad 2^{N \setminus \{i,j\}}$	$\hat{w}^S_{ij}$	$s_i - s_j$
$S \quad i \quad S \quad j \quad S \quad ij \quad S$	0	0
$S \quad i \quad S \quad ij \quad S \quad j \quad S$	1	2
$S \quad i \quad S \quad ij \quad S \quad S \quad j$	2	4
$S \quad i \quad S \quad j \quad S \quad S \quad ij$	0	0
$S \quad i \quad S \quad S \quad j \quad S \quad ij$	1	2
$S \quad i \quad S \quad S \quad ij \quad S \quad j$	2	4
$S \quad S \quad i \quad S \quad j \quad S \quad ij$	0	0
$S \quad S \quad i \quad S \quad ij \quad S \quad j$	1	2
$S \quad S \quad ij \quad S \quad i \quad S \quad j$	0	0
$S \quad ij \quad S \quad i \quad S \quad j \quad S$	0	0
$S \quad ij \quad S \quad i \quad S \quad S \quad j$	1	2
$S \quad ij \quad S \quad S \quad i \quad S \quad j$	0	0

Table 3.6: Bz-distance $\hat{w}^S_{ij}$ and ordinal Banzhaf scores $s_i - s_j$ for $\bar{d}^S_{ij}(\ ) = 1$ (the symmetric case $\bar{d}^S_{ij}(\ ) = -1$ is omitted).

□

Therefore, if a power relation is linear order then comparing any two individuals using the Banzhaf-distance based majority rule yields the same ranking as the ordinal Banzhaf solution.

Example 26. *Consider the power relation in Example 17.*

$$1 \quad 2 \quad 12 \quad 3 \quad 13 \quad 23 \quad 234 \quad 34 \quad 14(\ldots)$$
$$(\ldots) \quad 4 \quad 24 \quad 124 \quad 1234 \quad 134 \quad 123.$$

In order to rank individuals 1 *and* 2 *in the given power relation, it is sufficient to compute the Bz-distance for each of the four CP-comparisons* 1 2, 13 23, 14 24, *and* 234 134. *These values are reported in Table 3.7.*

S U_{12}	S 1 vs. S 2		$\overline{d}_{12}^{S}$	$\hat{w}_{12}^{S}$
	1	2	1	0
3	13	23	1	0
4	14	24	1	1
34	134	234	−1	2

Table 3.7: CP-comparisons on of Example 1.

By Definition 3.4.1 it results that $2P_A$ 1*.*

We showed that the CP-majority and the ordinal Banzhaf solution belong to the family of weigthed majority relations. Note that one can obtain other members of this family by assigning other non-negative real values to a weight scheme such (for instance, the size of coalitions, etc.).

3.5 Conclusion and Future Works

In this chapter we have studied the problem of ranking individuals given an ordinal ranking over the set of coalitions formed by them. Following the analogy with cooperative games, we have extended the classical notion of Banzhaf value to our ordinal framework. We have analyzed the ordinal Banzhaf solution using a property-driven approach and we have compared its fundamental features with the ones of another solution from the literature, the CP-majority relation. Finally, we have introduced a new family of relations over the set of individuals that includes the ordinal Banzhaf solution and the CP-majority one, and many others.

Since we have characterized the ordinal Banzhaf solution over the domain of all linear orders, as a direction for future work it would be interesting to investigate how the ordinal Banzhaf solution can be extended to other families of power relations, and to see which axioms characterize this solution on those classes. Another open problem is to study and axiomatically characterize ordinal versions of other semi-values [Carreras et al., 2003] like, for instance, the Shapley value [Shapley, 1953].

4

Weighted Ranking Rules

Abstract

In this chapter, we introduce social ranking rules as weighted extensions of *ceteris paribus* majority in order to rank more than two individuals. Different ways to define weighted extension of the *ceteris paribus* majority rule result in families of solutions that are in inclusion relation with each other. The inclusion relation between families forms a tree in which each edge between any two families indicates the inclusion relation between them. Corresponding to each edge of the tree, we define axioms that characterize social ranking rules belonging to one family as members of the other family of solutions.

Contents

4.1 Introduction

In this chapter, we establish a new approach in order to rank individuals when an ordinal ranking over coalitions formed by them is given. Recalling the *ceteris paribus* majority principle in Chapter 2, it suggests an interpretation of our ranking problem along the lines of a virtual election, with groups of individuals (coalitions) playing the role of voters. However, it differs from a classical voting scenario in which candidates can also be voters. One can argue that the corresponding ranking rule does not take into account an important part of information about the power relation. For example, it assigns the same voting power to coalitions (as voters), while coalitions are formed by different combinations of individuals.

Example 27. *Consider the power relation* 145   245   1234   23   12   13   35   14   24*. The ceteris paribus majority rule ranks individual* 1 *higher than individual* 2 *since, referring to corresponding CP-comparisons (*145   245*,* 14   24 *and* 23   13*), individual* 1 *performs better than individual* 2 *by joining two coalitions, while individual* 2 *performs better than individual* 1 *when it joins one coalition. However, based on the context of the ranking problem, coalition* 45 *may have different voting power than coalitions* 4 *and* 3*. Also, since coalition* 3 *compares more individuals than coalition* 4 *(coalition* 3 *compares all possible pairs of individuals* 1*,* 2*,* 5 *while coalition* 4 *only compares* 1 *and* 2*), they could have different voting powers.*

The possibility that, following the *ceteris paribus majority* principle, coalitions have different voting power is valid in many real settings. For example, suppose the president of a company wants to compare employees based on the evaluations made by committees of the employees. Each committee can compare any two employees that do not belong to the committee, by saying that one employee performs better than the other one or they are indifferent. Let's assume the company president follows the majority rule in order to combine committees' evaluations about any two employees. The company president can weight evaluations made by a committee according to the members inside the committee and (or) the other employees that get compared by the committee. This approach is easy to justify. Suppose committees follow a voting method in order to do evaluation. If there are some "dictatorial members", who impose their opinions to others, in the committee, then the company president may decide to give less values to the comparisons made by the committee because she knows there are some members in the committee whose opinion might be different. On the other hand, if all the members in a committee respect democratic ways of decision making, then the company president, probably, decides to give higher worth to the evaluation made by the committee because she knows the evaluation has a big support of the members in the committee. In another setting, suppose the employees are divided between different projects, let's say projects A, B, and C. If a committee with members who work on project A evaluates employees who work on projects B or C, then the company president may value it less because they do not work on the same topic, or she may value it more by the justification that the committee can look from outside, and it is more effective in comparing employees. In a still different approach, if the evaluation process in committees is based on majority voting, the company president may tend to give more weight to evaluation done by committees of bigger size, due to the larger number or proponents of the evaluation. It is also possible for the company president to assess the worth of evaluations done by a committee by its level of participation in the process of evaluation, which is reflected in the number of comparisons made by the committee.

All these considerations suggest to define weighted versions of *ceteris paribus* majority rule, in which each coalition, as a voter, is weighted by a weight function. Depending on the settings of the ranking problem, the weight assigned to a coalition is a function of different factors like the coalition (its internal structure) and (or) the comparisons made by the coalition, or size of coalition and (or) the number of comparisons made by the coalition.

So far, we have seen the necessity of weighting coalitions when a *ceteris paribus* majority principle is followed to rank individuals given a power relation. In order to rank more than two individuals, the aggregation function that we use is based on the work of [Terzopoulou and Endriss, 2019] where they focus on the normative characterisation of voting methods under which each agent has a weight that depends only on the size of her ballot, i.e., on the number of pairs of alternatives for which she chooses to report a relative ranking. They have designed a weight rule that selects an acyclic preference over alternatives which maximizes the sum of cumulative weights assigned to each pair in a preference profile expressed by the voters.

As we have seen, the weight function in a social ranking rule depends on some factors related to the structure of power relation (not necessarily complete), which is based on the settings of the ranking problem. Using alternative factors to define a weight function results in specific social ranking rules. Because there are infinite ways to define a weight function given a set of arguments, the corresponding social ranking rules, with together, form a *family* of social ranking rules.

Different families of weighted *ceteris paribus* majority rules have inclusion relation with each other. The inclusion relation among these families of solutions forms a tree. The leafs of the tree show weighted extensions of *ceteris paribus* majority rules, in which the weight function is uniquely defined. The main goal in this chapter is to analyse the properties that characterize the inclusion relation between any two families of social ranking rules in the mentioned tree structure.

The rest of this chapter is structured as follows. In Section 4.2, we review the necessary notations and concepts in this chapter. Section 4.3 introduces the framework and defines different families of weighted CP-majority rules that we study in this chapter. In section 4.4, we axiomatically analyse families of weighted CP-majority rules and explore the relation between them. Finally Section 4.5 concludes the chapter.

4.2 Preliminaries

In this section, we borrow the basic notations about power relations and social ranking rules from Chapters 2 and 3.

Following the principle of *ceteris paribus* majority in order to rank individuals given a power relation in Chapter 2, we define set of all pair-wise comparisons made by a coalition S 2^N in power relation as below:

$$_S = \{(i, j)|i \quad S \quad j \quad S \text{ s.t } i, j \quad N, i, j \; / \; S, i = j\}$$

and we call it the information set of coalitions S in power relation . For simplicity of notation, during this chapter, instead of referring to members of an information set as ordered pairs like (i, j) we denote them as ij. Therefore, we simply write $_S = \{ij|i \quad S \quad j \quad S \text{ s.t } i, j \quad N, i, j \; / \; S, i = j\}$. Also, we refer to the informative part of the power relation as $^I = \{ \; _S : \; S \quad 2^N\}$. Regarding to the informative part of power relations, the concept of “identical except” describes similarity between any two power relations.

Definition 4.2.1 (Identical-Except)**.** *Two power relations ,* $\mathcal{B}(2^N)$ *are called identical except for a set of coalitions* $\mathcal{S}$ *iff for all coalitions* $S \; / \; \mathcal{S}$ *it holds that* $_S = \; _S$.

In the next section, we introduce our ranking model as an extended version of *ceteris paribus* majority, and we explore ways that the ranking models differ depending on various definitions of weight function.

4.3 The model

In this section, we establish the settings of the ranking problem, and we extend the concept of *ceteris paribus* majority to a weighted rank rule in order to rank more than two individuals.

We first start by defining the general family of weighted *ceteris paribus* majority rules. Given a set N of individuals and a power relation $\mathcal{B}(2^N)$, let’s refer to $|R \cap \; _S|$ as the similarity between a linear order R and an information set $_S$. A weighted *ceteris paribus* majority rule maps each power relation to those linear orders which maximize a weighted sum of corresponding similarities. Formally, it is defined as follows.

Definition 4.3.1 (Weighted Ceteris Paribus (CP)-majority rule)**.** *A weighted Ceteris Paribus(CP)-majority rule is a function* F_w *that maps any given power relation* $\mathcal{B}(2^N)$ *to a subset of linear orders over the set* N *of individuals, i.e.,* $F_w : \mathcal{B}(2^N) \to 2^{\mathcal{L}(N)}$*:*

$$F_w(\;) = \underset{R \;\; \mathcal{L}(N)}{\text{argmax}} \sum_{S \;\; 2^N} w(S, \; _S) \cdot |R \cap \; _S|. \tag{4.1}$$

In this definition, w refers to a binary weight function that assigns to any ordered pair of coalition and its information set $((\; _S, S))$ a positive real number, $w : 2^{N \times N} \times 2^N \to \mathbb{R}^+$. This positive number expresses how each pair of individuals $ij \quad _S$ in the information set of coalition S is weighted by the ranking system. Equivalently, we refer to the value of weight function as the weight of coalition S in the power relation .

We refer to the family of all weighted CP-majority rules in Equation 6.1 as $\mathcal{F}_{w_{(I,C)}}$. Each member of the family is a ranking rule given by a specific definition of the weight function. The subscript $w_{(I,C)}$ denotes the structure of the weight function for the members in the family, which depends on both the information sets of coalitions like S and the coalitions themselves. Also, note that the outcome of the social ranking rule defined in Equation 6.1 is a set of linear orders.

Example 28. *Consider a set of three individuals $N = \{1, 2, 3\}$. The set of all linear orders over the set N is given as $\mathcal{L}(N) = \{1\,R\,2\,R\,3, 1\,R\,3\,R\,2, 2\,R\,1\,R\,3, 2\,R\,3\,R\,1, 3\,R\,1\,R\,2, 3\,R\,2\,R\,1\}$. Now, suppose a power relation is defined as $12 \quad 13 \quad 23 \quad 1 \quad 2$. In this case, the information sets $_1 = \{23, 32\}$, $= \{12\}$, $_2 = \{13\}$, and $_3 = \{12\}$ are extracted from the power relation. Let's assume the weight function in Equation 6.1 is defined with respect to coalitions and their information sets as below:*

$$w(\ _1, 1) = \frac{1}{2}, w(\ _3, 3) = 1, w(\ _2, 2) = \frac{1}{4}, w(\ \ ,\ \ ) = 0.$$

Note that to compute the weight value for each coalition, its arguments are the information set of the coalition and the coalition itself. Therefore, the weighted CP-majority rule for the given power relation can be reformulated as

$$F_w(\ \ ) = \underset{R\ \ \mathcal{L}(N)}{\operatorname{argmax}}[\frac{1}{2} \cdot |R \cap \{23, 32\}| + 1 \cdot |R \cap \{12\}| + \frac{1}{4} \cdot |R \cap \{13\}| + 0 \cdot |R \cap \{12\}|].$$

It is easy to verify that there exist more than one linear order maximizing the value of $[\frac{1}{2} \cdot |R \cap \{23, 32\}| + 1 \cdot |R \cap \{12\}| + \frac{1}{4} \cdot |R \cap \{13\}|]$. These linear orders, specifically, should prefer 1 over 3 and 1 over 2. Therefore $F_w(\ \ ) = \{1\,R\,2\,R\,3, 1\,R\,3\,R\,2\}$.

Remark 4. *We assume that the weight functions in the family of all weighted CP-majority rules are symmetrical, in the sense that if a pair of individuals in an information set is reversed, then the weight assigned to the pairs will not change. More precisely, consider two power relations $,\ \ \mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, ...i\ , j\ \}\ \ N(\ \ \mathbb{N})$. For a coalition S where $i_1, j_1, ..., i\ , j\ \ / S$, and information sets $_S = \{i_1 j_1, i_2 j_2, ..., i\ j\ \}$ and $_S = \{j_1 i_1, i_2 j_2, ..., i\ j\ \}$, we have $w(\ _S, S) = w(\ _S, S)$.*

Due to the description of the ranking problem, it is possible that the weight function depends on various factors. Definition 6.0.3 illustrates the most general case where the weight function hinges on the members inside the coalitions (their identities, interaction among them,...) and also the information sets related to each coalition (the identity of individuals getting compared by the coalition, their relevance to each other,...). However, in more specific situations, as mentioned in the example of employees' evaluation in the Introduction, the ranking system may assign weights to coalitions that depend on a part of the information provided by coalitions and their information sets.

Let's come back to the example of employees evaluation. Suppose the rules in the company allow committees to freely evaluate pairs of employees, and also assume that the evaluation of employees by the president is based on the number of times one employee ranked higher than the other by different committees. In this situation, the information set of each committee (the employees they compared) can affect the competence of the committee in doing specific comparisons. For example, in the case that the employees are distributed among three different projects A,B, and C, the committee whose comparisons made over employees working on different projects may be less weighted by the company president because she believes the committee is not able to do its evaluation on different projects simultaneously. On the other hand, she may assign bigger weight to the committee since she believes looking employees from outside of a project brings more insight about their performance. Following this approach the aggregation function in Definition 6.0.3 should be weighted by a weight function that depends on the information set of coalitions ($\mathcal{I}_S$).

By the same reasoning, the weight function in Definition 6.0.3 can be tailored to many other settings of a ranking problem.

All the possible families of the weighted CP-majority rules, that we are interested in, are illustrated in Figure (4.1). Each node of the tree represents a family of weighted *ceteris paribus* majority rules whose weight function is a function of the arguments indicated by the node. Any downward edge identifies the inclusion relation between the two linked nodes. For example, the node with ($\mathcal{I}_S$) refers to the set of all weighted *ceteris paribus* majority rules, in which the weight function depends on the information sets of the coalitions in a given power relation $\succeq$. In this node, a weight function assigns to any information set a positive real number: $w : 2^{N \times N} \to \mathbb{R}^+$. All such ranking rules form a family of weighted CP-majority rules which is distinct from the others by the weight function of its members. Let's refer to such weight functions as w_I. We illustrate the family of all weighted *ceteris paribus* majority rules imposed by the weight functions as $\mathcal{F}_{w_I}$. $F_{w_I} \in \mathcal{F}_{w_I}$ denotes a member of this family:

$$F_{w_I}(\succeq) = \underset{R \in \mathcal{L}(N)}{\operatorname{argmax}} \sum_{S \in 2^N} w_I(\mathcal{I}_S) \cdot |R \cap \mathcal{I}_S|. \tag{4.2}$$

Note that by Remark 4, the weight functions of the family of weighted *ceteris paribus* majority rules $\mathcal{F}_{w_I}$ are symmetrical:

Definition 4.3.2. *A weight function $w_I : 2^{N \times N} \to \mathbb{R}^+$ is symmetrical if for a set of individuals $i_1, j_1, i_2, j_2, \ldots i_k, j_k \in N$, any two coalitions S, S' with $i_1, j_1, i_2, j_2, \ldots i_k, j_k \notin S, S'$, and two information sets $\mathcal{I}_S = \{i_1 j_1, i_2 j_2, \ldots, i_k j_k\}$ and $\mathcal{I}_{S'} = \{j_1 i_1, i_2 j_2, \ldots, i_k j_k\}$, it holds that $w_I(\mathcal{I}_S) = w_I(\mathcal{I}_{S'})$.*

In the following, we list the weighted rank rules corresponding to nodes of the tree in Figure 4.1.

- Node (S): the weight function of the members of the concerning family associates to each subset of individuals (coalition) a positive real number as its weight: $w_C : 2^N \to \mathbb{R}^+$. The family of these weighted CP-majority rules is indicated by $\mathcal{F}_{w_C}$, also its members are specified by F_{w_C}:

$$F_{w_C}(\succeq) = \underset{R \in \mathcal{L}(N)}{\operatorname{argmax}} \sum_{S \in 2^N} w_C(S) \cdot |R \cap \mathcal{I}_S|. \tag{4.3}$$

- Node ($|\mathcal{I}_S|, |S|$): the weight function for the members of the related family is $w_{(\#I,\#C)} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$, that assigns to each pair of natural numbers, illustrating the ordered pair of sizes of information sets and sizes of coalition, a positive real number. We indicate the family of these weighted CP-majority rules as $\mathcal{F}_{w_{(\#I,\#C)}}$, and each member $F_{w_{(\#I,\#C)}} \in \mathcal{F}_{w_{(\#I,\#C)}}$ is defined as:

$$F_{w_{(\#I,\#C)}}(\succeq) = \underset{R \in \mathcal{L}(N)}{\operatorname{argmax}} \sum_{S \in 2^N} w_{(\#I,\#C)}(|\mathcal{I}_S|, |S|) \cdot |R \cap \mathcal{I}_S|. \tag{4.4}$$

- Node $(|\ _S|)$: for the members of the corresponding family, the weight function assigns to any natural number, representing the size of the information set of a coalition S, a non negative real number, $w_{\#I} : \mathbb{N} \to \mathbb{R}^+$. We show the family of these rank rules as $\mathcal{F}_{w_{\#I}}$, and each $F_{w_{\#I}} \in \mathcal{F}_{w_{\#I}}$ is as follows:

$$F_{w_{\#I}}(\ ) = \operatorname*{argmax}_{R \in \mathcal{L}(N)} \sum_{S \in 2^N} w_{\#I}(|\ _S|) \cdot |R \cap\ _S|. \tag{4.5}$$

- Node $(|S|)$: the weight function is defined as $w_{\#C} : \mathbb{N} \to \mathbb{R}^+$, that assigns to each natural number, referring to the size of coalition, a positive real number. We indicate the family of these ranking rules as $\mathcal{F}_{w_{\#C}}$, and each member $F_w \in \mathcal{F}_{w_{\#C}}$ is defined as below:

$$F_{w_{\#C}}(\ ) = \operatorname*{argmax}_{R \in \mathcal{L}(N)} \sum_{S \in 2^N} w_{\#C}(|S|) \cdot |R \cap\ _S|. \tag{4.6}$$

- Node $w_{\#I}(|\ _S|) = 1$: this node refers to a weighted CP-majority rule whose weight function is considered to be the constant function over the size of information sets, $w_{\#I}(|\ _S|) = 1$. We refer to this weighted CP-majority rule as $F^c_{(w_{\#I})}$, $w_{\#I}(|\ _S|) = 1,\ S \in 2^N$ and is given as below:

$$F^c_{w_{\#I}}(\ ) = \operatorname*{argmax}_{R \in \mathcal{L}(N)} \sum_{S \in 2^N} |R \cap\ _S|. \tag{4.7}$$

- Node $w_{\#I}(|\ _S|) = \frac{1}{|\ _S|}$: it corresponds to a weighted CP-majority rule with weight function as $w_{\#I}(|\ _S|) = \frac{1}{|\ _S|}$. This weighted CP-majority rule is denoted by $F^p_{w_{\#I}}$, $w_{\#I}(|\ _S|) = \frac{1}{|\ _S|},\ S \in 2^N$, and is defined as:

$$F^p_{w_{\#I}}(\ ) = \operatorname*{argmax}_{R \in \mathcal{L}(N)} \sum_{S \in 2^N} \frac{1}{|\ _S|} \cdot |R \cap\ _S|. \tag{4.8}$$

- Node $w_{w_{(\#I,\#C)}}(|\ _S|, |S|) = \frac{|S|}{|\ _S|}$: it refers to a weighted CP-majority rule with weight function as $w_{w_{(\#I,\#C)}}(|\ _S|, |S|) = \frac{|S|}{|\ _S|}$. We indicate this weighted CP-majority rule as $F^p_{w_{(\#I,\#C)}}$ as follows:

$$F^p_{w_{(\#I,\#C)}}(\ ) = \operatorname*{argmax}_{R \in \mathcal{L}(N)} \sum_{S \in 2^N} \frac{|S|}{|\ _S|} \cdot |R \cap\ _S|. \tag{4.9}$$

- Node $w_{w_{\#C}}(|S|) = |S|$: this node represents the weighted CP-majority rule whose weight function is considered to be the identity function $w_{w_{\#C}}(|S|) = |S|$. This weighed CP-majority rule is shown by $F^I_{w_{\#C}}$, and is defined as:

$$F^I_{w_{\#C}}(\ ) = \operatorname*{argmax}_{R \in \mathcal{L}(N)} \sum_{S \in 2^N} |S| \cdot |R \cap\ _S|. \tag{4.10}$$

One plausible interpretation of the hierarchical structure of the tree is that the members of the family of social ranking rules which are closer to the root rank individuals with more deliberation (by using more information) in the power relation. For instance, in the first layer of the tree, the family of CP-majority rules $\mathcal{F}_{w_{(I,C)}}$ utilizes the whole information provided by the informative part of power relations, i,e., the coalitions and the information sets. However, social ranking rules in the second layer of the tree do not use some parts of information in power relations. As an example, weight functions of members in the family $\mathcal{F}_{w_I}$ do not depend on coalitions, and the members of the family $\mathcal{F}_{w_{\#C}}$ neglect the information sets as well as the possible interaction between individuals in coalitions.
The following examples illustrates how the precision in ranking individuals changes by choosing different social ranking rules from families of ranking rules in different layers of the tree.

Example 29. *Consider the following power relation, which is incomplete,*

$$12 \quad 23 \quad 13 \quad 35 \quad 14 \quad 24 \quad 25, 35 \quad 45, 25 \quad 45$$

and is defined over the power set of $N = \{1, 2, 3, 4, 5\}$. *There are five coalitions that form the CP-comparisons in the power relation,* 1, 2, 3, 4 *and* 5, *with the information sets as* $_1 = \{23, 34, 24\}$, $_4 = \{12\}$, $_3 = \{21, 25, 15\}$, $_5 = \{32, 34, 24\}$, *and* $_2 = \{34, 45, 35, 13, 14, 15\}$. *Let's assume the members of the family* $\mathcal{F}_{w_{(I,C)}}$ *are used in order to rank the individuals. In this case, based on how the weight function is defined, there are three possibilities for the pairs* (1, 2) *or* (2, 1) *to appear in the ranking result: if the weight function assigns more weight to coalition* 4 *than coalition* 3, *then the final ranking result will contain the pair* (1, 2)*; if coalition* 3 *is weighted more than coalition* 4, *then the pair* (2, 1) *will appear in the ranking; if the two coalitions are weighted the same then the corresponding social ranking yields linear orders containing either* (1, 2) *or* (2, 1). *In the same way, there are three possibilities for the pairs of individuals* (2, 3) *and* (3, 2) *based on the weight assigned to coalitions* 1 *and* 5. *Therefore, based on how the weight function is defined, there is a large number of possibilities to rank individuals. However, if the ranking rule is a member of the family* $\mathcal{F}_{w_I}$, *then since the coalitions* 1 *and* 5 *have the same weight (they have symmetrical information sets) there is only one possibility for the ranking of* 2 *and* 3*: the corresponding social rankings engender linear orders containing either* (2, 3) *or* (3, 2)*. Also, depending on weights assigned to coalitions* 3 *and* 4, *there are still three possibilities for the ranking over individuals* 1 *and* 2. *Now, if the members of the family* $\mathcal{F}_{w_{\#C}}$ *are used to rank individuals, then there is only one possibility for the ranking of individuals* 1 *and* 2*: applying the members of the family results in linear orders containing either* (1, 2) *or* (2, 1), *since the weight functions assign the same weight to coalitions* 3 *and* 4 *(they have the same size).*

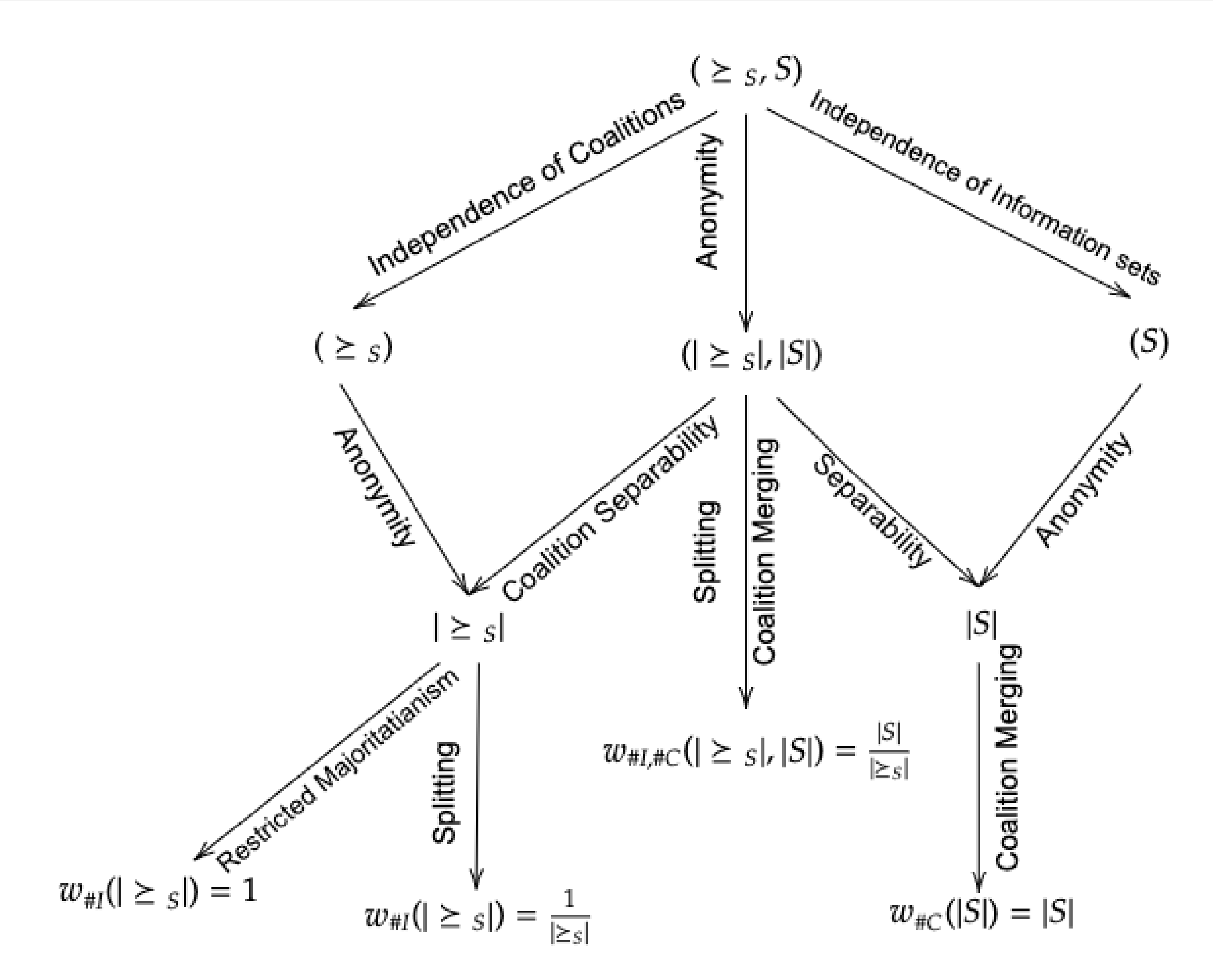

Figure 4.1: Being sub-family relation between different families.

The main goal in this chapter is to characterize the inclusion relation between any two families of the weighted CP-majority rules, represented by the downward edges in the tree. Each edge of the tree corresponds to at least one axiom (as indicated in Figure (4.1)). In the next section, we precisely define each axiom and use them to characterize the different social ranking rules as members of specific families.

4.4 Axiomatic Analysis

This section is devoted to study the tree structure in Figure 4.1 from property-driven approach. We define axioms corresponding to each edge of the tree which characterize the inclusion relation between each two families of the weighted CP-majority rules. First, we informally introduce axioms as listed below:

- Independence of Information set: the axiom *independence of information set* provides sufficient and necessary condition for the members of family $\mathcal{F}_{w_C}$ as a sub-family of $\mathcal{F}_{w_{(I,C)}}$. It states that coalitions with compatible preferences over a set of individuals in two different power relations, should have been able to combine their preferences and form a new power relation, without changing in the social ranking of the individuals.

- Independence of Coalitions: the axiom *independence of coalitions* characterizes the inclusion relation between the two families $\mathcal{F}_{w_{(I,C)}}$ and $\mathcal{F}_{w_I}$. It states that coalitions should be able to change their members, without any change in the social ranking of individuals.

- Anonymity: the axiom *anonymity* is the one that we use in order to characterize inclusion relation between weighted *ceteris paribus* majority rules whose weight functions depend on the ordered pairs of coalitions and their information sets, or coalitions, or information sets and those whose weight functions are based on the cardinality of the mentioned factors. More precisely, depending on the domain in which the axiom is defined, social ranking rules that satisfy *anonymity* do not take into consideration the names of individuals and the presence of interactions between them. We use this axiom to characterize the inclusion relation between $\mathcal{F}_{w_{(I,C)}}$ and $\mathcal{F}_{w_{(\#I,\#C)}}$ ($\mathcal{F}_{w_{(\#I,\#C)}} \quad \mathcal{F}_{w_{(I,C)}}$), the inclusion relation between $\mathcal{F}_{w_C}$ and $\mathcal{F}_{w_{\#C}}$ ($\mathcal{F}_{w_{\#C}} \quad \mathcal{F}_{w_C}$), and the inclusion relation between $\mathcal{F}_{w_I}$ and $\mathcal{F}_{w_{\#I}}$ ($\mathcal{F}_{w_{\#I}} \quad \mathcal{F}_{w_I}$).

- Coalition Separability: to characterize the inclusion relation between $\mathcal{F}_{w_{(\#I,\#C)}}$ and $\mathcal{F}_{w_{\#I}}$ ($\mathcal{F}_{w_{\#I}} \quad \mathcal{F}_{w_{(\#I,\#C)}}$) we use the axiom coalition separability. Consider a set of coalitions that have some common members. This axiom states that if they have the same information sets then one should expect that reducing the size of coalitions by removing the repeated members in the coalitions will not change the final ranking.

- Separability: this axiom is used to characterize the inclusion relation between $\mathcal{F}_{w_{(\#I,\#C)}}$ and $\mathcal{F}_{w_{\#C}}$ ($\mathcal{F}_{w_{\#C}} \quad \mathcal{F}_{w_{(\#I,\#C)}}$). It says that if a set of coalitions of the same size have mutually compatible preferences over individuals, then representing all of them as one coalition of that size doing all the comparisons should not change the social ranking.

- Splitting: to characterize the belonging relation between the family $\mathcal{F}_{w_{\#I}}$ and the member $\mathcal{F}^p_{w_{\#I}}$ ($\mathcal{F}^p_{w_{\#I}} \quad \mathcal{F}_{w_{\#I}}$) we benefit from this axiom. The idea is that if several coalitions have mutually compatible preferences, it should be possible for them to form a pre-election pact and all report the union of their individual preference sets, and it should not change the outcome of ranking individuals.

- Coalition Merging: it is used to characterize $\mathcal{F}^{I}_{w_{\#C}}$ as a member of $\mathcal{F}_{w_{\#C}}$ ($\mathcal{F}^{I}_{w_{\#C}}$ $\mathcal{F}_{w_{\#C}}$). This axiom gives the idea that when a group of coalitions have same information sets (preferences), then they should be able to merge together and form a coalition containing all the members in the previous coalitions with the same information set, without changing the ranking over individuals.

- Restricted Majoritarianism: the social ranking rule $\mathcal{F}^{c}_{w_{\#I}}$ is characterized as a member of the family $\mathcal{F}_{w_{\#I}}$ by this axiom which expresses one of the fundamental normative approaches in ranking individuals (if in more *ceteris paribus* comparisons i performs better than j, then it should be ranked higher).

- Also the two axioms Splitting and Coalition Merging can be used in order to characterize the social ranking rule $\mathcal{F}^{p}_{w_{(\#I,\#C)}}$ as a member of the family of $\mathcal{F}_{w_{(\#I,\#C)}}$ ($\mathcal{F}^{p}_{w_{(\#I,\#C)}}$ $\mathcal{F}_{w_{(\#I,\#C)}}$).

In the following sections we formalize the axioms and illustrate how they characterize some members of different families of social ranking rules.

4.4.1 Independence of Information Set

In the example of employees' evaluation in a company, based on the settings of the problem, the company president may aggregate committees' evaluations by weighting them using all features of the committees (in our framework the members inside the committees and the employees in their information sets). If the president follows a *ceteris paribus* majority principle to aggregate committees' evaluations and codify the features related to committees as weights, then the most appropriate ranking rules that could be used by the president are those belonging to $\mathcal{F}_{w_{(I,C)}}$. However, based on the structure of the company and the methods used by the committees to rank employees there could be situations where the company president should weight the comparisons made by a committee (or equivalently weight the committee) based on the members inside the committee. This happens when the way that the committee decides on employees' evaluation is important. For instance, if there are some employees in the committee with dictatorial traits then they impose their opinion to others, which may diminish the worth of their evaluation. The pertinent social ranking rules in this case are the members of $\mathcal{F}_{w_C}$. The axiom *Independence of Information set* provides necessary and sufficient condition for social ranking rules to be members of the family $\mathcal{F}_{w_C}$. Suppose coalitions compare a set of individuals in two power relations (and), the axiom asserts that if coalitions are able to combine their information sets in two power relations and and form a new power relation, then the social ranking of individuals should be the intersection of individuals' ranking in the two power relations.

Definition 4.4.1 (Independence of Information Set)**.** *A social ranking rule F satisfies the axiom independence of information set iff for any two power relations , $\mathcal{B}(2^N)$ and a set of individuals $\{i_1, ..., i\}$ N (N) if the two power relations are identical except for $\{S_1, ..., S_k\}$ with $\{i_1, ..., i\}$ / $S_1, ..., S_k$ such that $_{S_1} = \{i_{t_1} i_{t_2}, ..., i\ _1 i\ _2\}, ...,$ $_{S_k} = \{i_{t_k} i_{t_{k+1}}, ..., i\ _k i\ _{k+1}\}$ and $_{S_1} = \{i_{t_1} i_{t_2}, ..., i\ _1 i\ _2\}, ...,$ $_{S_k} = \{i_{t_k} i_{t_{k+1}}, ..., i\ _k i\ _{k+1}\}$ (1 ≤*

$t_1, t_1, \ _1, \ _1, ..., t_{k+1}, t_{k+1}, \ _{k+1}, \ _{k+1}$ ≤) and if a power relation E exists for which $E_{S_1} = \ _{S_1}$ $\ _{S_1}$,..., $E_{S_k} = \ _{S_k}$ $\ _{S_k}$ ($E_{S_1} = \{i_{t_1} i_{t_2}, ..., i_{\ 1} i_{\ 2}, i_{t_1} i_{t_2}, ..., i_{\ 1} i_{\ 2}\}, ..., E_{S_k} = \{i_{t_k} i_{t_{k+1}}, ..., i_{\ k} j_{\ k+1}, i_{t_k} j_{t_{k+1}}, ..., i_{\ k} j_{\ k+1}\}$), then it holds that $F(E) = F(\ ) \cap F(\ )$.

Theorem 4.4.2. *The unique social ranking rules in $\mathcal{F}_{w_{(I,C)}}$ that satisfy independence of information set are social ranking rules in $\mathcal{F}_{w_C}$.*

Proof. ( )(Existence) We first prove that any social ranking rule F_w $\mathcal{F}_{w_C}$ satisfies the axiom. Consider two power relations , $\mathcal{B}(2^N)$ and a set of individuals $\{i_1, ..., i\}$ N (N), and suppose the two power relations are identical except for $\{S_1, ..., S_k\}$ with $\{i_1, ..., i\}$ / $S_1, ..., S_k$ such that $\ _{S_1} = \{i_{t_1} i_{t_2}, ..., i_{\ 1} i_{\ 2}\}, ..., \ _{S_k} = \{i_{t_k} i_{t_{k+1}}, ..., i_{\ k} i_{\ k+1}\}$ and $\ _{S_1} = \{i_{t_1} i_{t_2}, ..., i_{\ 1} i_{\ 2}\}, ..., \ _{S_k} = \{i_{t_k} i_{t_{k+1}}, ..., i_{\ k} i_{\ k+1}\}$ ($1 \leq t_1, t_1, \ _1, \ _1, ..., t_{k+1}, t_{k+1}, \ _{k+1}, \ _{k+1} \leq$). Also, assume a power relation E exists which is identical to and except for the same set $\{S_1, ..., S_k\}$ of coalitions, and $E_{S_1} = \{i_{t_1} i_{t_2}, ..., i_{\ 1} i_{\ 2}, i_{t_1} i_{t_2}, ..., i_{\ 1} i_{\ 2}\}, ..., E_{S_k} = \{i_{t_k} i_{t_{k+1}}, ..., i_{\ k} i_{\ k+1}, i_{t_k} i_{t_{k+1}}, ..., i_{\ k} i_{\ k+1}\}$. Since the social ranking rule F_w belongs to the family $\mathcal{F}_{w_C}$, the weights assigned to pairs of individuals depends only on the coalitions. If we focus on the parts of the power relations that are different, each pair of individuals (i_t, i_r) ($1 \leq r, t \leq$) which belongs to a linear order R $F_w(\ ) \cap F_w(\ )$ has the same weight in linear order R $F_w(E)$. This is because the pair (i_t, r_r) materializes in the linear order R if for a coalition S 2^N, $i_t i_r$ $\ _S$, and $i_t i_r$ $\ _S$, which respectively means $i_t i_r$ E_S. Also, because the value of weight function only depends on the coalitions, the weight of the pair of individuals $i_t i_r$ remains the same. Therefore, we have $F_w(E) = F_w(\ ) \cap F(\ )$.

( )(Uniqueness) Now we prove that any member F_w $\mathcal{F}_{w_{(I,C)}}$ that satisfies *independence of information set* belongs to family $\mathcal{F}_{w_C}$. We prove that the weight function of the social ranking rule F_w meets the condition $w_{(I,C)}(\ _S, S) = w_{(I,C)}(\ _S, S)$ for any two power relations , $\mathcal{B}(2^N)$ and any S 2^N.

To this purpose, consider a set of individuals $\{i_1, j_1, i_2, j_2, ..., i, j\}$ N (N) and two power relations and with informative parts respectively as $^I = \{\ _{S_1}, \ _{S_2}, \ _{S_3}\}$ and $^I = \{\ _{S_1}, \ _{S_2}\}$ such that $\ _{S_1} = \{i_1 j_1, j_2 i_2\}$, $\ _{S_2} = \{j_1 i_1, i_2 j_2\}$, and $\ _{S_3} = \{j_2 i_2, i_1 j_1\}$. Also assume $\ _{S_1} = \{i_1 j_1\}$ and $\ _{S_2} = \{i_2 j_2, i_3 j_3, ..., i\ j\}$. Now let us define another power relation E such that $E^I = \{E_{S_1}, E_{S_2}, E_{S_3}\}$ with $E_{S_1} = \{i_1 j_1, j_2 i_2\}$, $E_{S_2} = \{j_1 i_1, i_2 j_2, ..., i\ j\}$, and $E_{S_3} = \{j_2 i_2, i_1 j_1\}$. Note that any information set in E is obtained from the union of the corresponding information sets in and , and since F_w satisfies the axiom *independence of information set* then it holds that $F_w(E) = F_w(\ ) \cap F_w(\ )$. It is obvious that applying the social ranking rule on the power relation results a linear order $R = \{(i_1, j_1), (i_2, j_2), ..., (i, j)\}$ $F_w(\ )$. Let's assume applying the social ranking rule on the power relation yields a linear order R $F_w(\ )$. The only way that the intersection $F_w(\ ) \cap F_w(\ )$ is not empty, is that R $F_w(\ )$ contains ordered pairs (i_1, j_1) and (i_2, j_2). This happens when for the power relation we have

$$w_{(I,C)}(\ _{S_1}, S_1) + w_{(I,C)}(\ _{S_3}, S_3) \geq w_{(I,C)}(\ _{S_2}, S_2) \geq w_{(I,C)}(\ _{S_1}, S_1) + w_{(I,C)}(\ _{S_3}, S_3). \tag{4.11}$$

Therefore the intersection $F_w(\ ) \cap F_w(\ )$ contains linear orders like R $F_w(E)$ with

$\{(i_1, j_1), (i_2, j_2)\}$ R . This occurs when for the power relation E we have

$$w_{(I,C)}(\mathrm{E}_{S_1}, S_1) + w_{(I,C)}(\mathrm{E}_{S_3}, S_3) \geq w_{(I,C)}(\mathrm{E}_{S_2}, S_2) \geq w_{(I,C)}(\mathrm{E}_{S_1}, S_1) + w_{(I,C)}(\mathrm{E}_{S_3}, S_3). \quad (4.12)$$

According to 4.11 and 4.12, and since $w_{(I,C)}(\ _{S_1}, S_1)$=$w_{(I,C)}(\mathrm{E}_{S_1}, S_1)$ and $w_{(I,C)}(\ _{S_3}, S_3)$=$w_{(I,C)}(\mathrm{E}_{S_3}, S_3)$ it holds that

$$w_{(I,C)}(\ _{S_2}, S_2) = w_{(I,C)}(\mathrm{E}_{S_2}, S_2) \ ; \quad _{S_2} \quad _{S_2}, | \ _{S_2} \cap \ _{S_2} | \geq 2. \quad (4.13)$$

To conclude the proof, let's define another power relation in which $_{S_2} = \{i_3 j_3, ..., i \ j \ \}$. By Equation 4.13 it holds that $w_{(I,C)}(\mathrm{E}_{S_2}, S_2) = w_{(I,C)}(\ _{S_2}, S_2)$, which by transitivity results $w_{(I,C)}(\ _{S_2}, S_2) = w_{(I,C)}(\ _{S_2}, S_2)$. □

4.4.2 Independence of Coalitions

In the example of employees' evaluation, as we said, in the extreme case the president wants to weight committees with respect to the members inside the committees and the employees that they compare. However, based on the settings of the problem, the weight function may only be a function of the information set. This happens, for instance, when employees are assigned to different projects and weighting committees' evaluations depends on how relevant are the comparisons made by committees, in the sense that the evaluated employees belong to same project or not. In such cases, social ranking rules belonging to $\mathcal{F}_{w_I}$ are the convenient tools to aggregate the evaluations made by the committees.

The main goal, in this section, is to characterize the family of $\mathcal{F}_{w_I}$ as a sub-family of $\mathcal{F}_{w_{(I,C)}}$. We define an axiom called *Independence of coalitions*, which states that coalitions should be able to change their members without any effect on social ranking.

Definition 4.4.3 (Independence of coalitions)**.** *A social ranking rule F satisfies independence of coalitions iff for any two power relations* , $\mathcal{B}(2^N)$ *and a set of individuals* $\{i_1, j_1, ..., i \ , j \ \} = A$ N*, if the two power relations are identical except for* $\{S_1, ..., S_k, S_1, ..., S_k\}$ *such that* S_t $N \setminus \{A$ $S_t\}$ *for* $1 \leq t \leq k$ *with* $_{S_1} = \ _{S_2} = ... = \ _{S_k} = \{i_1 j_1, ..., i \ j \ \}$ *and* $_{S_1} = ... = \ _{S_k} =$ *, and* $_{S_1} = ... = \ _{S_k} =$ *and* $_{S_1} = ... = \ _{S_k} = \{i_1 j_1, ..., i \ j \ \}$*, then it holds that* $F(\ ) = F(\ )$*.*

The following example clarifies the definition of *independence of coalitions*.

Example 30. *Consider a set* $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *of individuals and a power relation with the informative part as* $^I = \{ \ _{234}, \ _{12}, \ _5\}$ *such that* $_{234} = \{67, 87\}$, $_{12} = \{34, 56\}$*, and* $_5 = \{12, 32\}$*. Suppose a social ranking rule satisfies the axiom independence of coalitions. It states that each coalition should be able to change its combination without any change in social ranking, as long as its preferences do not change. Consequently, if we define another power relation in which the coalitions* 234*,* 12*, and* 5 *are transformed to other coalitions by removing or adding individuals to them (individuals that are not in their information sets), the social ranking of individuals should not change. Therefore, the power relation can be defined with the informative part as* $^I = \{ \ , \ _{127}, \ _{578}\}$ *where* $= \{67, 87\}$, $_{127} = \{34, 56\}$*, and*

$_{578} = \{12, 32\}$. *Then applying the social ranking rule that satisfies independence of coalitions on the two power relations and should result the same social rankings.*

In the following theorem we prove that a unique subset of rules in $\mathcal{F}_{w_{(I,C)}}$ satisfy *Independence of coalitions*, and the subset is $\mathcal{F}_{w_I}$.

Theorem 4.4.4. *The unique social ranking rules in* $\mathcal{F}_{w_{(I,C)}}$ *that satisfy independence of coalitions are social ranking rules in* $\mathcal{F}_{w_I}$.

Proof. ( )(Existence) For this part it is easy to verify that the members of $\mathcal{F}_{w_I}$ satisfy the axiom *independence of coalitions*. To this purpose, consider any social ranking rule $F_w \quad \mathcal{F}_{w_I}$, any two power relation $\ ,\quad \mathcal{B}(2^N)$ and assume $\{i_1, j_1, ..., i\ , j\ \} = A \quad N$. Suppose the two power relations are identical except for a set $\{S_1, ..., S_k, S_1, ..., S_k\}$ such that $_{S_1} = \ _{S_2} = ... = \ _{S_k} = \{i_1 j_1, ..., i\ j\ \}$, as well as $_{S_1} = ... = \ _{S_k} =$ and $_{S_1} = \ _{S_1}, \ _{S_2} = \ _{S_2}, ..., \ _{S_k} = \ _{S_k}$ where $S_t = N \setminus A \quad S_t$ for $1 \leq t \leq k$. Focusing on the parts of the two power relations that are different, any pair of individuals $i_r j_r$ ($1 \leq r \leq$) is weighted the same in both power relations. If for a coalition $S \quad \{S_1, ..., S_k\}$ we have $i_r j_r \quad _S$, then in power relation we have $i_r j_r \quad _T$ ($T \quad \{S_1, ..., S_k\}$) where $_S = \ _T$. Since the weight function in Equation 4.2 depends only on the information sets, then it holds that $F_w(\ ) = F_w(\ )$.

( )(Uniqueness) For the uniqueness part of the proof it is sufficient to show that given a social ranking rule $F_w \quad \mathcal{F}_{w_{(I,C)}}$ that satisfies the axiom, two power relations $\ ,\quad \mathcal{B}(2^N)$ exist such that, $F_w(\ ) = F_w(\ )$ requires that for any two coalition $S_1, S_2 \quad 2^N$ we have $w_{(I,C)}(\ _{S_1}, S_1) = w_{(I,C)}(\ _{S_2}, S_2)$ whenever $_{S_1} = \ _{S_2}$. To do so, let's define the power relations and and a set $\{i_1, j_1, ..., i\ , j\ \} = A \quad N$. Suppose the informative part of the power relation is given by $^I = \{\ _{S_1}, \ _{S_2}\}$ such that $_{S_1} = \{i_1 j_1, i_2 j_2, ..., i\ j\ \}$ and $_{S_2} = \{j_1 i_1, i_2 j_2, ..., i\ j\ \}$. Also, assume for the power relation we have $^I = \{\ _{S_1}, \ _{S_2}\}$ in which $_{S_1} = \{j_1 i_1, i_2 j_2, ..., i\ j\ \}$ and $_{S_2} = \{i_1 j_1, i_2 j_2, ..., i\ j\ \}$. Considering values of the weight function for $w_{(I,C)}(\ _{S_1}, S_1)$ and $w_{(I,C)}(\ _{S_2}, S_2)$ in $F_w(\ )$ in Equation 6.1, there are two possibilities: $w_{(I,C)}(\ _{S_1}, S_1) \geq w_{(I,C)}(\ _{S_2}, S_2)$ or $w_{(I,C)}(\ _{S_1}, S_1) \leq w_{(I,C)}(\ _{S_2}, S_2)$. First assume

$$w_{(I,C)}(\ _{S_1}, S_1) \geq w_{(I,C)}(\ _{S_2}, S_2) \tag{4.14}$$

which results in a linear order R belonging to $F_w(\ )$ with $(i_1, j_1) \quad R$. As the social ranking rule F_w satisfies *independence of coalition* and since $S_1 \quad N \setminus \{A \quad S_2\}$ and $S_2 \quad N \setminus \{A \quad S_1\}$, then we have $F_w(\ ) = F_w(\ )$, and, therefore, there must exists a linear order $R \quad F_w(\ )$ such that $(i_1, j_1) \quad R$. This occurs when

$$w_{(I,C)}(\ _{S_2}, S_2) \geq w_{(I,C)}(\ _{S_1}, S_1). \tag{4.15}$$

By the main assumption that the weight functions are symmetrical (Remark 4), we have $w_{(I,C)}(\ _{S_1}, S_1) = w_{(I,C)}(\ _{S_1}, S_1)$ and $w_{(I,C)}(\ _{S_2}, S_2) = w_{(I,C)}(\ _{S_2}, S_2)$. By substituting the values with those in 4.14 and 4.15 we have $w_{(I,C)}(\ _{S_1}, S_1) = w_{(I,C)}(\ _{S_2}, S_2)$ when $S_1 = S_2$ and $_{S_1} = \ _{S_2}$. The same argument holds for the case where $w_{(I,C)}(\ _{S_1}, S_1) \leq w_{(I,C)}(\ _{S_2}, S_2)$, which concludes the proof □

A weaker version of the axiom *Independence of coalitions* is used in Section 4.4.5 in order to characterize the family of $\mathcal{F}_{w_{\#I}}$ as a sub-family of $\mathcal{F}_{w_{(\#I,\#C)}}$. More precisely, to characterize the social ranking rules in $\mathcal{F}_{w_{\#I}}$ in the domain of social ranking rules belonging to $\mathcal{F}_{w_{(\#I,\#C)}}$ we can use the axiom independence of coalitions, since by the definition of the two axioms *coalition separability* and *independence of coalitions*, whenever the axiom *coalition separability* holds the axiom *independence of coalitions* holds as well.

4.4.3 Anonymity

In this section, we introduce the axiom *Anonymity* which can be used to characterize three families of the weighted CP-majority rules in Figure 4.1.

Consider a ranking problem, like the example of employees' evaluation in a company. As discussed in the previous sections, some interpretations of the problem drive the company president to weigh committees based on the members inside them, inside their information sets, or inside both of them. On the other hand, some interpretations of the ranking problem persuade the president to use ranking rules that assign weights to committees based on their size, size of their information sets, or size of the both. The *anonymity* axiom states a property of the social ranking rules that, according to the context of the ranking problem, does not take into consideration the effect of names and possible interactions among individuals on their ranking. It considers only the effect of the size of relevant sets on the worth of coalitions.

More precisely, the ultimate meaning of the axiom is that the name of individuals are not important. Therefore, availability of one "specific" individual in a coalition does not increase or decrease weight of the coalition, and also the ranking system is not biased to rank one individual higher or lower than the others. The *anonymity* axiom is defined as follows.

Definition 4.4.5. *A social ranking rule satisfies anonymity if and only of for any two power relations and and any permutation $\pi : N \to N$ such that is obtained from by permuting the individuals in using π, then it holds that $F(\ ) = \sigma_\pi(F(\ ))$ (where $\sigma_\pi : 2^{\mathcal{L}(N)} \to 2^{\mathcal{L}(N)}$ such that for $L_1, ..., L_k$ $\mathcal{L}(N)$ (K N) $\sigma_\pi(\{L_1, ..., L_k\})$ is obtained from $\{L_1, ..., L_k\}$ by permuting the individuals using π).*

Example 31. *Consider a set of six individuals $N = \{1, 2, 3, 4, 5, 6\}$ and the power relation as*

$$145 \quad 245 \quad 136 \quad 236 \quad 436 \quad 23 \quad 12 \quad 13 \quad 35 \quad 14 \quad 24.$$

The informative part of this power relation is the set $^I = \{\ _{45},\ _{36},\ _{1},\ _{2},\ _{3},\ _{4}\}$, *and* $_1 = \{23, 24, 34\}$, $_2 = \{31, 14, 34\}$, $_3 = \{21, 25, 15\}$, $_4 = \{12\}$, $_{45} = \{12\}$, $_{36} = \{14, 24, 12\}$.

The anonymity axiom refers to social ranking rules that do not take into account the names of the members and the interaction between them. Therefore, if we permute the individuals in the set N with the function π as $\pi(1) = 2, \pi(2) = 3, \pi(3) = 4, \pi(4) = 5, \pi(5) = 6, \pi(6) = 1$, a power relation with the informative part $^I = \{\ _{2},\ _{3},\ _{4},\ _{5},\ _{56},\ _{41}\}$ *forms where* $_2 = \{34, 35, 45\}$, $_3 = \{42, 25, 45\}$, $_4 = \{32, 36, 26\}$, $_5 = \{23\}$, $_{56} = \{23\}$, $_{14} = \{25, 35, 23\}$.
The weighted CP-majority rule that satisfies anonymity ranks individuals in the set N just based

on their performance when no interaction is assumed between them. For instance, if in power relation individual 1 *is ranked higher than* 2*, then in power relation individual* 2 *is ranked over* 3 *since in power relation individuals* 2 *and* 3 *perform the same as individuals* 1 *and* 2 *in power relation .*

In the following sections, we use the axiom anonymity to characterize different inclusion relations between families of CP-weighed majority rule.

Anonymity on the family of weighted CP-majority rules whose weight function depends on coalitions

In this part, we analyse the *anonymity* axiom as the property of the social ranking rules in the family of $\mathcal{F}_{w_C}$.

The members of the family $\mathcal{F}_{w_C}$ which satisfy such a property are those that do not make difference between the names of individuals in the coalitions and do not consider interaction among them (these are the members of $\mathcal{F}_{w_{\#C}}$). The interpretation of the axiom is that, changing the names of individuals, (individuals who form coalitions to compare the other individuals), should not affect the final ranking of the individuals. On the domain of $\mathcal{F}_{w_C}$, the *anonymity* axiom in Definition 4.4.5 can be reduced to one in which the permutations $\pi : N \rightarrow N$ are restricted to those that only permute the individuals in the coalitions, and keep the individuals in their information sets as they are.

The following theorem indicates that the *Anonymity* axiom as defined in Definition 4.4.5 provides sufficient and necessary condition for the members of $\mathcal{F}_{w_C}$ to be members of the sub-family of $\mathcal{F}_{w_{\#C}}$.

Theorem 4.4.6. *The unique social ranking rules in $\mathcal{F}_{w_C}$ that satisfy anonymity are social ranking rules in $\mathcal{F}_{w_{\#C}}$.*

Proof. ( )(Existence) It is easy to check that each member of F_w $\mathcal{F}_{w_{\#C}}$ satisfies anonymity. ( )(Uniqueness) For the other direction, let's assume a member F_w $\mathcal{F}_{w_C}$ satisfies anonymity. We prove that it is a member of $\mathcal{F}_{w_{\#C}}$. To do that, for specific power relations and and permutations π and σ_π, we show that $F( ) = \sigma_\pi(F( ))$ results $w_C(S) = w_C(S)$ when $|S| = |S |$, which is the property of the weight functions of the members of the family $\mathcal{F}_{w_{\#C}}$. Consider a power relation with informative part as $^I = \{ \; _{S_1}, \ldots, \; _{S_k}, \; _{S_1}, \ldots, \; _{S_k}\}$ (k N) and a set of individuals $\{i_1, j_1, i_2, j_2, \ldots, i \;, j \;\}$ N (N) such that coalitions $S_1, S_2, \ldots, S_k, S_1, S_2, \ldots, S_k$ are of the same size and $i_1, j_1, i_2, j_2, \ldots, i \;, j \;\; / \; S_1, \ldots, S_k, S_1, \ldots, S_k$. Also, assume $_{S_t} = \{i_1 j_1, i_2 j_2, \ldots, i \; j \;\}$ and $_{S_t} = \{j_1 i_1, i_2 j_2, \ldots, i \; j \;\}$ for any $1 \leq t \leq k$. Let's define another power relation with informative part as $^I = (\; _{S_1}, \ldots, \; _{S_k}, \; _{S_1}, \ldots, \; _{S_k})$ such that $_{S_t} = \{i_1 j_1, \ldots, i \; j \;\}$ and $_{S_t} = \{j_1 i_1, i_2 j_2, \ldots, i \; j \;\}$ for $1 \leq t \leq k$. It is possible to obtain the power relation from the power relation using a permutation π that maps any individual in $\{i_1, j_1, i_2, j_2, \ldots, i \;, j \;\}$ to itself except that $\pi(i_1) = j_1$ and $\pi(j_1) = i_1$, also it permutes S_1 with S_1, S_2 with S_2,...,S_k with S_k. Since we assume that the social ranking F_w satisfies anonymity, then it holds that $F_w(\ ) = \sigma_\pi(F_w(\ ))$, where:

$$F_w(\ ) = \underset{R \ \mathcal{L}(N)}{\text{argmax}}[w_C(S_1) \cdot |R \cap \ _{S_1}| + ... + w_C(S_k) \cdot |R \cap \ _{S_k}| + w_C(S_1) \cdot |R \cap \ _{S_1}| + ... + w_C(S_k) \cdot |R \cap \ _{S_k}|].$$

According to power relation and definition of $F_w(\ )$ there are two possibilities for any linear order $R \quad \sigma_\pi(F_w(\ )) = F_w(\ )$: either $(i_1, j_1) \quad R$ or $(j_1, i_1) \quad R$. First, we consider the case that $(j_1, i_1) \quad R$, and we prove that $w_C(S_1) = w_C(S_1)$. The same result will be hold for the case where $(i_1, j_1) \quad R$. By assuming $(j_1, i_1) \quad R$ and by definition of π and σ_π it holds that $(i_1, j_1) \quad R \quad F_w(\ )$. By definition $F_w(\ )$, we have $(i_1, j_1) \quad R \quad F_w(\ )$ when

$$w_C(S_1) + w_C(S_2) + ... + w_C(S_k) \geq w_C(S_1) + w_C(S_2) + ... + w_C(S_k). \tag{4.16}$$

Also, since $\sigma_\pi(F_w(\ )) = F_w(\ )$, we have $(j_1, i_1) \quad R \quad F_w(\ )$. According to definition of $F_w(\ )$:

$$F_w(\ ) = \underset{R \ \mathcal{L}(N)}{\text{argmax}}[w_C(S_1) \cdot |R \cap \ _{S_1}| + ... + w_C(S_k) \cdot |R \cap \ _{S_k}| + w_C(S_1) \cdot |R \cap \ _{S_1}| + ... + w_C(S_k) \cdot |R \cap \ _{S_k}|]$$

it holds that

$$w_C(S_1) + w_C(S_2) + ... + w_C(S_k) \geq w_C(S_1) + w_C(S_2) + ... + w_C(S_k). \tag{4.17}$$

From inequalities (4.16 and 4.17) it holds that $w_C(S_1) + w_C(S_2) + ... + w_C(S_k) = w_C(S_1) + w_C(S_2) + ... + w_C(S_k)$. For the special case of $k = 1$ we have $w_C(S) = w_C(S \ )$. The same argument holds for the second possibility that $(i_1, j_1) \quad R$. Since S and S are coalitions of the same size obtained from any combination of individuals, we conclude that $w_C(S) = w_C(S \ )$ when $S = S$ and $|S| = |S \ |$. □

Anonymity on weighted CP-majority rules whose weight function depends on the information sets

In this part, we explore *anonymity* axiom as a unique property of the members of $\mathcal{F}_{w_{\#I}}$ as a sub-family of $\mathcal{F}_{w_I}$.

The anonymity axiom in the domain of the social ranking rules belonging to $\mathcal{F}_{w_I}$ uniquely specifies the social ranking rules which ranks individuals according to their performance and not their names.

Following theorem provides the main result of this part and validates the claim that anonymity axiom characterizes the social ranking rules belonging to $\mathcal{F}_{w_{\#I}}$ as members of $\mathcal{F}_{w_I}$.

Theorem 4.4.7. *The unique social ranking rules in $\mathcal{F}_{w_I}$ that satisfy anonymity are social ranking rules in $\mathcal{F}_{w_{\#I}}$.*

Proof. ( ) (Existence) That the members of $\mathcal{F}_{w_{\# I}}$ satisfy anonymity is easy to verify.
( ) (Uniqueness) For the other direction, it is sufficient to prove that for any F_w $\mathcal{F}_{w_I}$ that satisfies anonymity, its weight function meets the condition $w_I(\ _S) = w_I(\ _S)$ whenever for two power relations and and two coalitions S, S 2^N it holds $|\ _S| = |\ _S|$, which is the main characteristic of the members belonging to the family of $\mathcal{F}_{w_{\# I}}$. We do the proof in two steps. In the first step, we prove that for specific power relations and and permutations π and σ_π, $F(\ ) = \sigma_\pi(F(\ ))$ results $w_I(\ _S) = w_I(\ _S)$ for the same sized information sets $_S$ and $_S$ $(S, S$ $2^N)$ when two individuals $\{i_1, j_1\}$ N exist such that $i_1 j_1$ $_S$ and $j_1 i_1$ $_S$. In the second step, we extend the result over any two information sets $_S$ and $_S$ of the same size, given any two power relations and .

Step 1: Consider a power relation that for the coalitions $S_1, ..., S_k, S_1, ..., S_k$, with $|S_t| = |S_t|, 1 \leq t \leq k$ (k N), its informative part is $^I = \{\ _{S_1}, ..., \ _{S_k}, \ _{S_1}, ..., \ _{S_k}\}$. Also consider a set of individuals $\{i_1, j_1, i_1, j_1, ..., i\ , j\ , i\ , j\ \}$ N (N) such that $i_1, j_1, i_1, j_1 ..., i\ , j\ , i\ , j$ / $S_1, ..., S_k, S_1, ..., S_k$ and $_{S_t} = \{i_1 j_1, i_2 j_2 ..., i\ j\ \}$ and $_{S_t} = \{j_1 i_1, i_2 j_2, ..., i\ j\ \}$ for $1 \leq t \leq k$. Let's define another power relation with the informative part as $^I = \{\ _{S_1}, ... \ _{S_k}, \ _{S_1}, ..., \ _{S_k}\}$ in which $_{S_t} = \{i_1 j_1, i_2 j_2 ..., i\ j\ \}$ and $_{S_t} = \{j_1 i_1, i_2 j_2, ..., i\ j\ \}$ for $1 \leq t \leq k$.

It is easy to verify that is obtained from by applying a bijection $\pi : N \to N$ that permutes i_1 with j_1, i_2 with i_2, j_2 with j_2,...,i with i , j with j , and that keeps the coalitions the same. Since the social ranking rule F_w satisfies anonymity it means that $F_w(\ ) = \sigma_\pi(F_w(\ ))$. By definition of the power relation and also $F_w(\ )$:

$$F_w(\ ) = \underset{R\ \mathcal{L}(N)}{\text{argmax}}[w_I(\ _{S_1}) \cdot |R \cap \ _{S_1}| + ... + w_I(\ _{S_k}) \cdot |R \cap \ _{S_k}| + w_I(\ _{S_1}) \cdot |R \cap \ _{S_1}| + ... + w_I(\ _{S_k}) \cdot |R \cap \ _{S_k}|]$$

there are two possibilities for any linear order R $F_w(\ )$: either (i_1, j_1) R or (j_1, i_1) R. Let us first assume (j_1, i_1) R. We prove that $w_I(\ _S) = w_I(\ _S)$ when $i_1 j_1$ $_S$ and $j_1 i_1$ $_S$. The same result will be hold for the case where (i_1, j_1) R. From (j_1, i_1) R $\sigma_\pi(F_w(\ ))$ we have (i_1, j_1) R $F_w(\ )$, which holds when

$$w_I(\ _{S_1}) + w_I(\ _{S_2}) + ... + w_I(\ _{S_k}) \geq w_I(\ _{S_1}) + w_I(\ _{S_2}) + ... + w_I(\ _{S_k}). \tag{4.18}$$

On the other hand, since (j_1, i_1) R $F_w(\ ) = \sigma_\pi(F_w(\ ))$, by definition of $F_w(\ )$:

$$F_w(\pi(\ )) = \underset{R\ \mathcal{L}(N)}{\text{argmax}}[w_I(\ _{S_1}) \cdot |R \cap \ _{S_1}| + ... + w_I(\ _{S_k}) \cdot |R \cap \ _{S_k}| + w_I(\ _{S_1}) \cdot |R \cap \ _{S_1}| + ... + w_I(\ _{S_k}) \cdot |R \cap \ _{S_k}|]$$

we have

$$w_I(\ _{S_1}) + w_I(\ _{S_2}) + ... + w_I(\ _{S_k}) \leq w_I(\ _{S_1}) + w_I(\ _{S_2}) + ... + w_I(\ _{S_k}). \tag{4.19}$$

Since $_{S_t} = \ _{S_t}$ and $_{S_t} = \ _{S_t}$ $(1 \leq t \leq k)$, and because we have both (4.18) and (4.19) it holds that $w_I(\ _{S_1}) + \ldots + w_I(\ _{S_k}) = w_I(\ _{S_1}) + \ldots + w_I(\ _{S_k})$. For the special case of $k = 1$ we have that $w_I(\ _{S_1}) = w_I(\ _{S_1})$ when $|\ _{S_1}| = |\ _{S_1}|$ and $i_1 j_1 \ \ _{S_1}$ and $j_1 i_1 \ \ _{S_1}$. The same result holds when we assume $(i_1, j_1) \ R$.

Since the anonymity axiom is defined on the domain of $\mathcal{F}_{w_I}$, where that changes in the members of coalitions does not change the weights of coalitions, we have

$$w_I(\ _T) = w_I(\ _S) \text{ when } |\ _T| = |\ _S|, \text{ and } i_1 j_1 \ \ _T \text{ and } j_1 i_1 \ \ _S \ (S, T \ \ S^N). \quad (4.20)$$

Step 2: Consider a set $\{i_1, j_1, i_1, j_1, i_1, j_1, \ldots, i, j, i, j, i, j\} \ N$ of individuals and a power relation $\ \mathcal{B}(2^N)$ with $_S = \{i_1 j_1, i_2 j_2, \ldots, i\, j\}$ and $_T = \{j_1 i_1, i_2 j_2, \ldots, i\, j\}$, and define a permutation $\pi : N \rightarrow N$ with $\pi(i_1) = i_2, \pi(j_1) = j_2, \pi(i_2) = i_3, \pi(j_2) = j_3, \ldots, \pi(i) = i_1, \pi(j) = j_1$. Also $\pi(i_2) = i_2, \pi(j_2) = j_2 \ldots, \pi(i) = i, \pi(j) = j$. Suppose applying the permutation on the information sets of power relation yields another power relation for which $_S = \{i_1 j_1, i_2 j_2, \ldots, i\, j\}$ and $_T = \{j_2 i_2, i_2 j_2, \ldots, i\, j\}$. By Equation 4.20 we know that $w_I(\ _S) = w_I(\ _T)$ and $w_I(\ _S) = w_I(\ _T)$. Also it is easy to verify that $w_I(\ _S) = w_I(\ _S)$. These equalities result that $w_I(\ _T) = w_I(\ _T)$ when $|\ _T| = |\ _T|$. Finally, since the axiom anonymity is defined on the domain of $\mathcal{F}_{w_I}$, in which the weights of coalitions do not depend on their members, it concludes that $w_I(\ _S) = w_I(\ _T)$ wherever $|\ _S| = |\ _T|$ where $S, T \ \ 2^N$. □

Anonymity on the family of weighted CP-majority rules whose weight function depends on both coalitions and the information sets

In this part, we analyse the *anonymity* axiom as a property that distinguishes the members of the family $\mathcal{F}_{w_{(\#I,\#C)}}$ from the other members of the general family of weighted CP-majority rules $\mathcal{F}_{w_{(I,C)}}$.

Recalling the example of evaluating employees, the company president might consider all possible factors in order to aggregate the evaluations made by the committees and benefit from the social ranking rules in the family of $\mathcal{F}_{w_{(I,C)}}$. However, if the structure of the company be such that all the employees are considered to be the same, for instance, the same work experience and the same expertise, the company president can only consider the number of employees who participate in the evaluation process (like the number of members in a committee, and the number of comparison that the committee make). In these cases, members of the family $\mathcal{F}_{w_{(\#I,\#C)}}$ could be used.

In the following theorem, we characterize the family of $\mathcal{F}_{w_{(\#I,\#C)}}$ as a sub-family of $\mathcal{F}_{w_{(I,C)}}$.

Theorem 4.4.8. *The unique social ranking rules in $\mathcal{F}_{w_{(I,C)}}$ that satisfy anonymity are social ranking rules in $\mathcal{F}_{w_{(\#I,\#C)}}$.*

Proof. ( ) (Existence) That the members of $\mathcal{F}_{w_{(\#I,\#C)}}$ satisfy anonymity is easy to prove.
( ) (Uniqueness) For the other direction, we prove every ranking rule $F_w \ \mathcal{F}_{w_{(I,C)}}$ that satisfies anonymity is a member of the sub-family $\mathcal{F}_{w_{(\#I,\#C)}}$. We prove $w_{(I,C)}(\ _S, S) = w_{(I,C)}(\ _T, T)$ for two power relations and and coalitions $S, T \ \ 2^N$, when $|\ _S| = |\ _T|$ and $|S| = |T|$.

We do the proof in two steps. In the first step, we show that for specific power relations and and permutations π and σ_π, the equality $w_{(I,C)}(\ _S, S) = w_{(I,C)}(\ _T, T)$ holds for any coalitions $S, T \quad 2^N$ with $|S| = |T|$ and information sets $_S$ and $_T$ with $| \ _S| = | \ _T|$ when individuals $i_1, j_1 \quad N$ exist with $i_1 j_1 \quad _S$ and $j_1 i_1 \quad _T$. In the second step, we extend the result to any two power relations and and prove that $w_{(I,C)}(\ _S, S) = w_{(I,C)}(\ _T, T)$ when $| \ _S| = | \ _T|$ and $|S| = |T|$.

Step 1: Consider the power relation with informative part $^I = \{ \ _{S_1} \ ..., \ _{S_k}, \ _{S_1} , ..., \ _{S_k}\}$ $(k \quad$ N) where $|S_t| = |S_t|$ for $1 \leq t \leq k$, and consider a set of individuals $\{i_1, j_1, i_1, j_1, ..., i \ , j \ , i \ , j \ \} \quad N$ (N) such that $i_1, j_1, i_1, j_1..., i \ , j \ , i \ , j \quad /$ $S_1, ..., S_k, S_1, ..., S_k$ and $_{S_t} = \{i_1 j_1, i_2 j_2 ..., i \ j \ \}$ and $_{S_t} = \{j_1 i_1, i_2 j_2, ..., i \ j \ \}$ for $1 \leq t \leq k$. Also, let's define another power relation with informative part $^I = \{ \ _{S_1} \ ..., \ _{S_k}, \ _{S_1} , ..., \ _{S_k}\}$ in which $_{S_t} = \{i_1 j_1, i_2 j_2 ..., i \ j \ \}$ and $_{S_t} = \{j_1 i_1, i_2 j_2, ..., i \ j \ \}$ for $1 \leq t \leq k$.

It is possible to obtain the power relation from the power relation by defining a permutation $\pi : N \to N$ that permutes i_1 with j_1, i_2 with i_2, j_2 with j_2,...,i with i , and j with j . Also assume π permutes S_1 with S_1,...,S_k with S_k. Since we assume that F_w satisfies anonymity it holds that $F_w(\ ) = \sigma_\pi(F_w(\ ))$. According to power relation and definition of $F_w(\ )$:

$$F_w(\ ) = \underset{R \ \mathcal{L}(N)}{\text{argmax}}[w_{(I,C)}(\ _{S_1}, S_1) \cdot |R \cap \ _{S_1}|$$
$$+...+w_{(I,C)}(\ _{S_k}, S_k) \cdot |R \cap \ _{S_k}| + w_{(I,C)}(\ _{S_1}, S_1) \cdot |R \cap \ _{S_1}| + ... + w_{(I,C)}(\ _{S_k}, S_k) \cdot |R \cap \ _{S_k}|]$$

for any $R \quad F_w(\ )$ there are two possibilities: either $(i_1, j_1) \quad R$ or $(j_1, i_1) \quad R$. We prove that each possibility results in $w_{(I,C)}(\ _S, S) = w_{(I,C)}(\ _T, T)$ when $|S| = |T|$ and $| \ _S| = | \ _T|$, and individuals $\{i_1, j_1\} \quad N$ exist with $i_1 j_1 \quad _S$ and $j_1 i_1 \quad _T$. Let us first assume $(j_1, i_1) \quad R \quad \sigma_\pi(F_w(\ )) = F_w(\ )$, which respectively means $(i_1, j_1) \quad F_w(\ )$. This happens when

$$w_{(I,C)}(\ _{S_1}, S_1) + w_{(I,C)}(\ _{S_2}, S_2) + ... + w_{(I,C)}(\ _{S_k}, S_k) \geq w_{(I,C)}(\ _{S_1}, S_1) + w_{(I,C)}(\ _{S_2}, S_2) + ... + w_{(I,C)}(\ _{S_k}, S_k). \quad (4.21)$$

Also, since $(j_1, i_1) \quad F_w(\ )$, by definition $F_w(\ )$:

$$F_w(\ ) = \underset{R \ \mathcal{L}(N)}{\text{argmax}}[w_{(I,C)}(\ _{S_1}, S_1) \cdot |R \cap \ _{S_1}| + ... + w_{(I,C)}(\ _{S_k}, S_k) \cdot |R \cap \ _{S_k}|$$
$$+ w_{(I,C)}(\ _{S_1}, S_1) \cdot |R \cap \ _{S_1}| + ... + w_{(I,C)}(\ _{S_k}, S_k) \cdot |R \cap \ _{S_k}|]$$

we have

$$w_{(I,C)}(\ _{S_1}, S_1) + w_{(I,C)}(\ _{S_2}, S_2) + ... + w_{(I,C)}(\ _{S_k}, S_k) \leq w_{(I,C)}(\ _{S_1}, S_1) + w_{(I,C)}(\ _{S_2}, S_2) + ... + w_{(I,C)}(\ _{S_k}, S_k). \quad (4.22)$$

Based on the two inequalities 4.21 and 4.22, and since the information sets $_{S_t} = \ _{S_t}$ $(1 \leq t \leq k)$, we have $w_{(I,C)}(\ _{S_1}, S_1) + w_{(I,C)}(\ _{S_2}, S_2) + ... + w_{(I,C)}(\ _{S_k}, S_k) = w_{(I,C)}(\ _{S_1}, S_1) + w_{(I,C)}(\ _{S_2}, S_2) + ... + w_{(I,C)}(\ _{S_k}, S_k)$. Considering the special case where $k = 1$, it results that

$$w_{(I,C)}(\ _{S_1}, S_1) = w_{(I,C)}(\ _{S_2}, S_2) \ when \ |\ _{S_1}| = |\ _{S_2}| \ and \ |S_1| = |S_2| \qquad i_1 j_1 \quad _{S_1} \ and \ j_1 i_1 \quad _{S_2}. \tag{4.23}$$

The same result holds when we assume $(i_1, j_1) \quad R \quad F(\pi(\ )) = \pi(F(\ ))$.

Step 2: Consider a set $\{i_1, j_1, i_1, j_1, i_1, j_1, ..., i_1, j\ , i\ , j\ , i\ , j\ \} \quad N$ of individuals and a power relation $\quad \mathcal{B}(2^N)$ with informative part $^I = \{S_1, S_2\}$ such that $|S_1| = |S_2|$ and $_{S_1} = \{i_1 j_1, i_2 j_2, ..., i\ j\ \}$ and $_{S_2} = \{j_1 i_1, i_2 j_2, ..., i\ j\ \}$. Let's define a permutation $\pi : N \rightarrow N$ that permutes S_1 with S_2 and $\pi(i_1) = i_2, \pi(j_1) = j_2, \pi(i_2) = i_3, \pi(j_2) = j_3, ..., \pi(i\ ) = i_1, \pi(j\ ) = j_1$. Also $\pi(i_2) = i_2, \pi(j_2) = j_2 ..., \pi(i\ ) = i\ , \pi(j\ ) = j\ $. Suppose applying the permutation on the information sets of the power relation yields another power relation for which $_{S_2} = \{i_1 j_1, i_2 j_2, ..., i\ j\ \}$ and $_{S_1} = \{j_2 i_2, i_2 j_2, ..., i\ j\ \}$. By Equation 4.23 we know that $w_{(I,C)}(\ _{S_1}, S_1) = w_{(I,C)}(\ _{S_2}, S_2)$ and $w_{(I,C)}(\ _{S_1}, S_1) = w_{(I,C)}(\ _{S_2}, S_2)$. Also it is easy to verify that $w_{(I,C)}(\ _{S_1}, S_1) = w_{(I,C)}(\ _{S_1}, S_1)$. These equalities result that $w_{(I,C)}(\ _{S_2}, S_2) = w_{(I,C)}(\ _{S_2}, S_2)$ when $|\ _{S_2}| = |\ _{S_1}|$ and $|S_1| = |S_2|$. □

4.4.4 Separability

In this section, we analyse a property of the members of $\mathcal{F}_{w_{\#C}}$ as a sub-family of $\mathcal{F}_{w_{(\#I,\#C)}}$. Particularly, we introduce an axiom called *separability* which characterizes social ranking rules in $\mathcal{F}_{w_{\#C}}$ as a subset of social ranking rules in $\mathcal{F}_{w_{(\#I,\#C)}}$.

This property states if a set of coalitions of the same size have mutually compatible preferences over individuals, then all of them can be represented by one coalition of that size doing the comparisons, without any change in the ranking of individuals.

Definition 4.4.9 (Separability)**.** *A social ranking rule F satisfies the axiom separability iff for any two power relations $\ ,\ \quad \mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, ..., i\ , j\ \} \quad N(\quad N)$ if the two power relations are identical except for a set of coalitions $\{S_1, ..., S\ , S\}$ with $i_1, j_1, i_2, j_2, ..., i\ , j \quad / \ S_1, ..., S\ , S$ and $|S_1| = |S_2| = ... = |S\ | = |S|$ such that $_{S_1} = \{i_1 j_1\}, \ _{S_2} = \{i_2 j_2\}, ..., \ _{S} = \{i\ j\ \}, \ _{S} = $ and $_{S_1} = ... = \ _{S} = $ and $_{S} = \{i_1 j_1, ..., i\ j\ \}$, then it holds that $F(\ ) = F(\ )$.*

An explanation of the axiom is provided in the example below.

Example 32. *Consider a set $N = \{1, 2, ..., 8\}$ of individuals and assume a power relation is given over the power set of N. Suppose the informative part of the power relation is $^I = \{\ _{12}, \ _{34}\}$ such that $_{12} = \{56\}$ and $_{34} = \{78\}$. Let's assume that due to the context of the ranking problem, the decision maker ranks individuals in the set N using social ranking rules belonging to $\mathcal{F}_{w_{(\#I,\#C)}}$. The separability axiom states that the two coalitions 12 and 34 should be able to reach an agreement and choose a coalition with the same size (of size two) and* *delegate their comparisons to it. Accordingly, another power relation can be formed with the informative part as* $^{I} = \{ \; _{23}\}$ *such that* $_{23} = \{56, 78\}$*. Applying the social ranking rule satisfying separability on the power relation , gives the same result as applying the social ranking on .*

The main result of this section is a characterization theorem that uniquely characterizes the the family of social ranking rules $\mathcal{F}_{w_{\#C}}$ as a sub-family of $\mathcal{F}_{w_{(\#I,\#C)}}$ ($\mathcal{F}_{w_{\#C}} \quad \mathcal{F}_{w_{(\#I,\#C)}}$).

Theorem 4.4.10. *The unique social ranking rules in* $\mathcal{F}_{w_{(\#I,\#C)}}$ *that satisfy separability are social ranking rules in* $\mathcal{F}_{w_{\#C}}$*.*

Proof. ( )(Existence) We first prove that any member of $\mathcal{F}_{w_{\#C}}$ as F_w satisfies separability. To do so, consider any two power relations $,\quad \mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, ..., i \;, j \;\}$ N (N), and assume the two power relations are identical except for a set of coalitions $\{S_1, ..., S \;, S\}$ with $i_1, j_1, i_2, j_2, ..., i \;, j \quad / \; S_1, ..., S \;, S$ and $|S_1| = |S_2| = ... = |S \;| = |S|$ such that $_{S_1} = \{i_1 j_1\}, \; _{S_2} = \{i_2 j_2\}, ..., \; _{S} = \{i \; j \;\}, \; _{S} = \;, \; _{S_1} = ... = \; _{S} = \;$, and $_{S} = \{i_1 j_1, ..., i \; j \;\}$. We prove that for these two power relations it holds that $F_w(\;) = F_w(\;)$. Since the weight function in Equation 6.5 is not depend on the information sets, all the coalitions $S_1, ..., S \;, S$ (or, equivalently the pairs of individuals in their information sets) have the same weight. Therefore, focusing on the parts that are different in the two power relations, each pair $i_t j_t$ for $1 \leq t \leq$ is weighted the same in both $F_w(\;)$ and $F_w(\;)$. Therefore, because the other parts of the power relations are the same we conclude that for the two power relations and , as defined, we have $F_w(\;) = F_w(\;)$.

( )(Uniqueness) For the other direction, suppose a weighted CP-majority rule F_w $\mathcal{F}_{w_{(\#I,\#C)}}$ satisfies the axiom separability. We prove that it should be a member of $\mathcal{F}_{w_{\#C}}$. We do that by showing that its weight function has the property that for any two coalitions $S, S \quad 2^N$ we have $w_{(\#I,\#C)}(| \; _{S}|, |S|) = w_{(\#I,\#C)}(| \; _{S}|, |S \;|)$ when $|S| = |S \;|$. Consider a set of individuals $\{i_1, j_1, i_2, j_2, ..., i \;, j \;\} \quad N$ (N), and suppose two power relations and are given with informative parts as $^{I} = \{ \; _{S}, \; _{S_1}, \; _{S_2}, ..., \; _{S}\}$ and $^{I} = \{ \; _{S}, \; _{S}\}$ such that $|S_1| = |S_2| = ... = |S_k| = |S| = |S \;|$ and $_{S} = \{i_1 j_1\}, \; _{S_1} = \{j_1 i_1\}, \; _{S_2} = \{i_2 j_2\}, ..., \; _{S} = \{i \; j \;\}$, $_{S} = \{i_1 j_1\}$ and $_{S} = \bigcup_{i=1} \; _{S_i}$ ($_{S} = \{j_1 i_1, i_2 j_2, ..., i \; j \;\}$), while $_{S_1} = \; _{S_2} = ... = \; _{S} = \;$. Then according to separability it holds that $F_w(\;) = F_w(\;)$. Note that

$$F_w(\;) = \underset{R \;\; \mathcal{L}(N)}{\operatorname{argmax}}[w_{(\#I,\#C)}(| \; _{S}|, |S|) \cdot |R \cap \{i_1 j_1\}| + w_{(\#I,\#C)}(| \; _{S_1}|, |S_1|) \cdot |R \cap \{j_1 i_1\}|$$
$$+ w_{(\#I,\#C)}(| \; _{S_2}|, |S_2|) \cdot |R \cap \{i_2 j_2\}| + \cdots + w_{(\#I,\#C)}(| \; _{S}|, |S \;|) \cdot |R \cap \{i \; j \;\}|] \quad (4.24)$$

and

$$F_w(\;) = \underset{R \;\; \mathcal{L}(N)}{\operatorname{argmax}}[w_{(\#I,\#C)}(| \; _{S}|, |S|) \cdot |R \cap \{i_1 j_1\}|$$
$$+ w_{(\#I,\#C)}(| \; _{S}|, |S \;|) \cdot |R \cap \{j_1 i_1, i_2 j_2, ..., i \; j \;\}|] \quad (4.25)$$

According to (4.24) and as we have $w_{(\#I,\#C)}(|\ \ _S|, |S|) = w_{(\#I,\#C)}(|\ \ _{S_1}|, |S_1|)$ (since $|\ \ _S| = |\ \ _{S_1}|$ and $|S| = |S_1|$), there are some $R, R\ \ \ F_w(\ ) = F_w(\ )$ such that $(i_1, j_1)\ \ R$ and $(j_1, i_1)\ \ R$. By (4.25) this only happens when $w_{(\#I,\#C)}(|\ \ _S|, |S|) = w_{(\#I,\#C)}(|\ \ _S|, |S\ |)$. Note that in this equality $|S| = |S\ |$ and $|\ \ _S| = |\ \ _S|$. Therefore, we have proved that $w_{(\#I,\#C)}(|\ \ _S|, |S|) = w_{(\#I,\#C)}(|\ \ _S|, |S\ |)$ when $|S| = |S\ |$, no matter what is the size of their information sets. □

4.4.5 Coalition Separability

Consider the domain of all weighted CP majority rules belonging to the family of $\mathcal{F}_{w_{(\#I,\#C)}}$. In this section, we analyse the properties of the members of the sub-family $\mathcal{F}_{w_{\#I}}$. Specially, we define an axiom called *coalition separability* that uniquely characterizes the members of $\mathcal{F}_{w_{\#I}}$ in the domain of $\mathcal{F}_{w_{(\#I,\#C)}}$.

There is an intuitive interpretation for the axiom *coalition separability*: considering the social ranking rules in the domain of $\mathcal{F}_{w_{(\#I,\#C)}}$, if overlapped coalitions have the same preferences over a set of individuals, then all these coalitions should be able to reduce their size by removing the repeated individuals in the coalitions, while the ranking result does not change.

The formal definition of the axiom is given as below.

Definition 4.4.11 (Coalition Separability)**.** *A social ranking rule F satisfies coalition separability iff for any two power relations $\ ,\ \ \ \mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, ..., i\ , j\ \}\ \ \ N$, if the two power relations are identical except for $\{S_1, S_2, ..., S_k, S_1 \setminus S, ..., S_k \setminus S\}$ with $\bigcap_{t=1}^{k} S_t = S$ such that $\ _{S_1} = \ _{S_2} = ... = \ _{S_k} = \{i_1 j_1, ..., i\ j\ \}$ and $\ _{S_1 \setminus S} = ... = \ _{S_k \setminus S} = \ ,\ \ _{S_1} = ... = \ _{S_k} = \ $, and $\ _{S_1 \setminus S} = ... = \ _{S_k \setminus S} = \{i_1 j_1, ..., i\ j\ \}$, then it holds that $F(\ ) = F(\ )$.*

The following example illustrates the use of the axiom.

Example 33. *Consider a set of eight individuals $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and suppose the power relation is given with the informative part $^I = \{\ _{12},\ _{134},\ _{14}\}$, for which $\ _{12} = \ _{134} = \ _{14} = \{56, 67, 57\}$. Let's assume a social ranking rule F, from the domain of $\mathcal{F}_{w_{(\#I,\#C)}}$, satisfies coalition separability. If coalitions 12, 134 and 14 decide to reduce their size by removing individual 1 (that is repeated) and to form a power relation with the informative part as $^I = \{\ _2,\ _{34},\ _4\}$ such that $\ _2 = \ _{34} = \ _4 = \{56, 67, 57\}$, then applying F on the power relation should provide the same social ranking as when it applies on .*

The following theorem validates the main result of this section, and characterizes the members of the family $\mathcal{F}_{w_{\#I}}$ as a sub-family of $\mathcal{F}_{w_{(\#I,\#C)}}$.

Theorem 4.4.12. *The only social ranking rules in the family $\mathcal{F}_{w_{(\#I,\#C)}}$ that satisfy coalition separability are $\mathcal{F}_{w_{\#I}}$. ($\mathcal{F}_{w_{\#I}}\ \ \ \mathcal{F}_{w_{(\#I,\#C)}}$).*

Proof. ( )(Existence) Consider any member $F_w\ \ \ F_{w_{\#I}}$. We first prove that it satisfies coalition separability. To this aim, consider two power relations $\ ,\ \ \ \mathcal{B}(2^N)$ such that for any set of individuals $\{i_1, j_1, ..., i\ , j\ \}\ \ \ N$ (N), power relations and are identical except for

$\{S_1, S_2, ..., S_k, S_1 \setminus S, ..., S_k \setminus S\}$ where $\bigcap_{t=1}^{k} S_t = S =$. Suppose $_{S_1} = {}_{S_2} = ... = {}_{S_k} = \{i_1 j_1, ..., i\ j\}$ and $_{S_1 \setminus S} = ... = {}_{S_k \setminus S} =$. Also assume $_{S_1} = ... = {}_{S_k} =$ and $_{S_1 \setminus S} = ... = {}_{S_k \setminus S} = \{i_1 j_1, ..., i\ j\}$. We prove that $F_w(\ ) = F_w(\ )$. Since the social ranking rule F_w belongs to $F_{w_{\# I}}$, the weight function in Equation 6.4 does not depend on the sizes of coalitions. Let's focus on the parts that the two power relations are different. Since coalitions have the same information sets, each pair of individuals $i_t j_t$ $(1 \leq t \leq \ )$ is weighted the same in both power relations: in power relation its weight is $k \times C$ where C indicates the same weight assigned to each coalition $S_1, S_2, ...S_k$. In the same way, in power relation , the pair of individuals has the weight $k \times C$ where C indicates the weight of coalitions $S_1 \setminus S, ..., S_k \setminus S$. Also the pairs have the same weight in the other parts of power relations. Therefore, for any two described power relations and we have $F_w(\ ) = F_w(\ )$.

( )(Uniqueness) For the other direction, suppose a member F_w $\mathcal{F}_{w_{(\# I, \# C)}}$ satisfies coalition separability. We prove that its weight function meets the condition that for any two coalitions S, T 2^N it holds $w_{(\# I, \# C)}(|\ _S|, |S|) = w_{(\# I, \# C)}(|\ _T|, |T|)$ when $|\ _S| = |\ _T|$. In order to do that, consider two power relations , $\mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, i_1, j_1, ..., i\ , j\ , i\ , j\ \}$ N. Assume the informative parts for is $^I = \{\ _{S_1}, ..., \ _{S_k}, \ _{S_1}, ..., \ _{S_k}\}$, $|S_t| = |S_t|$ for t $\{1, ..., k\}$, and $_{S_1} = ... = {}_{S_k} = \{i_1 j_1, i_2 j_2, ..., i\ j\}$ and $_{S_1} = ... = {}_{S_k} = \{j_1 i_1, i_2 j_2, ..., i\ j\}$. Also suppose the informative part for the power relation is given by $^I = (\ _{S_1 \setminus S}, ..., \ _{S_k \setminus S}, \ _{S_1}, ..., \ _{S_k})$, $\bigcap_{t=1}^{k} S_t = S =$, such that $_{S_1 \setminus S} = {}_{S_2 \setminus S} = ... = {}_{S_k \setminus S} = \{i_1 j_1, ..., i\ j\}$ and $_{S_1} = ... = {}_{S_k} = \{j_1 i_1, i_2 j_2, ..., i\ j\}$. Since F_w satisfies coalition separability, it holds that $F_w(\ ) = F_w(\ )$. Also, by definition of $F(\ )$, in Equation 6.3, and since coalitions S_t and S_t $(1 \leq t \leq \ )$ have the same weights $(|S_t| = |S_t|$ and $|\ _{S_t}| = |\ _{S_t}|$, $1 \leq t \leq \ )$, for some R, R $F(\ ) = F(\ )$, we have (i_1, j_1) R and (j_1, i_1) R (this is because R and R are linear orders). This happens only when $w_{(\# I, \# C)}(|\ _{S_1 \setminus S}|, |S_1 \setminus S|) + w_{(\# I, \# C)}(|\ _{S_2 \setminus S}|, |S_2 \setminus S|) + ... + w_{(\# I, \# C)}(|\ _{S_k \setminus S}|, |S_k \setminus S|) = w_{(\# I, \# C)}(|\ _{S_1}|, |S_1|) + ... + w_{(\# I, \# C)}(|\ _{S_k}|, |S_k|)$. For the special case when $k = 1$ it holds that $w_{(\# I, \# C)}(|\ _{S_1 \setminus S}|, |S_1 \setminus S|) = w_{(\# I, \# C)}(|\ _{S_1}|, |S_1|)$ in which $|\ _{S_1 \setminus S}| = |\ _{S_1}|$. Since $S_1 \setminus S$ and S_1 are totally different coalitions, in general we have proved that for any two coalitions S, T 2^N it holds $w_{(\# I, \# C)}(|\ _S|, |S|) = w_{(\# I, \# C)}(|\ _T|, |T|)$ when $|\ _S| = |\ _T|$. □

4.4.6 Coalition Merging

In this section, we consider the set of all social ranking rules in $\mathcal{F}_{w_{\# C}}$, and we look for a property that distinguishes the member $F^I_{w_{\# C}}$ from the other members of the family. The incentive to use $F^I_{w_{\# C}}$ can be seen in many practical ranking problems. In the example of employees' evaluation, suppose the ranking setting, conducted by the company president, forces committees to use majority voting in order to aggregate the opinion of their members to evaluate employees. In this case, we can expect that the company president assigns more weight to committees of bigger size , because she believes their evaluations are supported by more number of employees (the members of the committee). On possible social ranking rule to be used in such cases is $F^I_{w_{\# C}}$.

The axiom that we introduce in this section is called *coalition merging*. This axiom refers to the idea that if a group of coalitions have the same preferences, then before starting the ranking process they can decide to join together and form a single coalition with the same preferences as before. If a social ranking rule satisfies *coalition merging*, then the coalitions expect that their decision (to merge together) does not change the ranking result.

The axiom *coalition merging* is formally defined as follows.

Definition 4.4.13 (Coalition Merging)**.** *A social ranking F satisfies coalition merging iff for any two power relations , $\mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, i_2, j_2, ..., i\ , j\ \}$ N, N if the two power relations are identical except for a set $\{S_1, ..., S_k, \ {}_{i=1}^{k}S_i\}$ such that $S_1, ..., S_k$ are disjoint coalitions of the same size and $i_1, j_1, i_2, j_2, ..., i\ , j$ / $S_1, ..., S_k$, ${}_{i=1}^{k}S_i$ = and ${}_{S_1}$ = ... = ${}_{S_k}$ = $\{i_1j_1, ..., i\ j\ \}$ while ${}_{S_1}$ = ... = ${}_{S_k}$ = and ${}_{i=1}^{k}S_i$ = $\{i_1j_1, ..., i\ j\ \}$ then it holds that $F(\ ) = F(\ )$.*

An example of using this axiom is provided below.

Example 34. *Consider a set of eight individuals $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and suppose the power relation $\mathcal{B}(2^N)$ is given with the informative part ${}^I = \{\ {}_{12},\ {}_{4},\ {}_{56}\}$ such that ${}_{12} = {}_{4} = {}_{56} = \{37, 38, 78\}$. If we use the social ranking rule F $\mathcal{F}_{w_{\#C}}$ which satisfies the coalition merging axiom, then merging the coalitions* 12, 4, *and* 56 *in order to form a new power relation with the informative part as ${}^I = \{\ {}_{12456}\}$ with ${}_{12456} = \{37, 38, 78\}$ should not change the social ranking of individuals.*

The following theorem verifies that the weighted CP-majority rule $F^I_{w_{\#C}}$ can be characterised using coalition merging as a member of $\mathcal{F}_{w_{\#C}}$.

Theorem 4.4.14. *The only member of the family $\mathcal{F}_{w_{\#C}}$ satisfying coalition merging is a weighted CP-majority rule $F^I_{w_{\#C}}$ ($F^I_{w_{\#C}}(\ ) = \operatorname{argmax}_{R\ \mathcal{L}(N)} \sum_{S\ 2^N} |S| \cdot |R \cap\ {}_S|$).*

Proof. ( )(Existence) We first prove that $F^I_{w_{\#C}}$ satisfies coalition merging. For this purpose, suppose two power relations and are given, and assume they are identical except for the set $\{S_1, S_2, ..., S_k, \ {}_{i=1}^{k}S_i\}$ of coalitions in which $S_1, S_2, ..., S_k$ are disjoint coalitions of the same size. Also, for a set of individuals $\{i_1, j_1, i_2, j_2, ..., i\ , j\ \}$ N, N suppose we have ${}_{i=1}^{k}S_i$ = and ${}_{S_1}$ = ... = ${}_{S_k}$ = $\{i_1j_1, i_2j_2, ..., i\ j\ \}$ while ${}_{S_1}$ = ... = ${}_{S_k}$ = and ${}_{i=1}^{k}S_i$ = $\{i_1j_1, ..., i\ j\ \}$, we prove that $F^I_{w_{\#C}}(\ ) = F^I_{w_{\#C}}(\ )$. Referring to the definition of $F^I_{w_{\#C}}(\ )$ in 6.6, the weight function is $w_{\#C}(|S|) = |S|$. Let's focus on parts of the two power relations that are different. Each pair of individuals i_tj_t ($1 \leq t \leq\ $) has the same weight in both power relations and : suppose the size of coalitions $S_1, ..., S_k$ is equal to n. In the power relation each pair of individuals shows up in any of the k coalitions, and therefore its weight is $k \times n$. Also in the power relation each pair shows up only in the coalition ${}_{i=1}^{k}S_i$ whose size is equal to $k \times n$. Therefore each pair in is weighted $k \times n$ as well. Note that in the other parts of the power relations, that are the same, each pair weights the same in both power relation. Therefore, for any two power relation and we have $F^I_{w_{\#C}}(\ ) = F^I_{w_{\#C}}(\ )$.

( )(Uniqueness) For the other direction, consider two power relations and such that the informative part of is the set $\{ _{S_1}, ..., _{S_k}, _{S_1}, ..., _{S_k}\}$ for which $\{i_1j_1, i_2j_2, ..., i\ j\ \} = _{S_1} = _{S_2} = ... = _{S_k}$, $|S_1| = ... = |S_k| = |S_1| = ... = |S_k| = 1$ and $\{j_1i_1, i_2j_2, ..., i\ j\ \} = _{S_1} = ... = _{S_k}$. Also assume the informative part of the power relation is given as $\{ _{n=1}^{k} S_n, _{S_1}, ..., _{S_k}\}$ such that $_{n=1}^{k} S_n = \{i_1j_1, i_2j_2, ..., i\ j\ \}$ and $_{S_n} = _{S_n}$ for $n \quad \{1, ..., k\}$. Since we assumed that the weighted CP-majority rule $F_w \quad \mathcal{F}_{w_{\#C}}$ satisfies coalition merging then $F_w(\ ) = F_w(\ )$. Also because the pairs i_1j_1 and j_1i_1 exist in the power relation and since in $F_w(\ )$ we have $w_C(S_t) = w_C(S_t)(1 \leq t \leq k)$, linear orders $R, R \quad F_w(\ ) = F_w(\ )$ are plausible such that $\{(i_1, j_1), ..., (i\ j\ )\} \quad R$ and $\{(j_1i_1), (i_2, j_2), ..., (i\ j\ )\} \quad R$. This happens when in $F_w(\ )$ we have $w_{\#C}(|S_1|) + w_{\#C}(|S_2|) + ... + w_{\#C}(|S_k|) = w_{\#C}(| _{n=1}^{k} S_i|)$, which can be summarized as $k \times w_{\#C}(|S_1|) = w_{\#C}(| _{n=1}^{k} S_i|)$. Since $|S_1| = 1$ it holds that $w_{\#C}(| _{n=1}^{k} S_i|) = | _{n=1}^{k} S_i|$. □

4.4.7 Splitting

In this section, we consider the social ranking rule $F^p_{w_{\#I}}$ as defined in Equation 6.7. What motivates the use of such social ranking rules is the ranking scenarios in which a coalition that made larger number of comparisons (the bigger size of its information set) deserves to have a smaller weight. An example is the employees' evaluation when the larger number of comparisons made by a committee may indicate that the committee does comparisons regardless of its expertise. In the same way, in such scenarios a committee that does less number of comparisons merits to have a bigger weight.

In this section, we restrict our attention to the social ranking rules that weight coalitions according to the size of their information sets ($\mathcal{F}_{w_{\#I}}$), and we define an axiom that characterizes the particular social ranking rule $\mathcal{F}^p_{w_{\#I}}$. The axiom is called *splitting* axiom. Based on this axiom, the coalitions that have compatible preferences (information sets) over individuals, can agree on to have the same preferences as the union of their compatible information sets.
This axiom is formally defined as below.

Definition 4.4.15 (Splitting). *A ranking rule F satisfies splitting if and only if for any two given power relations , $\mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, i_2, j_2, ..., i\ , j\ \} \quad N, \quad N$ if the two power relations are identical except for a set of coalitions of the same size $\{S_1, ..., S\ \}$ such that $i_1, j_1, i_2, j_2, ..., i\ , j \quad / \ S_1, ..., S$ and $\{i_1j_1\} = _{S_1}, \{i_2j_2\} = _{S_2}, ..., \{i\ j\ \} = _S$ while $\{i_1j_1, i_2j_2, ..., i\ j\ \} = _{S_1} = ... = _S$ then it holds that $F(\ ) = F(\ )$.*

Example 35. *Consider a set N of individuals $N = \{1, 2, 3, ..., 7\}$ and suppose the power relation is given with the informative part of $^I = \{ _{12}, _{34}, _{23}\}$ in which $_{12} = \{56\}$, $_{34} = \{67\}$, and $_{23} = \{57\}$. Since the coalitions have compatible preferences over the set of individuals, they might reach a pre-agreement to all indicate the same preferences as the union of their current preferences, i.e., they transform the power relation to with the informative part as $^I = \{ _{12}, _{34}, _{23}\}$ in which $_{12} = _{34} = _{23} = \{56, 67, 57\}$. If the ranking rule $F \quad \mathcal{F}_{w_{\#I}}$ satisfies splitting, then applying it on both power relations should give the same social ranking of the individuals.*

Now we can present the main result of this section to characterize $\mathcal{F}^p_{w_{\#I}}$ as a member of $\mathcal{F}_{w_{\#I}}$.

Theorem 4.4.16. *The only weighted ranking rule of the family $\mathcal{F}_{w_{\#I}}$ that satisfies splitting is $F^p_{w_{\#I}}$ ($F^p_{w_{\#I}}(\ ) = \underset{R \ \mathcal{L}(N)}{\text{argmax}} \sum_{S \ 2^N} \frac{1}{| \ _S|} \cdot |R \cap \ _S|$).*

Proof. ( )(Existence) Suppose F_w is the weighted CP-majority rule $F^p_{w_{\#I}}$, we will prove that it satisfies splitting. Consider two power relations and and a set of individuals $\{i_1, j_1, i_2, j_2, ..., i \ , j \ \}$ N. Let's assume the two power relations are identical except for a set $\mathcal{S} = \{S_1, ..., S \ \}$ such that the information sets of coalitions in the set $\mathcal{S}$ in power relation are disjoint singleton pairs of individuals, i.e., $_{S_1} = \{i_1 j_1\}$, $_{S_2} = \{i_2 j_2\}, ...,$ $_S = \{i \ j \ \}$. Also suppose for the power relation , the information set of each coalition S $\mathcal{S}$ is the union of all the information sets in the set $\mathcal{S}$, i.e., $_S = \{i_1 j_1, i_2 j_2, ..., i \ j \ \}$ for all S $\mathcal{S}$. By the definition of the weight function for $F^p_{w_{\#I}}$, we have that $w_{\#I}(| \ _S|) = \frac{1}{| \ _S|}$, we know that each pair of individual in each power relation has the same weight. Focusing on the parts of the two power relation that are different, consider a pair of individuals $i_t j_t$, $1 \le t \le$. This pair has the weight of one in the power relation since it shows up only in one coalition S whose information set has the size of one. Also it has the weight of one in the power relation because it appears in information sets corresponding to the coalitions in $\mathcal{S}$ each one with weight of $\frac{1}{}$. Therefore we conclude that $F_w(\ ) = F_w(\ )$.

( )(Uniqueness) For the other direction, we prove that if a weighted ranking rule (F_w $\mathcal{F}_{w_{\#I}}$) satisfies splitting axiom then it should be $F^p_{w_{\#I}}$. Suppose the two power relations , $\mathcal{B}(2^N)$ and a subset $\{i_1, j_1, i_2, j_2, ..., i \ , j \ \}$ N of individuals are given. Now assume the informative part of power relation is $^I = \{ \ _S, \ _{S_1}, ..., \ _S \}$ with $_S = \{i_1 j_1\}$, $_{S_1} = \{j_1 i_1\}$, $_{S_2} = \{i_2 j_2\}, ...,$ $_S = \{i \ j \ \}$. Also consider power relation with informative part as $^I = \{ \ _S, \ _{S_1}, ..., \ _S \}$ with $_S = \{i_1 j_1\}$ and $\{j_1 i_1, i_2 j_2, ..., i \ j \ \} = \ _{S_1} = ... = \ _S$. First since the ranking rule satisfies splitting then it holds that $F_w(\ ) = F_w(\ )$. Also as both pairs $i_1 j_1$ and $j_1 i_1$ are materialized in the two power relations and since they have the same weight value in the power relation we must have R, R $F_w(\ ) = F_w(\ )$ for some $\{i_1 j_1, i_2 j_2, ..., i \ j \ \}$ R and $\{j_1 i_1, i_2 j_2, ..., i \ j \ \}$ R. This forces the weight function to satisfy $w_{\#I}(| \ _S|) = w_{\#I}(| \ _{S_1}|) + ... + w_{\#I}(| \ _S|)$ which equivalently means $w_{\#I}(1) = \underbrace{w_{\#I}(\ ) + ... + w_{\#I}(\ )}$ or $w_{\#I}(1) = \ \cdot w_{\#I}(\ )$, by the assumption that the weights are normalized ($w_{\#I}(1) = 1$) it concludes the proof. □

4.4.8 Constant-Weighted Rank Rule

As we have seen, ranking individuals based on pair-wise comparisons is the most intuitive way to rank individuals. However, reckless use of it over the set of all binary relations often causes the problematic ranking results [De Condorcet, 1785]. In Chapter 2, the concept of *social single peakedness* is introduced as a restriction over the domain of all binary relations to guarantee transitivity of the final ranking. In this section, the axiom *restricted majoritarianism*, emphasizes that the social ranking of interest should go by the concept of majority on the limited scope of

independence. In the example of employees' evaluation suppose there are two employees, let's say 1 and 2, who get compared to each others and not to the others (they might have different skills than others and it does not make sense to compare them with the others). In this case, if the number of times that 1 is evaluated better than 2 is larger than the number of times 2 is evaluated better than 1, then 1 should be ranked higher than 2.
The main result of this section indicates that the *restricted majoritarianism* characterizes the weighted CP-majority rule $\mathcal{F}^{p}_{w_{\#I}}$ as a member of $\mathcal{F}_{w_{\#I}}$.
Before formally introducing this axiom we need some preliminary definitions:

Definition 4.4.17 (Independence)**.** *Individuals i, j N are independent of the power relation if for any information set $_S, S$ $2^{N\setminus\{i,j\}}$, we have ij $_S$ or ji $_S$ but ik, ki, jk, kj / $_S$ for $k = i, j$ and S 2^N.*

Definition 4.4.18 (CP-majority set)**.** *The CP-majority set of a power relation is a set $m_{cp}($ $) = \{ij|$ $\sum_{S\ 2^N} d^S_{ij}($ $) >$ $\sum_{S\ 2^N} d^S_{ji}($ $)\}$*

Reminder: Note that $d^S_{ij}($ $)$ is one if i S j S, it is minus one if j S i S and it is zero if i S j S in power relation .

Definition 4.4.19 (Restricted Majoritarianism)**.** *A weighted rank rule $\mathcal{F}_w$ satisfies restricted majoritarianism iff for any set N of individuals and power relation $\mathcal{B}(2^N)$ and all alternatives i, j N that are independent of it holds that: if ij $m_{cp}($ $)$ then ij R for all R $\mathcal{F}_w($ $)$.*

Theorem 4.4.20. *The only member of the family $\mathcal{F}_{w_{\#I}}$ that satisfies restricted majoritarianism coincides with $F^c_{w_{\#I}}$ ($F^c_{w_{\#I}}($ $) = \operatorname{argmax}_{R\ \mathcal{L}(N)} \sum_{S\ 2^N} |R \cap$ $_S|$).*

Proof. ( )(Existence) That the weighted CP-majority rule $F^c_{w_{\#I}}$ satisfies restricted majoritarianism is easy to verify.

( )(Uniqueness) For the other direction, consider an arbitrary member of the family of $\mathcal{F}_{w_{\#I}}$ which satisfies restricted majoritarianism. Let's show the member as F_w, to prove that F_w coincides with $F^c_{w_{\#I}}$ it suffices to show that its weight function $w_{\#I}$ satisfies the equality $w_{\#I}($ $) = w($ $)$ for all , N.

Let's assume > and suppose for a sufficiently large set of individuals N we have $i_1, j_1, i_2, j_2, ..., i\ , j$ / $S_1, ..., S_k, S_1, ..., S_{k+1}$ for k N and also assume the power relation with informative part of $^I = \{$ $_{S_1}, ...,$ $_{S_k},$ $_{S_1}, ...,$ $_{S_{k+1}}\}$ is given such that $_{S_t} = \{i_1 j_1, i_2 j_2, ..., i\ j\ \}$ for t $\{1, ..., k\}$ and $_{S_t} = \{j_1 i_1, i_2 j_2, ..., i\ j\ \}$ for all t $\{1, ..., k+1\}$. Clearly, i_1, j_1 are independent of the power relation and $j_1 i_1$ $m_{cp}($ $)$, therefore by restricted majoritarianism it holds that $j_1 i_1$ R for all R $F_w($ $)$, which implies that $w_{\#I}(|$ $_{S_1}$ $|) + ... + w_{\#I}(|$ $_{S_k}$ $|) < w_{\#I}(|$ $_{S_1}$ $|) + ... + w_{\#I}(|$ $_{S_{k+1}}$ $|)$. Since $|$ $_{S_i}$ $| = |$ $_{S_j}$ $| =$ for all i, j $\{1, ..., k\}$ and also $|$ $_{S_i}$ $| = |$ $_{S_j}$ $| =$ for all i, j $\{1, ..., k+1\}$ we can shorten the inequality by writing $\frac{w}{w} < 1 + \frac{1}{k}$. Letting k goes to infinity (by considering very large set N of individuals) results that

$$\frac{w}{w} < 1 + \quad , 0 < \quad < 1 \tag{4.26}$$

Now consider another power relation with the informative part as $^{I} = \{ \; _{S_1}, ..., \; _{S_{k+1}}, ..., \; _{S_1}, ..., \; _{S_k}\}$ for which $_{S_t} = \{i_1 j_1, i_2 j_2, ..., i \; j \; \}$ for $t \; \{1, ..., k+1\}$ and $_{S_t} = \{j_1 i_1, i_2 j_2, ..., i \; j \; \}$ for $t \; \{1, ..., k\}$. Clearly, in this power relation i_1, j_1 are independent of the power relation , and also $i_1 j_1 \; m_{cp}(\;)$. Again, by restricted majoritarianism it holds that $i_1 j_1 \; R$ for all $R \; F_w(\;)$. This implies that $w_{\#I}(\; _{S_1}) + ... + w_{\#I}(\; _{S_{k+1}}) > w_{\#I}(\; _{S_1}) + ... + w_{\#I}(\; _{S_k})$. Since $| \; _{S_i}| = | \; _{S_j}|$ for all $i, j \; \{1, ..., k+1\}$ and $| \; _{S_i}| = | \; _{S_j}|$ for all $i, j \; \{1, ..., k\}$ the inequality can be written as $\frac{w}{w} > 1 - \frac{1}{k+1}$. Again letting k goes to infinity (by considering very large set N of individuals) results that

$$\frac{w}{w} > 1 - \; , 0 < \; < 1 \tag{4.27}$$

Note that the size of each information set for $\{ \; _{S_1}, ..., \; _{S_k}\}$ is equal to the size of each information set for $\{ \; _{S_1}, ..., \; _{S_{k+1}}\}$, and the same equality holds for each information set for $\{ \; _{S_1}, ... \; _{S_{k+1}}\}$ and $\{ \; _{S_1}, ..., \; _{S_k}\}$. Satisfying restricted majoritarianism for the weighted rank rule means that both (4.26) and (4.27) hold and this can happen when $w_{\#I}(\;) = w_{\#I}(\;)$ for any $\; > \; , \; , \;$ N. Symmetrically the same result holds for $\; > \; , \; , \;$ N. By the definition of the weight function we conclude that $w_{\#I}(\;) = w_{\#I}(\;)$ for all $\; , \;$ N.

□

4.4.9 Combination of Splitting and Coalition merging

In this section, we want to characterize the social ranking rule $\mathcal{F}^p_{w_{(\#I,\#C)}}$ as a member of the family of social ranking rules $\mathcal{F}_{w_{(\#I,\#C)}}$. The main incentive to study the members of the family of $\mathcal{F}_{w_{(\#I,\#C)}}$ is that in our context, considering coalitions as voters, each voter has a different set of alternatives to rank. Particularly, coalitions of bigger size have less alternative to compare, while coalitions of smaller size have a larger number of alternatives to compare. Therefore, considering just the sizes of information sets, as mentioned in Section 4.4.7, cannot truly indicate the degree to which coalitions are experienced or the extent to which coalitions do relevant comparisons. In this case, the social ranking rules that weight coalitions based on their size and size of their information sets can provide an appropriate ranking over the individuals. In some specific ranking scenarios, like in the example of employees' evaluation, the company president may want to assign more weight to bigger coalitions that do less number of comparisons. The justification of this weighting is that coalitions of bigger size doing small number of comparisons may do it with more deliberation. In such cases, the social ranking rule $F^p_{w_{(\#I,\#C)}}$ is worth to be considered.

The main result of this section is a characterization theorem in which we utilize the previously mentioned axioms *coalition merging* and *splitting* in order to characterize the social ranking rule $F^p_{w_{(\#I,\#C)}}$ as a member of $\mathcal{F}_{w_{(\#I,\#C)}}$.

Theorem 4.4.21. *The unique social ranking rule in $\mathcal{F}_{w_{(\#I,\#C)}}$ that satisfies splitting and coalition merging is $F^p_{w_{(\#I,\#C)}}$ ($F^p_{w_{(\#I,\#C)}}(\;) = \operatorname{argmax}_{R \; \mathcal{L}(N)} \sum_{S \; 2^N} \frac{|S|}{| \; _S|} \cdot |R \cap \; _S|$).*

Proof. ( )(Existence) In this part we prove that the social ranking rule $F^p_{w_{(\#I,\#C)}}$ satisfies the axioms splitting and coalition merging. We first prove that the social ranking rule satisfies Splitting. To do so, consider two power relations , $\mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, i_2, j_2, ..., i\ , j\ \}$ N, N, and suppose the two power relations are identical except for a set of coalitions of the same size $\{S_1, ..., S\ \}$ such that $i_1, j_1, i_2, j_2, ..., i\ , j$ / $S_1, ..., S$ and $\{i_1 j_1\} = {}_{S_1}, \{i_2 j_2\} = {}_{S_2}, ..., \{i\ j\ \} = {}_{S}$ while $\{i_1 j_1, i_2 j_2, ..., i\ j\ \} = {}_{S_1} = ... = {}_{S}$. Let's focus on parts of the two power relations that are different. In Equation 6.8 the weight function is defined as $w_{(\#I,\#C)}(|\ {}_S|, |S|) = \frac{|S|}{|\ {}_S|}$. The coalitions in the power relations have the same size, and let's indicate it with constant value C. Each pair of individuals $i_t j_t$ $(1 \leq t \leq\ )$ in power relation is weighted by C since the information sets of coalitions have the size of one. Also each pair of individuals in the power relation is also weighted by C because each pair $i_t j_t$ shows up times (in coalitions), and its weight in each coalition is $\frac{C}{}$. Also the weights of the coalitions are the same in the other parts of the power relation. Therefore, each pair of individuals $i_t j_t$ $(1 \leq t \leq\ )$ has the same size in both power relations and, as a result, it holds that $F^p_{w_{(\#I,\#C)}}(\ ) = F^p_{w_{(\#I,\#C)}}(\ )$.

Now let's consider two power relations , $\mathcal{B}(2^N)$ and a set of individuals $\{i_1, j_1, i_2, j_2, ..., i\ , j\ \}$ N, N also suppose the two power relations are identical except for a set $\{S_1, ..., S_k, \ {}^k_{i=1} S_i\}$, where $S_1, S_2, ..., S_k$ are disjoint coalitions with the same sizes. Also suppose $i_1, j_1, i_2, j_2, ..., i\ , j$ / $S_1, ..., S_k,$ ${}_{{}^k_{i=1} S_i} =$ and ${}_{S_1} = ... = {}_{S_k} = \{i_1 j_1, ..., i\ j\ \}$ while ${}_{S_1} = ... = {}_{S_k} =$ and ${}_{{}^k_{i=1} S_i} = \{i_1 j_1, ..., i\ j\ \}$. We indicate that the weighted CP-majority rule $F^p_{w_{(\#I,\#C)}}$ satisfies coalition merging by proving that $F^p_{w_{(\#I,\#C)}}(\ ) = F^p_{w_{(\#I,\#C)}}(\ )$. According to Equation (6.8), each pair of individuals $i_t j_t$ $(1 \leq t \leq\ )$ in power relation is weighted by $k \times \frac{C}{}$ where C is the size of coalitions $S_1, ..., S_k$. This is because each pair shows up in k different coalitions, and the weight of each coalition is $\frac{C}{}$. Also the weight of each pair in the power relation is equal to $\frac{k \times C}{}$ since each pair shows up only in the coalition ${}^k_{i=1} S_i$ whose size is $k \times C$. Also the pairs of individuals are weighted the same in other parts of the power relations. As a result, for the given power relations we have that $F^p_{w_{(\#I,\#C)}}(\ ) = F^p_{w_{(\#I,\#C)}}(\ )$.

( )(Uniqueness) For the other direction, consider two power relations , $\mathcal{B}(2^N)$ and a set of individuals $i_1, j_1, i_2, j_2, ..., i\ , j$ N. Let's assume F_w $F_{w_{(\#I,\#C)}}$ satisfies the two axioms, we prove that it coincides with $F^p_{w_{(\#I,\#C)}}$ and suppose the power relation is given with the informative part as ${}^I = \{\ {}_{S_{11}}, ..., {}_{S_{1k}}, {}_{S_{21}}, ..., {}_{S_{2k}}, ..., {}_{S\ 1}, ..., {}_{S\ k}, {}_{S}\}$ for $k,$ N for which $|S_{11}| = ... = |S_{1k}| = ... = |S\ _1| = ... = |S\ _k| = 1, |S| = k$ and ${}_{S_{11}} = ... = {}_{S_{1k}} = \{i_1 j_1\}$, ${}_{S_{21}} = ... = {}_{S_{2k}} = \{i_2 j_2\}$,..., ${}_{S\ 1} = ... = {}_{S\ k} = \{i\ j\ \}$ and ${}_{S} = \{j_1 i_1\}$. Also consider the power relation with informative part ${}^I = \{\ {}_{S_1}, ..., {}_{S}, {}_{S}\}$ such that $S_1 = \bigcup_{t=1}^{k} S_{1t}, ..., S\ = \bigcup_{t=1}^{k} S\ _t$ with ${}_{S_1} = \{i_1 j_1\}$, ${}_{S_2} = \{i_2 j_2\}, ...,$ ${}_{S} = \{i\ j\ \}$ and ${}_{S} = \{j_1 i_1\}$. Since F_w satisfies coalition merging, it holds that $F_w(\ ) = F_w(\ )$. Both pairs $i_1 j_1$ and $j_1 i_1$ are materialized in the power relation and since $w_{(\#I,\#C)}(|\ {}_{S_1}|, |S_1|) = w_{(\#I,\#C)}(|\ {}_S|, |S|)$, for some R, R $F_w(\ ) = F_w(\ )$ we have $i_1 j_1$ R and $j_1 i_1$ R. This happens only when

$\underbrace{w_{(\#I,\#C)}(1,1) + \ldots + w_{(\#I,\#C)}(1,1)}_{k} = w_{(\#I,\#C)}(1,k)$ or simply

$$k \times w_{(\#I,\#C)}(1,1) = w_{(\#I,\#C)}(1,k). \tag{4.28}$$

Now consider another power relation $= (_{S_1}, _{S_2}, \ldots, _{S}, _{S})$ such that $_{S_1} = \ldots = _{S} = \{i_1 j_1, i_2 j_2, \ldots, i\ j\ \}$, and $_{S} = \{j_1 i_1\}$. Again since F_w satisfies splitting and coalition merging, on one hand it holds that $F_w(\ ) = F_w(\ )$, and on the other hand $F_w(\ ) = F_w(\ )$. Therefore we also have $F_w(\ ) = F_w(\ )$.
We know that for some $R, R \quad F_w(\ )$ we have $i_1 j_1 \quad R$ and $j_1 i_1 \quad R$, this is also the case for $F_w(\ )$ (for some $R, R \quad F_w(\ )$, we have $i_1 j_1 \quad R$ and $j_1 i_1 \quad R$). This holds only when $\underbrace{w_{(\#I,\#C)}(\ ,k) + \ldots + w_{(\#I,\#C)}(\ ,k)}_{} = w_{(\#I,\#C)}(1,k)$ or simply $\times w_{(\#I,\#C)}(\ ,k) = w_{(\#I,\#C)}(1,k)$. Since the weight functions are not depend on the power relations (we only deal with sizes of coalitions and their information sets), we can replace value of $w_{(\#I,\#C)}(1,k)$ in equation (4.28). This results in $k \times w_{(\#I,\#C)}(1,1) = \times w_{(\#I,\#C)}(\ ,k)$. If we normalize weights and give the value of one to $w_{(\#I,\#C)}(1,1)$, then we have $\frac{k}{} = w_{(\#I,\#C)}(\ ,k)$ which is the definition of weight function for $F^{p}_{w_{(\#I,\#C)}}$. □

4.5 Conclusion

In this chapter, we have designed another way to rank individuals given a power relation over coalitions formed by them. Considering the problem as an electoral system, we have defined weighted social ranking rules which are extended from the *ceteris paribus* majority rules in Chapter 2. Particularly, as an input they take a binary relation over subsets of individuals, and as an output they result in a set of linear order over the set of individuals. Assigning weights to *ceteris paribus* comparisons initiates a debate on how the weight functions can form. Based on different possibilities to define weight functions we have formed different families of weighted *ceteris paribus* majority rules that are in inclusion relation with each other. The inclusion relation among families of weighted *ceteris paribus* majority rules form a tree structure graph, and the main results in this chapter concern properties that characterize inclusion relation among the families of solutions.

5 Conclusion

We have studied the problem of ranking individuals when ordinal rankings over coalitions formed by them is given. We have defined several approaches inspired from classical voting theory and cooperative game theory in order to deal with the ranking problem. In addition to design procedures to rank individuals, the majority of our work focused on axiomatic study of social ranking rules.

Summary of contributions

Let us present in more detail the scope of our contributions.

Chapter 1 has been devoted to review the literature on subjects of study that we use in order to analyse the ranking problem. We have explored problems in social choice theory and investigated the importance of axiomatic study of solutions in the context. Particularly, we have reviewed some solutions in the contexts of voting theory, cooperative game theory, and ranking sets of individuals, and study them from property driven approach.

In Chapter 2, we have formally defined the ranking problem. Given a binary relation over a set of coalitions, which is called *power relation*, we are looking for a total preorder over the set of individuals, which we call *social ranking rule*. As our first attempt in order to solve the problem, based on a *ceteris paribus* principle, we have defined a solution that goes by the concept of majority. More precisely, we have defined a social ranking rule called *ceteris paribus majority* rule, and we have analysed the solution from a property-driven approach. Particularly, the notion of *ceteris paribus* transforms the problem into a virtual election in which voters are coalitions. We have introduced three axioms *equality of coalitions*, *neutrality*, and *positive responsiveness*, which are inspired from the axiomatic studies in classical voting theory. One of the main results of the chapter is a theorem that characterizes *ceteris paribus majority* rule as a unique solution that satisfies the three mentioned axioms.

From the classical voting theory we know that following the concept of majority in order to rank more than two individuals may result in a Condorcet paradox. In order to avoid possibility of cycles in social rankings, we have proposed a restriction on power relations, called *social* *single peakedness*, which is inspired from single peakedness in voting theory.

Finally, the last section of the chapter has been devoted to investigate the possibility of using incompleteness of a power relation as a source of information. Particularly, we have explored a scenario in which each coalition, based on the number of comparisons made in a power relation, has a *level of information*. We have defined a social ranking rule called *Informative ceteris paribus majority* rule. It ranks individuals following a lexicographic approach over different classes of coalitions with different levels of information.

Another approach to rank individuals is presented in Chapter 3, which is based on classical solution concepts in cooperative game theory. This ranking method is motivated by showing that cardinal solution concepts, like Banzhaf index, are very sensitive to small changes in valuation of coalitions. We have extended the game theoretic notion of marginal contribution to the ordinal framework of power relations, and we have defined a social ranking rule called *ordinal Banzhaf relation*. The solution goes by the concept of majority over the *ordinal marginal contribution* of individuals. Therefore, the axiomatic study of the solution is inspired from the simple majority in the classical voting theory, and the axiomatic study of *ceteris paribus* majority rule in Chapter 2. Particularly, we have defined three axioms *coalitional anonymity*, *neutrality*, and *monotonicity*. One of the main results of the chapter is a theorem that provides a characterization of *ordinal Banzhaf solution* over the domain of power relations as linear orders. The similarity between the axiomatic study of the ordinal Banzhaf solution and the *ceteris paribus* majority rule triggered us to study the similarities and differences of the two social ranking rules in more detail. We have proved that, over the domain of power relations as linear orders, the two social ranking rules belong to a family of rules in which each member is a weighed version of *ceteris paribus* majority rule.

Chapter 4 has been devoted to axiomatic study of families of social ranking rules. The approach proposed in the chapter is based on a weighted extension of the *ceteris paribus* majority rule in order to rank more than two individuals. It assigns to each binary relation over a set of coalitions, a set of linear orders over individuals. The extended version is based on the interpretation of the ranking problem as a virtual election, which is different from classical elections: groups of individuals (coalitions) playing the role of voters; and candidates can also be voters. These differences have motivated us to consider different weight values for coalitions (as voters). The weights depend on different factors related to structure of the power relation. Different ways to define weights lead to families of social ranking rules. We have explored the relation between families of solutions, which results in a tree structure in Figure 4.1. The main goal of the chapter is to analyze the properties that uniquely characterize a family of social ranking rules as a subset of another family of social ranking rule. Specially, for each edge of the tree we have defined at least one axiom, and we have proved the related characterization theorem.

Some Extensions

The main goal of the thesis is to study different solutions from a property-driven approach. Therefore, it is worthy to study some of the possible extensions related to each chapter of the thesis.

Axiomatic study of informative *ceteris paribus* majority. In Chapter 2, we have proposed a social ranking rule that takes into account incompleteness of a power relation in order to rank individuals, which is called informative *ceteris paribus* majority solution. Based on this solution, coalitions are classified into different equivalence classes based on their *level of information*. It applies *ceteris paribus* majority in a lexicographic order to equivalence classes of coalitions (from the high-level informed classes to low-level informed classes). It can be interesting to define a set of axioms that uniquely characterize the solution when we restrict our attention to set of all social ranking rules that are based on a *ceteris paribus* majority principle. Since informative *ceteris paribus* majority follows the concept of majority over different equivalence classes, we can reformulate the two axioms *equality of coalitions* and *positive responsiveness* by restricting them to equivalence classes. The informative *ceteris paribus* majority rule satisfies the same *neutrality* axiom as we have defined for the *ceteris paribus* majority rule. Finally, in order to catch the lexico-graphic nature of the solution, it is possible to define an axiom that states changes in the comparisons made by coalitions in low-level information classes do not affect the social ranking of the individuals.

Extending the axiomatic study of ordinal Banzhaf relation. In Chapter 3, we have axiomatically characterized the ordinal Banzhaf index in domain of linear orders. However, it is possible to extend the characterization theorem to domain of all weak orders, where indifference between coalitions are allowed. The ordinal Banzhaf relation still satisfies the two axioms *coalitional anonymity* and *neutrality* in domain of weak orders. However, we need to reformulate the *monotonicity* axiom to take into account presence of indifference between coalitions. As we have mentioned in Chapter 3, Theorem 3.3.4 is restricted to the domain of all linear orders due to technical difficulties related to the uniqueness part of the proof. To avoid such difficulties, it is possible to define a property to avoid indifferences. The idea of the axiom is that: suppose in a power relation , adding individuals i and j to two coalitions (let's say S $2^{N \setminus \{i\}}$ and T $2^{N \setminus \{j\}}$) does not change the performances of coalitions (S $\{i\}$ T $\{j\}$), then forming another power relation in which, for instance, individual i improves the performance of coalition S (S $\{i\}$ A S), and individual j decreases the performance of coalition T (T A T $\{j\}$) should not change ranking over the individuals. Such an axiom, which is demanding, transforms each power relation consisting of indifference between its marginalistic comparisons to a power relation with no indifference between the marginalistic comparisons. This axiom extends the proof regardless of the presence of indifference in power relations.

Weaker versions of axioms in Chapter 4. Axioms proposed in Chapter 4 are not the only axioms that characterize corresponding social ranking rules. For some of the axioms, we can investigate weaker versions that characterize the same social ranking rules as the main axiom.

- Separability: we have proved a theorem that says the unique social ranking rules in the domain of $\mathcal{F}_{W_{(\#I,\#C)}}$ that satisfy separability are the members of $\mathcal{F}_{W_{\#C}}$. Separability states that if a set of coalitions of the same size have mutually compatible information sets (of size one), then all of them can be represented just by one coalition of that size doing comparisons, without any change in the ranking of individuals. However, it is possible to define a weaker version of the axiom by relaxing the assumption that the compatible information sets must be of size one. Since whenever separability axiom holds its weaker version holds as well, following the sketch of the proof in Theorem 4.4.10, it is easy to prove that the weaker version of separability uniquely characterizes family of $\mathcal{F}_{w_{\#C}}$ as a sub-family of $\mathcal{F}_{w_{(\#I,\#C)}}$. Finally, by making the axiom any weaker than this version, the theorem will failed to be proved. Suppose we make the axiom weaker by relaxing the size restriction of the compatible information sets. In this case, only the "existence" part of the theorem will be hold, and not the "uniqueness" part.

- Coalition separability: we have proved that the only social ranking rules in the family $\mathcal{F}_{w_{(\#I,\#C)}}$ that satisfy coalition separability are $\mathcal{F}_{w_{\#I}}$. Coalition separability states that if a set of coalitions have the same set of preferences over individuals, then one should expect that reducing the size of coalitions by removing the repeated members in the coalitions will not change the final ranking. A weaker version of the axiom is plausible by relaxing the assumption that coalitions must have same preferences over individuals. Following the sketch of the proof in Theorem 4.4.12, and since whenever coalition separability holds the weaker version of it holds as well, we can conclude that the weaker version of coaliiton separability uniquely characterizes members of $\mathcal{F}_{w_{\#I}}$ as members of $\mathcal{F}_{w_{(\#I,\#C)}}$.

  The other way to define the appropriate property that uniquely characterizes the members of the family $\mathcal{F}_{w_{\#I}}$ as members of the bigger family $\mathcal{F}_{w_{(\#I,\#C)}}$ is that given a power relation , any permutation that permutes coalitions with the same information sets should not change the ranking result. This axiom recalls the idea of *coalitional anonymity* axiom, which is extensively studied in Chapter 3. The extended version of *coalitional anonymity* characterizes the inclusion relation, although it assumes more than what needed when the domain of attention is the family of $\mathcal{F}_{w_{(\#I,\#C)}}$.

- Coalition merging: the axiom coalition merging has been used to characterize the social ranking rule $F^{I}_{w_{\#C}}$ as a member of the family $\mathcal{F}_{w_{\#C}}$. The axiom states that a group of coalitions of the same size with same information sets (preferences) should be able to merge together and form a coalition containing all the members in the previous coalitions with the same information set, without changing the ranking over individuals. One can define a weaker version of the axiom, by relaxing the assumption that coalitions must have the same size. Again, since whenever coalition merging holds the weaker version holds, it is possible to characterize the social ranking rule $F^{I}_{w_{\#C}}$ as a member of $\mathcal{F}_{w_{\#C}}$.

Future Works

A complementary approach to support the social ranking methods proposed in the thesis is to use them in real applications. Many real settings are conceivable to apply the ranking methods. For instance consider the following ranking scenario.

Consider a problem where a marketing agency wants to offer a collection of products to its customers, and the agency wants the collection to bring more satisfaction to its customers. Customers are satisfied with a proposed collection of products if they can easily select one product among the others, i.e, with confidence that the chosen product is what they have been looking for. This happens when the proposed bundle of products contains various range of products from the view point of the customers. However, each customer has a restricted attention to some specific features of the products. For instance, suppose the marketing agency suggests a collection of laptops to the customers. Customers have different priorities to select a laptop. For some of them the computational ability of laptops is important, while for the others their gaming ability is more essential. Since the marketing agency is not completely aware of the tastes of its customers, the question is which collection of products should be proposed to bring more satisfaction to the customers. One way to answer the question is to rank products by the amount of variety they bring to different bundles of products when joining them, and use the information to form appropriate collections. To do so, it is possible to propose a limited numbers of bundles to the customers, and evaluate their average level of satisfaction. Since the level of satisfaction cannot be estimated precisely with numbers, we just consider the case that one bundle is more satisfactory or less satisfactory than another one. This information forms a power relation. By applying the ranking methods proposed in the thesis, we can rank products based on their contribution in making bundles of products more diverse or less diverse. In the example of laptops, consider a case where the marketing agency has a set of four different laptops $\{A, B, C, D\}$, and it wants to propose the most appropriate bundle of laptops to its customers, i.e., the collection that brings more satisfaction to the customers. In a very simple case, suppose the agency evaluates the satisfaction of customers by proposing them the four bundles AB and BD, ABD, and AC, and suppose the power relation is formed that indicates the average level of satisfaction of customers about the bundles: $AB \quad AC \quad ABD \quad BD$. Suppose a social ranking rule R assigns to a given power relation a set of linear orders over the products, such that each linear order indicates the possible placement of products on a line based on a similarity scale. For instance, given the power relation, the social ranking rule considers A and B as two sides of a spectrum (since AB is the more diverse collection of products), and puts D between A and B (because the bundle ABD is less diverse) and close to B (since the bundle BD is not diverse enough). Using the ranking over products, it is possible to modify the power relation by making it more complete or by avoiding some of the bundles to form. For instance, by the positioning of products in the example of laptops one can infer that the bundle AD is diverse enough to be proposed to the customers. In order that the modification be consistent with the power relation, we need to look for ranking methods (if exist) which are the exact inverse of the methods we applied on the power relation to rank products. This application covers both problems of lifting ranking from individuals to ranking over subsets of individuals and mapping ranking over subsets of individuals to tanking over individuals.

Another application for our social ranking solutions is in the context of belief aggregation. In many of the recent applications the problem of measuring the effect of each belief in making a belief base inconsistent is modelled as a cooperative game. In this game, the characteristic function of each coalition (belief base) is the amount of inconsistency in the belief base. Using semi-values like Shapley value, one can measure the quota of each belief in making the belief base inconsistent [Hunter and Konieczny, 2010]. However since in many real situation quantifying the inconsistency is not straight-forward one can only assume that one belief base is ordinally more consistent or less consistent than another one and measure the "ordinal marginal contribution" of each belief to make a belief base inconsistent. For instance, suppose an agent exposed by a set of beliefs and she wants to regularly update her beliefs. To do so, the agent can imagine hypothetical power relation in which coalitions are formed by adding different beliefs to her current belief base. Given the power relation the agent can rank the beliefs based on the inconsistency they bring about, and add the one which is more consistent with her current belief base. [Serramia et al., 2020]

6
Résumé Long en Français

La conception de procédures visant à classer les personnes en fonction de leur comportement dans les différents groupes est d'une grande importance dans de nombreuses situations pratiques. Le problème se pose dans différents scénarios exposé par trois thépries: la théorie du choix social, la théorie des jeux coopératifs ou la théorie de la décision multi-attributs, et en voici quelques exemples. Comparer les chercheurs d'un département scientifique en tenant compte de leur impact dans différentes équipes [Papapetrou et al., 2011] ; trouver les partis politiques les plus influents dans un parlement en se basant sur les alliances passées au sein de coalitions majoritaires alternatives [Marošević and Soldo, 2018] ; évaluer les attributs en fonction de leur influence dans un contexte de décision à attributs multiples, où l'indépendance des attributs n'est pas vérifiée en raison des interactions mutuelles (voir [Bouyssou and Marchant, 2007] pour une discussion sur les coalitions de critères gagnantes, [Boutilier et al., 2004] pour les CP-nets concernés par la dépendance conditionnelle qualitative et l'indépendance des déclarations de préférence selon une interprétation *ceteris paribus*) ; et la quantification de la productivité des individus en présence d'un travail d'équipe en tenant compte du fait que la contribution d'un individu à une équipe peut également dépendre de la productivité de l'individu, puisque les individus les plus productifs apportent plus d'expertise, de compétences en matière de constitution d'équipes et de visibilité, et qu'ils contribuent davantage en moyenne [Flores-Szwagrzak and Treibich, 2020].

Dans de nombreuses applications du monde réel, une évaluation précise du "pouvoir" des coalitions peut être difficile, voire impossible à faire, en raison d'un ensemble de facteurs inconnus : existence de données incertaines, complexité de l'analyse, informations manquantes ou difficultés de mise à jour, etc. Dans de telles situations, mesurer l'importance des individus à l'aide des indices de pouvoir classiques n'est pas toujours simple. Dans ce cas, il peut être intéressant de ne considérer que les informations ordinales concernant les comparaisons binaires entre les coalitions. Supposons par exemple que le directeur d'un département souhaite évaluer les performances des professeurs sur la base de leur contribution dans les groupes scientifiques. Supposons également que la seule information fournie au directeur soit qu'un groupe est plus performant qu'un autre ou que les deux groupes ont le même niveau de performance. Cette hypothèse est valable car il n'est pas possible d'évaluer les performances des groupes scientifiques par des chiffres ; les performances d'un groupe scientifique dépendent d'une combinaison de facteurs tels que le nombre de publications faites par le groupe, l'importance des sujets pour le département, le nombre de citations, la qualité de leurs articles, et de nombreux autres facteurs qui peuvent être difficiles à quantifier.

Certains scientifiques ont modélisé le manque d'information dans de telles situations avec des méthodes probabilistes [Suijs et al., 1999], ou en estimant la valeur des coalitions à l'aide d'intervalles [Branzei et al., 2010]. Toutefois, ces méthodes ne sont pas toujours applicables en raison de divers types d'incertitude. Dans cette thèse, nous suivons la même approche que dans [Moretti and Öztürk, 2017] et [Bernardi et al., 2017], et modélisons la valeur des coalitions de manière ordinale en utilisant une relation binaire qui est définie sur l'ensemble des coalitions. Par conséquent, tout au long de la thèse, nous apportons des réponses à la question générale de savoir comment obtenir un classement sur un ensemble fini de N individus (appelé *classement social*), en fonction d'un classement des éléments d'un ensemble de pouvoirs 2^N (appelé *relation de pouvoir*, normalement désigné par   ou   ).

Dans le problème de l'évaluation des professeurs, supposons qu'avec un ensemble de cinq professeurs $N = \{1, 2, 3, 4, 5\}$, le directeur du département veut les classer. Supposons également que l'information fournie au directeur soit la relation de pouvoir $\{2, 4, 5\} \quad \{1, 3\} \quad \{1, 2\} \quad \{2, 3\} \quad \{2, 4, 5\} \quad \{3, 5\} \quad \{2, 4\} \quad \{2, 5\} \quad \{1, 4\}$ qui indique la performance relative des différents groupes scientifiques. Par exemple, $\{2, 4, 5\} \quad \{1, 3\}$ signifie qu'un groupe composé de trois professeurs 2, 4, et 5 fonctionne strictement mieux qu'un groupe de professeurs 1 et 3, et $\{2, 4, 5\} \quad \{3, 5\}$ signifie que les performances des groupes correspondants sont les mêmes.

Notre objectif n'est pas seulement de définir un classement social sur un ensemble d'individus. En effet, la majorité de nos recherches portent sur un ensemble de propriétés (axiomes) que les règles de classement social devraient satisfaire. À notre connaissance, la question a d'abord été introduite par [Marichal and Roubens, 1998], mais elle a été formellement étudiée par [Moretti, 2015] et [Moretti and Öztürk, 2017], qui analysent les solutions de classement social selon une approche axée sur les propriétés. Ils évaluent l'effet des propriétés de base dans la combinaison du classement social, et montrent que la combinaison par paire de ces propriétés naturelles aboutit soit à l'impossibilité (c'est-à-dire qu'il n'existe pas de classement social), soit à l'aplatissement (c'est-à-dire que tous les individus sont classés de manière égale), soit à la dictature. Dans le même cadre, [Bernardi et al., 2017] a caractérisé axiomatiquement une solution de classement social basée sur l'idée que les individus les plus influents sont ceux qui apparaissent le plus souvent dans les plus hautes positions du classement des coalitions. Une approche plus pratique de ce problème a été étudiée dans [Fayard and Escoffier, 2018] dans laquelle les auteurs mettent en œuvre une règle de classement social proposée au chapitre 2 afin de trouver une approximation du nombre minimum de coalitions à supprimer pour satisfaire la transitivité. Afin d'explorer de nouvelles méthodes pour classer les individus en fonction d'un classement ordinal sur leurs coalitions, nous utilisons dans cette thèse des notions différentes de la théorie classique des choix sociaux et de la théorie des jeux coopératifs.

Parallèlement à l'objectif de cette thèse, dans le chapitre 1 nous faisons une revue de la littérature sur les contextes liés à la thèse. Nous discutons de l'étude axiomatique et de ses composantes. Nous décrivons également le type de résultats attendus de l'étude et leur importance. Nous passons en revue les études axiomatiques qui ont été réalisées dans les contextes de la théorie du vote, de la théorie des jeux coopératifs et du classement d'ensembles d'objets. Enfin, nous étudions les progrès récents de notre problème de classement.

Le chapitre 2 présente notre première approche pour résoudre le problème du classement. Dans ce chapitre, nous étudions l'utilisation du principe de majorité *ceteris paribus* comme solution de classement social. Selon cette méthode de classement, deux personnes sont classées à l'aide d'informations provenant de *ceteris paribus* (c'est-à-dire, toutes choses égales par ailleurs) des comparaisons sur toutes les coalitions possibles. Cela suggère une interprétation de notre problème sur le modèle d'une élection virtuelle, avec des groupes d'individus (coalitions) jouant le rôle d'électeurs. Malheureusement, la solution de la majorité *ceteris paribus* peut conduire à un *condorcet-like paradox*. Par conséquent, une restriction du domaine sur la famille de classement des coalitions est proposée pour garantir la transitivité du classement sur les individus. Le chapitre se termine par une discussion sur les interprétations possibles du caractère incomplet de la relation de pouvoir. Nous proposons une nouvelle règle de classement social pour prendre en considération une interprétation spécifique du caractère incomplet.

Le chapitre 3 présente une autre méthode de classement des individus. La nouvelle solution est définie par l'extension des notions de contribution marginale et de l'indice de Banzhaf dans les jeux coopératifs classiques, et est appelée ordinale de Banzhaf. Nous limitons notre attention aux relations de pouvoir en tant qu'ordres linéaires, et nous caractérisons la *solution ordinale de Banzhaf* résultante au moyen d'un ensemble d'axiomes inspirés de ceux introduits dans le chapitre 2. La similitude entre l'étude axiomatique des deux solutions nous motive à explorer plus en détail les similitudes et les différences des solutions.

Le chapitre 4 de la thèse est consacré à l'étude axiomatique des familles de règles majoritaires *pondérées ceteris paribus*. Les règles de classement social de ce chapitre sont une extension pondérée de la règle de majorité *ceteris paribus* pour classer plus de deux individus. Selon l'interprétation des problèmes de classement, les poids attribués aux coalitions (électeurs) peuvent être fonction des coalitions, de l'ensemble des individus comparés par la coalition, de leur combinaison, de leur taille, etc. Comme les fonctions de pondération peuvent être définies d'une infinité de façons, chaque interprétation aboutit à une famille spécifique de règles de classement social pondérées qui est une sous-famille d'autres familles. La relation d'inclusion entre les familles forme un arbre dont les arcs montrent les relations d'inclusion correspondantes entre les familles, et le but principal du chapitre est d'analyser chaque famille de solutions comme un sous-ensemble d'une autre famille de solutions par l'étude axiomatique de leurs propriétés.

Les contributions de la thèse sont publiées dans les actes de conférences internationales, à savoir IJCAI 2018 [Haret et al., 2018] et IJCAI 2019 [Khani et al., 2019].

Chapitre 1: Revue de la Littérature

La théorie des choix sociaux permet d'étudier et d'analyser comment combiner les préférences des individus pour parvenir à une décision collective ou *social welfare* dans un certain sens. L'intérêt croissant pour le choix social provient de son lien étroit avec d'autres domaines de l'informatique, ainsi que d'un vaste échange entre eux. Une des raisons d'importance provient des notions importées de l'informatique dans le contexte du choix social afin de résoudre des problèmes, comme le calcul de la complexité des méthodes de classement social ou, par exemple, la spécification des méthodes de vote où la manipulation est difficilement plausible. D'autre part, les techniques développées dans la théorie du choix social peuvent être utilisées afin de résoudre des problèmes dans le contexte de l'informatique et de l'intelligence artificielle [Brandt et al., 2016b]. Un exemple est l'application de la théorie du choix social pour développer des systèmes de classement de pages dans les moteurs de recherche Internet [Chevaleyre et al., 2007], [Tennenholtz, 2004]. Comme nous le verrons, dans le corps de la thèse, nous bénéficions d'une des idées originales de la théorie du choix social afin de trouver des solutions à un problème de classement complexe.

Les thèmes abordés dans le choix social peuvent être classés selon deux axes distincts : la nature du problème de choix social que nous traitons et le type de techniques formelles ou calculatoire étudiées [Chevaleyre et al., 2007]. Dans cette thèse, nous nous concentrons principalement sur trois d'entre elles : *théorie du vote*, *formation de la coalition*, et *classement des ensembles d'objets*.

- **Théorie du vote**. Étant donné un ensemble d'électeurs et un ensemble de candidats, la question qui intéresse la théorie du vote est de savoir comment agréger l'opinion des électeurs (représentée par leurs bulletins de vote sur un ensemble de candidats) pour trouver un classement par rapport aux candidats, ou pour trouver le meilleur candidat. Cette question se pose dans de nombreux domaines comme les affaires, les organisations sociales ou la politique. L'origine de cette question remonte à l'époque des Romains, lorsque Pline et d'autres sénateurs devaient décider du sort d'un certain nombre de prisonniers : acquittement (A), bannissement (B), ou condamnation à mort (C). Bien que l'option A, privilégiée par Pline, ait le plus grand nombre de partisans, elle n'a pas la majorité absolue. L'un des partisans de la punition sévère a alors stratégiquement décidé de retirer la proposition C, laissant ses anciens partisans soutenir l'option B, qui est évidemment la gagnante du concours de majorité entre A et B. Si les sénateurs avaient voté sur les trois options en utilisant la règle de la majorité, alors l'option A aurait gagné. Cet exemple illustre plusieurs caractéristiques intéressantes des règles de vote. Par exemple, il peut être interprété comme démontrant un manque d'équité de la règle de la majorité relative : même si une majorité d'électeurs estime que A est inférieur à l'une des autres options (à savoir, B), A gagne quand même [Brandt et al., 2016a]. En fait, lorsqu'il n'y a que deux candidats, le choix du meilleur candidat est simple. Cela va dans le sens de la *majorité*. Cependant, lorsqu'il y a plus de deux candidats, il n'y a pas de façon évidente de choisir le meilleur candidat. Différentes méthodes sont proposées, chacune d'entre elles tenant compte d'un certain sens de l'équité. Les écrits du philosophe, poète et missionnaire catalan Ramon Llull (1232-1316) sur les règles de vote sont un autre indice sur l'utilisation de la règle de vote au Moyen Âge. Il soutenait l'idée que les résultats des élections devraient être basés sur des concours à la majorité directe entre deux candidats. La règle à laquelle il a fait référence semble être celle connue aujourd'hui sous le nom de règle Copeland [Copeland, 1951], selon laquelle le candidat qui remporte le plus grand nombre de concours à la majorité par paire est élu.

  D'autres tentatives dans le domaine des règles de vote sont les travaux de l'ingénieur français Jean-Charles de Borda (1733-1799) et du philosophe et mathématicien français

Marquis de Condorcet (1743-1794). La discussion entre ces deux scientifiques a motivé Borda à proposer une méthode de vote, aujourd'hui connue sous le nom de règle Borda. Selon cette règle, le vainqueur d'une élection est choisi en donnant à chaque candidat, un nombre de points correspondant au nombre de candidats classés plus bas [Brandt et al., 2016a]. Il a fait valoir la supériorité de sa méthode sur la règle de la pluralité par un exemple indiquant que la règle Borda est une règle de vote *single-winner*.

Bien que les premiers travaux sur le contexte de la prise de décision collective et la théorie du vote se soient limités à concevoir diverses règles de vote et à comparer leurs avantages et leurs inconvénients, cette manière de procéder a changé grâce aux travaux fondateurs de Kenneth Arrow en 1963, dans lesquels il a adopté une vision plus large et mis en évidence certaines propriétés communes à toutes les règles de vote proposées. Arrow a expliqué les motivations philosophiques et économiques pour définir différentes règles de vote en termes mathématiques comme des axiomes [Arrow, 1963].

- **Formation de coalitions**. La formation de coalitions étant considérée comme un processus de choix impliquant plus de deux individus [van Deemen, 1991], la théorie du choix social est importante dans ce contexte : le processus de formation de coalitions est considéré comme un problème d'agrégation des préférences dans lequel chaque acteur a une préférence par rapport à d'éventuelles coalitions, et la question est de savoir comment agréger ces préférences afin de former les coalitions. Une autre question importante se pose après la formation des coalitions : comment répartir entre ses membres le partage des bénéfices ou des coûts d'une coalition. Pour ce faire, il est nécessaire d'établir une sorte de classement de l'ensemble des acteurs en fonction de leurs performances au sein de la coalition. Ces problèmes sont étudiés dans le domaine de la théorie des jeux coopératifs [van Deemen, 1991]. La théorie des jeux coopératifs analyse comment des coalitions d'individus peuvent se former, et comment ils doivent répartir le partage des bénéfices ou des coûts de leur coopération. La notion de jeux coopératifs a été introduite pour la première fois dans les essais de Von Neumann Morgensten [von Neumann and Morgenstern, 2007] comme une tentative de distinguer deux approches des jeux coopératifs et non coopératifs.

  Les jeux simples sont des jeux coopératifs dans lesquels les coalitions sont divisées en deux ensembles, l'ensemble des coalitions gagnantes et les autres coalitions. Les jeux simples sont utilisés comme modèle pour les situations de vote binaire : dans le cas où il y a deux candidats, les agents en faveur du plus attrayant forment une coalition gagnante et les autres seront la coalition perdante. Les décisions des gagnants concernent l'ensemble des joueurs et les perdants sont obligés de prendre ces décisions pour acquises, que les effets des décisions des gagnants leur soient favorables ou non [van Deemen, 1991]. Un exemple est le jeu du vote à la majorité. Dans ce jeu, seule une coalition majoritaire d'électeurs peut gagner, c'est-à-dire déterminer une alternative gagnante. L'avantage d'une telle abstraction est que des jeux simples peuvent être étudiés sans se référer à des règles spécifiques comme la majorité, la pluralité, etc. Des indices de pouvoir comme l'indice de Shapley-Shubik [Shapley, 1953], et l'indice de Banzhaf [Banzhaf III, 1964] sont introduits afin de mesurer le pouvoir des agents dans chaque coalition. Une situation réelle dans laquelle les indices de pouvoir sont utilisés est le problème que certains pays nordiques ont dû affronter pour rejoindre ou non l'Union européenne. Le problème est apparu parce que les pays nordiques accordent traditionnellement une grande priorité (par rapport à l'Union européenne) à la protection de l'environnement. Par conséquent, la question était de savoir comment les nouveaux membres de l'Union européenne pouvaient influer sur les normes environnementales de l'Union européenne, par exemple en les adaptant à leurs propres normes. La valeur de Shapley apporte une solution appropriée en évaluant le pouvoir de chaque membre lorsqu'il rejoint l'UE [Holzinger, 1995].

- **Classement des ensembles d'objets**. Imaginez un ensemble donné d'énoncés, chacun d'eux avec un degré de plausibilité, et supposez que l'objectif soit de choisir un ensemble d'énoncés plus plausibles que les autres. À première vue, la réponse à ce problème semble simple : il suffit d'ordonner les énoncés en fonction de leur degré de plausibilité, puis de sélectionner les énoncés en haut du classement comme étant l'ensemble d'énoncés qui est hautement plausible [Packard, 1981]. Toutefois, il convient de noter que dans de nombreux cas, une combinaison de deux déclarations plausibles n'est pas nécessairement plausible car la combinaison de deux déclarations plausibles peut former un ensemble de déclarations incohérentes. La considération d'avoir une incohérence lorsque deux énoncés plausibles sont combinés a été le début d'une étude de terrain dont le but est de classer des ensembles d'objets lorsqu'un classement sur l'ensemble des objets est fourni. Dans [Barberà et al., 2004], les auteurs classent le problème de classement en trois classes de *incertitude complète*, *ensembles d'opportunités*, et *ensembles comme résultats finaux*. Ces catégories sont définies en fonction de l'objectif visé par le traitement des ensembles d'objets. Différentes réponses sont fournies pour le problème selon la catégorie dans laquelle se situe le problème de classement.

Axiomatisation

Plusieurs aspects du choix social ont été étudiés afin de qualifier les solutions pour un problème de choix social donné [Brandt et al., 2016a]. L'optimisation d'une solution en termes de charge de calcul est l'une des questions importantes pour résoudre des problèmes concrets. L'évaluation des solutions en fonction de leur complexité afin de résoudre un problème est largement étudiée dans le contexte du choix social. Cependant, l'approche classique pour étudier les solutions dans la théorie du choix social consiste à analyser des ensembles de propriétés ou d'axiomes qu'elles satisfont. Dans la terminologie du choix social, ce type d'études est appelé *étude axiomatique* [Thomson, 2001]. Jusqu'à récemment, l'étude axiomatique était la principale méthode d'investigation dans quelques branches de l'économie et de la théorie des jeux, telles que la théorie du choix social et de l'utilité. Cependant, ces dernières années, cette méthode s'est développée spécialement dans deux domaines, celui de la théorie du marchandage et celui de la formation de coalitions. De tells développement jettent une nouvelle lumière sur les techniques axiomatiques et, de nos jours, elles sont utilisées de manière plus ciblée afin de comparer différentes solutions ou même de trouver de nouvelles solutions qui satisfont à des propriétés accetable et remarquables [Thomson, 2001].

L'étude axiomatique est motivée par la nécessité de différencier les solutions qui sont plausibles pour un problème de choix social. Comme mentionné dans la section précédente, pour un domaine spécifique de problèmes, plusieurs solutions intuitivement attrayantes peuvent exister. L'étude axiomatique nous permet de valider l'existence de solutions selon des interprétations simples et naturelles. De plus, lorsqu'il n'existe pas de solution selon une interprétation intuitive et naturelle, l'étude axiomatique peut nous aider à en trouver une. En fonction de domaine spécifique de recherche, l'étude axiomatique commence par une liste de propriétés souhaitables pour le domaine et finit par décrire les solutions au problème aussi précisément que possible en utilisant les propriétés données. Elle permet également d'étudier les relations logiques entre les axiomes, et de voir comment des changements dans le domaine des problèmes peuvent affecter les axiomes. Normalement, l'étude axiomatique des solutions conduit à un théorème de caractérisation qui est une description de la solution en fonction de propriétés données, bien que l'objectif ultime de l'étude axiomatique aille plus loin que cela. En fait, le but de l'étude axiomatique devrait être de comprendre et de décrire les implications de la liste des propriétés aussi précisément que possible [Thomson, 2001].

Deux raisons motivent l'analyse des solutions dans une perspective axiomatique. La première raison est que bien que la définition des solutions ne soit pas une tâche lourde en soi, se concentrer uniquement sur la définition des solutions nous empêche d'explorer tout l'espace des solutions pour un problème donné. Il peut y avoir d'autres solutions qui satisfont des propriétés beaucoup plus attrayantes que celles définies actuellement, et nous ne pouvons pas les réaliser sans définir les propriétés et les combiner. Ainsi, l'étude axiomatique nous aide à avoir une vision plus large de l'espace de toutes les solutions possibles, et à grandir le spectre du champ de les solutions afin de trouver des solutions optimales. Il convient également de mentionner que l'étude axiomatique nous libère parfois de la recherche sans effort de solutions [Thomson, 2001]. Par exemple, la méthode de vote la plus intuitive dans le cas de deux candidats est la règle de la majorité qui satisfait à un ensemble de propriétés bénignes, mais si l'on passe aux domaines avec trois candidats ou plus, la majorité peut entraîner un classement cyclique sur les candidats, ce qui n'est pas approprié dans de nombreux scénarios pratiques. En conséquence, on peut chercher d'autres solutions qui satisfont aux exigences mentionnées tout en évitant les cycles dans les résultats du classement. Cependant, [Arrow, 1963] met en evidence un "improsibility theory" exprimant la non-disponibilité de ce type de solutions!

La deuxième raison pour laquelle l'etude axiomatique est intéressent et que parfois, par intuition, une solution peut être reconnue pour donner les bonnes réponses, alors que d'autres solutions peuvent exister, tout aussi efficaces pour ces exemples. Ces solutions peuvent être obtenues par une évaluation axiomatique des propriétés.

Il faut également noter concernant l'étude axiomatique, que dans de nombreux cas, la caractérisation est faite pour une solution spécifique. Par exemple, des solutions comme la règle de la majorité et Borda sont largement étudiées et appliquées dans la littérature de la théorie du choix social. On peut se demander quelles sont les propriétés satisfaites par ces solutions qui les rendent si pratiques ? La réponse à cette question est en quelque sorte clarifiée par la caractérisation effectuée par [May, 1952]. Ce type de caractérisation est légitime lorsque la solution est largement utilisée dans la pratique ou dans la littérature théorique, comme la règle de la majorité dans la théorie du vote ou la valeur de Shapley dans la théorie des jeux coopératifs.

Comme nous le verrons dans les prochains chapitres, la plupart des solutions introduites pour résoudre le problème de classement dans cette thèse s'inspirent de solutions bien connues et largement utilisées dans d'autres contextes comme la théorie du vote et la théorie des jeux coopératifs. Par conséquent, il est utile de comprendre comment les propriétés des solutions changent lorsqu'elles s'adaptent à d'autres cadres, et cela permet de mieux définir de nouvelles solutions spécifiques à assez neuvoaux cadres.

Ce que nous avons mentionné à propos de l'importance de l'étude axiomatique concernait les mérites de la caractérisation de solutions uniques, cependant, la caractérisation de familles de solutions mérite davantage. En fait, en considérant une grande famille de solutions (qui ont une structure commune), nous pouvons définir quelques petits ensembles d'axiomes, même un seul axiome, et voir quels membres de la famille les satisfont. Ces axiomes donnent lieu à des théorèmes qui analysent la relation d'inclusion entre deux familles de solutions. L'étude axiomatique de la relation d'inclusion peut être poursuivie jusqu'à atteindre un seul membre dont l'appartenance à une famille est caractérisée par un ensemble d'axiomes. La définition répétée d'axiomes pour les membres de familles de solutions hiérarchiques se ramifie dans plusieurs directions, et chaque ramification mène à une seule solution. Pour clarifier, considérez la famille de toutes les méthodes de notation dans la théorie du vote. Ces méthodes de notation attribuent un score aux candidats selon certains systèmes de notation spécifiques. En fonction du contexte dans lequel les méthodes de notation sont appliquées, il existe une infinité de systèmes de notation permettant de noter les candidats [Chebotarev and Shamis, 1998]. Une sous-famille de ces règles de notation contient des méthodes de vote qui notent les candidats en fonction de leur position dans les préférences des électeurs. Cette sous-famille comporte également une infinité de systèmes de notation. Par exemple, dans le système original de Borda, la notation des candidats dépend du nombre de candidats en lice dans le processus de vote : s'il y a cinq candidats, le candidat le mieux classé dans les préférences d'un électeur obtient la note de cinq, le candidat en deuxième position obtient la note de quatre, et ainsi de suite jusqu'à ce que le dernier candidat positionné obtienne la note de un. D'autres variantes de la règle de notation peuvent commencer à noter à partir de zéro au lieu de un (dans l'exemple de cinq candidats, on attribue la note zéro au dernier candidat placé et la note quatre au candidat le mieux classé). Une autre règle de notation permet de noter les candidats proportionnellement à leur position dans une préférence donnée (le candidat le mieux classé obtient la note de $\frac{1}{1}$, le second obtient la note de $\frac{1}{2}$ et ainsi de suite) [Fraenkel and Grofman, 2014]. A partir de cette sous-famille, d'autres sous-familles sont plausibles, et la structure hiérarchique peut être poursuivie pour n'atteindre qu'un seul membre (par exemple, la règle Borda). L'étude axiomatique dans ce contexte, consiste à analyser la relation d'inclusion entre différentes familles, comme la famille des règles de notation générales et celles avec des systèmes de notation liés au positionnement des candidats, ou, par exemple, à caractériser une méthode de notation spécifique comme Borda, en tant que membre de la sous-famille des règles de notation avec système de notation basé sur le positionnement des candidats.

Solutions de Classement Social

Le problème du classement des individus par rapport aux coalitions formées par eux, est d'abord étudié par [Moretti and Öztürk, 2017], à partir d'une approche axée sur la propriété. Les auteurs de l'article étudient axiomatiquement le problème et explorent des solutions satisfaisant un ensemble de propriétés significatives.

Étant donné un ensemble de N individus, les auteurs indiquent le classement sur les coalitions formées par les individus comme , qui est un préordre total (complet et transitif) sur l'ensemble de puissance de N (2^N). Ce classement est appelé une relation de pouvoir. Les auteurs indiquent l'ensemble de tous les préordres totaux sur l'ensemble des individus par $\mathcal{T}^N$, et l'ensemble des préordres totaux sur l'ensemble des coalitions par $\mathcal{T}^{2^N}$. Le problème de classement consiste à trouver un préordre total sur l'ensemble des N individus (classement social) lorsqu'une relation de pouvoir sur des sous-ensembles de N est donnée. Plus formellement, une règle de classement social est définie comme une fonction $\rho : \mathcal{T}^{2^N} \rightarrow \mathcal{T}^N$ qui fait correspondre un préordre total sur un ensemble de coalitions à un préordre total sur l'ensemble des individus. Dans ce cas, pour deux individus quelconques i et j, la notation $i\rho(\ )j$ fait référence à la faible préférence de i pour j. Les auteurs présentent deux axiomes des méthodes de classement et analysent l'effet des axiomes sur la définition des méthodes de classement. Ces propriétés sont énumérées comme suit.

Dominance. Une règle de classement social $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \rightarrow \mathcal{T}^N$ satisfait *dominance* sur $\mathcal{C}^{2^N} \quad \mathcal{T}^{2^N}$ si et seulement si pour tous $\quad \mathcal{T}^{2^N}$, et pour chacun des deux individus $i, j \quad N$, si i domine j dans , alors $i\rho(\ )j$ (et non $j\rho(\ )i$ si i domine strictement j dans).

Cet axiome indique que si chaque coalition contenant un individu spécifique comme i est toujours classée plus haut que la coalition S lorsque i est remplacé par un autre individu j, alors i devrait être classé plus haut que j.

Une règle de classement social $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \rightarrow \mathcal{T}^N$ satisfait *Indépendance des coalitions non pertinentes* si et seulement si

$$i\rho(\ )j \Leftrightarrow i\rho(\ )j$$

Le deuxième axiome est *symmetry*. Cet axiome exclut les méthodes de classement qui classent les individus en fonction de leur nom et non de leurs performances.

Symétrie . Une règle de classement social $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \rightarrow \mathcal{T}^N$ satisfait *symétrie* si et seulement si

$$i\rho(\ )j \Leftrightarrow p\rho(\ )q$$

pour tous $i, j, p, q \quad N$ et $\quad \mathcal{C}^{2^N}$ tels que $|D^k_{ij}(\ )| = |D^k_{pq}(\ )|$ et $|D^k_{ji}(\ )| = |D^k_{qp}(\ )|$ pour tout $k = 0, 1, ..., n-2$ ($D^k_{ij}(\ ) = \{S \quad 2^{N\setminus\{i,j\}}, |S| = k, S \quad \{i\} \quad S \quad \{j\}\}$).

Les auteurs examinent si les deux méthodes de classement intuitif, *primitive* et *complete primitif*, satisfont *Dominance* et *Symétrie*. Ils ont prouvé que, pour un ensemble de trois individus, les classements sociaux primitifs et primitifs complémentaires satisfont les axiomes dominance et symétrie. Étant donné une relation de pouvoir $\quad \mathcal{T}^{2^N}$, un classement social $\rho : \mathcal{C}^{2^N}$, un classement social $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \to \mathcal{T}^N$ est appelé primitif si et seulement si pour tout individu $i, j \quad N$ il tient que $i\rho(\ )j \Leftrightarrow \{i\} \quad \{j\}$. De même, si pour une coalition S, le complément de S est défini comme $S \ = N \setminus S$, alors le classement social $\rho : \mathcal{C}^{2^N} \quad \mathcal{T}^{2^N} \to \mathcal{T}^N$ est appelé *complete primitive* si et seulement si pour deux individus quelconques $i, j \quad N$ nous avons $i\rho(\ )j \Leftrightarrow \{j\} \quad \{i\}$.

L'un des résultats importants de l'article est un théorème qui illustre l'incompatibilité de la symétrie et de la dominance des axiomes lorsqu'il y a plus de trois individus à classer. Ils montrent que la combinaison de ces propriétés naturelles conduit soit à l'impossibilité (c'est-à-dire qu'il n'existe pas de classement social), soit à l'aplatissement (c'est-à-dire que tous les individus sont classés de la même façon), soit à la dictature (c'est-à-dire que, le classement social est imposé par la comparaison relative des coalitions d'une taille donnée) [Moretti and Öztürk, 2017].

Theorem 6.0.1. *Let $|N| > 3$. Il n'y a pas de solution de classement social $\rho : \mathcal{T}^{2^N} \to \mathcal{T}^N$ qui satisfait la dominance et la symétrie sur $\mathcal{T}^{2^N}$.*

Dans le même cadre, [Bernardi et al., 2017] caractérise axiomatiquement une solution de classement social basée sur l'idée que les individus les plus influents sont ceux qui apparaissent le plus souvent dans les plus hautes positions du classement des coalitions. Suivant ce raisonnement, ils fixent un ensemble de propriétés qu'une solution doit satisfaire, et explorent les méthodes de classement possibles qui les satisfont.

Neutralité. Une règle de classement social ρ satisfait à la neutralité si et seulement si pour deux individus quelconques et une relation de pouvoir $\quad \mathcal{T}^{2^N}$, elle stipule que

$$i\rho(\ )j \Leftrightarrow \sigma(i)\rho(\ _\sigma)\sigma(j)$$

où σ est une bijection sur N telle que pour toute relation de pouvoir $\quad \mathcal{T}^{2^N}$, $\ _\sigma$ est défini comme suit

$$\sigma(i) \quad _\sigma \sigma(j) \Leftrightarrow i \quad j.$$

L'axiome de la neutralité repose sur l'idée qu'une solution doit préserver le rang des individus dans une société par rapport aux permutations des noms des individus.

Anonymat de Coalition. Une règle de classement social ρ satisfait *anonymat de coalition*, si et seulement si pour deux relations de pouvoir $\ , \quad \mathcal{T}^{2^N}$, deux individus quelconques $i, j \quad N$, et bijection π sur $2^{N\setminus\{i,j\}}$ nous avons que $i\rho(\ )j \Leftrightarrow i\rho(\ )j$ quand pour tous $S, T \quad 2^{N\setminus\{i,j\}}$ on a

$$S \quad \{i\} \quad T \quad \{j\} \Leftrightarrow \pi(S) \quad \{i\} \quad T \quad \{j\}.$$

L'axiome indique essentiellement que le classement entre deux éléments quelconques i, j doit être indépendant des autres éléments.

Monotonie. Nous disons qu'une solution ρ est monotone si pour toute relation de pouvoir $\mathcal{T}^{2^N}$, chaque individu i, j N tel que $i\rho( )j$ et $j\rho( )i$, et toute relation de pouvoir $\mathcal{T}^{2^N}$ qui est obtenue en améliorant strictement le classement d'un sous-ensemble contenant i mais pas j, en appliquant la solution de classement social sur classe i strictement mieux que j ($i\rho( )j$ et non $j\rho( )i$).

Indépendance par rapport au ensemble moins bon. Considérons qu'une relation de pouvoir $\mathcal{T}^{2^N}$ est donnée comme

$$S_1 \quad S_2 \quad \ldots \quad S_{2^n-1},$$

dans lequel $S_1, S_2, \ldots, S_{2^N-1}$ 2^N, et disons Σ_1 Σ_2 $\ldots$ Σ est un ordre dans lequel les sous-ensembles S_t sont regroupés dans la classe d'équivalence Σ_k. Nous disons qu'une règle de classement social ρ satisfait *Indépendance par rapport au ensemble moins bon* si pour toute relation de pouvoir avec l'ordre associé Σ_1 Σ_2 $\ldots$ Σ (> 2), et i, j N tel que i est strictement meilleur que j ($i\rho( )j$ et non $j\rho( )i$) nous devrions avoir i est strictement meilleur que j dans la relation de pouvoir ($i\rho( )j$ et non $j\rho( )i$) lorsque $\mathcal{T}^{2^N}$ est obtenu à partir de en subdivisant Σ par $T_1, T_2, \ldots, T_m$:

$$\Sigma_1 \text{ A } \Sigma_2 \text{ A } \ldots \text{ A } \Sigma_{-1} \text{ A } T_1 \text{ A } \ldots \text{ A } T_m$$

Cet axiome éclaire une méthode pour classer les individus en fonction d'une relation de pouvoir. L'axiome donne plus d'importance aux sous-ensembles classés plus haut dans la relation de pouvoir. Une interprétation possible de cette propriété est l'évaluation des professeurs sur la base de leur collaboration scientifique dans différents groupes. Une fois qu'un ordre total entre deux professeurs est établi sur la base de leur productivité scientifique dans tous les groupes, l'utilisation éventuelle d'un critère secondaire pour l'évaluation des groupes (par exemple, l'offre éducative d'une équipe) n'affectant que les coalitions ayant la plus faible productivité scientifique, peut ne pas avoir d'impact sur un ordre total défini selon les critères d'évaluation les plus importants [Bernardi et al., 2017].

Ils ont défini une règle de classement social appelée *solution d'excellence lexicographique* qui suit la notion d'ordonnancement lexicographique sur les classes d'équivalence des sous-ensembles dans une relation de pouvoir donnée. Sur la base de cette solution, pour classer les individus i et j ayant une relation de pouvoir , en partant de la classe de sous-ensembles la mieux classée, on compte le nombre de fois que chacun des i et j apparaît dans les sous-ensembles de la classe. Le fait de trouver une différence entre les nombres de présence des deux individus met fin au processus, et le classement de l'un apparaît plus haut que celui de l'autre. Si nous constatons une indifférence entre les nombres de présence des deux individus, le processus se poursuit pour les autres classes d'équivalence. Si l'indifférence se produit pour toutes les classes d'équivalence, alors les deux individus sont considérés comme indifférents.

Les auteur ont prouvé un théorème qui caractérise la solution de l'excellence lexicographique en utilisant les quatre axiomes mentionnés [Bernardi et al., 2017].

Theorem 6.0.2. *La solution d'excellence lexicographique est la solution unique qui remplit les axiomes neutralité, anonymat de coalition, monotonie, et Indépendance par rapport au ensemble moins bon.*

La majorité des travaux sur le problème du classement des individus lors de l'établissement de classements par rapport aux coalitions formées par eux sont effectués sur la base de l'étude axiomatique. Une approche plus pratique dans ce contexte est donnée dans [Fayard and Escoffier, 2018], qui est basée sur une solution introduite dans le chapitre 2. Comme la solution introduite ne garantit pas la transitivité du classement sur les individus, dans leur document, les auteurs mettent en œuvre une règle de classement social pour trouver une approximation de l'endroit où un nombre minimum de coalitions sont supprimées afin de satisfaire la transitivité. Ils ont appelé la solution *CP-majorité avec coalitions maximales*. Une autre approche empirique est fournie dans [Allouche et al., 2020]. Dans cet article, les auteurs étudient la manipulabilité des règles de classement social lorsque chaque individu préfère améliorer sa position dans le classement social. Afin d'être cohérent dans la thèse, fixons les notations que nous utilisons dans les chapitres.

Chapitre 2: Ceteris Paribus Règle de la majorité

Dans ce chapitre, nous présentons notre première approche pour résoudre le problème du classement des individus lorsqu'un classement ordinal sur les coalitions formées par eux est donné. Ce problème est d'une grande pertinence dans le contexte de la théorie de la décision, de la théorie du choix social et de la théorie des jeux. Considérons, par exemple, le problème de l'estimation de la "puissance" des pays dans le processus de prise de décision collective au sein d'un parlement international, ou l'évaluation de l'influence d'une croyance sur la cohérence ou l'incohérence des bases de croyance [Hunter and Konieczny, 2010]. Cependant, dans de telles situations, comme dans beaucoup d'autres, la valeur de chaque groupe (ou coalition) est, en général, difficilement quantifiable, les seules informations disponibles sur la force relative des groupes étant purement ordinales. Ainsi, nous supposons qu'étant donné un ensemble d'individus, nous avons en entrée un classement ordinal sur des sous-ensembles (coalitions) d'individus, et en sortie nous recherchons un classement sur l'ensemble des individus. Étant donné que l'objectif de la procédure de classement est de comparer deux individus quelconques, nous supposons que le classement sur les individus est un préordre complet (transitif et réflexif).

Prenant l'exemple d'un département scientifique où les chercheurs collaborent à différents projets en formant des équipes. Supposons que certaines promotions soient disponibles et que le directeur du département veuille les répartir entre les chercheurs qui obtiennent de meilleurs résultats en collaboration avec d'autres. Pour ce faire, le directeur doit être en mesure de classer les chercheurs du meilleur au moins bon. Supposons que la seule information fournie au directeur est la performance relative des équipes ou des coalitions : une équipe est meilleure qu'une autre, ou les deux équipes ont la même performance. Nous appelons le classement sur les coalitions *relation de pouvoir*. Plus précisément, étant donné un ensemble de N individus, si nous interprétons des sous-ensembles d'individus comme des coalitions, une relation de pouvoir ( ) est une relation binaire sur l'ensemble des coalitions, qui indique leur performance relative ($2^N \times 2^N$). Notez que, dans ce chapitre, nous n'imposons aucune propriété sur les relations binaires dans le domaine d'une solution (l'ensemble des relations de pouvoir).

Chapitre 3: Solution Banzhaf Ordinale

Dans la théorie de la prise de décision et du choix social, un certain nombre d'études sont consacrées au classement des individus en fonction des performances des coalitions qu'ils forment. Comme nous l'avons étudié au chapitre 1, les indices de pouvoir de l'individu, comme l'indice de Banzhaf [Banzhaf III, 1964] et la valeur de Shapley [Shapley, 1953] sont décrits à partir de la nécessité de mesurer le pouvoir *a priori* de l'individu dans certains jeux coopératifs (jeux simples). Ces indices de pouvoir sont basés sur le rôle que chaque individu peut obtenir lorsqu'il rejoint une coalition, role est codifié avec la notion de *contribution marginale* . Ces méthodes peuvent être utilisées dans diverses applications, par exemple pour trouver les objets les plus "précieux", lorsque les préférences d'un utilisateur sont définies par rapport à leurs combinaisons ; ou pour comparer l'influence des différents pays au sein d'un conseil international (par exemple, le Conseil de l'Union européenne).

Dans la théorie des jeux coopératifs, certaines hypothèses sont faites de manière conventionnelle. Par exemple, on suppose que les coalitions sont quantifiables et que leurs valeurs sont monotones, en ce sens que si une coalition fait partie d'une autre coalition, la valeur de la première coalition sera inférieure ou égale à la valeur de la seconde. Cependant, dans de nombreuses situations pratiques, il n'est pas possible de calculer la valeur des coalitions de manière quantitative, de même la monotonie ne tient pas nécessairement. Par exemple, la valeur d'une coalition peut diminuer en raison de l'adhésion de nouveaux membres lorsque le coût de la communication et de la coopération entraîne des frais généraux, ou lorsque certaines personnes au sein des coalitions ne sont pas amies et ont une synergie négative entre elles. Ces possibilités nous incitent à supposer l'existence d'une relation binaire sur des ensembles de coalitions.

Dans ce chapitre, comme dans le chapitre 2, nous supposons qu'une relation binaire sur des sous-ensembles d'individus est donnée, et nous cherchons une cartographie pour transformer le classement sur des sous-ensembles d'individus en un classement sur l'ensemble des individus, qui est un préordre complet. En suivant le concept principal de majorité, dans cette section, nous utilisons une partie des comparaisons dans la relation de pouvoir qui indique d'une certaine manière la version ordinale des contributions marginales classiques des individus [Banzhaf III, 1964]. Dans ce chapitre, nous appelons la règle de classement social *relation ordinale de Banzhaf*. Pour cette solution, nous fournissons une caractérisation axiomatique qui s'inspire principalement de l'étude axiomatique du chapitre 2 pour la solution majoritaire *ceteris paribus* sur un ensemble de deux individus.

La règle de la majorité CP et la solution ordinale de Banzhaf suggèrent toutes deux une interprétation de notre problème de classement social sur le modèle d'une élection virtuelle, avec des groupes d'individus (coalitions) jouant le rôle d'électeurs : selon la solution de la majorité CP, une coalition S préfère l'individu i à l'individu j si $S \quad \{i\} \quad S \quad \{j\}$, c'est-à-dire que la coalition $S \quad \{i\}$ est "plus forte" que la coalition $S \quad \{j\}$; selon la solution ordinale de Banzhaf, la coalition S approuve un individu i si $S \quad \{i\} \quad S$, c'est-à-dire que la contribution marginale de i à $S \quad \{i\}$ est positive. Selon cette interprétation, nous proposons une nouvelle famille de relations sur les éléments de N que nous appelons relations *majorité pondérée*. Nous enquêtons sur certains membres de la famille et nous montrons que la majorité CP et la solution Banzhaf ordinale sont des cas particuliers de cette famille, lorsque la relation de pouvoir est un ordre linéaire sur les coalitions.

Les contributions de ce chapitre sont publiées dans les actes de la conférence internationale IJCAI-19 [Khani et al., 2019].

Chapitre 4: Famille de Règles de Classement Pondéré

Dans ce chapitre, nous établissons une nouvelle approche afin de classer les individus lorsqu'un classement ordinal sur les coalitions qu'ils forment existe. Rappelant le principe de majorité *ceteris paribus* du chapitre 2, il suggère une interprétation de notre problème de classement sur le modèle d'une élection virtuelle, avec des groupes d'individus (coalitions) jouant le rôle d'électeurs. Toutefois, il diffère d'un scénario de vote classique dans lequel les candidats peuvent également être des électeurs. On peut avancer que la règle de classement correspondante ne tient pas compte d'une partie importante des informations sur la relation de pouvoir. Par exemple, elle attribue le même pouvoir de vote aux coalitions (en tant qu'électeurs), alors que les coalitions sont formées par différentes combinaisons d'individus. En outre, étant donné que les individus dans une relation de pouvoir peuvent jouer différents rôles en tant que membres de l'électorat ou en tant qu'alternative (sur la base de la comparaison ceteris paribus), l'identité des individus comparés par une coalition peut affecter la valeur des comparaisons effectuées par cette coalition.

Example 36. *Considérez la relation de pouvoir* 145 245 1234 23 12 13 35 14 24*. La règle de la majorité ceteris paribus classe l'individu* 1 *plus haut que l'individu* 2 *puisque, en se référant aux comparaisons de la CP correspondantes (*145 245*,* 14 24 *et* 23 13*), l'individu* 1 *obtient de meilleurs résultats que l'individu* 2 *en rejoignant deux coalitions, tandis que l'individu* 2 *obtient de meilleurs résultats que l'individu* 1 *lorsqu'il rejoint une coalition. Cependant, en fonction du contexte du problème de classement, la coalition* 45 *peut avoir un pouvoir de vote différent de celui des coalitions* 4 *et* 3*. De plus, comme la coalition* 3 *compare plus d'individus que la coalition* 4 *(la coalition* 3 *compare toutes les paires possibles d'individus* 1*,* 2*,* 5 *alors que la coalition* 4 *ne compare que* 1 *et* 2*), ils pourraient avoir des pouvoirs de vote différents.*

La possibilité que, selon le principe de la majorité *ceteris paribus*, les coalitions aient un pouvoir de vote différent est valable dans de nombreux contextes réels. Par exemple, supposons que le président d'une entreprise veuille comparer les employés sur la base des évaluations faites par les comités des employés. Chaque comité peut comparer deux employés qui n'en font pas partie, en disant qu'un employé est plus performant que l'autre ou qu'ils sont indifférents. Supposons que le président de l'entreprise applique la règle de la majorité afin de combiner les évaluations des comités concernant deux employés quelconques. Le président de la société peut pondérer les évaluations faites par un comité en fonction des membres du comité et (ou) des autres employés qui sont comparés par le comité. Cette approche est facile à justifier. Supposons que les comités suivent une méthode de vote afin de procéder à l'évaluation. S'il y a des "membres statutaires" qui imposent leurs opinions aux autres membres du comité, le président de l'entreprise peut décider d'accorder moins de valeur aux comparaisons faites par le comité parce qu'il sait que certains membres du comité peuvent avoir une opinion différente. D'autre part, si tous les membres d'un comité respectent les modes de décision démocratiques, le président de la société décide probablement d'accorder une plus grande valeur à l'évaluation faite par le comité, car il sait que cette évaluation bénéficie d'un grand soutien de la part des membres du comité. Dans un autre contexte, supposons que les employés soient répartis entre différents projets, disons les projets A, B et C. Si un comité dont les membres travaillent sur le projet A évalue les employés qui travaillent sur les projets B ou C, le président de l'entreprise peut alors accorder moins de valeur à l'évaluation parce qu'ils ne travaillent pas sur le même sujet, ou il peut l'apprécier davantage en justifiant que le comité peut regarder de l'extérieur, et que cette evaluation est plus efficace pour comparer les employés. Dans une approche encore différente, si le processus d'évaluation dans les comités est basé sur le vote à la majorité, le président de l'entreprise peut avoir tendance à donner plus de poids à l'évaluation faite par des comités de plus grande taille, en raison du plus grand nombre des partisans de l'évaluation. Il est également possible pour le président de l'entreprise de mesurer la valeur des évaluations effectuées par un comité en fonction de son niveau de participation au processus d'évaluation, ce qui se reflète dans le nombre de comparaisons effectuées par le comité.

Toutes ces considérations suggèrent de définir des versions pondérées de la règle de majorité *ceteris paribus*, dans laquelle chaque coalition, en tant qu'électeur, est pondérée par une fonction de pondération. Selon les paramètres du problème de classement, le poids attribué à une coalition est fonction de différents facteurs comme la coalition (sa structure interne) et (ou) les comparaisons effectuées par la coalition, ou la taille de la coalition et (ou) le nombre de comparaisons effectuées par la coalition.

Jusqu'à présent, nous avons vu la nécessité de pondérer les coalitions lorsqu'un principe de majorité *ceteris paribus* est suivi pour classer les individus ayant une relation de pouvoir. Pour classer plus de deux individus, la fonction d'agrégation que nous utilisons s'appuie sur les travaux de [Terzopoulou and Endriss, 2019] où ils se concentrent sur la caractérisation normative des modes de scrutin selon lesquels chaque agent a un poids qui ne dépend que de la taille de son bulletin de vote, c'est-à-dire du nombre de paires d'alternatives pour lesquelles il choisit de déclarer un classement relatif. Ils ont conçu une règle de pondération qui sélectionne une préférence acyclique par rapport aux alternatives et qui maximise la somme des poids cumulés attribués à chaque paire dans un profil de préférence exprimé par les électeurs.

Nous pensons que cette règle de pondération, que nous appellerons règle de classement social, correspond le mieux à notre modèle puisque, quelle que soit la relation de pouvoir, elle donne toujours un classement collectif acyclique sur un ensemble d'individus.

Comme nous l'avons vu, la fonction de pondération dans une règle de classement social dépend de certains facteurs liés à la structure de la relation de pouvoir (pas nécessairement complète), qui est basée sur les paramètres du problème de classement. L'utilisation de facteurs alternatifs pour définir une fonction de pondération donne lieu à des règles de classement social spécifiques. Comme il existe une infinité de façons de définir une fonction de pondération en fonction d'un ensemble d'arguments, les règles de classement social correspondantes forment, ensemble, une *famille* de règles de classement social.

Les différentes familles de règles de majorité *ceteris paribus* pondérées ont une relation d'inclusion les unes avec les autres. La relation d'inclusion entre ces familles de solutions forme un arbre. Les feuilles de l'arbre présentent des extensions pondérées des règles de majorité *ceteris paribus*, dans lesquelles la fonction de pondération est définie de manière unique. L'objectif principal de ce chapitre est d'analyser les propriétés qui caractérisent la relation d'inclusion entre deux familles de règles de classement social dans l'arborescence mentionnée.

Afin de définir le modèle de classement dans ce chapitre, nous appelons une relation de pouvoir un ensemble d'"ensembles d'informations". L'ensemble d'informations d'une coalition $S \quad 2^N$ dans la relation de pouvoir est défini comme suit :

$$_S = \{(i, j)|i \quad S \quad j \quad S \text{ s.t } i, j \quad N, i, j \ / S, i = j\}$$

C'est pourquoi nous formulons une famille générale de règles de classement par poids comme une règle de classement de type Kemeny : Étant donné un ensemble N d'individus et une relation de pouvoir $\mathcal{B}(2^N)$, appelons $|R \cap \quad _S|$ la similarité entre un ordre linéaire R et un ensemble d'informations $_S$. Une règle de majorité pondérée *ceteris paribus* met en correspondance chaque relation de pouvoir avec les ordres linéaires qui maximisent une somme pondérée de similarités correspondantes. Formellement, elle est définie comme suit.

Definition 6.0.3 (Règle de majorité pondérée Ceteris Paribus(CP))**.** *Une règle de majorité pondérée Ceteris Paribus(CP) est une fonction F_w qui fait correspondre une relation de pouvoir donnée $\mathcal{B}(2^N)$ à un sous-ensemble d'ordres linéaires sur l'ensemble N des individus, c'est-à-dire $F_w : \mathcal{B}(2^N) \rightarrow 2^{\mathcal{L}(N)}$:*

$$F_w(\quad) = \underset{R \quad \mathcal{L}(N)}{\text{argmax}} \sum_{S \quad 2^N} w(S, \quad _S) \cdot |R \cap \quad _S|. \tag{6.1}$$

Dans cette définition, w fait référence à une fonction de pondération binaire qui attribue à toute paire de coalition ordonnée et à son ensemble d'informations $((\quad _S, S))$ un nombre réel positif, $w : 2^{N \times N} \times 2^N \rightarrow \mathbb{R}^+$. Ce nombre positif exprime la façon dont chaque paire d'individus $ij \quad _S$ dans l'ensemble d'informations de la coalition S est pondérée par le système de classement. De même, nous désignons la valeur de la fonction de pondération comme étant le poids de la coalition S dans la relation de pouvoir .

La définition de la famille générale des règles de classement des poids ainsi que la structure des relations de pouvoir ouvrent la possibilité de spécifier différentes sous-familles de règles de classement des poids où la fonction de pondération dépend d'une partie spécifique des informations relatives aux ensembles d'informations et aux coalitions.

La figure 6.1 représente une structure arborescente dont les nœuds sont différentes façons possibles de définir les règles de classement des poids. Cette structure met en évidence la relation "être un sous-ensemble de" entre deux règles de pondération quelconques.

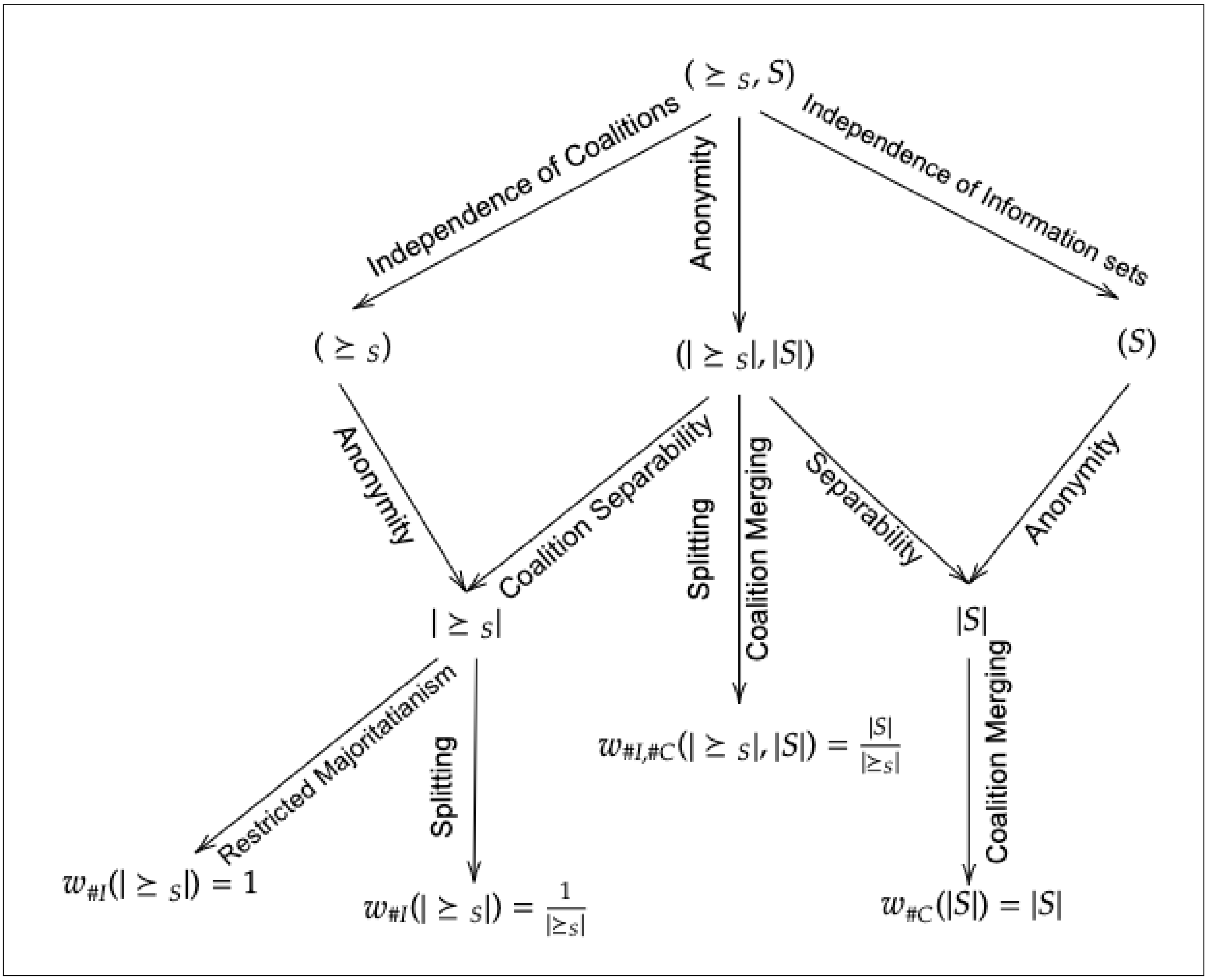

Figure 6.1: Being sub-family relation between different families.

La liste suivante illustre les règles de classement des poids relatives à chaque nœud de l'arbre:

- Noeud (S) : la fonction de pondération des membres de la famille concernée associe à chaque sous-ensemble d'individus (coalition) un nombre réel positif comme son poids : $w_C : 2^N \to \mathbb{R}^+$. La famille de ces règles de majorité CP pondérées est indiquée par $\mathcal{F}_{w_C}$, ses membres sont également spécifiés par F_{w_C} :

$$F_{w_C}(\succeq) = \underset{R \in \mathcal{L}(N)}{\text{argmax}} \sum_{S \in 2^N} w_C(S) \cdot |R \cap \succeq_S|. \tag{6.2}$$

- Noeud $(|\succeq_S|, |S|)$: la fonction de pondération pour les membres de la famille apparentée est $w_{(\#I,\#C)} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ qui attribue à chaque paire de nombres naturels un nombre réel positif. Chaque paire de nombre naturels illustre la paire ordonnée de tailles d'ensembles d'informations et de tailles de coalition. Nous indiquons la famille de ces règles de majorité CP pondérée comme étant $\mathcal{F}_{w_{(\#I,\#C)}}$, et chaque membre $F_{w_{(\#I,\#C)}} \quad \mathcal{F}_{w_{(\#I,\#C)}}$ est défini comme :

$$F_{w_{(\#I,\#C)}}(\quad) = \operatorname{argmax}_{R \quad \mathcal{L}(N)} \square_{S \quad 2^N} w_{(\#I,\#C)}(|\quad_S|, |S|) \cdot |R\cap \quad_S|. \tag{6.3}$$

- Nœud $(|\quad_S|)$: pour les membres de la famille correspondante, la fonction de pondération attribue à tout nombre naturel, représentant la taille de l'ensemble d'informations d'une coalition S, un nombre réel non négatif, $w_{\#I} : \mathbb{N} \to \mathbb{R}^+$. Nous montrons la famille de ces règles de rang comme $\mathcal{F}_{w_{\#I}}$, et chaque $F_{w_{\#I}} \quad \mathcal{F}_{w_{\#I}}$ est le suivant :

$$F_{w_{\#I}}(\quad) = \operatorname{argmax}_{R \quad \mathcal{L}(N)} \square_{S \quad 2^N} w_{\#I}(|\quad_S|) \cdot |R\cap \quad_S|. \tag{6.4}$$

- Nœud $(|S|)$: la fonction de pondération est définie comme $w_{\#C} : \mathbb{N} \to \mathbb{R}^+$, qui attribue à chaque nombre naturel, se référant à la taille de la coalition, un nombre réel positif. Nous indiquons la famille de ces règles de classement comme étant $\mathcal{F}_{w_{\#C}}$, et chaque membre $F_w \quad \mathcal{F}_{w_{\#C}}$ est défini comme ci-dessous :

$$F_{w_{\#C}}(\quad) = \operatorname{argmax}_{R \quad \mathcal{L}(N)} \square_{S \quad 2^N} w_{\#C}(|S|) \cdot |R\cap \quad_S|. \tag{6.5}$$

- Nœud $w_{\#I}(|\quad_S|) = 1$: ce nœud se réfère à une règle de majorité CP pondérée dont la fonction de pondération est considérée comme la fonction constante sur la taille des ensembles d'informations, $w_{\#I}(|\quad_S|) = 1$. Nous appelons cette règle de majorité CP pondérée $F^c_{(w_{\#I})}$, $w_{\#I}(|\quad_S|) = 1, \quad S \quad 2^N$ et elle s'écrit comme ci-dessous :

$$F^c_{w_{\#I}}(\quad) = \operatorname{argmax}_{R \quad \mathcal{L}(N)} \square_{S \quad 2^N} |R\cap \quad_S|. \tag{6.6}$$

- Noeud $w_{\#I}(|\quad_S|) = \frac{1}{|\quad_S|}$: il correspond à une règle de majorité CP pondérée avec la fonction de pondération $w_{\#I}(|\quad_S|) = \frac{1}{|\quad_S|}$. Cette règle de majorité CP pondérée est désignée par $F^p_{w_{\#I}}$, $w_{\#I}(|\quad_S|) = \frac{1}{|\quad_S|}, \quad S \quad 2^N$, et elle est défini comme suit:

$$F^p_{w_{\#I}}(\quad) = \operatorname{argmax}_{R \quad \mathcal{L}(N)} \square_{S \quad 2^N} \frac{1}{|\quad_S|} \cdot |R\cap \quad_S|. \tag{6.7}$$

- Noeud $w_{w_{(\#I,\#C)}}(|\quad_S|, |S|) = \frac{|S|}{|\quad_S|}$: il se réfère à une règle de majorité CP pondérée avec la fonction de pondération $w_{w_{(\#I,\#C)}}(|\quad_S|, |S|) = \frac{|S|}{|\quad_S|}$. Nous indiquons cette règle de majorité CP pondérée sous la forme $F^p_{w_{(\#I,\#C)}}$ comme suit :

$$F^p_{w_{(\#I,\#C)}}(\quad) = \operatorname{argmax}_{R \quad \mathcal{L}(N)} \square_{S \quad 2^N} \frac{|S|}{|\quad_S|} \cdot |R\cap \quad_S|. \tag{6.8}$$

- Nœud $w_{w_{\#C}}(|S|) = |S|$: ce nœud représente la règle de majorité CP pondérée dont la fonction de pondération est considérée comme la fonction d'identité $w_{w_{\#C}}(|S|) = |S|$. Cette règle de majorité CP pondérée est indiquée par $F^{I}_{w_{\#C}}$, et est définie comme suit:

$$F^{I}_{w_{\#C}}(\quad) = \underset{R \quad \mathcal{L}(N)}{\text{argmax}} \sum_{S \quad 2^N} |S| \cdot |R \cap \quad _S|. \tag{6.9}$$

Notez que selon les paramètres de notre problème de classement, une des familles de règles de classement par poids peut être utilisée afin de classer les individus.

Une autre chose importante à propos de la Figure 6.1 est l'étiquetage des bords de l'arbre qui indique les axiomes qui peuvent caractériser une famille de règles de classement par poids comme une sous-famille d'autres règles. Le reste du chapitre est consacré à la justification de l'utilisation des axiomes ainsi que des théorèmes qui caractérisent les familles de solutions.

Étant donné l'arborescence de 6.1, dans le dernier chapitre de la thèse, l'auteur a étudié la possibilité de définir des axiomes afin de caractériser chaque famille de solutions comme une sous-famille des autres. La liste des axiomes ainsi que leur interprétation est présentée ci-dessous. Les proves relatives aux théorèmes de caractérisation se trouvent dans le chapitre 4.

Axiomes

- Independence of Information set : l'axiome *independence of information set* fournit une condition suffisante et nécessaire pour les membres de la famille $\mathcal{F}_{w_C}$ en tant que sous-famille de $\mathcal{F}_{w_{(I,C)}}$. Elle indique que les coalitions ayant des préférences compatibles sur un ensemble d'individus se trouvant dans deux relations de pouvoir différentes, auraient dû être en mesure de combiner leurs préférences et de former une nouvelle relation de pouvoir, sans modifier le classement social des individus.

- Independence of Coalitions: l'axiome *independence of coalitions* caractérise la relation d'inclusion entre les deux familles $\mathcal{F}_{w_{(I,C)}}$ et $\mathcal{F}_{w_I}$. Elle indique que les coalitions doivent pouvoir changer leurs membres, sans que le classement social des individus ne soit modifié.

- Anonymity : l'axiome *anonymity* est celui que nous utilisons pour caractériser la relation d'inclusion entre les règles de majorité *ceteris paribus* pondérées dont les fonctions de pondération dépendent des paires ordonnées de coalitions et de leurs ensembles d'informations, ou coalitions, ou ensembles d'informations et celles dont les fonctions de pondération sont basées sur la cardinalité des facteurs mentionnés. Plus précisément, selon le domaine dans lequel l'axiome est défini, les règles de classement social qui satisfont à l'*anonymity* ne prennent pas en considération les noms des individus et la présence d'interactions entre eux. Nous utilisons cet axiome pour caractériser la relation d'inclusion entre $\mathcal{F}_{w_{(I,C)}}$ et $\mathcal{F}_{w_{(\#I,\#C)}}$ ($\mathcal{F}_{w_{(\#I,\#C)}} \quad \mathcal{F}_{w_{(I,C)}}$), la relation d'inclusion entre $\mathcal{F}_{w_C}$ et $\mathcal{F}_{w_{\#C}}$ ($\mathcal{F}_{w_{\#C}} \quad \mathcal{F}_{w_C}$), et la relation d'inclusion entre $\mathcal{F}_{w_I}$ et $\mathcal{F}_{w_{\#I}}$ ($\mathcal{F}_{w_{\#I}} \quad \mathcal{F}_{w_I}$).

- Coalition Separability : pour caractériser la relation d'inclusion entre $\mathcal{F}_{w_{(\#I,\#C)}}$ et $\mathcal{F}_{w_{\#I}}$ ($\mathcal{F}_{w_{\#I}} \quad \mathcal{F}_{w_{(\#I,\#C)}}$) nous utilisons l'axiome *coalition separability*. Considérons un ensemble de coalitions qui ont quelques membres communs. Cet axiome indique que si elles ont les mêmes ensembles d'informations, on doit s'attendre à ce que la réduction de la taille des coalitions par la suppression des membres répétés dans les coalitions ne change pas le classement final.

- Separability : cet axiome est utilisé pour caractériser la relation d'inclusion entre $\mathcal{F}_{w_{(\#I,\#C)}}$ et $\mathcal{F}_{w_{\#C}}$ ($\mathcal{F}_{w_{\#C}} \quad \mathcal{F}_{w_{(\#I,\#C)}}$). Elle indique que si un ensemble de coalitions de même taille a des préférences mutuellement compatibles sur des individus, le fait de les représenter toutes comme une seule coalition de cette taille effectuant toutes les comparaisons ne devrait pas modifier le classement social.

- Splitting : nous utilisons cet axiome pour caractériser la relation d'appartenance entre la famille $\mathcal{F}_{w_{\#I}}$ et le membre $\mathcal{F}^p_{w_{\#I}}$ ($\mathcal{F}^p_{w_{\#I}} \quad \mathcal{F}_{w_{\#I}}$). L'idée est que si plusieurs coalitions ont des préférences mutuellement compatibles, il devrait leur être possible de former un pacte préélectoral et de signaler l'union de leurs ensembles de préférences individuelles, et cela ne devrait pas changer le résultat du classement des individus.

- Coalition Merging : il est utilisé pour caractériser $\mathcal{F}^I_{w_{\#C}}$ comme membre de $\mathcal{F}_{w_{\#C}}$ ($\mathcal{F}^I_{w_{\#C}} \quad \mathcal{F}_{w_{\#C}}$). Cet axiome donne l'idée que lorsqu'un groupe de coalitions dispose des mêmes ensembles d'informations (préférences), elles devraient pouvoir fusionner et former une coalition contenant tous les membres des coalitions précédentes disposant du même ensemble d'informations, sans modifier le classement des individus.

- Restricted Majoritarianism : la règle de classement social $\mathcal{F}^c_{w_{\#I}}$ est caractérisée comme un membre de la famille $\mathcal{F}_{w_{\#I}}$ par cet axiome qui exprime une des approches normatives fondamentales dans le classement des individus (si dans plus de comparaisons *ceteris paribus* i fait mieux que j, alors il devrait être classé plus haut).

- Les deux axiomes *separability* et *coalition merging* peuvent également être utilisés pour caractériser la règle de classement social $\mathcal{F}^p_{w_{(\#I,\#C)}}$ comme un membre de la famille de $\mathcal{F}_{w_{(\#I,\#C)}}$ ($\mathcal{F}^p_{w_{(\#I,\#C)}} \quad \mathcal{F}_{w_{(\#I,\#C)}}$).

Chapitre 5: Les Travaux à Venir

Une approche complémentaire pour soutenir les méthodes de classement social proposées dans la thèse consiste à les utiliser dans des applications réelles. De nombreux contextes réels sont envisageables pour appliquer les méthodes de classement. Considérons par exemple le scénario de classement suivant:

Imaginez un problème où une agence de marketing veut offrir une collection de produits à ses clients, et où l'agence veut que la collection apporte plus de satisfaction à ses clients. Les clients sont satisfaits d'une collection de produits proposée s'ils peuvent facilement sélectionner un produit parmi les autres, c'est-à-dire en étant sûrs que le produit choisi correspond à ce qu'ils recherchent. C'est le cas lorsque le lot de produits proposé contient une gamme variée de produits du point de vue des clients. Toutefois, chaque client ne peut accorder qu'une attention limitée à certaines caractéristiques spécifiques des produits. Par exemple, supposons que l'agence de commercialisation propose aux clients une collection d'ordinateurs portables. Les clients ont des priorités différentes pour choisir un ordinateur portable. Pour certains d'entre eux, la capacité de calcul des ordinateurs portables est importante, tandis que pour les autres, leur capacité de jeu est plus essentielle. Comme l'agence de marketing n'est pas totalement consciente des goûts de ses clients, la question est de savoir quelle collection de produits devrait être proposée pour apporter plus de satisfaction aux clients. Une façon de répondre à cette question est de classer les produits en fonction de la variété qu'ils apportent aux différents lots de produits lorsqu'ils les rejoignent, et d'utiliser ces informations pour former des collections appropriées. Pour ce faire, il est possible de proposer un nombre limité de lots aux clients et d'évaluer leur niveau moyen de satisfaction. Comme le niveau de satisfaction ne peut être estimé précisément à l'aide de chiffres, nous considérons simplement le cas où une liasse est plus ou moins satisfaisante qu'une autre. Cette information forme une relation de pouvoir. En appliquant les méthodes de classement proposées dans la thèse, nous pouvons classer les produits en fonction de leur contribution à la diversification ou à l'hétérogénéité des lots de produits. Dans l'exemple des ordinateurs portables, considérons le cas où l'agence de marketing possède un ensemble de quatre ordinateurs portables différents $\{A, B, C, D\}$, et qu'elle veut proposer à ses clients le lot d'ordinateurs portables le plus approprié, c'est-à-dire la collection qui apporte le plus de satisfaction aux clients. Dans un cas très simple, supposons que l'agence évalue la satisfaction des clients en leur proposant les quatre offres groupées AB et BD, ABD et AC, et supposons que se forme la relation de pouvoir qui indique le niveau moyen de satisfaction des clients à propos des offres groupées : $AB \quad AC \quad ABD \quad BD$. Supposons qu'une règle de classement social R attribue à une relation de pouvoir donnée un ensemble d'ordres linéaires sur les produits, de sorte que chaque ordre linéaire indique le placement possible des produits sur une ligne en fonction d'une échelle de similarité. Par exemple, étant donné la relation de pouvoir, la règle de classement social considère A et B comme les deux côtés d'un spectre (puisque AB est la collection de produits la plus diversifiée), et place D entre A et B (parce que l'ensemble ABD est moins diversifié) et près de B (parce que l'ensemble BD n'est pas assez diversifié). En utilisant le classement des produits, il est possible de modifier le rapport de force en le rendant plus complet ou en évitant que certains des lots ne se forment. Par exemple, en positionnant les produits dans l'exemple des ordinateurs portables, on peut déduire que l'offre groupée AD est suffisamment diversifiée pour être proposée aux clients. Afin que la modification soit cohérente avec la relation de pouvoir, nous devons rechercher des méthodes de classement (si elles existent) qui sont l'inverse exact des méthodes que nous avons appliquées sur la relation de pouvoir pour classer les produits. Cette application couvre à la fois les problèmes de passage du classement des individus au classement sur des sous-ensembles d'individus et de mise en correspondance du classement sur des sous-ensembles d'individus avec le classement sur des individus.

Une autre application de nos solutions de classement social se situe dans le contexte de l'agrégation des croyances. Dans de nombreuses applications récentes, le problème de la mesure de l'effet de chaque croyance sur l'incohérence d'une base de croyances est modélisé comme un jeu coopératif. Dans ce jeu, la fonction caractéristique de chaque coalition (base de croyance) est le degré d'incohérence de la base de croyance. En utilisant des demi-valeurs comme la valeur de Shapley, on peut mesurer la part de chaque croyance dans l'incohérence de la base de croyance [Hunter and Konieczny, 2010]. Cependant, comme dans de nombreuses situations réelles, la quantification de l'incohérence n'est pas simple, on peut seulement supposer qu'une base de croyances est ordinairement plus ou moins cohérente qu'une autre et mesurer la "contribution marginale ordinale" de chaque croyance pour rendre une base de croyances incohérente. Supposons par exemple qu'un agent exposé par un ensemble de croyances souhaite mettre régulièrement à jour ses croyances. Pour ce faire, l'agent peut imaginer une relation de pouvoir hypothétique dans laquelle des coalitions se forment en ajoutant différentes croyances à sa base de croyances actuelle. Compte tenu de la relation de pouvoir, l'agent peut classer les croyances en fonction de l'incohérence qu'elles provoquent et ajouter celle qui est la plus cohérente avec sa base de croyances actuelle [Serramia et al., 2020].

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RÉSUMÉ

La conception de procédures visant à classer les personnes en fonction de leur comportement dans des groupes est d'une grande importance dans de nombreuses situations. Le problème se pose dans une variété de scénarios de la théorie du choix social, de la théorie des jeux coopératifs ou de la théorie de la décision multi-attributs. Cependant, dans de nombreuses applications du monde réel, une évaluation précise sur les "coalitions de pouvoir" peut être difficile pour de nombreuses raisons. Dans ce cas, il peut être intéressant de ne considérer que les informations ordinales concernant les comparaisons binaires entre les coalitions. L'objectif de cette thèse est d'étudier le problème de la recherche d'un classement ordinal sur l'ensemble N d'individus (appelé classement social),en lui attribuant un rang ordinal par rapport à son ensemble de pouvoir (appelé relation de pouvoir). Pour ce faire, nous utilisons des notions de la théorie de vote classique et la théorie des jeux coopératifs. Nous avons principalement défini des concepts de solution nommés règle de majorité *ceteris paribus*, et l'indice ordinal Banzhad, qui sont respectivement inspirées de la théorie de vote classique et de la théorie des jeux coopératifs. Comme la majorité de notre travail de thèse consiste à étudier des solutions à partir d'une approche fondée sur la propriété, nous étudions axiomatiquement les solutions en reformulant les axiomes dans la théorie classique du vote. Enfin, l'exploration des extensions pondérées de la règle de la majorité *ceteris paribus* pour classer plus de deux personnes, engendre une étude des familles de solutions pondérées.

MOTS CLÉS

Indices de Pouvoir Ordinale, Classement Sociale, Étude Axiomatique, La majorité de Ceteris Paribus, Relation Ordinalel Banzhaf,Règles Pondérées.

ABSTRACT

The design of procedures aimed at ranking individuals according to how they behave in various groups is of great importance in many practical situations. The problem occurs in a variety of scenarios coming from social choice theory, cooperative game theory or multi-attribute decision theory, and examples include: comparing researchers in a scientific department by taking into account their impact across different teams; finding the most influential political parties in a parliament based on past alliances within alternative majority coalitions; rating attributes according to their influence in a multi-attribute decision context, where independence of attributes is not verified because of mutual interactions. However, in many real world applications, a precise evaluation on the coalitions' "power" may be hard for many reasons (e.g., uncertain data, complexity of the analysis, missing information or difficulties in the update, etc.). In this case, it may be interesting to consider only ordinal information concerning binary comparisons between coalitions. The main objective of this thesis is to study the problem of finding an ordinal ranking over the set N of individuals (called *social ranking*), given an ordinal ranking over its power set (*called power relation*). In order to do that, during the thesis we use notions in classical voting theory and cooperative game theory. Mainly, we have defined solution concepts named *ceteris paribus* majority rule, and ordinal Banzhad index, which are respectively inspired from classical voting theory and cooperative game theory. Since the majority of our work in the thesis is to study solutions from property-driven approach, we axiomatically study the solutions by reformulating axioms in classical voting theory. Finally, exploring weighted extensions of the *ceteris paribus* majority rule to rank more than two individuals result in an axiomatic study of families of weighted solutions.

KEYWORDS

Ordinal power indices, Social ranking, Axiomatic Study, Ceteris Paribus majority, ordinal Banzhaf relation, weighted rules.