Construction of blowup solutions for the Complex Ginzburg-Landau equation with critical parameters

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Abstract

We construct a solution for the Complex Ginzburg-Landau (CGL) equation in a general critical case, which blows up in finite time T only at one blow-up point. We also give a sharp description of its profile. In a first part, we construct formally a blow-up solution. In a second part we give the rigorous proof. The proof relies on the reduction of the problem to a finite dimensional one, and the use of index theory to conclude. The interpretation of the parameters of the finite dimension problem in terms of the blow-up point and time allows to prove the stability of the constructed solution. We would like to mention that the asymptotic profile of our solution is different from previously known profiles for CGL or for the semilinear heat equation.

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1 Introduction

We consider the following Complex Ginzburg-Landau (CGL) equation

$$\begin{aligned} &u_t = (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u + \alpha u, \\ &u(.,0) = u_0 \in L^\infty(\mathbb{R}^N, \mathbb{C}) \end{aligned} \quad \text{(CGL)} \qquad (1)$$

where $\delta, \beta, \alpha \in \mathbb{R}$.

This equation, most often considered with a cubic nonlinearity ($p = 3$), has a long history in physics (see Aranson and Kramer [AK02]). The Complex Ginzburg-Landau

(CGL) equation is one of the most studied equations in physics. It describes a lot of phenomena including nonlinear waves, second-order phase transitions, and superconductivity. We note that the Ginzburg-Landau equation can be used to describe the evolution of amplitudes of unstable modes for any process exhibiting a Hopf bifurcation (see for example Section VI-C, page 37 and Section VII, page 40 from [AK02] and the references cited therein). The equation can be considered as a general normal form for a large class of bifurcations and nonlinear wave phenomena in continuous media systems. More generally, the Complex Ginzburg-Landau (CGL) equation is used to describe synchronization and collective oscillation in complex media.

The study of blow-up, collapse or chaotic solutions of equation (1) appears in many works; in the description of an unstable plane Poiseuille flow, see Stewartson and Stuart [SS71], Hocking, Stewartson, Stuart and Brown [HSSB72] or in the context of binary mixtures in Kolodner and *al*, [KBS88], [KSAL95], where the authors describe an extensive series of experiments on traveling-wave convection in an ethanol/water mixture, and they observe collapse solution that appear experimentally.
For our purpose, we consider CGL independently from any particular physical context and investigate it as a mathematical model in partial differential equations with $p > 1$.

We note also that the interest on the study of singular solutions in CGL comes also from the analogies with the three-dimensional Navier-Stokes. The two equations have the same scaling properties and the same energy identity (for more details see the work of Plecháč and Šverák [PŠ01]; the authors in this work give some evidence for the existence of a radial solution which blow up in a self-similar way). Their argument is based on matching a numerical solution in an inner region with an analytical solution in an outer region. In the same direction we can also cite the work of Rottschäfer [Rot08] and [Rot13].

The Cauchy problem for equation (1) can be solved in a variety of spaces using the semigroup theory as in the case of the heat equation (see [Caz03, GV96, GV97]).
We say that $u(t)$ blows up or collapse in finite time $T < \infty$, if $u(t)$ exists for all $t \in [0, T)$ and $\lim_{t\to T} \|u(t)\|_{L^\infty} = +\infty$. In that case, T is called the blow-up time of the solution. A point $x_0 \in \mathbb{R}^N$ is said to be a blow-up point if there is a sequence $\{(x_j, t_j)\}$, such that $x_j \to x_0$, $t_j \to T$ and $|u(x_j, t_j)| \to \infty$ as $j \to \infty$. The set of all blow-up points is called the blow-up set.
Let us now introduce the following definition;

Definition 1.1 *The parameters (β, δ) are said to be critical (resp. subcritical, resp. supercritical) if $p - \delta^2 - \beta\delta(p+1) = 0$ (resp. > 0, resp. < 0). In addition to that, we also define some critical constants as follows:*

$$p_{cri} = \begin{cases} \sqrt{\frac{p(2p-1)}{p-2}} & \text{if } p > 2 \\ +\infty & \text{if } p \in (1, 2] \end{cases}, \tag{2}$$

and

$$b_{cri}^2 = \frac{(p-1)^4(p+1)^2\delta^2}{16(1+\delta^2)(p(2p-1)-(p-2)\delta^2)((p+3)\delta^2+p(3p+1))} > 0, \tag{3}$$

for all $\delta \in (-p_{cri}, p_{cri})$.

Remark 1.2 *We choose p_{cri} in such way the denominator in expression* (3) *is strictly positif.*

An extensive literature is devoted to the blow-up profiles for CGL when $\beta = \delta = 0$ (which is the nonlinear heat equation), see Velázquez [Vel92, Vel93a, Vel93b] and Zaag [Zaa02a, Zaa02b, Zaa02c] for partial results). In one space dimension, given a a blow-up point, this is the situation:

- either

$$\sup_{|x-a|\leq K\sqrt{(T-t)\log(T-t)}} \left|(T-t)^{\frac{1}{p-1}}u(x,t) - f\left(\frac{x-a}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| \to 0, \quad (4)$$

- or for some $m \in \mathbb{N}$, $m \geq 2$, and $C_m > 0$

$$\sup_{|x-a|<K(T-t)^{1/2m}} \left|(T-t)^{\frac{1}{p-1}}u(x,t) - f_m\left(\frac{C_m(x-a)}{(T-t)^{1/2m}}\right)\right| \to 0, \quad (5)$$

  as $t \to T$, for any $K > 0$, where

$$\begin{aligned} f(z) &= \left(p-1+b_0 z^2\right)^{-\frac{1}{p-1}} \text{ with } b_0 = \tfrac{(p-1)^2}{4p}, \\ f_m(z) &= \left(p-1+|z|^{2m}\right)^{-\frac{1}{p-1}}. \end{aligned} \quad (6)$$

If $(\beta, \delta) \neq (0,0)$, some results are available in the *subcritical* case by Zaag [Zaa98] ($\beta = 0$) and Masmoudi and Zaag [MZ08] ($\beta \neq 0$). More precisely, if

$$p - \delta^2 - \beta\delta(p+1) > 0,$$

then, the authors construct a solution of equation (1), which blows up in finite time $T > 0$ only at the origin such that for all $t \in [0, T)$,

$$\left\|(T-t)^{\frac{1+i\delta}{p-1}}|\log(T-t)|^{-i\mu}u(x,t) - \left(p-1+\frac{b_{sub}|x|^2}{(T-t)|\log(T-t)|}\right)^{-\frac{1+i\delta}{p-1}}\right\|_{L^\infty} \leq \frac{C_0}{1+\sqrt{|\log(T-t)|}}, \quad (7)$$

where

$$b_{sub} = \frac{(p-1)^2}{4(p-\delta^2-\beta\delta(1+p))} > 0 \text{ and } \mu = -\frac{2b_{sub}\beta}{(p-1)^2}(1+\delta^2). \quad (8)$$

Note that this result was previously obtained formally by Hocking and Stewartson [HS72] ($p = 3$) and mentioned later in Popp et *al* [PSKK98] (see those references for more blow-up results often aproved numerically, in various regimes of the parameters).

In the *critical* case, few results are known about blow-up solutions for the equation. Up to our knowledge, there are two blow-up results in the critical case: one formal by

Popp and *al* [PSKK98], when $p = 3$ and the result by the second and thirs author in [NZ18] when $\beta = 0$.

Let us now give the formal result, when $p = 3$, given by Popp and *al* [PSKK98] (equations (44) and (64)) in the *critical* case $(3 - 4\delta\beta - \delta^2 = 0)$: the authors obtained

$$(T-t)^{\frac{1+i\delta}{p-1}} u(x,t) \sim e^{i\psi(t)} \left(2 + \frac{b_{PSKK}|x|^2}{(T-t)|\log(T-t)|^{\frac{1}{2}}}\right)^{-\frac{1+i\delta}{2}}. \tag{9}$$

where

$$b_{PSKK} = 2\left(\sqrt{\frac{3}{2\delta^2}(\delta^2+5)(\delta^2+1)(15-\delta^2)}\right)^{-1}. \tag{10}$$

and $\psi(t)$ is given by equation (40) in [PSKK98]. We can clearly see that this profile exist only for $\delta^2 < 15$, when $p = 3$.

Remark 1.3 *We will see later, in Section 2, that we obtain the formally the same b_{PSKK}, for any $p > 1$. Moreover, the constant in* (10) *will be proven to be true in the rigorous proof.*

As for the result of [NZ18], we simply say that as far as statements are concerned, it is a particular case of our new result given in Theorem 1 below (just take $\beta = 0$). However, we stress the fact that the profs of $\beta = 0$ and $\beta \neq 0$ are very different technically.

1.1 Statement of our result

Our main claim is to construct a solution $u(x,t)$ of (1) in the critical case ($\beta \neq 0$ and $p - \beta\delta(p+1) - \delta^2 = 0$) that blows up in some finite time T, in the sense that

$$\lim_{t\to T} \|u(.,t)\|_{L^\infty} = +\infty.$$

We also prove the stability of the constructed solution. This is our first statement:

Theorem 1 (Blow-up profiles for equation (1)**)** *Let us consider the critical case where $p - \delta^2 - \beta\delta(p+1) = 0$, $\beta \neq 0$ and $\delta \in (0, p_{cri})$ with p_{cri} defined as in* (2)*. Then, there exists a unique constant μ depending on p and δ such that equation* (1) *has a solution $u(x,t)$, which blows up in finite time T, only at the origin. Moreover:*

(i) For all $t \in [0,T)$,

$$\left\|(T-t)^{\frac{1+i\delta}{p-1}} e^{-i\nu\sqrt{|\log(T-t)|}} |\log(T-t)|^{-i\mu} u(.,t) - \varphi_0\left(\frac{.}{\sqrt{(T-t)}|\log(T-t)|^{1/4}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C_0}{1+|\log(T-t)|^{\frac{1}{4}}}, \tag{11}$$

where

$$\varphi_0(z) = \left(p - 1 + b_{cri} z^2\right)^{-\frac{1+i\delta}{p-1}}, \tag{12}$$

with b_{cri} is defined as in (3),

$$\nu = -\frac{4b\beta(1+\delta^2)}{(p-1)^2} \text{ and } a = 2\kappa(1-\beta\delta)\frac{b}{(p-1)^2}. \tag{13}$$

(ii) For all $x \neq 0$, $u(x,t) \to u^*(x) \in C^2(\mathbb{R}^N\backslash\{0\})$ and

$$u^*(x) \sim e^{i\nu\sqrt{2|\log|x||}}|2\log|x||^{i\mu_{cri}}\left[\frac{b_{cri}|x|^2}{\sqrt{2|\log|x||}}\right]^{-\frac{1+i\delta}{p-1}} \quad \text{as } x \to 0. \tag{14}$$

Theorem 2 (First order terms) *Following Theorem 1, we claim that the solution decomposes in self similar variables*

$$W(y,t) = (T-t)^{\frac{1+i\delta}{p-1}}u(x,t), \;\; y = \frac{x}{\sqrt{T-t}},$$

as follows: For $M > 0$

$$\sup_{|y|<M|\log(T-t)|^{\frac14}} \Bigg|W(y,t)e^{-i\nu\sqrt{|\log(T-t)|}}|\log(T-t)|^{-i\mu}e^{i\theta(t)} - \left\{\varphi_0\left(\frac{y}{|\log(T-t)|^{1/4}}\right) + \frac{a(1+i\delta)}{|\log(T-t)|^{\frac12}} + \frac{1}{|\log(T-t)|}\mathcal{F}(y)\right\}\Bigg| \leq \frac{C}{|\log(T-t)|^{\frac32}}(1+|y|^5), \tag{15}$$

and $\theta(t) \to \theta_0$ as $t \to T$, such that

$$|\theta(t) - \theta_0| \leq \frac{C}{|\log(T-t)|^{\frac14}}$$

with

$$\varphi_0(z) = \left(p - 1 + b_{cri}z^2\right)^{-\frac{1+i\delta}{p-1}}, \tag{16}$$

where b_{cri} is defined as in (3)*, ν and a are given by* (13) *and $\mathcal{F}(y)$ is a function defined as follows*

$$\mathcal{F}(y) = \mathcal{A}_0h_0(y) + \mathcal{A}_2h_2(y) + \tilde{\mathcal{A}}_2\tilde{h}_2(y), \tag{17}$$

where $\mathcal{A}_0$, $\mathcal{A}_2$ and $\tilde{\mathcal{A}}_2$ depend only on β and δ and are given by (76) *in Definition 4.1 and $h_0(y)$, $h_2(y)$ and $\tilde{h}_2(y)$ will be given in Lemma 3.2.*

Remark 1.4 *Theorem 2 is true only on the case of $\beta \neq 0$.*

Remark 1.5 *We will give the proof only in one dimension. Our proof remains valid in higher dimensions, with exactly the same ideas, and purely technical differences, that we omit to keep this (already long) paper in reasonable length. Indeed, the computation of the eigenfunctions of $\mathcal{L}_{\beta,\delta}$ (see* (49)*) and the projection of equation* (48) *on the eigenspaces become much more complicated when $N \geq 2$.*

Remark 1.6 *In this paper we will treat the case $\beta \neq 0$, which is far from being a simple adaptation of the case $\beta = 0$, treated in [NZ18]. Indeed, the techniques are different. There is an additional difficulty arising from the linearized operator $\mathcal{L}_{\beta,\delta}$ (see* (49)*), which is not diagonalisable as in the case $\beta = 0$. To avoid this problem, we will use ideas from the work of Masmoudi and Zaag [MZ08]. There is another difficulty, coming from the criticality of the problem, which makes the projection of the linearized equation quite complicated especially for the neutral mode.*

Remark 1.7 *We will consider CGL, given by* (1)*, only when $\alpha = 0$. The case $\alpha \neq 0$ can be done as in [EZ11]. In fact, when $\alpha \neq 0$, exponentially small terms will be added to our estimates in self-similar variable (see* (18) *below), and that will be absorbed in our error terms, since our trap $\mathcal{V}_A(s)$ defined in Definition 4.1 is given in polynomial scales.*

Remark 1.8 *The derivation of the blow-up profile* (12) *can be understood through a formal analysis, using the matching asymptotic expansions. This method was used by Galaktionov, Herrero and Velázquez [VGH91] to derive all the possible behaviors of the blow-up solution given by (4,5) in the heat equation ($\beta = \delta = 0$). This formal method fails in the determination of b_{cri} when $\delta \neq 0$, because of the complexity of the system of ODE in that case.*

We will use in this case the formal method used by Popp and al in [PSKK98]. We note that this method was used by Hocking and Stewartson [HS72] as well as Masmoudi and Zaag [MZ08] ($\delta \neq 0$) to obtain the profile in the subcritical case of CGL, and also by Berger and Kohn for the nonlinear heat equation ($\beta = \delta = 0$).

Remark 1.9 *The exhibited profile* (12) *is new in two respects:*

- *The scaling law in the critical case is $\sqrt{(T-t)|\log(T-t)|^{\frac{1}{2}}}$ instead of the laws of subcritical case, $\sqrt{(T-t)|\log(T-t)|}$, (see* (7)*).*
- *The profile function: $\varphi_0(z) = (p-1+b_{cri}|z|^2)^{-\frac{1+i\delta}{p-1}}$ is different from the profile of the subcritical case , namely $f(z) = (p-1+b_{sub}|z|^2)^{-\frac{1+i\delta}{p-1}}$, in the sense that $b_{cri} \neq b_{sub}$ (see* (8) *and* (3)*).*

Remark 1.10 *In the subcritical case $p - \delta^2 - \beta\delta(p+1) > 0$ ($p - \delta^2 > 0$, when $\beta = 0$), the final profile of the CGL is given by*

$$u^*(x) \sim |2\log|x||^{i\mu} \left[\frac{b}{2}\frac{|x|^2}{|\log|x||}\right]^{-\frac{1+i\delta}{p-1}} \quad \text{as } x \to 0$$

with

$$b_{sub} = \frac{(p-1)^2}{4(p-\delta^2-\beta\delta(p+1))} \text{ and } \mu = -\frac{2b\beta}{(p-1)^2}(1+\delta^2).$$

In the critical case $p - \delta^2 - (p+1)\beta\delta = 0$, the final profile is given by (14).

As a consequence of our techniques, we show the stability of the constructed solution with respect to perturbations in initial data. More precisely, we have the following result.

Theorem 3 (Stability of the solution constructed in Theorem 1) *Let us denote by $\hat{u}(x,t)$ the solution constructed in Theorem 1 and by $\hat{T}$ its blow-up time. Then, there exists a neighborhood $\mathcal{V}_0$ of $\hat{u}(x,0)$ in L^∞ such that for any $u_0 \in \mathcal{V}_0$, equation (1) has a unique solution $u(x,t)$ with initial data u_0, and $u(x,t)$ blows up in finite time $T(u_0)$ at one single blow-up point $a(u_0)$. Moreover estimate (11) is satisfied by $u(x-a,t)$ and*

$$T(u_0) \to \hat{T}, \;\; a(u_0) \to 0 \text{ as } u_0 \to \hat{u}_0 \text{ in } L^\infty(\mathbb{R}^N, \mathbb{C}).$$

Remark 1.11 *We will not give the proof of Theorem 3 because the stability result follows from the reduction to a finite dimensional case as in [MZ97] (see Theorem 2 and its proof in Section 4) and [MZ08] (see Theorem 2 and its proof in Section 6) with the same argument. Hence, we only prove the existence result (Theorem 1) and kindly refer the reader to [MZ97] and [MZ08] for the proof of the stability.*

Let us give an idea of the method used to prove the results. We construct the blow-up solution with the profile in Theorem 1, by following the method of [MZ97], [BK94], though we are far from a simple adaptation, since we are studying the critical problem, which make the technical details harder to elaborate. This kind of methods has been applied for various nonlinear evolution equations. For hyperbolic equations, it has been successfully used for the construction of multi-solitons for semilinear wave equation in one space dimension (see [CZ13]). For parabolic equations, it has been used in [MZ08] and [Zaa01] for the Complex Ginzburg Landau (CGL) equation with no gradient structure, the critical harmonic heat flow in [RS13], the two dimensional Keller-Segel equation in [RS14] and the nonlinear heat equation involving nonlinear gradient term in [EZ11], [TZ19]. Recently, this method has been applied for various non variational parabolic system in [NZ15] and [GNZ17, GNZ18b, GNZ18a, GNZ19], for a logarithmically perturbed nonlinear equation in [NZ16, Duo19b, Duo19a, DNZ19]. We also mention a result for a higher order parabolic equation [GNZ20], two more results for equation involving non local terms in [DZ19, AZ19].

Unlike in the subcritical case of [MZ08] and [Zaa01], the criticality of the problem induces substantial changes in the blow-up profile as pointed-out in the comments following Theorem 1. Accordingly, its control requires special arguments. So, working in the framework of [MZ97], [MZ08], [NZ18], some crucial modifications are needed. In particular, we have overcome the following challenges:

- The prescribed profile was not known before and is not obvious to obtain. See Section 2 for a formal approach to justify such a profile.
- The profile is different from the profile in [MZ97] and [MZ08], therefore new estimates are needed.
- In order to handle the new scaling, we introduce a new shrinking set to trap the solution, see Definition 4.1. Finding such set is not trivial, in particular in the critical case, where we need much more details in the expansions of the rest term (see Appendix D).

Then, following [MZ97], the proof is divided in two steps. First, we reduce the problem to a finite dimensional case. Second, we solve the finite time dimensional problem and conclude by contradiction using index theory. More precisely, the proof is performed in the framework of the similarity variables defined below in (18). We linearize the self-similar solution around the profile φ_0 and we obtain q (see (42) below). Our goal is to guarantee that $q(s)$ belongs to some set $\mathcal{V}_A(s)$ (introduced in Definition 4.1), which shrinks to 0 as $s \to +\infty$. The proof relies on two arguments:

- The linearized equation gives two positives mode; $\tilde{Q}_0$ and $\tilde{q}_1$, one zero modes ($\tilde{q}_2$) and an infinite dimensional negative part. The negative part is easily controlled by the effect of the heat kernel. The control of the zero mode is quite delicate (see Part 2: Proof of Proposition 4.10, page 45). Consequently, the control of q is reduced to the control of its positive modes.
- The control of the positive modes $\tilde{Q}_0$ and $\tilde{q}_1$ is handled thanks to a topological argument based on index theory (see the argument at page 24).

The organization of the rest of this paper is as follows. In Section 2, we explain formally how we obtain the profile. In Section 3, we give a formulation of the problem in order to justify the formal argument. Section 4 is divided in two subsections. In Subsection 4.1 we give the proof of the existence of the profile assuming technical details. In particular, we construct a shrinking set and give an example of initial data giving rise to the blow-up profile. Subsection 4.2 is devoted to the proof of technical results which are needed in the proof of existence. Finally, in Section 5, we give the proof of Theorem 1 and Theorem 2.

2 Formal approach

The aim of this section is to explain formally how we derive the behavior given in Theorem 1. In particular, how to obtain the profile φ_0 in (12), the parameter b_{cri} in (3). In fact, let us consider CGL, given by (1) in the case where $\alpha = 0$ (the case $\alpha \neq 0$ is the same, thanks to what we mentioned before in Remark 1.7).

Firstly, we consider an arbitrary $T > 0$ and introduce the following self-similar variable transformation of equation (1), defined by the following

$$w(y,s) = (T-t)^{\frac{1+i\delta}{p-1}} u(x,t), \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -\log(T-t). \tag{18}$$

As a matter of fact, if $u(x,t)$ satisfies (1) for all $(x,t) \in \mathbb{R}^N \times [0,T)$, then $w(y,s)$ satisfies for all $(x,t) \in \mathbb{R}^N \times [-\log T, +\infty)$ the following equation

$$\frac{\partial w}{\partial s} = (1+i\beta)\Delta w - \frac{1}{2} y.\nabla w - \frac{1+i\delta}{p-1} w + (1+i\delta)|w|^{p-1} w, \tag{19}$$

for all $(y,s) \in \mathbb{R}^N \times [-\log T, +\infty)$. Thus constructing a solution $u(x,t)$ for the equation (1) that blows up at $T < \infty$ like $(T-t)^{-\frac{1}{p-1}}$ is reduced to constructing a global solution $w(y,s)$ for equation (19) such that

$$0 < \varepsilon \leq \lim_{s\to\infty} \|w(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{\varepsilon}. \tag{20}$$

Inspired by the work of Popp and *al* [PSKK98] and Nouaili and Zaag in [NZ18], we consider a new form of w. More precisely, we suppose that $w(y,s) = w(|y|,s)$ and $w\left(rs^{\frac{1}{4}}, s\right) = D(r,s)$, where $r = |y|$. Then, it is easy to see that D satisfied the following

$$\begin{aligned} \partial_s D(r,s) &= (1+i\beta)D''(r,s)\frac{1}{\sqrt{s}} + \left(\frac{1}{4s} - \frac{1}{2}\right) rD'(r,s) - \frac{1+i\delta}{p-1}D(r,s) \\ &+ (1+i\delta)|D|^{p-1}D. \end{aligned} \tag{21}$$

In addition to that, we also assume that we can write D under the following form

$$D(r,s) = R(r,s)e^{i\varphi(r,s)}, \tag{22}$$

where R and φ are real functions. In particular, using equation (22), we obtain the following system, satisfied by R and φ

$$\left\{ \begin{aligned} \partial_s R &= \tfrac{1}{\sqrt{s}}\left[R'' - R(\varphi')^2 - \beta(2R'\varphi' + R\varphi'')\right] + R'r\left(\tfrac{1}{4s} - \tfrac{1}{2}\right) - \tfrac{R}{p-1} + |R|^{p-1}R, \\ \partial_s \varphi &= \tfrac{1}{\sqrt{s}}\left[\varphi'' - \beta(\varphi')^2 + \tfrac{1}{R}\left(2R'\varphi' + \beta R''\right)\right] + \varphi' r\left(\tfrac{1}{4s} - \tfrac{1}{2}\right) - \tfrac{\delta}{p-1} + \delta|R|^{p-1}. \end{aligned} \right. \tag{23}$$

In the following, we will consider the following ansatz, inspired by the work of Popp and *al* [PSKK98]

$$\begin{aligned} R(r,s) &= R_0(r) + \frac{R_1(r)}{\sqrt{s}} + \frac{R_2(r)}{s} + ... \\ \varphi(r,s) &= \Phi(s) + \varphi_0(r) + \frac{\varphi_1(r)}{\sqrt{s}} + \frac{\varphi_2(r)}{s} + ... \end{aligned}$$

where $\Phi(s) = \nu\sqrt{s} + \mu \ln s$ and ν, μ unknown.

Beside that, we introduce the notation that $f' = \partial_r f$. By looking at the leading order, we formally derive that R_0 and φ_0 should satisfy

$$-\frac{1}{2}R_0' r - \frac{R_0}{p-1} + |R_0|^{p-1}R_0 = 0, \tag{24}$$

and

$$-\frac{1}{2}\varphi_0' r - \frac{\delta}{p-1} + \delta|R_0|^{p-1} = 0. \tag{25}$$

Hence, we can explicitly solve (24) and obtain the following

$$R_0(r) = \left(p-1+br^2\right)^{-\frac{1}{p-1}}, \tag{26}$$

for some constant $b \in \mathbb{R}$. However, our aim is finding a global profile, so b will be chosen as a positive constant. From (25) and (26), we deduce that

$$\varphi_0(r) = -\frac{\delta}{p-1}\ln\left(p-1+br^2\right). \tag{27}$$

We now look at the order $\frac{1}{\sqrt{s}}$ in the system (23) and we deduce the following

$$-\frac{1}{2}R_1' r - \frac{R_1}{p-1} + p|R_0|^{p-1}R_1 + R_0'' - R_0\varphi_0'^2 - \beta\left(2R_0'\varphi_0' + R_0\varphi_0''\right) = 0, \tag{28}$$

and

$$-\frac{1}{2}\varphi_1' r + \varphi_0'' - \beta\varphi_0'^2 + R_0^{-1}(2R_0'\varphi_0' + \beta R_0'') + \delta(p-1)|R_0|^{p-2}R_1 - \frac{\nu}{2} = 0. \tag{29}$$

We now solve (28), and obtain

$$R_1(r) = \frac{r^2}{(p-1+br^2)^{\frac{p}{p-1}}} \tag{30}$$

$$\times \left[-\frac{2b(\delta\beta-1)}{p-1}r^{-2} + \frac{8b^2(p-(p+1)\delta\beta-\delta^2)}{(p-1)^3}\left(\ln|r| - \frac{\ln(p-1+br^2)}{2}\right) + \mathcal{C}\right] \tag{31}$$

R_1 and its derivatives should be bounded at least in some inner region around the pulse center. Thus, the contribution $\log|r|$ are suppressed. Consequently, we obtain

$$p - (p+1)\delta\beta - \delta^2 = 0, \tag{32}$$

which is the critical condition in our paper. Besides that, the constant $\mathcal{C}$ is an unknown constant, depending on β, δ and we particularly know from [NZ18] that $\mathcal{C}(0,\delta) = 2\frac{pb^2}{(p-1)^3}$.

We now solve (29) and obtain

$$\varphi_1(r) = \left[-\nu - \frac{4b\beta(1+\delta^2)}{(p-1)^2}\right]\ln|r| + \frac{2\beta(1+\delta^2)b}{(p-1)^2}\ln(p-1+br^2)$$
$$-\frac{2b}{(p-1)^2}\left((p+3)\delta + \beta(2p+\delta^2(p-3)) + \frac{\mathcal{C}\delta(p-1)^3}{2b^2}\right)(p-1+br^2)^{-1}.$$

By the regularity of φ_1 at 0, the contribution of $\log|r|$ need to be removed. This gives us the following condition

$$\nu = -\frac{4b\beta(1+\delta^2)}{(p-1)^2}. \tag{33}$$

At the order $\frac{1}{s}$, we get

$$-\frac{1}{2}R_2' r - \frac{R_2}{p-1} + p|R_0|^{p-1}R_2 + F_3 = 0, \tag{34}$$

and

$$-\frac{1}{2}\varphi_2' r + F_4(r) = 0, \tag{35}$$

where

$$F_3(r) = R_1'' - 2R_0\varphi_0'\varphi_1' - R_1\varphi_0'^2 - 2\beta R_0'\varphi_1' - 2\beta R_1'\varphi_0' - \beta R_0\varphi_1'' - \beta R_1\varphi_0''$$
$$+ \frac{1}{4}R_0' r + \frac{p(p-1)}{2}|R_0|^{p-3}R_0R_1^2. \tag{36}$$

$$F_4(r) = \varphi_1'' - 2\beta\varphi_0'\varphi_1' + 2\left(R_0^{-1}R_0'\varphi_1' + R_0^{-1}R_1'\varphi_0' - R_0'\varphi_0'R_0^{-2}R_1\right) - \mu \tag{37}$$
$$+ \beta\left(R_0^{-1}R_1'' - R_0''R_0^{-2}R_1\right) - \frac{1}{4}\varphi_0' r + \delta\left((p-1)|R_0|^{p-3}R_0R_2 + \frac{(p-1)(p-2)}{2}|R_0|^{p-3}R_1^2\right).$$

We solve (34), by using variation of constant and we obtain that

$$R_2 = H^{-1}(r)\left(\int F_3 \frac{2H}{r} H dr\right),$$

where

$$H(r) = \frac{(p-1+br^2)^{\frac{p}{p-1}}}{r^2}.$$

In particular, by a careful calculation we can write

$$\frac{2H}{r}F_3 = P(p,\delta,\beta)\frac{1}{r} + \text{ “regular term”}.$$

After integrating, we can see that $\frac{1}{r}$ will become $\ln r$. So, in order to remove this term, we need to have

$$P = 0.$$

The computation of P is straightforward though a bit lengthy, the interested reader will find details in Appendix E. Then P is given by the following formula;

$$\begin{aligned}
P &= 2\left\{-\frac{b}{2(p-1)} - \frac{b^3}{(p-1)^4}(\delta\beta-1)\left(\frac{-8\delta^4+(12p-4p^2)\delta^2+8p^3}{(p+1)\delta^2}\right)\right.\\
&+ \frac{b^3}{(p-1)^5}(\delta\beta-1)\left(\frac{(8p^2+20p-8)\delta^4+20p^3\delta^2-8p^2(p^2-1)}{(p+1)\delta^2}\right)\\
&+ \frac{b^3}{(p-1)^4}(\delta\beta-1)\left(\frac{4p(1+\delta^2)}{p+1}\right),\\
&+ \left.\frac{b^3}{(p-1)^5}(\delta\beta-1)\left(\frac{(-4p^2-16p-32)\delta^4+(-16p^3-68p^2)\delta^2-32p^4}{(p+1)\delta^2}\right)\right\}.
\end{aligned}$$

Then, the condition $P = 0$ is equivalent to the following

$$\begin{aligned}
&\frac{b}{2(p-1)}\\
&= \frac{b^3}{(p-1)^5}\frac{(\delta\beta-1)}{(p+1)\delta^2}\left((8p^2+8p-48)\delta^4+(8p^3-80p^2+8p)\delta^2-48p^4+8p^3+8p^2\right)\\
&= \frac{b^3}{(p-1)^5}\frac{8(\delta\beta-1)}{(p+1)\delta^2}\left((p^2+p-6)\delta^4+p(p^2-10p+1)\delta^2+p^2(-6p^2+p+1)\right).
\end{aligned}$$

Plugging the critical condition $p-(p+1)\delta\beta-\delta^2 = 0$ into the above equality, we obtain

$$b^2 = \frac{(p-1)^4(p+1)^2\delta^2}{-16(1+\delta^2)L}, \tag{38}$$

where

$$L = (p^2+p-6)\delta^4+p(p^2-10p+1)\delta^2+p^2(-6p^2+p+1).$$

As a matter of fact, in order to justify the positivity of b^2, we need to have

$$L < 0.$$

This condition is satisfied if and only if

$$\delta \in (-p_{cri}, p_{cri}), \delta \neq 0. \tag{39}$$

where

$$p_{cri} = \sqrt{\frac{p(2p-1)}{p-2}} \text{ if } p > 2, \text{and } p_{cri} = +\infty \text{ if } p \in (1,2]$$

which was already introduced in Definition 1.1.

In the following, we try to determine μ. We use (35), taking $r = 0$, we obtain

$$F_4(0) = 0.$$

By using the definition of F_4 we directly derive (see Appendix E for more details),

$$\begin{aligned} \mu &= \frac{b^2}{(p-1)^4}\left\{8(p+1)\delta + 8p\beta + (4p+8)\delta^2\beta + (16p-8)\delta\beta^2 + (8p-16)\delta^3\beta^2\right\} \\ &+ \frac{2\beta(1+\delta^2)\mathcal{C}}{p-1}. \end{aligned}$$

Note that in μ, there is the unknown constant $\mathcal{C}$ already introduced in (31).

Summary: From the above approach, we can formally derive the profile of our solution as in Theorem 1

$$w(y,s) \sim e^{i(\nu\sqrt{s}+\mu\ln s)}\left(p-1+b_{cri}\frac{|y|^2}{s}\right)^{-\frac{1+i\delta}{p-1}}.$$

In other word, we have

$$(T-t)^{-\frac{1+i\delta}{p-1}} e^{-i\left(\nu\sqrt{|\ln(T-t)|}+\mu\ln(|\ln(T-t)|)\right)} u(x,t) \sim \left(p-1+b_{cri}\frac{|x|^2}{(T-t)|\ln(T-t)|^{\frac{1}{2}}}\right)^{-\frac{1+i\delta}{p-1}},$$

where

$$\begin{aligned} \mu &= \frac{b^2}{(p-1)^4}\left\{8(p+1)\delta + 8p\beta + (4p+8)\delta^2\beta + (16p-8)\delta\beta^2 + (8p-16)\delta^3\beta^2\right\} \\ &+ \frac{2\beta(1+\delta^2)\mathcal{C}}{p-1}, \\ \nu &= -\frac{4b\beta(1+\delta^2)}{(p-1)^2}, \\ b_{cri} &= \frac{(p-1)^2(p+1)\delta}{\sqrt{16(1+\delta^2)(p(2p-1)-(p-2)\delta^2)((p+3)\delta^2+p(3p+1))}}. \end{aligned}$$

Remark 2.1 *We note, that coming at this level in our formal approach, we are not able to determine explicitly μ. But we will see in the rigorous proof that we can obtain the existence and uniqueness of μ (see equation* (142)*, page 50).*

3 Formulation of the problem

We recall that we consider CGL, given by (1), when $\alpha = 0$, as we mentioned before in Remark 1.7.
The preceding calculation is purely formal. However, the formal expansion provides us with the profile of the function ($w(y,s) = e^{i(\nu\sqrt{s}+\mu\ln s)}\left(\varphi_0(\frac{y}{s^{1/4}}) + ...\right)$). Our idea is to linearize equation (19) around that profile and prove that the linearized equation as well as the nonlinear equation have a solution that goes to zero as $s \to \infty$. Let us introduce $q(y,s)$ and $\theta(s)$ such that

$$w(y,s) = e^{i\left(\nu\sqrt{s}+\mu\log s+\theta(s)\right)}\left(\varphi(y,s) + q(y,s)\right), \tag{40}$$

where

$$\varphi(y,s) = \varphi_0\left(\frac{y}{s^{1/4}}\right) + (1+i\delta)\frac{a}{s^{1/2}} \equiv \kappa^{-i\delta}\left(p-1+b\frac{|y|^2}{s^{1/2}}\right)^{-\frac{1+i\delta}{p-1}} + (1+i\delta)\frac{a}{s^{1/2}}, \tag{41}$$

$$\nu = -\frac{4b\beta(1+\delta^2)}{(p-1)^2} \text{ and } a = 2\kappa(1-\beta\delta)\frac{b}{(p-1)^2}, \tag{42}$$

and the other constants b, μ will be defined in the rigorous proof.

In order to guarantee the uniqueness of the couple (q,θ) an additional constraint is needed, see (79) below; we will choose $\theta(s)$ such that we kill one the neutral modes of the linearized operator.

Note that $\varphi_0(z)$ has been exhibited in the formal approach and satisfies the following equation

$$-\frac{1}{2}z\cdot\nabla\varphi_0 - \frac{1+i\delta}{p-1}\varphi_0 + (1+i\delta)|\varphi_0|^{p-1}\varphi_0 = 0, \tag{43}$$

which makes $\varphi(y,s)$ an approximate solution of (19). In addition to that, if w satisfies this equation, then q satisfies the following equation

$$\frac{\partial q}{\partial s} = \mathcal{L}_\beta q - \frac{(1+i\delta)}{p-1}q + L(q,\theta',y,s) + R^*(\theta',y,s) \tag{44}$$

where

$$\begin{aligned}
\mathcal{L}_\beta q &= (1+i\beta)\Delta q - \tfrac{1}{2}y\cdot\nabla q,\\
L(q,\theta',y,s) &= (1+i\delta)\left\{|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi - i\left(\tfrac{\nu}{2\sqrt{s}} + \tfrac{\mu}{s} + \theta'(s)\right)q\right\},\\
R^*(\theta',y,s) &= R(y,s) - i\left(\tfrac{\nu}{2\sqrt{s}} + \tfrac{\mu}{s} + \theta'(s)\right)\varphi,\\
R(y,s) &= -\tfrac{\partial\varphi}{\partial s} + (1+i\beta)\Delta\varphi - \tfrac{1}{2}y\cdot\nabla\varphi - \tfrac{(1+i\delta)}{p-1}\varphi + (1+i\delta)|\varphi|^{p-1}\varphi.
\end{aligned} \tag{45}$$

Our aim is to find a $\theta \in C^1([-\log T, \infty), \mathbb{R})$ such that equation (48) has a solution $q(y,s)$ defined for all $(y,s) \in \mathbb{R}^N \times [-\log T, \infty)$ such that

$$q(y,s) = \frac{\mathcal{F}(y)}{s} + v(y,s),$$

where $\mathcal{F}$ is defined by (17) in Theorem 2 and

$$\|v(s)\|_{L^\infty} \to 0 \text{ as } s \to \infty.$$

From (43), one sees that the variable $z = \frac{y}{s^{1/4}}$ plays a fundamental role. Thus, we will consider the dynamics for $|z| > K$, and $|z| < 2K$ separately for some $K > 0$ to be fixed large.

3.1 The outer region where $|y| > Ks^{1/4}$

Let us consider a non-increasing cut-off function $\chi_0 \in C^\infty(\mathbb{R}^+, [0,1])$ such that $\chi_0(\xi) = 1$ for $\xi < 1$ and $\chi_0(\xi) = 0$ for $\xi > 2$ and introduce

$$\chi(y,s) = \chi_0\left(\frac{|y|}{Ks^{1/4}}\right), \tag{46}$$

where K will be fixed large. Let us define

$$q_e(y,s) = e^{\frac{i\delta}{p-1}s} q(y,s)\left(1 - \chi(y,s)\right), \tag{47}$$

and note that q_e is the part of $q(y,s)$, corresponding to the non-blowup region $|y| > Ks^{1/4}$. As we will explain in subsection (4.3.3), the linear operator of the equation satisfied by q_e is negative, which makes it easy to control $\|q_e(s)\|_{L^\infty}$. This is not the case for the part of $q(y,s)$ for $|y| < 2Ks^{1/4}$, where the linear operator has two positive eigenvalues, a zero eigenvalue in addition to infinitely many negative ones. Therefore, we have to expand q with respect to these eigenvalues in order to control $\|q(s)\|_{L^\infty(|y|<2Ks^{1/4})}$. This requires more work than for q_e. The following subsection is dedicated to that purpose. From now on, K will be fixed constant which is chosen such that $\|\varphi(s')\|_{L^\infty(|y|>Ks'^{1/4})}$ is small enough, namely $\|\varphi_0(z)\|^{p-1}_{L^\infty(|z|>K)} \leq \frac{1}{C(p-1)}$ (see subsection (4.3.3) below, for more details).

3.2 The inner region where $|y| < 2Ks^{1/4}$

If we linearize the term $L(q, \theta', y, s)$ in equation (44), then we can write (44) as

$$\frac{\partial q}{\partial s} = \mathcal{L}_{\beta,\delta} q - i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q + V_1 q + V_2 \bar{q} + B(q,y,s) + R^*(\theta', y, s), \tag{48}$$

where

$$
\begin{array}{lcl}
\mathcal{L}_{\delta,\beta} q & = & (1+i\beta)\Delta q - \frac{1}{2} y\cdot\nabla q + (1+i\delta)\Re q,\\
V_1(y,s) & = & (1+i\delta)\frac{p+1}{2}\left(|\varphi|^{p-1} - \frac{1}{p-1}\right),\ \ V_2(y,s) = (1+i\delta)\frac{p-1}{2}\left(|\varphi|^{p-3}\varphi^2 - \frac{1}{p-1}\right),\\
B(q,y,s) & = & (1+i\delta)\left(|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi - |\varphi|^{p-1}q - \frac{p-1}{2}|\varphi|^{p-3}\varphi(\varphi\bar{q}+\bar{\varphi}q)\right),\\
R^*(\theta',y,s) & = & R(y,s) - i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right)\varphi,\\
R(y,s) & = & -\frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y\cdot\nabla\varphi - \frac{(1+i\delta)}{p-1}\varphi + (1+i\delta)|\varphi|^{p-1}\varphi
\end{array}
\tag{49}
$$

Note that the term $B(q,y,s)$ is built to be quadratic in the inner region $|y| \le Ks^{1/4}$. Indeed, we have for all $K \ge 1$ and $s \ge 1$,

$$
\sup_{|y|\le 2Ks^{1/4}} |B(q,y,s)| \le C(K)|q|^2. \tag{50}
$$

Note also that $R(y,s)$ measures the defect of $\varphi(y,s)$ from being an exact solution of (19). However, since $\varphi(y,s)$ is an approximate solution of (19), one easily derives the fact that

$$
\|R(s)\|_{L^\infty} \le \frac{C}{\sqrt{s}}. \tag{51}
$$

Therefore, if $\theta'(s)$ goes to zero as $s \to \infty$, we expect the term $R^*(\theta',y,s)$ to be small, since (48) and (51) yield

$$
|R^*(\theta',y,s)| \le \frac{C}{\sqrt{s}} + |\theta'(s)|. \tag{52}
$$

Therefore, since we would like to make q go to zero as $s \to \infty$, the dynamics of equation (48) are influenced by the asymptotic limit of its linear term,

$$
\tilde{\mathcal{L}} + V_1 q + V_2 \bar{q},
$$

as $s \to \infty$. In the sense of distribution (see the definitions of V_1 and V_2 in (48) and φ (41)) this limit is $\tilde{\mathcal{L}}$.

3.3 Spectral properties of $\mathcal{L}_\beta$

Here, we will restrict to $N = 1$. We consider the Hilbert space $L^2_{|\rho_\beta|}(\mathbb{R}^N,\mathbb{C})$ which is the set of all $f \in L^2_{loc}(\mathbb{R}^N,\mathbb{C})$ such that

$$
\int_{\mathbb{R}^N} |f(y)|^2 |\rho_\beta(y)| dy < +\infty,
$$

where

$$
\rho_\beta(y) = \frac{e^{-\frac{|y|^2}{4(1+i\beta)}}}{(4\pi(1+i\beta))^{N/2}} \text{ and } |\rho_\beta(y)| = \frac{e^{-\frac{|y|^2}{4(1+\beta^2)}}}{(4\pi\sqrt{1+\beta^2})^{N/2}}. \tag{53}
$$

We can diagonalize $\mathcal{L}_\beta$ in $L^2_{|\rho_\beta|}(\mathbb{R}^N,\mathbb{C})$. Indeed, we can write

$$
\mathcal{L}_\beta q = \frac{1}{\rho_\beta} div(\rho_\beta \nabla q).
$$

We notice that $\mathcal{L}_\beta$ is formally "self-adjoint" with respect to the weight ρ_β. Indeed, for any v and w in $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$ satisfying $\mathcal{L}_\beta v$ and $\mathcal{L}_\beta w$ in $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$, it holds that

$$\int v \mathcal{L}_\beta w \rho_\beta dy = \int w \mathcal{L}_\beta v \rho_\beta dy. \tag{54}$$

If we introduce for each $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ the polynomial

$$f_\alpha(y) = c_\alpha \Pi_{i=1}^N H_{\alpha_i}\left(\frac{y_i}{2\sqrt{1+i\beta}}\right), \tag{55}$$

where H_n is the standard one dimensional Hermite polynomial and $c_\alpha \in \mathbb{C}$ is chosen so that the term of highest degree in f_α is $\Pi_{i=1}^N y_i^{\alpha_i}$, then, we get a family of eigenfunction of $\mathcal{L}_\beta$, "orthogonal" with respect to the weight ρ_β, in the sense that for any different α and $\sigma \in \mathbb{N}^N$

$$\begin{array}{ll} \mathcal{L}_\beta f_\alpha & = -\frac{\alpha}{2} f_\alpha, \\ \int_{\mathbb{R}} f_\alpha(y) f_\sigma(y) \rho_\beta(y) dy & = 0. \end{array} \tag{56}$$

Moreover, the family f_α is a basis for $L^2_{|\rho_\beta|}(\mathbb{R}^N, \mathbb{C})$ considered as a $\mathbb{C}$ vector space. Al the facts about the operator $\mathcal{L}_\beta$ and the family f_α can be found in Appendix A of [MZ08].

3.4 Spectral properties of $\mathcal{L}_{\beta,\delta}$

In the sequel, we will assume $N = 1$. Now, with the explicit basis diagonalizing $\mathcal{L}_\beta$, we are able to write $\mathcal{L}_{\beta,\delta}$ in a Jordan's block's. More precisely, we recall Lemma 3.1 from [MZ08]

Lemma 3.1 (Jordan block's decomposition of $\mathcal{L}_{\beta,\delta}$) *For all $n \in \mathbb{N}$, there exists two polynomials*

$$\begin{array}{ll} h_n & = i f_n + \sum_{j=0}^{n-1} d_{j,n} f_j, \text{ where } d_{j,n} \in \mathbb{C} \\ \tilde{h}_n & = (1+i\delta) f_n + \sum_{j=0}^{n-1} \tilde{d}_{j,n} f_j, \text{ where } \tilde{d}_{j,n} \in \mathbb{C}, \end{array} \tag{57}$$

of degree n such that

$$\begin{array}{l} \mathcal{L}_{\beta,\delta} h_n = -\dfrac{n}{2} h_n, \\ \mathcal{L}_{\beta,\delta} \tilde{h}_n = \left(1 - \dfrac{n}{2}\right) \tilde{h}_n + c_n h_{n-2}, \end{array} \tag{58}$$

with $c_n = n(n-1)\beta(1+\delta^2)$ (and we take $h_k \equiv 0$ for $k < 0$). The term of highest of h_n (resp. $\tilde{h}_n$) is iy^n (resp. $(1+i\delta)y^n$).

Proof : See the proof of Lemma 3.1 in [MZ08]. To prove that $c_n = n(n-1)$, we look at the imaginary part of order $n-1$ in the equation $\mathcal{L}_{\beta,\delta}\tilde{h}_n = \left(1 - \frac{n}{2}\right)\tilde{h}_n + c_n h_{n-2}$. A simple identification gives the result.

Remark: The semigroup and the fundamental solution generated by $(1+i\beta)\Delta v$ have the same regularizing effect independently from β.

Lemma 3.2 (The basis vectors of degree less or equal to 6) *we have*

$$\begin{array}{ll} h_0(y) = i, & \tilde{h}_0 = (1+i\delta), \\ h_1(y) = iy, & \tilde{h}_1 = (1+i\delta)y, \\ h_2(y) = iy^2 + \beta - i(2+\delta\beta), & \tilde{h}_2 = (1+i\delta)(y^2 - 2 + 2\beta\delta), \end{array}$$

$$
\begin{array}{ll}
h_4(y) = iy^4 + y^2(c_{4,2} + id_{4,2}) + c_{4,0} + id_{4,0} & , \\
c_{4,2} = 6\beta, & d_{4,2} = -6(2+\beta\delta) = -18 - 6(\beta\delta - 1), \\
c_{4,0} = -4\beta(3+\beta\delta), & d_{4,0} = 12 - 6\beta^2 + 12\beta\delta + 2\beta^2\delta^2,
\end{array}
$$

$$
\begin{array}{l}
\tilde{h}_4(y) = (1+i\delta)y^4 + y^2(12(\beta\delta - 1) + i\tilde{d}_{4,2}) + \tilde{c}_{4,0} + i\tilde{d}_{4,0}. \\
\tilde{c}_{4,2} = 12(\beta\delta - 1), \quad \tilde{d}_{4,2} = 0, \\
\tilde{c}_{4,0} = 6\beta^2(1+\delta^2) - 12(\beta\delta - 1), \quad \tilde{d}_{4,0} = -6\beta^2\delta(3\delta^2 + 7) - 12\delta(\beta\delta + 1)
\end{array}
$$

$$
\begin{array}{l}
h_6(y) = iy^6 + y^4(c_{6,4} + id_{6,4}) + y^2(c_{6,2} + id_{6,2}) + c_{6,0} + id_{6,0}, \\
c_{6,4} = 15\beta, \quad d_{6,4} = -15(2+\beta\delta), \\
c_{6,2} = -60\beta(3+\delta\beta), \quad d_{6,2} = -90\beta^2 + 180 + 180\beta\delta + 30\beta^2\delta^2, \\
c_{6,0} = 180\beta + 120\delta\beta^2 - 45\beta^3 + 15\beta^3\delta, \\
d_{6,0} = -180\beta\delta + 55\delta\beta^3 - 60\delta^2\beta^2 - 5\beta^3\delta^2 + 180\beta^2 - 120,
\end{array}
$$

$$
\begin{array}{l}
\tilde{h}_6(y) = (1+i\delta)y^6 + y^4(\tilde{c}_{6,4} + i\tilde{d}_{6,4}) + y^2(\tilde{c}_{6,2} + i\tilde{d}_{6,2}) + \tilde{c}_{6,0} + i\tilde{d}_{6,0}, \\
\tilde{c}_{6,4} = 30(\beta\delta - 1), \quad \tilde{d}_{6,4} = 0, \\
\tilde{c}_{6,2} = 90\beta^2(1+\delta^2) - 180(\beta\delta - 1), \\
\tilde{d}_{6,2} = -90\beta(1+\delta^2)(3\beta\delta + 4) + 180(\beta\delta - 1)(\delta - 2\beta), \\
\tilde{c}_{6,0} = -20\beta^2(1+\delta^2)(11\beta\delta + 21) + 120(\beta\delta - 1)(-2\beta^2 + \beta\delta + 1), \\
\tilde{d}_{6,0} = 270\beta(1+\delta^2)(2+\beta\delta) + \beta^2(1+\delta^2)(140\beta\delta^2 - 180\beta\delta + 390\delta) \\
+60(\beta\delta - 1)(2\beta^2\delta - \beta\delta^2 + 9\beta - 4\delta),
\end{array}
$$

Moreover, we have

$$
\begin{array}{l}
\mathcal{L}_{\beta,\delta}\tilde{h}_0 = \tilde{h}_0, \\
\mathcal{L}_{\beta,\delta}\tilde{h}_1 = \frac{1}{2}\tilde{h}_1, \\
\mathcal{L}_{\beta,\delta}\tilde{h}_2 = 2\beta(1+\delta^2)h_0 = 2i\beta(1+\delta^2), \\
\mathcal{L}_{\beta,\delta}\tilde{h}_4 = -\tilde{h}_4 + 12\beta(1+\delta^2)h_2, \\
\mathcal{L}_{\beta,\delta}\tilde{h}_6 = -2\tilde{h}_6 + 30\beta(1+\delta^2)h_4.
\end{array}
$$

Proof : The proof is straightforward though a bit lengthy.

Corollary 3.1 (Basis for the set of polynomials) *The family $(h_n\tilde{h}_n)_n$ is a basis of $\mathbb{C}[X]$, the $\mathbb{R}$ vector space of complex polynomials.*

3.5 Decomposition of q

For the sake of controlling q in the region $|y| < 2Ks^{1/4}$, we will expand the unknown function q (and not just χq where χ is defined in (46)) with respect to the family f_n and the with respect to the h_n. We start by writing

We start by writing

$$
q(y,s) = \sum_{n \le M} Q_n(s) f_n(y) + q_-(y,s), \tag{59}
$$

where f_n is the eigenfunction of $\mathcal{L}_\beta$ defined in (55), $Q_n(s) \in \mathbb{C}$, q_- satisfies

$$
\int q_-(y,s) h_n(y)\rho(y) dy = 0 \text{ for all } n \le M,
$$

and M is a fixed even integer satisfying

$$M \geq 4\left(\sqrt{1+\delta^2}+1+2\max_{i=1,2,y\in\mathbb{R},s\geq 1}|V_i(y,s)|\right), \tag{60}$$

with $V_{i=1,2}$ defined in (49). From (59), we have

$$\mathcal{Q}_n(s) = \frac{\displaystyle\int q(y,s)f_n(y)\rho_\beta(y)dy}{\int f_n(y)^2\rho_\beta(y)} \equiv F_n(q(s)), \tag{61}$$

The function $q_-(y,s)$ can be seen as the projection of $q(y,s)$ onto the spectrum of $\mathcal{L}_\beta$, which is smaller than $(1-M)/2$. We will call it the infinite dimensional part of q and we will denote it $q_- = P_{-,M}(q)$. We also introduce $P_{+,M} = Id - P_{-,M}$. Notice that $P_{-,M}$ and $P_{+,M}$ are projections. In the sequel, we will denote $P_- = P_{-,M}$ and $P_+ = P_{+,M}$. The complementary part $q_+ = q - q_-$ will be called the finite dimensional part of q. We will expand it as follows

$$q_+(y,s) = \sum_{n\leq M}\mathcal{Q}_n(s)f_n(y) = \sum_{n\leq M} q_n(s)h_n(y) + \tilde{q}_n(s)\tilde{h}_n(y), \tag{62}$$

where $\tilde{q}_n, q_n \in \mathbb{R}$. Finally, we notice that for all s, we have

$$\int q_-(y,s)q_+(y,s)\rho_\beta(y)dy = 0.$$

Our purpose is to project (48) in order to write an equation for q_n and $\tilde{q}_n$. For that we need to write down the expression of q_n and $\tilde{q}_n$ in terms of $\mathcal{Q}_n$. Since the matrix $(h_n,\tilde{h}_n)_{n\leq M}$ in the basis of (if_n, f_n) is upper triangular (see Lemma 3.2). The same holds for its inverse. Thus, we derive from (62)

$$\begin{aligned} q_n = {} & \mathrm{Im}\mathcal{Q}_n(s) - \delta\ \mathrm{Re}\mathcal{Q}_n(s) + \textstyle\sum_{j=n+1}^M A_{j,n}\mathrm{Im}\mathcal{Q}_j(s) + B_{j,n}\ \mathrm{Re}\mathcal{Q}_j(s) \equiv P_{n,m}(q(s)),\\ \tilde{q}_n(s) = {} & \mathrm{Re}\mathcal{Q}_n(s) + \textstyle\sum_{j=n+1}^M \tilde{A}_{j,n}\ \mathrm{Im}\mathcal{Q}_j(s) + \tilde{B}_{j,n}\ \mathrm{Re}\mathcal{Q}_j(s) \equiv \tilde{P}_{n,M}(q(s)), \end{aligned} \tag{63}$$

where all the constants are real. Moreover, the coefficient of $\mathrm{Im}\mathcal{Q}_n$ and $\mathrm{Re}\mathcal{Q}_n$ in the expression of q_n and $\tilde{q}_n$ are explicit. This comes from the fact that the same holds for the coefficient of if_n and f_n in the expansion of h_n and $\tilde{h}_n$ (see Lemma 3.1).
Note that the projector $P_{n,m}(q)$ and $\tilde{P}_{n,m}(q)$ are well-defined thanks to (61). We will project equation (48) on the different modes h_n and $\tilde{h}_n$. Note that from (59) and (62), that

$$q(y,s) = \left(\sum_{n\leq M} q_n(s)h_n(y) + \tilde{q}_n(s)\tilde{h}_n(y)\right) + q_-(y,s), \tag{64}$$

we should keep in mind that the presentation in (64) is unique.

4 Existence

In this section, we prove the existence of a solution $q(s), \theta(s)$ of problem (44)-(79) such that

$$\begin{aligned} & q(y,s) = \frac{1}{s}\left(\tilde{\mathcal{A}}_0\tilde{h}_0(y) + \mathcal{A}_2 h_2(y) + \tilde{\mathcal{A}}_2\tilde{h}_2(y)\right) + v(y,s), \\ & \text{with, for all } M>0 \quad \sup_{|y|<Ms^{\frac14}} |v(y,s)| \le C\frac{1+|y|^5}{s^{\frac32}}, \\ & \text{and } |\theta'(s)| \le \frac{CA^{10}}{s^{\frac54}} \text{ for all } s\in[-\log T, +\infty), \end{aligned} \tag{65}$$

where $\tilde{\mathcal{A}}_0$, $\mathcal{A}_2$ and $\tilde{\mathcal{A}}_2$ are given by (76) in Definition 4.1 and $h_0(y)$, $h_2(y)$ and $\tilde{h}_2(y)$ are given in Lemma 3.2.

Hereafter, we denote by C a generic positive constant, depending only on p, δ, β and K introduced in (46), itself depending on p. In particular, C neither depend on A nor on s_0, the constants that will appear shortly and throughout the paper and need to be adjusted for the proof.
We proceed in two subsections. In the first, we give the proof assuming the technical details. In the second subsection we give the proof of the technical details.

4.1 Proof of the existence assuming technical results

We work in the set of even functions to construct a blow-up solution. However, since we need to prove the stability of the constructed soluton in the set of all functions with no evenness assumption, we have to handle general functions.

Definition 4.1 (A set shrinking to zero) *For all $K>1$, $A\ge 1$ and $s\ge 1$, we define $\mathcal{V}_A(s)$ as the set of all $q\in L^\infty(\mathbb{R})$ such that*

$$\|q_e\|_{L^\infty(\mathbb{R})} \le \frac{A^{M+2}}{s^{\frac14}}, \qquad \left\|\frac{q_-(y)}{1+|y|^{M+1}}\right\|_{L^\infty(\mathbb{R})} \le \frac{A^{M+1}}{s^{\frac{M+2}{4}}},$$

$$|q_j|,\ |\tilde{q}_j| \le \frac{A^j}{s^{\frac{j+1}{4}}} \text{ for all } 5\le j\le M, \quad |q_0| \le \frac{1}{s^{\frac32}}, |\tilde{q}_1| \le \frac{A}{s^{\frac32}}, \quad |q_1| \le \frac{A^4}{s^{\frac32}}.$$

In addition to the the other modes will satisfy the following condition:

$$\begin{aligned} & |Q_4| \le \frac{A^7}{s^{7/4}} \text{ and } \left|\tilde{Q}_4\right| \le \frac{A^4}{s^{\frac74}}, \\ & |q_3| \le \frac{A^3}{s^{\frac32}} \text{ and } |\tilde{q}_3| \le \frac{A^3}{s^{\frac32}}, \\ & |Q_2| \le \frac{A^8}{s^{\frac74}} \text{ and } \left|\tilde{Q}_2\right| \le \frac{A^{10}}{s^{\frac54}}, \end{aligned}$$

and

$$\left|\tilde{Q}_0\right| \le \frac{A}{s^{\frac74}},$$

where

$$
\begin{aligned}
Q_4 &= q_4 - \left(\frac{1}{2} D_{4,2} \frac{\tilde{q}_2}{\sqrt{s}} + \left[\frac{C_{4,2} R^*_{2,1}}{2} + \frac{R^*_{4,2}}{2}\right] \frac{1}{s^{\frac{3}{2}}}\right), && (66)\\
&= q_4 - (\frac{\mathcal{B}_4}{s^{\frac{3}{2}}} + \mathcal{C}_4 \frac{\tilde{q}_2}{\sqrt{s}}), && (67)\\
\tilde{Q}_4 &= \tilde{q}_4 - \left(\tilde{D}_{4,2} \frac{\tilde{q}_2}{\sqrt{s}} + \frac{1}{s^{\frac{3}{2}}} \left[\tilde{C}_{4,2} R^*_{2,1} + \tilde{R}^*_{4,2}\right]\right), && (68)\\
&= \tilde{q}_4 - (\frac{\tilde{\mathcal{B}}_4}{s^{\frac{3}{2}}} + \tilde{\mathcal{C}}_4 \frac{\tilde{q}_2}{\sqrt{s}}), && (69)\\
\tilde{Q}_0 &= \tilde{q}_0 - \left(\frac{\tilde{q}_2}{\sqrt{s}} \left[\frac{\nu \tilde{L}_{0,2}}{2} - \tilde{D}_{0,2} - \frac{\tilde{\Theta}^*_{0,0} c_2}{\kappa}\right] - \frac{\tilde{R}^*_{0,1}}{s}\right) && (70)\\
&- \left(\frac{1}{s^{\frac{3}{2}}} \left[-\tilde{X}_0 + \frac{\nu \tilde{K}_{0,2} R^*_{2,1}}{2} - \tilde{C}_{0,2}.R^*_{2,1}\right]\right), && (71)\\
&= \tilde{q}_0 - (\frac{\tilde{\mathcal{A}}_0}{s} + \frac{\tilde{\mathcal{B}}_0}{s^{\frac{3}{2}}} + \tilde{\mathcal{C}}_0 \frac{\tilde{q}_2}{\sqrt{s}}), && (72)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_2 &= q_2 - \left(\frac{\tilde{q}_2}{\sqrt{s}} \left[D_{2,2} - \frac{\nu}{2}(1+\delta^2) + c_4 \tilde{D}_{4,2} + \frac{\Theta^*_{2,0} c_2}{\kappa}\right] + \frac{R^*_{2,1}}{s}\right)\\
&- \left(\frac{1}{s^{\frac{3}{2}}} \left[X_2 + c_4[\tilde{C}_{4,2} R^*_{2,1} + \tilde{R}^*_{4,2}] - D_{2,0}.\tilde{R}^*_{0,1}\right]\right), && (73)\\
&= q_2 - (\frac{\mathcal{A}_2}{s} + \frac{\mathcal{B}_2}{s^{\frac{3}{2}}} + \mathcal{C}_2 \frac{\tilde{q}_2}{\sqrt{s}}), && (74)\\
\tilde{Q}_2 &= \tilde{q}_2 - \frac{\tilde{\mathcal{A}}_2}{s}, && (75)
\end{aligned}
$$

and

$$
\tilde{\mathcal{A}}_0 = -\tilde{R}^*_{0,1}, \quad \tilde{\mathcal{A}}_2 = -\frac{R^*_{0,1}}{c_2}, \quad \mathcal{A}_2 = R^*_{2,1} \tag{76}
$$

with $R^*_{0,1}$, $\tilde{R}^*_{0,1}$, $R^*_{2,1}$ and c_2 are defined as in page 68 and Lemma 3.1, respectively. The constants $C_{i,j}$, $\tilde{C}_{i,j}$, $D_{i,j}$ and $\tilde{C}_{i,j}$ are defined in (113)*. The constants $K_{i,j}$, $\tilde{K}_{i,j}$, $L_{i,j}$ and $\tilde{L}_{i,j}$ are defined by* (94) *and* (96)*.*

Since $A \geq 1$, the sets $\mathcal{V}_A(s)$ are increasing (for fixed s) with respect to A in the sense of inclusions.
We also show the following property of elements of $\mathcal{V}_A(s)$:
For all $A \geq 1$, there exists $s_{01}(A) \geq 1$, such that for all $s \geq s_{01}$ and $r \in \mathcal{V}(A)$, we have

$$
\|r\|_{L^\infty(\mathbb{R})} \leq C(K) \frac{A^{M+2}}{s^{\frac{1}{4}}}, \tag{77}
$$

where C is a positive constant (see Claim 4.8 below for the proof).
By (77), if a solution q stays in $\mathcal{V}(A)$ for $s \geq s_{01}$, then it converges to 0 in $L^\infty(\mathbb{R})$.

The solution of equation (48) will be denoted by q_{s_0,d_0,d_1} or q when there is no ambiguity. We will show that if A is fixed large enough, then, s_0 is fixed large enough depending on A, we can fix the parameters $(d_0, d_1) \in [-2, 2]^2$, so that the solution $v_{s_0,d_0,d_1} \to 0$ as $s \to \infty$ in $L^\infty(\mathbb{R})$, that is (65) holds. Our construction is built on a careful choice of the initial data of q at a time s_0. We will choose it in the following form:

Definition 4.2 (Choice of initial data) *Let us define, for $A \geq 1$, $s_0 = -\log T > 1$ and $d_0, d_1 \in \mathbb{R}$, the function*

$$\begin{aligned}\psi_{s_0,d_0,d_1}(y) = & \left[\left(\frac{A}{s_0^{7/4}}\tilde{d}_0 + \frac{\tilde{\mathcal{A}}_0}{s_0} + \frac{\mathcal{B}_0}{s_0^{3/2}} + \frac{\mathcal{C}_0\tilde{\mathcal{A}}_2}{s_0^{3/2}}\right)\tilde{h}_0 + \frac{A}{s_0^{3/2}}\tilde{d}_1\tilde{h}_1(y) + d_0 h_0\right.\\ & + \frac{\tilde{\mathcal{A}}_2}{s_0}\tilde{h}_2 + \left(\frac{\mathcal{A}_2}{s_0} + \frac{\mathcal{B}_2}{s_0^{3/2}} + \frac{\mathcal{C}_2\tilde{\mathcal{A}}_2}{s_0^{3/2}}\right)h_2 \\ & \left. + \left(\frac{\tilde{\mathcal{B}}_4}{s_0^{3/2}} + \frac{\tilde{\mathcal{C}}_4\tilde{\mathcal{A}}_2}{s_0^{3/2}}\right)\tilde{h}_4 + \left(\frac{\mathcal{B}_4}{s_0^{3/2}} + \frac{\mathcal{C}_4\tilde{\mathcal{A}}_2}{s_0^{3/2}}\right)h_4\right]\chi(2y, s_0),\end{aligned} \tag{78}$$

where $s_0 = -\log T$ and h_i, $\tilde{h}_i$, $i = 0, 1, 2, 3, 4$ are given in Lemma 3.2, χ is defined by (46) and $d_0 = d_0(\tilde{d}_0, \tilde{d}_1)$ will be fixed later in (i) of Proposition 4.5. The constants $\tilde{\mathcal{A}}_i, \mathcal{A}_i$, $\tilde{\mathcal{B}}_i, \mathcal{B}_i$, $\tilde{\mathcal{C}}_i, \mathcal{C}_i$, for $i = 0, 2, 4$ are given by (66–75).

Remark 4.3 *Let us recall that we will modulate the parameter θ to kill one of the neutral modes, see equation* (79) *below. It is natural that this condition must be satisfied for the initial data at $s = s_0$. Thus, it is necessary that we choose d_0 to satisfy condition* (79), *see* (80) *below.*

So far, the phase $\theta(s)$ introduced in (40) is arbitrary, in fact as we will show below in Proposition 4.6. We can use a modulation technique to choose $\theta(s)$ in such a way that we impose the condition

$$P_{0,M}(q(s)) = 0, \tag{79}$$

which allows us to kill the neutral direction of the operator $\tilde{\mathcal{L}}$ defined in (48). Reasonably, our aim is then reduced to the following proposition:

Proposition 4.4 (Existence of a solution trapped in $\mathcal{V}_A(s)$) *There exists $A_2 \geq 1$ such that for $A \geq A_2$ there exists $s_{02}(A)$ such that for all $s_0 \geq s_{02}(A)$, there exists $(\tilde{d}_0, \tilde{d}_1)$ such that if q is the solution of* (48)-(79), *with initial data given by* (78) *and* (80), *then $v \in \mathcal{V}_A(s)$, for all $s \geq s_0$.*

This proposition gives the stronger convergence to 0 in $L^\infty(\mathbb{R})$ thanks to (77).
Let us first be sure that we can choose the initial data such that it starts in $\mathcal{V}_A(s_0)$. In other words, we will define a set where where will be selected the good parameters $(\tilde{d}_0, \tilde{d}_1)$ that will give the conclusion of Proposition 4.4. More precisely, we have the following result:

Proposition 4.5 (Properties of initial data) *For each $A \geq 1$, there exists $s_{03}(A) > 1$ such that for all $s_0 \geq s_{03}$:*
(i) $P_{0,M}(i\chi(2y, s_0)) \neq 0$ and the parameter $d_0(s_0, \tilde{d}_0, \tilde{d}_1)$ given by

$$\begin{aligned} d_0(s_0, \tilde{d}_0, \tilde{d}_1) &= -\frac{A}{s_0^{3/2}} \tilde{d}_1 \frac{P_{0,M}\left(\tilde{h}_1 \chi(2y, s_0)\right)}{P_{0,M}(i\chi(2y, s_0))} \\ &- \left(\frac{A}{s_0^{7/4}} \tilde{d}_0 + \frac{\tilde{\mathcal{A}}_0}{s_0} + \frac{\tilde{\mathcal{B}}_0}{s_0^{3/2}} + \frac{\tilde{\mathcal{C}}_0 \tilde{\mathcal{A}}_2}{s_0^{3/2}} \right) \frac{P_{0,M}\left(\tilde{h}_0 \chi(2y, s_0)\right)}{P_{0,M}(i\chi(2y, s_0))} \\ &- \left(\frac{\tilde{\mathcal{A}}_2}{s_0} \right) \frac{P_{0,M}\left(\tilde{h}_2 \chi(2y, s_0)\right)}{P_{0,M}(i\chi(2y, s_0))} \\ &- \left(\frac{\mathcal{A}_2}{s_0} + \frac{\mathcal{B}_2}{s_0^{3/2}} + \frac{\mathcal{C}_2 \tilde{\mathcal{A}}_2}{s_0^{3/2}} \right) \frac{P_{0,M}(h_2 \chi(2y, s_0))}{P_{0,M}(i\chi(2y, s_0))} \\ &- \left(\frac{\tilde{\mathcal{B}}_4}{s_0^{3/2}} + \frac{\tilde{\mathcal{C}}_4 \tilde{\mathcal{A}}_2}{s_0^{3/2}} \right) \frac{P_{0,M}\left(\tilde{h}_4 \chi(2y, s_0)\right)}{P_{0,M}(i\chi(2y, s_0))} \\ &- \left(\frac{\mathcal{B}_4}{s_0^{3/2}} + \frac{\mathcal{C}_4 \tilde{\mathcal{A}}_2}{s_0^{3/2}} \right) \frac{P_{0,M}(h_4 \chi(2y, s_0))}{P_{0,M}(i\chi(2y, s_0))} \end{aligned} \tag{80}$$

is well defined, where χ defined in (46) *and the constants $\tilde{\mathcal{A}}_i, \mathcal{A}_i, \tilde{\mathcal{B}}_i, \mathcal{B}_i, \tilde{\mathcal{C}}_i, \mathcal{C}_i$, for $i = 0, 2, 4$ are given by (66– 75).*
(ii) If ψ is given by (78) *and* (80) *with d_0 defined by* (80)*.Then, there exists a quadrilateral $\mathcal{D}_{s_0} \subset [-2, 2]^2$ such that the mapping*

$$(\tilde{d}_0, \tilde{d}_1) \to \left(\tilde{\Psi}_0 = \tilde{\psi}_0 - \left(\frac{\tilde{\mathcal{A}}_0}{s_0} + \frac{\tilde{\mathcal{B}}_0}{s_0^{3/2}} + \frac{\tilde{\mathcal{C}}_0 \tilde{\mathcal{A}}_2}{s_0^{3/2}} \right), \tilde{\psi}_1 \right)$$

(where ψ stands for $\psi_{s_0, \tilde{d}_0, \tilde{d}_1}$) is linear, one to one from $\mathcal{D}_{s_0}$ onto $[-\frac{A}{s_0^{7/4}}, \frac{A}{s_0^{7/4}}] \times [-\frac{A}{s_0^{3/2}}, \frac{A}{s_0^{3/2}}]$. Moreover it is of degree 1 on the boundary.
(iii) For all $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$, $\psi_e \equiv 0$, $\psi_0 = 0$, $|\tilde{\psi}_i| + |\psi_j| \leq CAe^{-\gamma s_0}$
for some $\gamma > 0$, for some $\gamma > 0$ and for all $3 \leq i \leq M$, $i \neq 4$ and $1 \leq j \leq M$, $j \neq 4$ and

$$|\tilde{\Psi}_i| + |\Psi_j| \leq CAe^{-\gamma s_0} \text{ for } i, j = \{2, 4\},$$

where $\tilde{\Psi}_i$ and Ψ_i are defined as in (66– 75).
Moreover , $\|\frac{\psi_-(y)}{(1+|y|)^{M+1}}\|_{L^\infty(\mathbb{R})} \leq C\frac{A}{s_0^{\frac{M}{4}+1}}$.
(iv) For all $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$, $\psi_{s_0, \tilde{d}_0, \tilde{d}_1} \in \mathcal{V}_A(s_0)$ with strict inequalities except for $(\tilde{\psi}_0, \tilde{\psi}_1)$.

The proof of previous proposition is postponed to subsection 4.2.

In the following, we find a local in time solution for equation (48) coupled with the condition (79).

Proposition 4.6 ***(Local in time solution and modulation for problem*** **(48)-(79)** ***with initial data*** (78)-(80)*) For all $A \geq 1$, there exists $T_3(A) \in (0, 1/e)$ such that for all*

$T \le T_3$, the following holds:
For all $(\tilde d_0, \tilde d_1) \in D_T$, there exists $s_{max} > s_0 = -\log T$ such that problem (48)-(79) *with initial data at $s = s_0$,*

$$(q(s_0), \theta(s_0)) = (\psi_{s_0, \tilde d_0, \tilde d_1}, 0),$$

where $\psi_{s_0,\tilde d_0,\tilde d_1}$ is given by (78) *and* (80)*, has a unique solution $q(s), \theta(s)$ satisfying $q(s) \in V_{A+1}(s)$ for all $s \in [s_0, s_{max})$.*

The proof of this proposition will be given later in page 27.

Let us now give the proof of Proposition 4.4.
Proof of Proposition 4.4: Let us consider $A \ge 1$, $s_0 \ge s_{03}$, $(\tilde d_0, \tilde d_1) \in \mathcal{D}_{s_0}$, where s_{03} is given by Proposition 4.5. From the existence theory (which follows from the Cauchy problem for equation (1)), starting in $\mathcal{V}_A(s_0)$ which is in $\mathcal{V}_{A+1}(s_0)$, the solution stays in $\mathcal{V}_A(s)$ until some maximal time $s_* = s_*(\tilde d_0, \tilde d_1)$. Then, either:
• $s_*(\tilde d_0, \tilde d_1) = \infty$ for some $(\tilde d_0, \tilde d_1) \in \mathcal{D}_{s_0}$, then the proof is complete.
• $s_*(\tilde d_0, \tilde d_1) < \infty$, for any $(\tilde d_0, \tilde d_1) \in \mathcal{D}_{s_0}$, then we argue by contradiction. By continuity and the definition of s_*, the solution on s_* is in the boundary of $\mathcal{V}_A(s_*)$. Then, by definition of $\mathcal{V}_A(s_*)$, one at least of the inequalities in that definition is an equality. Owing to the following proposition, this can happen only for the first two components $\tilde q_0, \tilde q_1$. Precisely we have the following result

Proposition 4.7 (Control of $q(s)$ by $(q_0(s), q_1(s))$ in $\mathcal{V}_A(s)$) . *There exists $A_4 \ge 1$ such that for each $A \ge A_4$, there exists $s_{04} \in \mathbb{R}$ such that for all $s_0 \ge s_{04}$. The following holds:*
If q is a solution of (48) *with initial data at $s = s_0$ given by* (78) *and* (80) *with $(\tilde d_0, \tilde d_1) \in \mathcal{D}_{s_0}$, and $q(s) \in \mathcal{V}(A)(s)$ for all $s \in [s_0, s_1]$, with $q(s_1) \in \partial \mathcal{V}_A(s_1)$ for some $s_1 \ge s_0$, then:*
*(i)**(Smallness of the modulation parameter θ defined in* (40)*)** For all $s \in [s_0, s_1]$,*

$$|\theta'(s)| \le \frac{CA^{10}}{s^{\frac{5}{4}}}.$$

*(ii) **(Reduction to a finite dimensional problem)** We have:*

$$\left(\tilde Q_0(s_1), \tilde q_1(s_1)\right) \in \partial \left(\left[-\frac{A}{s_1^{\frac{7}{4}}}, \frac{A}{s_1^{\frac{7}{4}}} \right] \times \left[-\frac{A}{s_1^{\frac{3}{2}}}, \frac{A}{s_1^{\frac{3}{2}}} \right] \right).$$

*(iii)**(Transverse crossing)** There exists $\omega \in \{-1, 1\}$ such that*

$$\omega \tilde Q_0(s_1) = \frac{A}{s_1^{\frac{7}{4}}} \text{ and } \omega \frac{d\tilde Q_0(s_1)}{ds}(s_1) > 0.$$

$$\omega \tilde q_1(s_1) = \frac{A}{s_1^{\frac{3}{2}}} \text{ and } \omega \frac{d\tilde q_1}{ds}(s_1) > 0.$$

Assume the result of the previous proposition, for which the proof is given below in page 27, and continue the proof of Proposition 4.4. Let $A \geq A_4$ and $s_0 \geq s_{04}(A)$. It follows from Proposition 4.7, part (ii) that $\left(\tilde{Q}_0, \tilde{q}_1(s_*)\right) \in \partial\left(\left[-\frac{A}{s_1^{\frac{7}{4}}}, \frac{A}{s_1^{\frac{7}{4}}}\right] \times \left[-\frac{A}{s_1^{\frac{3}{2}}}, \frac{A}{s_1^{\frac{3}{2}}}\right]\right)$, and the following function

$$\begin{aligned} \phi \quad &: \mathcal{D}_{s_0} \to \partial([-1,1]^2) \\ &(\tilde{d}_0, \tilde{d}_1) \to \left(\frac{s_*^{7/4}}{A}\tilde{Q}_0, \frac{s_*^{3/2}}{A}\tilde{q}_1\right)_{(\tilde{d}_0, \tilde{d}_1)}(s_*), \text{ with } s_* = s_*(\tilde{d}_0, \tilde{d}_1), \end{aligned}$$

is well defined. Then, it follows from Proposition 4.7, part (iii) that ϕ is continuous. On the other hand, using Proposition 4.5 (ii)-(iv) together with the fact that $q(s_0) = \psi_{s_0, \tilde{d}_0, \tilde{d}_1}$, we see that when $(\tilde{d}_0, \tilde{d}_1)$ is in the boundary of the rectangle $\mathcal{D}_{s_0}$, we have strict inequalities for the other components.
Applying the transverse crossing property given by (iii) of Proposition 4.7, we see that $q(s)$ leaves $\mathcal{V}_A(s)$ at $s = s_0$, hence $s_*(\tilde{d}_0, \tilde{d}_1) = s_0$. Using Proposition 4.5, part (ii), we see that the restriction of ϕ to the boundary is of degree 1. A contradiction, then follows from the index theory. Thus there exists a value $(\tilde{d}_0, \tilde{d}_1) \in \mathcal{D}_{s_0}$ such that for all $s \geq s_0$, $q_{s_0, d_0, d_1}(s) \in \mathcal{V}_A(s)$. This concludes the proof of Proposition 4.4.
Using (i) of Proposition 4.7, we get the bound on $\theta'(s)$. This concludes the proof of (65).

4.2 Proof of the technical results

This section is devoted to the proof of the existence result given by Theorem 1. We proceed in 4 steps, each of them making a separate subsection.

- In the first subsection, we give some properties of the shrinking set $\mathcal{V}_A(s)$ defined in Definition 4.1 and translate our goal of making $q(s)$ go to 0 in $L^\infty(\mathbb{R})$ in terms of belonging to $\mathcal{V}_A(s)$. We also give the proof of Proposition 4.5.
- In second subsection, we solve the local in time Cauchy problem for equation (48) coupled with some orthogonality condition.
- In the third subsection using the spectral properties of equation (48), we reduce our goal from the control of $q(s)$ (an infinite dimensional variable) in $\mathcal{V}_A(s)$ to control its two first components $(\tilde{Q}_0, \tilde{q}_1)$ a two variables in $\left[-\frac{A}{s^{\frac{7}{4}}}, \frac{A}{s^{\frac{7}{4}}}\right] \times \left[-\frac{A}{s^{\frac{3}{2}}}, \frac{A}{s^{\frac{3}{2}}}\right]$.
- In the fourth subsection, we solve the finite dimensional problem using the index theory and conclude the proof of Theorem 1 .

4.2.1 Properties of the shrinking set $\mathcal{V}_A(s)$ and preparation of initial data

In this subsection, we give some properties of the shrinking set defined in Definition 4.1. Let us first introduce the following claim:

Claim 4.8 (Properties of the shrinking set defined in Definition 4.1) *For all $r \in \mathcal{V}_A(s)$,*

(i) $\|r\|_{L^\infty(|y|<2Ks^{\frac{1}{4}})} \le C(K)\frac{A^{M+1}}{s^{\frac{1}{4}}}$ *and* $\|r\|_{L^\infty(\mathbb{R})} \le C(K)\frac{A^{M+2}}{s^{\frac{1}{4}}}$.

(ii) for all $y \in \mathbb{R}$, $|r(y)| \le C\frac{A^{M+1}}{s}(1+|y|^{M+1})$.

Proof: Take $r \in \mathcal{V}_A(s)$ and $y \in \mathbb{R}$.

(i) If $|y| \ge 2Ks^{\frac{1}{4}}$, then we have from the definition of r_e (47), $|r(y)| = |r_e(y)| \le \frac{A^{M+2}}{s^{\frac{1}{4}}}$.

Now, if $|y| < 2Ks^{\frac{1}{4}}$, since we have for all $0 \le j \le M$, $|\tilde{r}_j| + |r_j| \le C\frac{A^j}{s^{\frac{j+1}{4}}}$ from Definition 4.1 (use the fact that $M \ge 4$), we write from (64)

$$\begin{aligned}
|r(y)| &\le \left(\sum_{j\le M} |\tilde{r}_j||\tilde{h}_j| + |r_j||h_j|\right) + |r_-(y)|,\\
&\le C\sum_{j\le M}\frac{A^{M+1}}{s^{\frac{j+1}{4}}}(1+|y|)^j + \frac{A^{M+1}}{s^{\frac{M+2}{4}}}(1+|y|)^{M+1},\\
&\le C\sum_{j\le M}\frac{A^{M+1}}{s^{\frac{j+1}{4}}}(1+Ks^{\frac{1}{4}})^j + \frac{A^{M+1}}{s^{\frac{M+2}{4}}}(1+Ks^{\frac{1}{4}})^{M+1} \le C\frac{(KA)^{M+1}}{s^{\frac{1}{4}}},
\end{aligned} \tag{81}$$

which gives (i).

(ii) Just use (81) together with the fact that for all $0 \le j \le M$, $|\tilde{r}_j| + |r_j| \le C\frac{A^{M+1}}{s}$ from Definition 4.1. This ends the proof of Claim 4.8. ■

Let us now give the proof of Proposition 4.5.

Proof of Proposition 4.5 For simplicity, we write ψ instead of $\psi_{s_0,\tilde{d}_0,\tilde{d}_1}$. We note that, from Claim 4.8, (iv) follows from (ii) and (iii) by taking $s_0 = -\log T$ large enough (that is T is small enough). Thus, we only prove (i), (ii) and (iii). Consider $K \ge 1$, $A \ge 1$ and $T \le 1/e$. Note that $s_0 = -\log T \ge 1$.

The proof of (i) is a direct consequence of (iii) of the following claim

Claim 4.9 *There exists $\gamma = \frac{1}{32(1+\beta^2)} > 0$ and $T_2 < 1/e$ such that for all $K \ge 1$ and $T \le T_2$, if g is given by $(1+i\delta)\chi(2y,s_0)$, $(1+i\delta)y\chi(2y,s_0)$, $(1+i\delta)h_2(y)\chi(2y,s_0)$ or $i\chi(2y,s_0)$, then $\left\|\frac{g_-(y)}{1+|y|^{M+1}}\right\|_{L^\infty} \le \frac{C}{s_0^{\frac{M}{4}}}$ and all g_i, $\tilde{g}_i$ for $0 \le i \le M$ are less than $Ce^{-\gamma s_0}$.*

expect:

i) $|\tilde{g}_0 - 1| \le Ce^{-\gamma s_0}$ when $g = \tilde{h}_0(y)\chi(2y,s_0)$.

ii) $|\tilde{g}_1 - 1| \le Ce^{-\gamma s_0}$ when $g = \tilde{h}_1(y)\chi(2y,s_0)$.

iii) $|\tilde{g}_2 - 1| \le Ce^{-\gamma s_0}$ when $g = \tilde{h}_2(y)\chi(2y,s_0)$.

iv) $|g_0 - 1| \le Ce^{-\gamma s_0}$ when $g = h_0(y)\chi(2y,s_0)$.

v)$|g_2 - 1| \le Ce^{-\gamma s_0}$ when $g = h_2(y)\chi(2y,s_0)$.

vi)$|\tilde{g}_4 - 1| \le Ce^{-\gamma s_0}$ when $g = \tilde{h}_4(y)\chi(2y,s_0)$.

vii)$|g_4 - 1| \le Ce^{-\gamma s_0}$ when $g = h_4(y)\chi(2y,s_0)$.

Proof: In all cases, we write

$$g(y) = p(y) + r(y) \text{ where } p(y) = \tilde{h}_{j|j=0,1,2,4} \text{ or } h_{j|j=0,2,4} \text{ and } r(y) = p(y)(\chi(2y,s_0)-1). \tag{82}$$

From the uniqueness of the decomposition (64), we see that $p_- \equiv 0$ and al p_i, $\tilde{p}_i$ are zero except

$$\tilde{p}_j = 1, \text{ when } p(y) = \tilde{h}_j, \text{ for } j = 0, 1, 2 \text{ and } 4$$

and

$$p_j = 1, \text{ when } p(y) = h_j, \text{ for } j = 0, 2 \text{ and } 4$$

Concerning the cases $2|y| < Ks^{\frac{1}{4}}$ and $2|y| > Ks^{\frac{1}{4}}$, we have the definition of χ (46),

$$1 - \chi(2y, s) \le \left(\frac{2|y|}{Ks_0^{\frac{1}{4}}} \right)^{M-1},$$

$$|\rho_\beta(y)(1 - \chi(2y, s))| \le \sqrt{|\rho_\beta(y)|}\sqrt{\left|\rho_\beta\left(\frac{K}{2}s^{\frac{1}{4}}\right)\right|} \le Ce^{-\frac{K^2 s_0}{32(1+\beta^2)}}\sqrt{|\rho_\beta(y)|}.$$

Therefore, from (64) and (82), we see that

$$\begin{aligned} |r(y)| &\le C(1 + |y|^2)\left(\frac{2|y|}{Ks_0^{\frac{1}{4}}}\right)^{M-1} \le C\frac{(1+|y|^{M+1})}{s_0^{\frac{M}{4}}}, \\ |\mathcal{R}_j| + |r_j| + |\tilde{r}_j| &\le Ce^{-\frac{K^2\sqrt{s_0}}{32(1+\beta^2)}} \text{ for all } j \le M. \end{aligned} \tag{83}$$

Hence, using (83) and (59) and the fact that $|f_j(y)| \le C(1 + |y|)^M$, for all $j \le M$, we get also

$$|r_-(y)| \le C\frac{(1 + |y|)^M}{s_0^{\frac{M}{4}}}.$$

Using (82) and the estimates for $p(y)$ stated below, we conclude the proof of Claim 4.9 and (i) of Proposition 4.5.

(ii) of Proposition 4.5: From (78) and (80), we see that

$$\begin{pmatrix} \tilde{\Psi}_0 \\ \tilde{\psi}_1 \end{pmatrix} = G \begin{pmatrix} \tilde{d}_0 \\ \tilde{d}_1 \end{pmatrix} \text{ where } G = (g_{i,j})_{0 \le i,j \le 1}. \tag{84}$$

Using Claim 4.9, we see from (78) and (80) that

$$|d_0| \le C(|\tilde{d}_0| + |\tilde{d}_1|)e^{-\gamma s_0} \tag{85}$$

for T small enough. Using again Claim 4.9. We see that $G \to \begin{pmatrix} \frac{A}{s_0^{\frac{7}{4}}} & 0 \\ 0 & \frac{A}{s_0^{\frac{3}{2}}} \end{pmatrix}$ and

as $s_0 \to \infty$ (for fixed K and A), which concludes the proof of (ii) of Proposition 4.5.

(iii) of Proposition 4.5: Since $supp(\psi) \subset B(0, Ks_0^{\frac{1}{4}})$ by (78) and (80), we see that $\psi_e \equiv 0$ and that ψ_0 is zero from the definition of d_0 (78) and (80). Using the fact that $|\tilde{d}_{i,i=0,1}| \le 2$ and the bound on d_0 by (85), we see that the estimates on ψ_j and $\tilde{\psi}_j$ and ψ_- in (iii) follows from (78) and (80) and Claim 4.9. This concludes the proof of Proposition 4.5. ■

In the following we give the proof of Local in time solution for problem (48)-(79). In fact, we impose some orthogonality condition given by (79), killing the one of the zero eigenfunction of the linearized operator of equation (48).

Proof of Proposition 4.6: From solution of the local in time Cauchy problem for equation (1) in $L^\infty(\mathbb{R})$, there exists $s_1 > s_0$ such that equation (19) with initial data (at $s = s_0$) $\varphi(y, s_0) + \psi_{s_0,\tilde{d}_0,\tilde{d}_1}(y)$, where $\varphi(y, s)$ is given by (41) has a unique solution $w(s) \in C([s_0, s_1), L^\infty(\mathbb{R}))$. Now, we have to find a unique $(q(s), \theta(s))$ such that

$$w(y, s) = e^{i(\nu\sqrt{s}+\mu\log s+\theta(s))} (\varphi(y, s) + q(y, s)) \tag{86}$$

and (79) is satisfied. Since $f_0 = 1$ and $\int_{\mathbb{R}} \rho_\beta(y)dy = 1$, we use (63) to write (79) as follows

$$\begin{aligned} P_{0,M}(q) &= \Im\left(\int q(y, s)\rho_\beta(y)dy\right) - \delta\Re\left(\int q(y, s)\rho_\beta(y)dy\right) \\ &= \Im\left((1 - i\delta)\int q(y, s)\rho_\beta(y)dy\right) = 0, \end{aligned}$$

or using (86)

$$F(s, \theta) \equiv \Im\left((1 - i\delta)\int \left(e^{-i(\nu\sqrt{s}+\mu\log s+\theta(s))}w(y, s) - \varphi(y, s)\right)\rho_\beta(y)dy\right) = 0.$$

Note that

$$\frac{\partial F}{\partial\theta}(s, \theta) = -\Re\left((1 - i\delta)\int e^{-i(\nu\sqrt{s}+\mu\log s+\theta(s))}w(y, s)\rho_\beta(y)dy\right).$$

From (iii) in Proposition 4.5, $F(s_0, 0) = P_{0,M}(\psi_{s_0,\tilde{d}_0,\tilde{d}_1}) = 0$ and

$$\begin{aligned} \frac{\partial F}{\partial\theta}(s_0, 0) &= -\Re\left((1 - i\delta)\int(\varphi(y, s_0) + \psi_{s_0,\tilde{d}_0,\tilde{d}_1}(y))\rho_\beta(y)dy\right) \\ &= -\kappa + O\left(\frac{1}{s_0^{1/4}}\right) \text{ as } s_0 \to \infty, \end{aligned}$$

for fixed K and A.
Therefore, if T is small enough in terms of A, then $\frac{\partial F}{\partial\theta}(s_0, 0) \neq 0$, and from the implicit function Theorem, there exists $s_2 \in (s_0, s_1)$ and $\theta \in C^1([s_0, s_2), \mathbb{R})$ such that $F(s, \theta(s)) = 0$ for all $s \in [s_0, s_2)$.Defining $q(s)$ by (86) gives a unique solution of the problem (49)-(79) for all $s \in [s_0, s_2)$. Now, since we have from (iv) of Proposition 4.5, $q(s_0) \in V_A(s_0) \underset{\neq}{\subset} V_{A+1}(s_0)$, there exists $s_3 \in (s_0, s_2)$ such that for all $s \in [s_0, s_3)$, $q(s) \in V_{A+1}(s)$. This concludes the proof of Proposition 4.6. ■

4.2.2 Reduction to a finite dimensional problem

In the following we give the proof of Proposition 4.7:

The idea of the proof is to project equation (48) on the different components of the decomposition (64). More precisely, we claim that Proposition 4.7 is a consequence of the following

Proposition 4.10 *There exists $A_5 \geq 1$ such that for all $A \geq A_5$, there exists $s_5(A)$ such that the following holds for all $s_0 \geq s_5$:*
Assuming that for all $s \in [\tau, s_1]$ for some $s_1 \geq \tau \geq s_0$, $q(s) \in \mathcal{V}_A(s)$ and $q_0(s) = 0$, then the following holds for all $s \in [\tau, s_1]$:

*(i) (**Smallness of the modulation parameter**):*

$$|\theta'(s)| \leq \frac{CA^{10}}{s^{\frac{5}{4}}}.$$

*(ii) (**ODE satisfied by the expanding mode**): For $m = 0$ and 1, we have*

$$\left|\tilde{Q}_0'(s) - Q_0(s)\right| \leq \frac{C}{s^{\frac{7}{4}}},$$

and

$$\left|\tilde{q}_1' - \frac{1}{2}\tilde{q}_1\right| \leq \frac{C}{s^{\frac{3}{2}}}.$$

*(iii) (**ODE satisfied by the null mode**):*

$$\left|\tilde{Q}_2'(s) - \tilde{H}_1\frac{\tilde{Q}_2}{s}\right| \leq \frac{CA^8}{s^{\frac{9}{4}}},$$

where $\tilde{H}_1$ is a constant depending only on p, δ and less than $-\frac{3}{2}$.
*(iv) (**Control of negative modes**):*

$$|q_1(s)| \leq e^{-\frac{(s-\tau)}{2}}|q_1(\tau)| + \frac{CA^3}{s^{\frac{3}{2}}},$$

$$|Q_2(s)| \leq e^{-(s-\tau)}\,|Q_2(\tau)| + \frac{CA^7}{s^{\frac{7}{4}}},$$

$$|q_3| \leq e^{-\frac{3}{2}(s-\tau)}|q_3(\tau)| + \frac{CA^2}{s^{\frac{3}{2}}},$$

$$|\tilde{q}_3| \leq e^{-\frac{s-\tau}{2}}|\tilde{q}_3(\tau)| + \frac{CA^2}{s^{\frac{3}{2}}},$$

$$|Q_4(s)| \leq e^{-2(s-\tau)}\,|Q_4(\tau)| + \frac{CA^6}{s^{\frac{7}{4}}},$$

$$\left|\tilde{Q}_4(s)\right| \leq e^{-(s-\tau)}\left|\tilde{Q}_4(\tau)\right| + \frac{CA^3}{s^{\frac{7}{4}}},$$

$$|q_j(s)| \leq e^{-j\frac{(s-\tau)}{2}}|q_j(\tau)| + \frac{CA^{j-1}}{s^{\frac{j+1}{4}}}, \text{ for all } 5 \leq j \leq M,$$

$$|\tilde{q}_j(s)| \leq e^{-(j-2)\frac{(s-\tau)}{2}}|\tilde{q}_j(\tau)| + \frac{CA^{j-1}}{s^{\frac{j+1}{4}}}, \text{ for all } 5 \leq j \leq M,$$

$$\left\|\frac{q_-(y,s)}{1+|y|^{M+1}}\right\|_{L^\infty} \le e^{-\frac{M+1}{4}(s-\tau)}\left\|\frac{q_-(\tau)}{1+|y|^{M+1}}\right\|_{L^\infty} + C\frac{A^M}{s^{\frac{M+2}{4}}},$$

$$\|q_e(y,s)\|_{L^\infty} \le e^{-\frac{(s-\tau)}{2(p-1)}}\|q_e(\tau)\|_{L^\infty} + \frac{CA^{M+1}}{\tau^{\frac{1}{4}}}(1+s-\tau),$$

where $\tilde{Q}_0$, Q_2, $\tilde{Q}_2$, Q_4 and $\tilde{Q}_4$ are defined by (66-75).

The idea of the proof of Proposition 4.10 is to project equations (44) and (48) according to the decomposition (64). However because of the number of parameters and coordinates in (64), the computation become too long. That is why Subsection 4.3 is devoted to the proof of Proposition 4.10.

Let us now derive Proposition 4.7 from Proposition 4.10.

Proof of Proposition 4.7 assuming Proposition 4.10:
We will take $A_4 \ge A_5$. Hence, we can use the conclusion of Proposition 4.10.
(i) The proof follows from (i) of Proposition 4.10. Indeed by choosing T_4 small enough, we can make $s_0 = -\log T$ bigger than $s_5(A)$.

(ii) We notice that from Claim 4.8 and the fact that $q_0(s) = 0$, it is enough to prove that for all $s \in [s_0, s_1]$,

$$\left|\tilde{Q}_2(s)\right| = \left|\tilde{q}_2(s) - \frac{\mathcal{A}_2}{s}\right| < \frac{A^{10}}{s^{\frac{5}{4}}}. \tag{87}$$

$$\begin{array}{ll} \|q_e\|_{L^\infty(\mathbb{R})} \le \frac{A^{M+2}}{2s^{\frac{1}{4}}}, & \left\|\frac{q_-(y,s)}{1+|y|^{M+1}}\right\|_{L^\infty} \le \frac{A^{M+1}}{2s^{\frac{M+2}{4}}}, \\ |q_j|, |\tilde{q}_j| \le \frac{A^j}{2s^{\frac{j+1}{4}}} \text{ for all } 5 \le j \le M, & |q_1| \le \frac{A^4}{2s^{\frac{3}{2}}}, \\ |Q_2| \le \frac{A^8}{2s^{\frac{7}{4}}} & |q_3|, |\tilde{q}_3| \le \frac{A^3}{2s^{\frac{3}{2}}}, \\ |Q_4|, |\tilde{Q}_4| \le \frac{A^7}{2s^{\frac{7}{4}}}. & \end{array} \tag{88}$$

Let us first prove (87): Indeed, we will use a contradictory argument, we assume that there exists $s_* \in [s_0, s_1]$ such that

$$\tilde{Q}_2(s_*) = \left(\tilde{q}_2(s_*) - \frac{\mathcal{A}_2}{s_*}\right) = \omega\frac{A^{10}}{s_*^{5/4}} \text{ and for all } s \in [s_0, s^*[, \ \left|\tilde{q}_2(s) - \frac{\mathcal{A}_2}{s}\right| < \frac{A^{10}}{s^{5/4}},$$

where $\omega = \pm 1$. As a matter of fact, we can reduce to the positive case where $\omega = 1$ (the case $\omega = -1$ also work by the same way). Note by item (iv) in Proposition 4.5 that

$$\left|\tilde{q}_2(s_0) - \frac{\mathcal{A}_2}{s_0}\right| < \frac{A^{10}}{s_0^{\frac{5}{4}}},$$

thus $s_* > s_0$, and the interval $[s_0, s_*]$ is not empty.
Using the continuity of $\tilde{Q}_2$ and the definition of s_*, it is clearly that $\tilde{Q}_2(s_*)$ is the maximal value of $\tilde{Q}_2$ in $[s_* - \epsilon, s_*]$ with $\epsilon > 0$ and small enough in one hand, recalling, from (iii) Proposition 4.10 that

$$\left|\tilde{Q}_2' - \tilde{H}_1\frac{\tilde{Q}_2}{s}\right| \le \frac{CA^8}{s^{\frac{9}{4}}},$$

and from (140) that $\tilde{H}_1 \le -\frac{3}{2}$, we write

$$\tilde{Q}_2'(s_*) \le \tilde{H}_1 \frac{\tilde{Q}_2}{s} + \frac{A^8}{s^{9/4}} \le \frac{-3/2A^{10} + CA^8}{s^{9/4}} < 0, \tag{89}$$

for A large enough. Then, $\tilde{Q}_2$ has to decrease in $[s_* - \epsilon_1, s_*]$ which implies a contradiction with the assumption that $\tilde{Q}_2$ admits maximum at s_*. In other word, (87) holds.

Now, let us deal with (88). Define $\sigma = \log A$ and take $s_0 \ge \sigma$ (that is $T \le e^{-\sigma} = 1/A$) so that for all $\tau \ge s_0$ and $s \in [\tau, \tau + \sigma]$, we have

$$\tau \le s \le \tau + \sigma \le \tau + s_0 \le 2\tau \text{ hence } \frac{1}{2\tau} \le \frac{1}{s} \le \frac{1}{\tau} \le \frac{2}{s}. \tag{90}$$

We consider two cases in the proof.
Case 1: $s \le s_0 + \sigma$.
Note that (90) holds with $\tau = s_0$. Using (iv) of Proposition 4.10 and estimate (iii) of Proposition 4.5 on the initial data $q(., s_0)$ (where we use (90) with $\tau = s_0$), we write

$$\begin{array}{l}
|q_1(s)| \le CAe^{-\gamma_1 \frac{s}{2}} + \frac{CA^3}{s^{3/2}},\\
|Q_2(s)| \le CAe^{-\gamma_1 \frac{s}{2}} + \frac{CA^7}{s^{\frac{7}{4}}},\\
|q_3| \le CAe^{-\gamma_1 \frac{s}{2}} + \frac{CA^2}{s^{\frac{3}{2}}},\\
|\tilde{q}_3| \le CAe^{-\gamma_1 \frac{s}{2}} + \frac{CA^2}{s^{\frac{3}{2}}},\\
|Q_4(s)| \le CAe^{-\gamma_1 \frac{s}{2}} + \frac{CA^6}{s^{\frac{7}{4}}},\\
|\tilde{Q}_4(s)| \le CAe^{-\gamma_1 \frac{s}{2}} + \frac{CA^3}{s^{\frac{7}{4}}},\\
|\tilde{q}_j(s)| \le CAe^{-\gamma_1 \frac{s}{2}} + \frac{CA^{j-1}}{s^{\frac{j+1}{4}}} \text{ for all } 3 \le j \le M,\ \ j \ne 4\\
|q_j(s)| \le CAe^{-\gamma_1 \frac{s}{2}} + \frac{CA^{j-1}}{s^{\frac{j+1}{4}}} \text{ for all } 3 \le j \le M,\ \ j \ne 4\\
\left\| \frac{q_-(s)}{1+|y|^{M+1}} \right\|_{L^\infty} \le C\frac{A}{(\frac{s}{2})^{\frac{M}{4}+2}} + C\frac{A^M}{s^{\frac{M+2}{4}}},\\
\|q_e(s)\|_{L^\infty} \le \frac{CA^{M+1}}{(\frac{s}{2})^{\frac{1}{4}}}(1 + \log A).
\end{array} \tag{91}$$

Thus, if $A \ge A_6$ and $s_0 \ge s_6(A)$ (that is $T \le e^{-s_6(A)}$) for some positive A_6 and $s_6(A)$, we see that (88) holds.
Case 2: $s > s_0 + \sigma$.
Let $\tau = s - \sigma > s_0$. Applying (iv) of Proposition 4.10 and using the fact that $q(\tau) \in \mathcal{V}_A(\tau)$,

we write (we use (90) to bound any function of τ by a function of s)

$$\begin{aligned}
&|q_1(s)| \le e^{-\frac{\sigma}{2}} \frac{A^6}{\left(\frac{s}{2}\right)^{3/2}} + \frac{CA^3}{s^{3/2}},\\
&|Q_2(s)| \le e^{-\sigma} \frac{A^8}{\left(\frac{s}{2}\right)^{7/4}} + \frac{CA^7}{s^{7/4}},\\
&|q_3(s)| \le e^{-\frac{3\sigma}{2}} \frac{A^3}{\left(\frac{s}{2}\right)^{3/2}} + \frac{CA^2}{s^{3/2}},\\
&|\tilde{q}_3(s)| \le e^{-\frac{\sigma}{2}} \frac{A^3}{\left(\frac{s}{2}\right)^{3/2}} + \frac{CA^2}{s^{3/2}},\\
&|Q_4(s)| \le e^{-2\sigma} \frac{A^7}{\left(\frac{s}{2}\right)^{7/4}} + \frac{CA^6}{s^{7/4}},\\
&|\tilde{Q}_4(s)| \le e^{-\sigma} \frac{A^4}{\left(\frac{s}{2}\right)^{7/4}} + \frac{CA^3}{s^{7/4}},\\
&|\tilde{q}_j(s)| \le e^{-\frac{(j-2)\sigma}{2}} \frac{A^j}{\left(\frac{s}{2}\right)^{\frac{j+1}{4}}} + \frac{CA^{j-1}}{s^{\frac{j+1}{4}}} \text{ for all } 5 \le j \le M,\\
&|q_j(s)| \le e^{-\frac{j\sigma}{2}} \frac{A^j}{\left(\frac{s}{2}\right)^{\frac{j+1}{4}}} + \frac{CA^{j-1}}{s^{\frac{j+1}{4}}} \text{ for all } 5 \le j \le M,\\
&\left\| \frac{q_-(s)}{1+|y|^{M+1}} \right\|_{L^\infty} \le e^{-\frac{M+1}{4}\sigma} \frac{A^{M+1}}{\left(\frac{s}{2}\right)^{\frac{M+2}{4}}} + C\frac{A^M}{s^{\frac{M+2}{4}}},\\
&\|q_e(s)\|_{L^\infty} \le e^{-\frac{\sigma}{2(p-1)}} \frac{A^{M+2}}{\left(\frac{s}{2}\right)^{\frac{1}{4}}} + \frac{CA^{M+1}}{\left(\frac{s}{2}\right)^{\frac{1}{4}}}(1+\sigma).
\end{aligned} \tag{92}$$

For all the coordinates, it is clear that if $A \ge A_7$ and $s_0 \ge s_7(A)$ for some positive A_7 and $s_7(A)$, then (87) and (88) is satisfied (remember that $\sigma = \log A$).

Conclusion of (ii): If $A \ge \max(A_6, A_7, A_8)$ and $s_0 \ge \max(s_6(A), s_7(A), s_8(A))$, then (88) is satisfied. Since we know that $q(s_1) \in \partial V_A(s_1)$, we see from the definition of $V_A(s)$ that $(\tilde{Q}_0(s_1), \tilde{q}_1(s_1)) \in \partial[-\frac{A}{s_1^{7/4}}, \frac{A}{s_1^{7/4}}] \times [-\frac{A}{s_1^{3/2}}, \frac{A}{s_1^{3/2}}]$. This concludes the proof of (ii) of Proposition 4.7.

(iii) From (ii), there is $\omega = \pm 1$ such that $\tilde{Q}_0(s_1) = \omega \frac{A}{s_1^{7/4}}$ or $\tilde{q}_0(s_1) = \omega \frac{A}{s_1^{3/2}}$.

Using (ii) of Proposition 4.7, we see that

$$\omega \tilde{Q}_0'(s_1) \ge \omega \tilde{Q}_0(s_1) - \frac{C}{s_1^{7/4}},$$

or

$$\omega \tilde{q}_1'(s_1) \ge \frac{1}{2} \omega \tilde{q}_1(s_1) - \frac{C}{s_1^{3/2}}.$$

Taking A large enough gives $\omega \tilde{Q}_0'(s_1) > 0$ or $\omega \tilde{q}_1'(s_1) > 0$, and concludes the proof of Proposition 4.7. ■

4.3 Proof of Proposition 4.10

In this section, we prove Proposition 4.10. We just have to project equations (44) and (48) to get equations satisfied by the different coordinates of the decomposition (64). We proceed as Section 5 in [MZ08], taking into account the new scaling law $\frac{y}{s^{1/4}}$. We note that the projections of $V_1 q + V_2 \bar{q}$, B and R^* in (49), will need much more effort and this is due to the fact that we are dealing with the critical case when $\beta \neq 0$.

More precisely, the proof will be carried out in 3 subsections

- In the first subsection, we deal with equation (48) to write equations satisfied by $\tilde{q}_j$ and q_j. Then, we prove (i), (ii), (iii) and (iv) (expect the two last identities) of Proposition 4.10.
- In the second subsection, we first derive from equation (48) an equation satisfied by q_- and prove the last but one identity in (iv) of Proposition 4.10.
- In the third subsection, we project equation (44) (which is simpler than (48)) to write an equation satisfied by q_e and prove the last identity in (iv) of Proposition 4.10.

4.3.1 The finite dimensional part q_+

We proceed in 2 parts:

- In Part 1, we give the details of projections of equation (48) to get ODEs, satisfied by modes $\tilde{q}_j$ and q_j.
- In Part 2, we prove (i), (ii) and (iii) of Proposition 4.10, together with the estimates concerning $\tilde{q}_j$ and q_j in (iv).

Part 1: The projection of equation (48) on the eigenfunction of the operator $\mathcal{L}_{\beta,\delta}$ In the following, we will find the main contribution in the projections $\tilde{P}_{n,M}$ and $P_{n,M}$ of the six terms appearing in equation (48): $\partial_s q$, $\mathcal{L}_{\beta,\delta} q$, $-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q$, $V_1 q + V_2 \bar{q}$, $B(q, y, s)$ and $R^*(\theta', y, s)$. Most of the time, we give two estimates of error terms, depending on whether we use or not the fact that $q(s) \in \mathcal{V}_A(s)$.

First term: $\dfrac{\partial q}{\partial s}$.
From (63), its projection on $\tilde{h}_n$ and $\hat{h}_n$ is $\tilde{q}_n'$ and q_n' respectively:

$$\tilde{P}_{n,M}\left(\frac{\partial q}{\partial s}\right) = \tilde{q}_n' \text{ and } P_{n,M}\left(\frac{\partial q}{\partial s}\right) = q_n'. \tag{93}$$

Second term: $\mathcal{L}_{\beta,\delta} q$, where $\mathcal{L}_{\beta,\delta}$ is defined as in (49). We will use the following Lemma from [MZ08]:

Lemma 4.11 (Projection of $\mathcal{L}_{\beta,\delta}$ on $\tilde{h}_n$ and h_n for $n \leq M$)
a) If $n \leq M-2$, then

$$\left|P_{n,M}(\mathcal{L}_{\beta,\delta} q) - \left(-\frac{n}{2} q_n(s) + c_{n+2}\tilde{q}_{n+2}\right)\right| \leq C \left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty},$$

where c_n is given in Lemma 3.1. Moreover, we have the following
If $M-1 \leq n \leq M$, then

$$\left|P_{n,M}(\mathcal{L}_{\beta,\delta} q) + \frac{n}{2} q_n(s)\right| \leq C \left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}.$$

(b) If $n \leq M$, then the projection of $\mathcal{L}_{\beta,\delta}$ on $\tilde{h}_n$ satisfies

$$\left|\tilde{P}_{n,M}(\mathcal{L}_{\beta,\delta} q) - \left(1 - \frac{n}{2}\right)\tilde{q}_n(s)\right| \leq C \left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}.$$

Proof: The proof is quiet the same as the proof of Lemma 5.1 in [MZ08].

Using Lemma 4.11 and the fact that $q(s) \in V_A(s)$ (see Definition 4.1) in addition, then the error estimates can be improved as follows

Corollary 4.1 *For all $A \geq 1$, there exists $s_9 \geq 1$ such that for all $s \geq s_9(A)$, if $q(s) \in V_A(s)$, then:*

a) For $n = 0$, we have

$$|P_{0,M}(\mathcal{L}_{\beta,\delta} q) - c_2 \tilde{q}_2| \leq C \frac{A^{M+1}}{s^{\frac{M+2}{4}}}.$$

b) For $1 \leq n \leq M-1$, we have

$$\left| P_{n,M}(\mathcal{L}_{\beta,\delta} q) + \frac{n}{2} q_n(s) \right| \leq C \frac{A^{n+2}}{s^{\frac{n+3}{2}}}.$$

In particular, we have a smaller bound for $P_{2,M}(\mathcal{L}_{\beta,\delta} q)$:

$$|P_{2,M}(\mathcal{L}_{\beta,\delta} q) + q_2 - c_4 \tilde{q}_4| \leq \frac{A^{M+1}}{s^{\frac{M+2}{4}}}.$$

c) For $n = M$, we have

$$\left| P_{M,M}(\mathcal{L}_{\beta,\delta} q) + \frac{M}{2} q_M(s) \right| \leq C \frac{A^{M+1}}{s^{\frac{M+2}{2}}}.$$

d) For $0 \leq n \leq M$, we have

$$\left| \tilde{P}_{n,M}(\mathcal{L}_{\beta,\delta} q) - \left(1 - \frac{n}{2}\right) \tilde{q}_n(s) \right| \leq C \frac{A^{M+1}}{s^{\frac{M+1}{2}}}.$$

Third term: $-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q$**.** It is enough to project iq, from (63), we recall Lemma 5.3 from [MZ08]:

Lemma 4.12 (Projection of the term $-i(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s))q$ on h_n and $\tilde{h}_n$ for $n \leq M$) *Its projection on h_n is given by*

$$\begin{aligned} & P_{n,M}\left(-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q\right) \\ = \; & -\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right)\left(\delta q_n + (1+\delta^2)\tilde{q}_n + \sum_{j=n+1}^{M} K_{n,j} q_j + L_{n,j} \tilde{q}_j\right), \end{aligned}$$

where $K_{n,j}$ and $L_{n,j}$ defined by

$$K_{n,j} = P_{n,M}(i h_j), \tag{94}$$
$$L_{n,j} = P_{n,M}(i \tilde{h}_j). \tag{95}$$

Its projection on $\tilde{h}_n$ is given by

$$\tilde{P}_{n,M}\left(-i\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'(s)\right)q\right) = -\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'\right)\left(-q_n-\delta\tilde{q}_n+\sum_{j=n+1}^{M}\tilde{K}_{n,j}q_j+\tilde{L}_{n,j}\tilde{q}_j\right),$$

where $\tilde{K}_{n,j}$ and $\tilde{L}_{n,j}$ defined as follows

$$\tilde{K}_{n,j} = \tilde{P}_{n,M}(ih_j), \quad (96)$$

$$\tilde{L}_{n,j} = \tilde{P}_{n,M}(i\tilde{h}_j). \quad (97)$$

Using the fact that $q(s) \in \mathcal{V}_A(s)$ in addition, then the error estimates can be bounded from Definition 4.1 as follows:

Corollary 4.2 *For all $A \geq 1$, there exists $s_{10}(A) \geq 1$ such that for all $s \geq s_{10}(A)$, if $q \in \mathcal{V}_A(s)$ and $|\theta'(s)| \leq \frac{CA^{10}}{s^{\frac{5}{4}}}$, then:*

a) For all $1 \leq n \leq M$, we have

$$\left|P_{n,M}\left(-i\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'(s)\right)q\right)\right| \leq C\frac{A^n}{s^{\frac{n+5}{4}}}.$$

b) For $1 \leq n \leq M$, we have

$$\left|\tilde{P}_{n,M}\left(-i\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'(s)\right)q\right)\right| \leq C\frac{A^n}{s^{\frac{n+5}{4}}}.$$

In particular, when $n = 0, 2, 4$, we can get smaller bounds as follows:
c) For $n = 0$, we have the following in particular

$$\left|P_{0,M}\left(-i(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta')q\right)+(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta')\{\delta q_0+(1+\delta^2)\tilde{q}_0+K_{0,2}q_2+L_{0,2}\tilde{q}_2\}\right| \leq C\frac{A^4}{s^{\frac{7}{4}}},$$

$$\left|\tilde{P}_{0,M}\left(-i(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta')q\right)+\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'\right)\{-q_0-\delta\tilde{q}_0+\tilde{K}_{0,2}q_2+\tilde{L}_{0,2}\tilde{q}_2\}\right| \leq C\frac{A^4}{s^{\frac{7}{4}}},$$

d) For $n = 2$, we have

$$\left|P_{2,M}\left(-i(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta')q\right)+(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta')[\delta q_2+(1+\delta^2)\tilde{q}_2]\right| \leq C\frac{A^4}{s^{\frac{7}{4}}},$$

$$\left|\tilde{P}_{2,M}\left(-i\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'\right)q\right)+\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'(s)\right)\left(-q_2-\delta\tilde{q}_2+\tilde{K}_{2,4}q_4+\tilde{L}_{2,4}\tilde{q}_4\right)\right| \leq C\frac{A^6}{s^{\frac{9}{4}}},$$

e) For $n=3$, we have

$$\begin{aligned}\left|P_{3,M}\left(-i(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta')q\right)\right| &\leq C\frac{A^2}{s^{\frac{3}{2}}},\\ \left|\tilde{P}_{3,M}\left(-i\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'\right)q\right)\right| &\leq C\frac{A^2}{s^{\frac{3}{2}}},\end{aligned}$$

f) For $n=4$, we have

$$\begin{aligned}\left|P_{4,M}\left(-i(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta')q\right)\right| &\leq C\frac{A^5}{s^2},\\ \left|\tilde{P}_{4,M}\left(-i\left(\frac{\nu}{2\sqrt{s}}+\frac{\mu}{s}+\theta'\right)q\right)\right| &\leq C\frac{A^5}{s^2},\end{aligned}$$

Fourth term: $V_1q+V_2\bar{q}$.

We claim the following

Lemma 4.13 (Projection of V_1q and $V_2\bar{q}$) *(i) It holds that*

$$|V_i(y,s)| \leq C\frac{(1+|y|^2)}{s^{1/2}}, \text{ for all } y\in\mathbb{R} \text{ and } s\geq 1, \tag{98}$$

and for all $k\in\mathbb{N}^*$

$$V_i(y,s) = \sum_{j=1}^{k}\frac{1}{s^{j/2}}W_{i,j}(y)+\tilde{W}_{i,k}(y,s), \tag{99}$$

where $W_{i,j}$ is an even polynomial of degree $2j$ and $\tilde{W}_{i,k}(y,s)$ satisfies

$$\text{for all } s\geq 1 \text{ and } |y|\leq s^{1/4}, \ \left|\tilde{W}_{i,k}(y,s)\right| \leq C\frac{(1+|y|^{2k+2})}{s^{\frac{k+1}{2}}}. \tag{100}$$

(ii) The projection of V_1q and $V_2\bar{q}$ on $(1+i\delta)h_n$ and ih_n, and we have

$$\begin{aligned}&|\tilde{P}_n(V_1q)|+|\hat{P}_n(V_1q)|\\ &\leq \frac{C}{s^{1/2}}\sum_{j=n-2}^{M}(|\tilde{q}_j|+|\hat{q}_j|)+\sum_{j=0}^{n-3}\frac{C}{s^{\frac{n-j}{4}}}(|\tilde{q}_j|+|\hat{q}_j|)+\frac{C}{s^{1/2}}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty},\end{aligned} \tag{101}$$

and the same holds for $V_2\bar{q}$

Remark 4.14 *If $n\leq 2$, the first sum in* (101) *runs for $j=0$ to M and the second sum doesn't exist.*

If in addition $q(s)\in\mathcal{V}_A(s)$, then the error estimates can be bounded from Definition 4.1 as follows:

Corollary 4.3 *For all $A\geq 1$, there exists $s_{11}(A)\geq 1$ such that for all $s\geq s_{11}(A)$, if $q\in\mathcal{V}_A(s)$, then for $3\leq n\leq M$, we have*

$$\left|\tilde{P}_n(V_1q+V_2\bar{q})\right|+|P_n(V_1q+V_2\bar{q})| \leq \frac{CA^{n-2}}{s^{\frac{n+1}{4}}}.$$

Proof of Lemma 4.13:
(i) The estimates of V_1q and $V_2\bar{q}$ are the same, so we only deal with V_1q. Let $F(u) = \frac{(p+1)}{2}(1+i\delta)\left[|u|^{p-1} - \frac{1}{p-1}\right]$, where $u \in \mathbb{C}$ and consider $z = \frac{y}{s^{1/4}}$. Note that from (48) and (43), we have

$$V_1(y,s) = F(\varphi(y,s)), \text{ where } \varphi(y,s) = \varphi_0(\frac{y}{s^{1/4}}) + \frac{a}{s^{1/2}}(1+i\delta).$$

Note that there exist positive constant c_0 and s_0 such that $\varphi_0(z)|$ and $|\varphi(y,s)| = |\varphi_0(\frac{y}{s^{1/4}}) + \frac{a}{s^{1/2}}(1+i\delta)|$ are both larger than $\frac{1}{c_0}$ and smaller than c_0, uniformly in $|z| < 1$ and for $s \geq s_0$. Since $F(u)$ is C^∞ for $\frac{1}{c_0} \leq |u| \leq c_0$, we expand it around $u = \varphi_0(z)$ as follows: for all $s \geq s_0$ and $|z| < 1$,

$$\begin{aligned} \left|F\left(\varphi_0(z) + \tfrac{a}{s^{1/2}}(1+i\delta)\right) - F\left(\varphi_0(z)\right)\right| &\leq \frac{C}{s^{1/2}}, \\ \left|F\left(\varphi_0(z) + \tfrac{a}{s^{1/2}}(1+i\delta)\right) - F\left(\varphi_0(z)\right) - \sum_{j=1}^{n} \frac{1}{s^{j/2}} F_j(\varphi_0(z))\right| &\leq \frac{C}{s^{\frac{n+1}{2}}}, \end{aligned}$$

where $F_j(u)$ are C^∞. Hence, we can expand $F(u)$ and $F_j(u)$ around $u = \varphi_0(0)$ and write for all $s \geq s_0$ and $|z| < 1$,

$$\begin{aligned} &\left|F\left(\varphi_0(z) + \tfrac{a}{s^{1/2}}(1+i\delta)\right) - F\left(\varphi_0(0)\right)\right| \leq Cz^2 + \tfrac{C}{s^{1/2}}, \\ &\left|F\left(\varphi_0(z) + \tfrac{a}{s^{1/2}}(1+i\delta)\right) - F\left(\varphi_0(0)\right) - \sum_{l=1}^{n} c_{0,l} z^{2l} - \sum_{j=1}^{n} \sum_{l=0}^{n-j} \frac{c_{j,l}}{s^{j/2}} z^{2l}\right| \\ &\leq C|z|^{2n+2} + \sum_{j=1}^{n} \frac{C}{j^{1/2}} |z|^{2(n-j)+2} \frac{C}{s^{\frac{n+1}{2}}}. \end{aligned}$$

Since $F(\varphi_0(0)) = F(\kappa) = 0$ and $z = \frac{y}{s^{1/4}}$, this gives us estimates in (i), when $s \geq s_0$ and $|y| < s^{1/4}$. Since V_1 is bounded, the inequalities still valid when $|y| \geq s^{1/4}$ and then when $s \geq 1$.

(ii) Note first that it is enough to prove the bound (62) for the projection of V_iq onto h_n to get the same bound for $\tilde{P}_{n,M}(V_iq)$ and $P_{n,M}(V_iq)$. Since in addition, the proof for $V_2\bar{q}$ is the same as for V_1q, we only prove (101) for the projection of V_1q onto f_n. Using (64) and (61), we see that the projection is given by

$$\int f_n V_1 q \rho \quad = \int f_n V_1 q_- \rho + \sum_{j=0}^{M} \tilde{q}_j \int f_n \tilde{h}_j V_1 \rho_\beta + \sum_{j=0}^{M} q_j \int f_n h_j V_1 \rho_\beta. \tag{102}$$

The first term can be bounded by

$$\int f_n V_1 \left(\frac{1+|y|^2}{s^{1/2}}\right) |q_-||\rho_\beta| \leq \frac{C}{s^{1/2}} \left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}. \tag{103}$$

Now we deal with the second term. We only focus on the terms involving h_j .
If $j \geq n-2$, we use (98) to write $|\int f_n h_j V_1 \rho_\beta| \leq \frac{C}{s^{1/2}}$.
If $j \leq n-3$, then we claim that

$$\left|\int f_n h_j V_1 \rho_\beta\right| \leq \frac{C}{s^{\frac{n-j}{4}}}, \tag{104}$$

(this actually vanishes if j and n have different parities). It is clear that (101) follows from (102), (103) and (104).
Let us prove (104). Note that $k \equiv \left[\frac{n-j-1}{2}\right]$ (which is in $\mathbb{N}^*$ since $j \leq n-3$) is the largest integer such that $j+2k<n$. We use (99) to write

$$\begin{aligned}
\int f_n h_j V_1 \rho_\beta &= \int_{|y|<s^{1/4}} f_n h_j V_1 \rho_\beta + \int_{|y|>s^{1/4}} f_n h_j V_1 \rho_\beta, \\
&= \sum_{l=1}^{k} \frac{1}{s^{l/2}} \int_{|y|<s^{1/4}} f_n h_j W_{1,l} \rho_\beta + O\left(\frac{1}{s^{\frac{\left[\frac{n-j-1}{2}\right]+1}{2}}} \int (1+|y|^{n-j+1})|f_n||h_j|\rho_\beta dy\right) \\
&+ \textstyle\int_{|y|>s^{1/4}} h_n h_j V_1 \rho, \\
&= \sum_{l=1}^{k} \frac{1}{s^{l/2}} \int_{\mathbb{R}^N} h_n h_j W_{1,l} \rho + O\left(\frac{1}{s^{\frac{\left[\frac{n-j-1}{2}\right]+1}{2}}}\right) - \sum_{l=1}^{k} \frac{1}{s^l} \int_{|y|>s^{1/4}} f_n h_j W_{1,l} \rho_\beta \\
&+ \textstyle\int_{|y|>s^{1/4}} f_n h_j V_1 \rho_\beta,
\end{aligned} \tag{105}$$

since $deg(h_j W_{1,l}) = j+2l \leq j+2k < n = deg(h_n)$, h_n is orthogonal to $h_j W_{1,l}$ and

$$\int_{\mathbb{R}^N} f_n h_j W_{1,l} \rho_\beta = 0.$$

Since $|\rho_\beta(y)| \leq Ce^{-cs^{1/2}}$ when $|y| > s^{1/4}$, the integrals over the domain $|y| > s^{1/4}$ can be bounded by

$$Ce^{-cs^{1/2}} \int_{|y|>s^{1/4}} |f_n||h_j|(1+|y|^{2k})\sqrt{\rho_\beta} \leq Ce^{-cs^{1/2}}.$$

Using that $\left[\frac{n-j-1}{2}\right]+1 \geq \frac{n-j}{2}$, we deduce that (104) holds. Hence, we have proved (101) and this concludes the proof of Lemma 4.13. ■.

As a matter of fact, we need more refinements in the cases where $n=0$ and 2 for the terms $\tilde{P}_{2,M}(V_1 q)$, $\tilde{P}_{2,M}(V_2 \bar{q})$, $P_{0,M}(V_1 q)$ and $P_{0,M}(V_2 \bar{q})$. More precisely

Lemma 4.15 (Projection of $V_1 q$ and $V_2 \bar{q}$ on $\hat{h}_0, \tilde{h}_0, \hat{h}_2, \tilde{h}_2, \hat{h}_4$ and h_4) *Using the definition of V_1, V_2, the following hold:*
(i) It holds that for $i=1,2$

$$\forall s \geq 1 \text{ and } |y| < s^{1/4}, \quad \left|V_i(y,s) - \frac{1}{s^{1/2}} W_{i,1}(y) - \frac{1}{s} W_{i,2}(y)\right| \leq \frac{C}{s^{3/2}}(1+|y|^6), \tag{106}$$

where

$$
\begin{array}{rcl}
W_{1,1}(y) & = & -\frac{(p+1)b}{2(p-1)^2}(1+i\delta)(y^2-2(1-\delta\beta)),\\
W_{1,2}(y) & = & (1+i\delta)\frac{b^2(p+1)}{2(p-1)^4}\left\{(p-1)y^4-[2(1-\beta\delta)(p-2+\delta^2)]y^2\right.\\
& + & \left.2(p-2+\delta^2)(1-\beta\delta)^2\right\},\\
W_{2,1}(y) & = & -(1+i\delta)\frac{b}{2(p-1)^2}\,(p-1+2i\delta)\,(y^2-2(1-\beta\delta)),\\
W_{2,2}(y) & = & (1+i\delta)\frac{b^2}{2(p-1)^4}\left\{(p-2+2i\delta)(p-1+i\delta)y^4\right.\\
& - & [2(p-1)(p-2)+(2p-10)\delta^2+(8p-16)i\delta](1-\delta\beta)y^2\\
& + & \left.(1-\delta\beta)(2p^2+8ip\delta+4-16i\delta-10\delta^2-6p+2p\delta^2)\right\}.
\end{array}
\tag{107}
$$

(ii) The projection of V_1q and $V_2\bar{q}$ on $\tilde{h}_2$ satisfy

$$
\begin{array}{l}
\left|\tilde{P}_n(V_1q+V_2\bar{q})-\frac{1}{\sqrt{s}}\sum_{j\geq 0}[\tilde{C}_{n,j}q_j+\tilde{D}_{n,j}\tilde{q}_j]-\frac{1}{s}\sum_{j\geq 0}[\tilde{E}_{n,j}q_j+\tilde{F}_{n,j}\tilde{q}_j]\right|\\
\leq \frac{C}{s^{\frac{3}{2}}}\sum_{j\geq 0}[|\hat{q}_j|+|\tilde{q}_j|]+\frac{1}{\sqrt{s}}\left\|\frac{q_-(.,s)}{1+|y|^M}\right\|_{L^\infty},
\end{array}
\tag{108}
$$

and

$$
\begin{array}{l}
\left|P_n(V_1q+V_2\bar{q})-\frac{1}{\sqrt{s}}\sum_{j\geq 0}[C_{n,j}q_j+D_{2,j}\tilde{q}_j]-\frac{1}{s}\sum_{j\geq 0}[E_{n,j}q_j+F_{n,j}\tilde{q}_j]\right|\\
\leq \frac{C}{s^{\frac{3}{2}}}\sum_{j\geq 0}[|\hat{q}_j|+|\tilde{q}_j|]+\frac{1}{\sqrt{s}}\left\|\frac{q_-(.,s)}{1+|y|^M}\right\|_{L^\infty}.
\end{array}
\tag{109}
$$

where for all $n, j\geq 0$, we have

$$C_{n,j}=P_{n,M}(W_{1,1}h_j+W_{2,1}\bar{h}_j)\quad \tilde{C}_{n,j}=\tilde{P}_{n,M}(W_{1,1}h_j+W_{2,1}\bar{h}_j), \tag{110}$$

$$D_{n,j}=P_{n,M}(W_{1,1}\tilde{h}_j+W_{2,1}\bar{\tilde{h}}_j)\quad \tilde{D}_{n,j}=\tilde{P}_{n,M}(W_{1,1}\tilde{h}_j+W_{2,1}\bar{\tilde{h}}_j), \tag{111}$$

$$E_{n,j}=P_{i,M}(W_{1,2}h_j+W_{2,2}\bar{h}_j)\quad \tilde{E}_{n,j}=\tilde{P}_{i,M}(W_{1,2}h_j+W_{2,2}\bar{h}_j), \tag{112}$$

$$F_{n,j}=P_{n,M}(W_{1,2}\tilde{h}_j+W_{2,2}\bar{\tilde{h}}_j)\quad \tilde{F}_{n,j}=\tilde{P}_{n,M}(W_{1,2}\tilde{h}_j+W_{2,2}\bar{\tilde{h}}_j). \tag{113}$$

In addition, by using that fact that $q(s)\in\mathcal{V}_A(s)$, then the error estimates can be bounded from Definition 4.1 as follows;

Corollary 4.4 *For all $A\geq 1$, there exists $s_{12}(A)\geq 1$ such that for all $s\geq s_{12}(A)$, if*

$q(s) \in \mathcal{V}_A(s)$, *then*

$$\left|P_{0,M}(V_1 q + V_2\bar{q}) - \left(C_{0,0}\frac{q_0}{\sqrt{s}} + D_{0,0}\frac{\tilde{q}_0}{\sqrt{s}} + C_{0,2}\frac{q_2}{\sqrt{s}} + D_{0,2}\frac{\tilde{q}_2}{\sqrt{s}}\right)\right| \le C\frac{A^4}{s^{\frac{7}{4}}},$$

$$\left|\tilde{P}_{0,M}(V_1 q + V_2\bar{q}) - \left(\tilde{D}_{0,0}\frac{\tilde{q}_0}{\sqrt{s}} + \tilde{C}_{0,2}\frac{q_2}{\sqrt{s}} + \tilde{D}_{0,2}\frac{\tilde{q}_2}{\sqrt{s}}\right)\right| \le C\frac{A^4}{s^{\frac{7}{4}}},$$

$$\left|P_{2,M}(V_1 q + V_2\bar{q}) - \left(\frac{D_{2,0}\tilde{q}_0}{\sqrt{s}} + \frac{C_{2,2}q_2}{\sqrt{s}} + \frac{D_{2,2}\tilde{q}_2}{\sqrt{s}}\right)\right| \le C\frac{A^4}{s^{\frac{7}{4}}},$$

$$\left|\tilde{P}_{2,M}\left(V_1 q + V_2\bar{q}\right) - \frac{1}{\sqrt{s}}\left\{\tilde{q}_0\tilde{D}_{2,0} + q_2\tilde{C}_{2,2} + \tilde{q}_2\tilde{D}_{2,2} + q_4\tilde{C}_{2,4} + \tilde{q}_4\tilde{D}_{2,4}\right\}\right.$$
$$\left. - \quad \frac{1}{s}\left\{\tilde{q}_0\tilde{F}_{2,0} + q_2\tilde{E}_{2,2} + \tilde{q}_2\tilde{F}_{2,2}\right\}\right| \le C\frac{A^6}{s^{\frac{9}{4}}},$$

$$\left|P_{4,M}(V_1 q + V_2\bar{q}) - \left(C_{4,2}\frac{q_2}{\sqrt{s}} + D_{4,2}\frac{\tilde{q}_2}{\sqrt{s}}\right)\right| \le C\frac{A^4}{s^{\frac{7}{4}}},$$

$$\left|\tilde{P}_{4,M}(V_1 q + V_2\bar{q}) - \left(\tilde{C}_{4,2}\frac{q_2}{\sqrt{s}} + \tilde{D}_{4,2}\frac{\tilde{q}_2}{\sqrt{s}}\right)\right| \le C\frac{A^4}{s^{\frac{7}{4}}}.$$

and

$$|P_{3,M}(V_1 q + V_2\bar{q})| \le \frac{CA^4}{s^2},$$
$$\left|\tilde{P}_{3,M}(V_1 q + V_2\bar{q})\right| \le \frac{CA^4}{s^2}.$$

Proof of Lemma 4.15: (i) This is a simple, but lengthy computation that we omit. For more details see Appendix B. In addition to that, item (ii) directly follows Lemma 4.13, and item (i) and Definitions of the projection $P_{n,M}$ and $\tilde{P}_{n,M}$, defined as in (63). Finally, in order to know the exact formulas of the constants in item (ii), we kindly address the readers to Appendix B ■

Fifth term: $B(q,y,s)$ Let us recall from (49) that:

$$B(q,y,s) = (1+i\delta)\left(|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi - |\varphi|^{p-1}q - \frac{p-1}{2}|\varphi|^{p-3}\varphi(\varphi\bar{q}+\bar{\varphi}q)\right).$$

We have the following

Lemma 4.16 *The function $B = B(q,y,s)$ can be decomposed for all $s \ge 1$ and $|q| \le 1$ as*

$$\sup_{|y|<s^{1/4}}\left| B - \sum_{l=0}^{M}\sum_{\substack{0\le j,k\le M+1\\ 2\le j+k\le M+1}}\frac{1}{s^{l/2}}\left[B^l_{j,k}\left(\frac{y}{s^{1/4}}\right)q^j\bar{q}^k + \tilde{B}^l_{j,k}(y,s)q^j\bar{q}^k\right]\right|$$
$$\le C|q|^{M+2} + \frac{C}{s^{\frac{M+1}{2}}},$$

where $B^l_{j,k}(\frac{y}{s^{1/4}})$ *is an even polynomial of degree less or equal to* M *and the rest* $\tilde{B}^l_{j,k}(y,s)$ *satisfies*

$$\forall s \geq 1 \text{ and } |y| < s^{1/4},\ \left|\tilde{B}^l_{j,k}(y,s)\right| \leq C\frac{1+|y|^{M+1}}{s^{\frac{M+1}{2}}}.$$

Moreover,

$$\forall s \geq 1 \text{ and } |y| < s^{1/4},\ \left|B^l_{j,k}(\frac{y}{s^{1/4}}) + \tilde{B}^l_{j,k}(y,s)\right| \leq C.$$

On the other hand, in the region $|y| \geq s^{1/4}$*, we have*

$$|B(q,y,s)| \leq C|q|^{\bar{p}}, \tag{114}$$

for some constant C *where* $\bar{p} = \min(p,2)$*.*

Proof: See the proof of Lemma 5.9, page 1646 in [MZ08]. ■

Lemma 4.17 (The quadratic term $B(q,y,s)$**)** *For all* $A \geq 1$*, there exists* $s_{13} \geq 1$ *such that for all* $s \geq s_{13}$*, if* $q(s) \in \mathcal{V}_A(s)$*, then:*
a) the projection of $B(q,y,s)$ *on* h_n *and on* $\tilde{h}_n$*, for* $n \geq 3$ *satisfy*

$$\left|\tilde{P}_{n,M}(B(q,y,s))\right| + |P_{n,M}(B(q,y,s))| \leq C\frac{A^n}{s^{\frac{n+2}{4}}}. \tag{115}$$

b) For $n = 0,1,3,4$*, we have*

$$\left|\tilde{P}_{n,M}(B(q,y,s))\right| + |P_{n,M}(B(q,y,s))| \leq \frac{C}{s^2}, \tag{116}$$

c) For $n = 2$*, we have*

$$|P_{2,M}(B(q,y,s))| \leq \frac{C}{s^2}. \tag{117}$$

$$\left|\tilde{P}_{2,M}(B(q)) - \left(\tilde{B}_2\tilde{q}_2^2 + B_1\frac{\tilde{q}_2}{s} + \frac{B_2}{s^2}\right)\right| \leq \frac{CA^6}{s^{\frac{9}{4}}}, \tag{118}$$

where

$$\begin{aligned}
\tilde{B}_2 &= \frac{1}{\kappa}\left\{4(p-\delta^2) - \delta\beta(6+4p+2\delta^2)\right\},\\
B_1 &= \frac{1}{\kappa}\left\{(-7\delta^2\beta + p\beta - 6\beta)R^*_{2,1} - (p-\delta^2)\tilde{R}^*_{0,1}\right\},\\
B_2 &= \frac{(R^*_{2,1})^2}{s^2}\frac{(32-64\delta\beta)}{8\kappa}.
\end{aligned}$$

Proof : As a matter of fact, we only prove estimate (115). In addition to that, the proofs of (116), (117) and (118) will be given in the same way that using the Taylor expansion of B, established in Appendix C.

Let us start the proof of estimate (115) for the projection on h_n, since it implies the same estimate on $\tilde{P}_n$ and P_n through (63). We have

$$\int h_n B(q,y,s)\rho dy = \int_{|y|<s^{1/4}} h_n B(q,y,s)\rho dy + \int_{|y|>s^{1/4}} h_n B(q,y,s)\rho dy.$$

Using Lemma 4.16, we deduce that

$$\left|\int_{|y|<s^{1/4}} h_n B(q,y,s)\rho dy - \int_{|y|<s^{1/4}} h_n\rho \sum_{l=0}^{M} \sum_{\substack{0\le j,k\le M+1\\ 2\le j+k\le M+1}} \frac{1}{s^{l/2}}\left[B^l_{j,k}(\frac{y}{s^{1/4}})q^j\bar{q}^k + \tilde{B}^l_{j,k}(y,s)q^j\bar{q}^k\right]\right|$$
$$\le C\int_{|y|<s^{1/4}} |h_n||\rho|(|q|^{M+2}+\frac{1}{s^{\frac{M+1}{2}}}).$$

Let us write

$$B^l_{j,k}(\frac{y}{s^{1/4}}) = \sum_{i=0}^{M/2} b^{l,i}_{j,k}(\frac{y}{s^{1/4}})\left(\frac{y}{s^{1/4}}\right)^{2i},$$

$$q^j = \left(\sum_{m=0}^{M}\tilde{q}_m\tilde{h}_m + \hat{q}_m\hat{h}_m + q_-\right)^j,\quad q^k = \left(\sum_{m=0}^{M}\tilde{q}_m\tilde{h}_m + \hat{q}_m\hat{h}_m + q_-\right)^k,$$

where $b^{l,i}_{j,k}$ are the coefficients of the polynomials $B^l_{j,k}$. Using the fact that $\|q(s)\|_{L^\infty}\le 1$ (which holds for s large enough, from the fact that $q(s)\in\mathcal{V}_A(s)$ and Definition 4.1). We deduce that

$$|q^j - q^j_+| \le C(|q_-|^j + |q_-|).$$

Using that $q(s)\in\mathcal{V}_A(s)$ and the fact that $\sqrt{s}\ge 2A^2$, we deduce that in the region $|y|\le s^{1/4}$, we have $|q_-|\le \frac{1}{s^{1/4}}(\frac{A}{s^{1/4}})^{M+1}(1+|y|)^{M+1}$ and that

$$|q^j - (\sum_{m=0}^{M}\tilde{q}_m\tilde{h}_m + \hat{q}_m\hat{h}_m)^j| \le C\left(\frac{A}{s^{1/4}}\right)^{M+1}\frac{1}{s^{1/4}}(1+|y|)^{jM+j}.$$

In the same way, we have

$$|q^k - (\sum_{m=0}^{M}\tilde{q}_m\tilde{h}_m + \hat{q}_m\hat{h}_m)^k| \le C\left(\frac{A}{s^{1/4}}\right)^{M+1}\frac{1}{s^{1/4}}(1+|y|)^{kM+k},$$

hence, the contribution coming from q_- is controlled by the right-hand side of (115). Moreover for all j,k and l, we have

$$\left|\int_{|y|<s^{1/4}} h_n\rho B^l_{j,k}(\frac{y}{s^{1/4}})q^j_+\bar{q}^k_+ - \int h_n\rho B^l_{j,k}(\frac{y}{s^{1/4}})q^j_+\bar{q}^k_+\right| \le Ce^{-C\sqrt{s}}. \tag{119}$$

To compute the second term on the left had side of (119), we notice that $B^l_{j,k}(\frac{y}{s^{1/4}})q^j_+\bar{q}^k_+$ is a polynomial in y and that the coefficient of the term of degree n is controlled by the right had side of (115) since $q\in\mathcal{V}_A$.

Moreover, using that $\sqrt{s} \geq 2A^2$, we infer that $|q| \leq \frac{1}{s^{1/4}}(1+|y|)^{M+1}$ in the region $|y| \leq s^{1/4}$ and hence for all j, k and l, we have

$$\left|\int_{|y|<s^{1/4}} h_n \rho \frac{1}{s^{l/2}} \tilde{B}^l_{j,k}(y,s) q^j \bar{q}^k\right| \leq C \frac{1}{s^{\frac{l}{2}+\frac{M+1+j+k}{4}}}$$

and

$$\left|\int_{|y|<s^{1/4}} h_n \rho (|q|^{M+2} + \frac{1}{s^{\frac{M+1}{2}}})\right| \leq C \frac{1}{s^{\frac{M+2}{4}}}.$$

The terms appearing in these two inequalities are controlled by the right hand side of (115).
Using the fact that $\|q(s)\|_{L^\infty} \leq 1$ and (49), we remark that $|B(q,y,s)| \leq C$. Since $|\rho(y)| \leq Ce^{-s\sqrt{s}}$ for all $|y| > s^{1/4}$, it holds that

$$\left|\int_{|y|>s^{1/4}} h_n B(q,y,s) \rho dy\right| \leq Ce^{-C\sqrt{s}}.$$

This concludes the proof of Lemma 4.17. ■

Sixth term: $R^*(\theta', y, s)$
In the following, we expand R^* as a power series of $\frac{1}{s}$ as $s \to \infty$, uniformly for $|y| \leq s^{1/4}$.

Lemma 4.18 (Power series of R^* as $s \to \infty$) *For all $n \in \mathbb{N}$,*

$$R^*(\theta', y, s) = \Pi_n(\theta', y, s) + \tilde{\Pi}_n(\theta', y, s), \tag{120}$$

where,

$$\Pi_n(\theta', y, s) = \sum_{k=0}^{n-1} \frac{1}{s^{\frac{k+1}{2}}} P_k(y) - i\theta'(s) \left(\frac{a}{s^{1/2}}(1+i\delta) + \sum_{k=0}^{n-1} e_k \frac{y^{2k}}{s^{k/2}}\right), \tag{121}$$

and

$$\forall |y| < s^{1/4}, \quad \left|\tilde{\Pi}_n(\theta', y, s)\right| \leq C(1 + s|\theta'(s)|) \frac{(1+|y|^{2n})}{s^{\frac{n+1}{2}}}, \tag{122}$$

where P_k is a polynomial of order $2k$ for all $k \geq 1$ and $e_k \in \mathbb{R}$.
In particular,

$$\begin{aligned} &\sup_{|y|\leq s^{1/4}} \left| R^*(\theta', y, s) - \sum_{k=0}^{1} \frac{1}{s^{\frac{k+1}{2}}} P_k(y) + i\theta' \left[\kappa + \frac{(1+i\delta)}{s^{1/2}} \left(a - \frac{b\kappa y^2}{(p-1)^2}\right)\right]\right| \\ &\leq C\left(\frac{1+|y|^4}{s^{3/2}} + C|\theta'|\frac{y^4}{s^2}\right). \end{aligned} \tag{123}$$

Proof: Using the definition of φ (41), the fact that φ_0 satisfies (43) and (48), we see that R^* is in fact a function of θ', $z = \frac{y}{s^{1/4}}$ and s that can be written as

$$\begin{aligned} R^*(\theta', y, s) &= \frac{1}{4}\frac{z}{s}\nabla_z \varphi_0(z) + \frac{1}{2}\frac{a}{s^{3/2}}(1+i\delta) + \frac{1}{s^{1/2}}\Delta_z \varphi_0(z) \\ &- \frac{a(1+i\delta)^2}{(p-1)s^{1/2}} + \left(F\left(\varphi_0(z) + \frac{a}{s^{1/2}}(1+i\delta)\right) - F(\varphi_0)\right) \\ &- i\frac{\mu}{s}\left(\varphi_0(z) + \frac{a}{s^{1/2}}(1+i\delta)\right) - i\theta'(s)\left(\varphi_0(z) + \frac{a}{s^{1/2}}(1+i\delta)\right), \end{aligned} \tag{124}$$

where $F(u) = (1+i\delta)|u|^{p-1}u$.

Since $|z| < 1$, there exists positive c_0 and s_0 such that $|\varphi_0(z)|$ and $|\varphi_0(z) + \frac{a}{s^{1/2}}(1+i\delta)|$ are both larger that $\frac{1}{c_0}$ and smaller than c_0, uniformly in $|z| < 1$ and $s > s_0$. Since $F(u)$ is C^∞ for $\frac{1}{c_0} \le |u| \le c_0$, we expand it around $u = \varphi_0(z)$ as follows

$$\left| F\left(\varphi_0(z) + \frac{a}{s^{1/2}}(1+i\delta)\right) - F\left(\varphi_0(z)\right) - \sum_{j=1}^{n} \frac{1}{s^{j/2}} F_j\left(\varphi_0(z)\right) \right| \le C\frac{1}{s^{\frac{n+1}{2}}},$$

where $F_j(u)$ are C^∞. Hence, we can expand $F_j(u)$ around $u = \varphi_0(0)$ and write

$$\left| F\left(\varphi_0(z) + \frac{a}{s^{1/2}}(1+i\delta)\right) - F\left(\varphi_0(z)\right) - \sum_{j=1}^{n}\sum_{l=0}^{n-j} \frac{c_{j,l}}{s^{j/2}} z^{2l} \right| \le \sum_{j=1}^{n} \frac{C}{s^{\frac{j}{2}}}|z|^{2(n-j)+2} + \frac{C}{s^{\frac{n+1}{2}}},$$

Similarly, we have the following

$$\left| \frac{z}{s}\nabla_z\varphi_0(z) - \frac{|z|^2}{s}\sum_{j=0}^{n-2} d_j z^{2j} \right| \le \frac{C}{s}|z|^{2n},$$

$$\left| \frac{1}{s^{1/2}}\Delta_z\varphi_0(z) - \frac{1}{s^{1/2}}\sum_{j=0}^{n-1} b_j z^{2j} \right| \le \frac{C}{s^{1/2}}|z|^{2n} \text{ and } \left| \varphi_0(z) - \sum_{j=0}^{n-1} e_j z^{2j} \right| \le C|z|^{2n}.$$

Recalling that $z = \frac{y}{s^{1/4}}$, we get the conclusion of the Lemma. ■

In the following, we introduce $F_j(R^*)(\theta, s)$ as the projection of the rest term $R^*(\theta', y, s)$ on the standard Hermite polynomial, introduced in Lemma 3.1.

Lemma 4.19 (Projection of R^* on the eigenfunction of $\mathcal{L}$) *It holds that $F_j(R^*)(\theta', s) \equiv 0$ when j is odd, and $|F_j(R^*)(\theta', s)| \le C\frac{1+s|\theta'(s)|}{s^{\frac{j}{4}+\frac{1}{2}}}$, when j is even and $j \ge 4$.*

Proof: Since R^* is even in the y variables and f_j is odd when j is odd, $F_j(R^*)(\theta', s) \equiv 0$, when j is odd.
Now, when j is even, we apply Lemma 4.18 with $n = [\frac{j}{2}]$ and write

$$R^*(\theta', y, s) = \Pi_{\frac{j}{2}}(\theta', y, s) + O\left(\frac{1 + s|\theta'(s)| + |y|^j}{s^{\frac{j}{4}} + \frac{1}{2}}\right),$$

where $\Pi_{\frac{j}{2}}$ is a polynomials in y of degree less than $j-1$. Using the definition of $F_j(R^*)$ (projection on the h_j of R^*), we write

$$\begin{aligned}
\int_{\mathbb{R}^N} R^* h_j \rho &= \int_{|y|<s^{1/4}} R^* h_j \rho dy + \int_{|y|>s^{1/4}} R^* h_j \rho dy \\
&= \int_{|y|<s^{1/4}} \Pi_{\frac{j}{2}} h_j \rho dy + O\left(\int_{|y|<s^{1/4}} \frac{1 + s|\theta'(s)| + |y|^j}{s^{\frac{j}{4}} + \frac{1}{2}} h_j \rho dy\right) + \int_{|y|>s^{1/4}} R^* h_j \rho dy \\
&= \int_{\mathbb{R}^N} \Pi_{\frac{j}{2}} h_j \rho dy + O\left(\frac{1 + s|\theta'(s)|}{s^{\frac{j}{4}} + \frac{1}{2}}\right) + \int_{|y|>s^{1/4}} R^* h_j \rho dy + \int_{|y|>s^{1/4}} \Pi_{\frac{j}{2}} h_j \rho dy.
\end{aligned} \tag{125}$$

We can see that $\int_{\mathbb{R}^N} \Pi_{\frac{j}{2}} h_j \rho dy = 0$ because h_j is orthogonal to all polynomials of degree less than $j-1$. Then, note that both integrals over the domain $\{|y| > s^{1/4}\}$ are controlled by

$$\int_{s^{1/4}} \left(|R^*(\theta', y, s)| + 1 + |y|^j\right)(1 + |y|^j)\rho dy.$$

Using the fact that $R(y,s)$ measures the defect of $\varphi(y,s)$ from being an exact solution of (19). However, since φ is an approximate solution of (19), one easily derive the fact that

$$\begin{array}{l} \|R(s)\|_{L^\infty} \leq \frac{C}{\sqrt{s}}, \text{ and} \\ |R^*(\theta', y, s)| \leq \frac{C}{\sqrt{s}} + |\theta'(s)|. \end{array} \tag{126}$$

Using the fact that $|\rho(y)| \leq Ce^{\sqrt{-s}}$, for $|y| > s^{1/4}$, we can bound our integral by

$$C(1 + |\theta'(s)|) \int_{\mathbb{R}^N} (1 + |y|^j)^2 e^{c\sqrt{-s}} \sqrt{\rho} dy = C(j)(1 + |\theta'(s)|) e^{c\sqrt{-s}}.$$

This inequality gives us the result for $j \geq 4$. ■

Lemma 4.20 (Projection of R^* on the eigenfunction $\tilde{h}$ and h_n) *Let us consider R^* defined as in the above, then the following hold:*

(i) For $j \geq 4$ which is even, then $\tilde{P}_j(R^*)(\theta', s)$ and $P_j(R^*)(\theta', s)$ are $O\left(\frac{1+s|\theta'|}{s^{\frac{j}{4}+\frac{1}{2}}}\right)$.

(ii) For all j odd, we have $\tilde{P}_j(R^*)(\theta', s) = P_j(R^*)(\theta', s) = 0$.

(iii) For $j = 0$, we have

$$\begin{aligned} P_{0,M}(R^*(\theta'(s), s)) &= \frac{R^*_{0,0}}{s^{\frac{1}{2}}} + \frac{R^*_{0,1}}{s} + \frac{R^*_{0,2}}{s^{\frac{3}{2}}} \\ &+ \theta'(s)\left(-\kappa + \frac{\Theta^*_{0,0}}{\sqrt{s}} + O\left(\frac{1}{s}\right)\right) + O\left(\frac{1}{s^2}\right), \\ \tilde{P}_{0,M}(R^*(\theta'(s), s)) &= \frac{\tilde{R}^*_{0,0}}{s^{\frac{1}{2}}} + \frac{\tilde{R}^*_{0,1}}{s} + \frac{\tilde{R}^*_{0,2}}{s^{\frac{3}{2}}} \\ &+ \theta'(s)\left(\frac{\tilde{\Theta}^*_{0,0}}{\sqrt{s}} + O\left(\frac{1}{s}\right)\right) + O\left(\frac{1}{s^2}\right). \end{aligned}$$

(iv) For $j = 2$, we have

$$\begin{aligned} P_{2,M}(R^*(\theta'(s), s)) &= \frac{R^*_{2,1}}{s} + \frac{R^*_{2,2}}{s^{\frac{3}{2}}} + \theta'(s)\left(\frac{\Theta^*_{2,0}}{\sqrt{s}} + O\left(\frac{1}{s}\right)\right) + O\left(\frac{1}{s^2}\right), \\ \tilde{P}_{2,M}(R^*(\theta'(s), s)) &= \frac{\tilde{R}^*_{2,1}}{s} + \frac{\tilde{R}^*_{2,2}}{s^{\frac{3}{2}}} + \frac{\tilde{R}^*_{2,3}}{s^2} \\ &+ \theta'(s)\left(\frac{\tilde{\Theta}^*_{2,0}}{\sqrt{s}} + \frac{\tilde{\Theta}^*_{2,1}}{s} + O\left(\frac{1}{s^{\frac{3}{2}}}\right)\right) + O\left(\frac{1}{s^{\frac{5}{2}}}\right), \end{aligned}$$

where $R^*_{j,k}, \tilde{R}^*_{j,k}, \Theta^*_{j,k}, \tilde{\Theta}^*_{j,k}$ are constants, depending on p, δ, β only. For more details see page 68 and equation (175).

and the other constants are complicated, will we give them in the proof of this Lemma.

Proof: See Appendix D. □

Part 2: Proof of Proposition 4.10

In this part, we consider $A \geq 1$ and take s large enough so that Part 1 is satisfied.

+ The proof of item (i): We control $\theta'(s)$, from the projection of (48) on $h_0(y) = i$, we obtain

$$q_0' = c_2\tilde{q}_2 - P_{0,M}\left(\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'\right) q\right) + P_{0,M}(V_1 q + V_2\bar{q}) + P_{0,M}(B) + P_{0,M}(R^*(\theta'(s), s)), \tag{127}$$

where $c_2 = 2\beta(1+\delta^2)$, defined as in Lemma 3.1. In addition to that, from the fact that $q_0 \equiv 0$ by the modulation, we also obtain that

$$q_0' \equiv 0.$$

By the definition of the shrinking set $\mathcal{V}_A(s)$ in Definition 4.1, Corollary 4.4, Lemma 4.17 and Lemma 4.20, we obtain the following:

$$\kappa\theta'(s) = c_2\tilde{q}_2 + \frac{\tilde{q}_2}{\sqrt{s}}\left[D_{0,2} - \frac{\nu}{2}L_{0,2} + \frac{\Theta^*_{0,0}c_2}{\kappa}\right] + \frac{R^*_{0,1}}{s} \tag{128}$$
$$+ \quad \frac{1}{s^{\frac{3}{2}}}\left[\frac{\nu}{2}(1+\delta^2)\tilde{R}^*_{0,1} - \frac{\nu}{2}K_{0,2}R^*_{2,1} - D_{0,0}\tilde{R}^*_{0,1} + C_{0,2}R^*_{2,1} + R^*_{0,2} + \frac{\Theta^*_{0,0}R^*_{0,1}}{\kappa}\right] + O\left(\frac{1}{s^{\frac{7}{4}}}\right).$$

In addition to that, we derive again from the shrinking set that

$$\left|c_2\tilde{q}_2(s) + \frac{R^*_{0,1}}{s}\right| \leq \frac{CA^{10}}{s^{\frac{5}{4}}}.$$

Thus, we obtain

$$\left|\kappa\theta'(s)\right| \leq \frac{CA^{10}}{s^{\frac{5}{4}}}.$$

and item (i) of Proposition 4.10.

+ The proof of item (iii): Let us project equation (48) on $\tilde{h}_2$, we get

$$\begin{aligned} \tilde{q}_2' \quad &= \quad \tilde{P}_{2,M}(\mathcal{L}_{\beta,\delta}q) + \tilde{P}_{2,M}\left(-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right)q\right) + \tilde{P}_{2,M}(V_1 q + V_2\bar{q}) \\ &+ \quad \tilde{P}_{2,M}(B(q)) + \tilde{P}_{2,M}(R^*(\theta'(s), s)). \end{aligned} \tag{129}$$

Using the fact that $q(s) \in \mathcal{V}_A(s)$ for all $s \in [\tau, s_1]$, Corollary 4.4, Lemma 4.17 and Lemma 4.20, we can obtain some bounds for the terms in the right hand side of (129):

$$\tilde{P}_{2,M}(\partial_s q) = \partial_s \tilde{q}_2 \tag{130}$$

$$\left|\tilde{P}_{2,M}(\mathcal{L}_{\beta,\delta} q)\right| \le \frac{A^{M+1}}{s^{\frac{M+2}{4}}}, \tag{131}$$

and

$$\begin{aligned}
&\tilde{P}_{2,M}\left(-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q\right) \\
= \ & \tilde{q}_2^2 \left\{\frac{c_2\delta}{\kappa}\right\} \\
+ \ & \frac{\tilde{q}_2}{\sqrt{s}} \left\{\frac{\nu\delta}{2}\right\} \\
+ \ & \frac{\tilde{q}_2}{s} \Bigg\{\frac{\nu}{2}\left[c_4 \tilde{D}_{4,2} - (1+\delta^2)\frac{\nu}{2} + D_{2,2} + \frac{\Theta^*_{2,0} c_2}{\kappa}\right] - \frac{\nu}{2}\left[\frac{\tilde{K}_{2,4} D_{4,2}}{2} + \tilde{L}_{2,4}\tilde{D}_{4,2}\right] \\
+ \ & \mu\delta + \frac{c_2}{\kappa} R^*_{2,1} + \frac{\delta R^*_{0,1}}{\kappa}\Bigg\} \\
+ \ & \frac{1}{s^{\frac{3}{2}}}\left\{\frac{\nu}{2} R^*_{2,1}\right\} \\
+ \ & \frac{1}{s^2}\Bigg\{\frac{\nu}{2}\left[X_2 + c_4[\tilde{C}_{4,2} R^*_{2,1} + \tilde{R}^*_{4,2}] - D_{2,0}\tilde{R}_{0,1}\right] - \frac{\nu}{2}\left[\tilde{K}_{2,4}\left(\frac{C_{4,2} R^*_{2,1}}{2} + \frac{R^*_{4,2}}{2}\right)\right] \\
- \ & \frac{\nu}{2}\left[\tilde{L}_{2,4}\left(\tilde{C}_{4,2} R^*_{2,1} + \tilde{R}^*_{4,2}\right)\right] + \mu R^*_{2,1} + \frac{R^*_{0,1} R^*_{2,1}}{\kappa}\Bigg\} + O\left(\frac{A^8}{s^{\frac{9}{4}}}\right).
\end{aligned} \tag{132}$$

+ Let us now give the estimate for $\tilde{P}_{2,M}(V_1 q + V_2 \bar{q})$:

$$\begin{aligned}
&\tilde{P}_{2,M}\left(V_1 q + V_2 \bar{q}\right) \\
= \ & \frac{1}{\sqrt{s}}\left\{\tilde{q}_0 \tilde{D}_{2,0} + q_2 \tilde{C}_{2,2} + \tilde{q}_2 \tilde{D}_{2,2} + q_4 \tilde{C}_{2,4} + \tilde{q}_4 \tilde{D}_{2,4}\right\} \\
+ \ & \frac{1}{s}\left\{\tilde{q}_0 \tilde{F}_{2,0} + q_2 \tilde{E}_{2,2} + \tilde{q}_2 \tilde{F}_{2,2}\right\} + O\left(\frac{A^6}{s^{\frac{9}{4}}}\right).
\end{aligned} \tag{133}$$

+ In the following, we aim at estimating to $\tilde{P}_{2,M}(B(q))$: Using Lemma 4.17, we deduce then

$$\tilde{P}_{2,M}(B(q)) = \tilde{B}_2 \tilde{q}_2^2 + B_1 \frac{\tilde{q}_2}{s} + \frac{B_2}{s^2} + O\left(\frac{A^6}{s^{\frac{9}{4}}}\right), \tag{134}$$

where

$$\begin{aligned}
\tilde{B}_2 &= \frac{1}{\kappa}\left\{4(p-\delta^2) - \delta\beta(6 + 4p + 2\delta^2)\right\}, \\
B_1 &= \frac{1}{\kappa}\left\{(-7\delta^2\beta + p\beta - 6\beta) R^*_{2,1} - (p-\delta^2)\tilde{R}^*_{0,1}\right\}, \\
B_2 &= \frac{(R^*_{2,1})^2}{s^2}\frac{(32 - 64\delta\beta)}{8\kappa}.
\end{aligned}$$

+ Finally, we give the estimate of $\tilde{P}_{2,M}(R^*, s)$: From Lemma 4.20, we get

$$\tilde{P}_{2,M}(R^*) = \frac{\tilde{R}^*_{2,1}}{s} + \frac{\tilde{R}^*_{2,2}}{s^{\frac{3}{2}}} + \frac{\tilde{R}_{2,3}}{s^2} + \frac{\theta'(s)\kappa}{\sqrt{s}} \frac{-\delta b}{(p-1)^2} + \theta'(s)\kappa \frac{(p+1)\delta[12 - 6\delta\beta + 6\beta^2]b^2}{2(p-1)^4} \frac{1}{s}$$
$$+ \quad O\left(\frac{\theta'(s)}{s^{\frac{3}{2}}}\right) + O\left(\frac{1}{s^{\frac{5}{2}}}\right).$$

However, by the definition of ν and a, it follows that

$$\tilde{R}^*_{2,1} = 0.$$

In addition to that, we use (128) to deduce the following

$$\tilde{P}_{2,M}(R^*(\theta'(s))) = \frac{1}{s^{\frac{3}{2}}} \left\{ -\frac{\delta b}{(p-1)^2} R^*_{0,1} + \tilde{R}^*_{2,2} \right\}$$
$$+ \quad \frac{\tilde{q}_2}{\sqrt{s}} \left[-c_2 \frac{\delta b}{(p-1)^2} \right]$$
$$+ \quad \frac{\tilde{q}_2}{s} \left\{ -\frac{\delta b}{(p-1)^2} \left[D_{0,2} - \frac{\nu}{2} L_{0,2} + \frac{\Theta^*_{0,0} c_2}{\kappa} \right] + \frac{b^2}{2(p-1)^4} c_2 (p+1)\delta[12 - 6\delta\beta + 6\beta^2] \right\}$$
$$+ \quad \frac{1}{s^2} \left\{ -\frac{\delta b}{(p-1)^2} \left[\frac{\nu}{2}(1+\delta^2)\tilde{R}^*_{0,1} - \frac{\nu}{2} K_{0,2} R^*_{2,1} - D_{0,0}\tilde{R}^*_{0,1} + C_{0,2} R^*_{2,1} + R^*_{0,2} + \frac{\Theta^*_{0,0} R^*_{0,1}}{\kappa} \right] \right.$$
$$+ \quad \left. \tilde{R}^*_{2,3} + \frac{(p+1)\delta[12 - 6\delta\beta + 6\beta^2] R^*_{0,1} b^2}{2(p-1)^4} \right\} + O\left(\frac{A^4}{s^{\frac{9}{4}}}\right). \tag{135}$$

ODE of $\tilde{q}_2$

Adding estimates (130), (131), (132), (133), (134) and (135), we obtain the following

$$\tilde{q}_2' = \left\{ \frac{\nu\delta}{2} + \tilde{D}_{2,2} - c_2 \frac{\delta b}{(p-1)^2} \right\} \frac{\tilde{q}_2}{\sqrt{s}}$$
$$+ \quad \tilde{q}_2^2 \left\{ \frac{c_2\delta}{\kappa} + \frac{4(p-\delta^2) - \delta\beta(6 + 4p + 2\delta^2)}{\kappa} \right\}$$
$$+ \quad \frac{1}{s^{\frac{3}{2}}} \left\{ \frac{\nu}{2} R^*_{2,1} + \tilde{C}_{2,2} R^*_{2,1} - \tilde{D}_{2,0}\tilde{R}^*_{0,1} - \frac{\delta b}{(p-1)^2} R^*_{0,1} + \tilde{R}^*_{2,2} \right\}$$
$$+ \quad \frac{\tilde{q}_2}{s}\tilde{H}_1 + \frac{\tilde{H}_2}{s^2} + O\left(\frac{A^8}{s^{\frac{9}{4}}}\right), \tag{136}$$

where

$$\tilde{H}_1 = \frac{\nu}{2}\left[c_4\tilde{D}_{4,2} - (1+\delta^2)\frac{\nu}{2} + D_{2,2} + \Theta^*_{2,0}\frac{c_2}{\kappa}\right] - \frac{\nu}{2}\left[\frac{\tilde{K}_{2,4}D_{4,2}}{2} + \tilde{L}_{2,4}\tilde{D}_{4,2}\right]$$
$$+ \quad \mu\delta + \frac{c_2}{\kappa}R^*_{2,1} + \frac{\delta R^*_{0,1}}{\kappa}$$
$$+ \quad \tilde{D}_{2,0}\left[\frac{\nu\tilde{L}_{0,2}}{2} - \tilde{D}_{0,2} - \tilde{\Theta}^*_{0,0}\frac{c_2}{\kappa}\right] + \tilde{C}_{2,2}\left[D_{2,2} - \frac{\nu}{2}(1+\delta^2) + c_4\tilde{D}_{4,2} + \frac{\Theta^*_{2,0}c_2}{\kappa}\right]$$
$$+ \quad \frac{\tilde{C}_{2,4}D_{4,2}}{2} + \tilde{D}_{2,4}\tilde{D}_{4,2}$$
$$+ \quad \tilde{F}_{2,2} + B_1 - \frac{\delta b}{(p-1)^2}\left[D_{0,2} - \frac{\nu}{2}L_{0,2} + \Theta^*_{0,0}\frac{c_2}{\kappa}\right] + \frac{b^2}{2(p-1)^4}c_2(p+1)\delta[12 - 6\delta\beta + 6\beta^2],$$

$$\tilde{H}_2 = \frac{\nu}{2}\left[X_2 + c_4[\tilde{C}_{4,2}R^*_{2,1} + \tilde{R}^*_{4,2}] - D_{2,0}\tilde{R}_{0,1}\right] - \frac{\nu}{2}\left[\tilde{K}_{2,4}\left(\frac{C_{4,2}R^*_{2,1}}{2} + \frac{R^*_{4,2}}{2}\right)\right]$$
$$- \quad \frac{\nu}{2}\left[\tilde{L}_{2,4}\left(\tilde{C}_{4,2}R^*_{2,1} + \tilde{R}^*_{4,2}\right)\right] + \mu R^*_{2,1} + \frac{R^*_{0,1}R^*_{2,1}}{\kappa}$$
$$+ \quad \tilde{D}_{2,0}\left[-\tilde{X}_0 + \frac{\nu\tilde{K}_{0,2}R^*_{2,1}}{2} - \tilde{C}_{0,2}R^*_{2,1}\right] + \tilde{C}_{2,2}\left[X_2 + c_4(\tilde{C}_{4,2}R^*_{2,1} + \tilde{R}_{4,2}) - D_{2,0}\tilde{R}^*_{0,1}\right]$$
$$+ \quad \tilde{C}_{2,4}\left[\frac{C_{4,2}R^*_{2,1}}{2} + \frac{R^*_{4,2}}{2}\right] + \tilde{D}_{2,4}(\tilde{C}_{4,2}R^*_{2,1} + \tilde{R}^*_{4,2}) + \tilde{E}_{2,2}R^*_{2,1} - \tilde{F}_{2,0}\tilde{R}^*_{0,1}$$
$$+ \quad \frac{1}{8\kappa}(32 - 64\delta\beta)(R^*_{2,1})^2$$
$$- \quad \frac{\delta b}{(p-1)^2}\left[\frac{\nu}{2}(1+\delta^2)\tilde{R}^*_{0,1} - \frac{\nu}{2}K_{0,2}R^*_{2,1} - D_{0,0}\tilde{R}^*_{0,1} + C_{0,2}R^*_{2,1} + R^*_{0,2} + \frac{\Theta^*_{0,0}R^*_{0,1}}{\kappa}\right]$$
$$+ \quad \tilde{R}^*_{2,3} + \frac{(p+1)\delta[12 - 6\delta\beta + 6\beta^2]R^*_{0,1}b^2}{2(p-1)^4}.$$

We return to ODE (136), we need to cancel some coefficient in order to finish the proof.
+ **Cancellation of the coefficient of $\frac{1}{s}$**: It is equivalent to

$$\tilde{R}^*_{2,1} = 0,$$

which holds because of the definition of $\tilde{R}^*_{2,1}$, ν, a as in (42) and the critical condition as in Definition 1.1

+ **Cancellation of the coefficient of the coefficient of $\frac{\tilde{q}_2}{\sqrt{s}}$**: It is equivalent to the following

$$\frac{\nu\delta}{2} + \tilde{D}_{2,2} - c_2\frac{\delta b}{(p-1)^2} = 0,$$

which is alway satisfied because of the definitions of $\tilde{D}_{2,2}, \nu, c_2$ and the critical condition given Definition 1.1.

$$p - \beta\delta(p+1) - \delta^2 = 0.$$

+ **Cancellation of the coefficient of $\tilde{q}_2^2$**: It is due the critical condition.

+ **Cancellation of the coefficient of $\frac{1}{s^{3/2}}$**: It is equivalent to the following

$$\frac{\nu}{2}R_{2,1}^* + \tilde{C}_{2,2}R_{2,1}^* - \tilde{D}_{2,0}\tilde{R}_{0,1}^* - \frac{\delta b}{(p-1)^2}R_{0,1}^* + \tilde{R}_{2,2}^* = 0. \tag{137}$$

In fact, plugging the definition of constants in the right hand side and using the critical condition, we obtain the equation satisfied by b. We note that the resolution of this equation gives us the b_{cri}, the **same** obtained by the formal approach and given by formula (38).

After all this cancellation the equation satisfied by $\tilde{q}_2$ become

$$\tilde{q}_2' = \frac{\tilde{q}_2}{s}\tilde{H}_1 + \frac{\tilde{H}_2}{s^2} + O\left(\frac{A^8}{s^{\frac{9}{4}}}\right), \tag{138}$$

Using the critical condition, we can write $\tilde{H}_1$ as follows:

$$\tilde{H}_1 \tag{139}$$
$$= -\frac{1}{4}\frac{[\delta^8 + (1-2p)\delta^6 + (36-8p-5p^2)\delta^4 + (-6p+61p^2-6p^3)\delta^2 + 36p^4 - 6p^3 - 6p^2]}{(-p+6-p^2)\delta^4 + (-p+10p^2-p^3)\delta^2 - p^2 + 6p^4 - p^3}.$$

In addition to that, we claim to prove the following property: for all $p > 1$ and $\delta \in (-p_{cri}, p_{cri})$ defined as in Definition 1.1

$$\tilde{H}_1(p,\delta) \leq -\frac{3}{2}. \tag{140}$$

Indeed, let us consider

$$\begin{aligned}
\tilde{H}_1 + \frac{3}{2} &= -\frac{1}{4}\frac{\delta^2(\delta^6 + \delta^4 - 2\delta^4 p + p^2\delta^2 - 2p\delta^2 + p^2)}{(-p+6-p^2)\delta^4 + (-p+10p^2-p^3)\delta^2 - p^2 + 6p^4 - p^3} \\
&= -\frac{1}{4}\frac{\delta^2(1+\delta^2)(\delta^2-p)^2}{(-p+6-p^2)\delta^4 + (-p+10p^2-p^3)\delta^2 - p^2 + 6p^4 - p^3} \\
&= \frac{1}{4}\frac{\delta^2(1+\delta^2)(\delta^2-p)^2}{((p-2)\delta^2 - p(2p-1))((p+3)\delta^2 + p(3p+1))} \leq 0,
\end{aligned}$$

provided that $\delta \in (-p_{cri}, p_{cri})$.

We recall $\tilde{Q}_2$ defined as in (75), then, we derive from (136) that

$$\tilde{Q}_2' = \tilde{H}_1\frac{\tilde{Q}_2}{s} + \frac{\left(\tilde{\mathcal{A}}_2(\tilde{H}_1+1) + \tilde{H}_2\right)}{s^2} + O\left(\frac{A^8}{s^{9/4}}\right).$$

As a matter of fact, in order to end this part, we will choose $\mu = \mu_{cri}$, which cancel order $\frac{1}{s^2}$ in the above ODE. As the calculation are lengthy and to keep our work in a reasonable length, we will only prove the existence and unicity of such μ. Thus why we will try to find the contribution of μ in the term

$$\tilde{\mathcal{A}}_2(\tilde{H}_1+1)+\tilde{H}_2.$$

In fact, we claim to to write

$$\tilde{\mathcal{A}}_2(\tilde{H}_1+1)+\tilde{H}_2 = a_0(p,\delta)\mu + a_1(p,\delta,\beta), \tag{141}$$

where $a_0 \neq 0$. We kindly direct the reader to see the proof of (141) in item (b), Appendix F. We now directly derive from (141) that

$$\mu = -\frac{a_1(p,\delta,\beta)}{a_0(p,\delta)} \equiv \mu_{cri}, \tag{142}$$

this gives us the existence and unicity of μ and we finally obtain

$$\tilde{Q}_2' = \frac{\tilde{H}_1}{s}\tilde{Q}_2 + O\left(\frac{A^8}{s^{9/4}}\right),$$

which implies item (iii) of Proposition 4.10 .

– The proof of item (ii) of Proposition 4.10: We start to the ODE of $\tilde{q}_0$. In fact, using again the shrinking set $\mathcal{V}_A$, Corollary 4.4, Lemma 4.17 and Corollary 4.20, since $q(s) \in \mathcal{V}_A(s)$ for all $s \in [\tau, s_1]$, we get

$$\begin{aligned}
\tilde{q}_0' &= \tilde{q}_0 + \frac{\delta\nu}{2\sqrt{s}}\tilde{q}_0 - \frac{\nu\tilde{K}_{0,2}}{2\sqrt{s}}q_2 - \frac{\nu\tilde{L}_{0,2}}{2\sqrt{s}}\tilde{q}_2 + \tilde{D}_{0.0}\frac{\tilde{q}_0}{\sqrt{s}} + \tilde{C}_{0.2}\frac{q_2}{\sqrt{s}} + \tilde{D}_{0.2}\frac{\tilde{q}_2}{\sqrt{s}} \\
&+ \frac{\tilde{\Theta}^*_{0,0}}{\kappa}\left(c_2\frac{\tilde{q}_2}{\sqrt{s}} + \frac{R^*_{0,1}}{s^{\frac{3}{2}}}\right) + \frac{\tilde{R}^*_{0,1}}{s} + \frac{\tilde{R}^*_{0,2}}{s^{\frac{3}{2}}} + O\left(\frac{1}{s^{\frac{7}{4}}}\right).
\end{aligned} \tag{143}$$

Let us recall $\tilde{Q}_0$ defined as in Definition 4.1 and using the fact that $q \in V_A(s)$ and the ode (138), we can derive the following

$$\left|\tilde{Q}_0'(s) - \tilde{Q}_0(s)\right| \le \frac{C}{s^{\frac{3}{2}}}.$$

Next, considering $\tilde{q}_1$ and proceeding similarly as for $\tilde{q}_0$, we can deduce that

$$\left|\tilde{q}_1'(s) - \frac{1}{2}\tilde{q}_1(s)\right| \le \frac{C}{s^{\frac{3}{2}}}.$$

This finishes the proof of item (ii).

+ The proof of item (iv): *Estimates of* q_1, q_2, q_j *and* $\tilde{q}_j$ *for* $3 \le j \le M$*:* We first project equation (48) on $\tilde{h}_j$ and h_j

$$\begin{aligned}
\tilde{q}_j' &= \tilde{P}_{j,M}(\mathcal{L}_{\beta,\delta}) + \tilde{P}_{j,M}\left(-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right)q\right) \\
&+ \tilde{P}_{j,M}(V_1 q + V_2\bar{q}) + \tilde{P}_{j,M}(B) + \tilde{P}_{j,M}(R^*), \\
q_j' &= P_{j,M}(\mathcal{L}_{\beta,\delta}) + P_{j,M}\left(-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right)q\right) \\
&+ P_{j,M}(V_1 q + V_2\bar{q}) + P_{j,M}(B) + \tilde{P}_{j,M}(R^*).
\end{aligned}$$

Using the definition of the shrinking set $\mathcal{V}_A$ Definition 4.1 and Corollary 4.4, Lemma 4.17, Corollary 4.20 from Part 1 and the fact that $q(s) \in \mathcal{V}_A(s)$, we see that for all $s \in [\tau, s_1]$, we have

$$\begin{array}{ll} \left|q_1^{'} + \frac{1}{2}q_1\right| \leq C\frac{A^3}{s^{\frac{3}{2}}}, & \\ \left|q_j^{'} + \frac{j}{2}q_j\right| \leq C\frac{A^{j-1}}{s^{\frac{j+1}{4}}}, & \text{if} \quad 5 \leq j \leq M, \\ \left|\tilde{q}_j^{'} + \frac{j-2}{2}\tilde{q}_j\right| \leq C\frac{A^{j-1}}{s^{\frac{j+1}{4}}}, & \text{if} \quad 5 \leq j \leq M. \end{array} \quad (144)$$

In particular, for $q_2, q_3, \tilde{q}_3, q_4$ and $\tilde{q}_4$, we have the following

$$q_2' = -q_2 + \frac{R_{2,1}^*}{s} + \frac{R_{2,2}^*}{s^{\frac{3}{2}}} + c_4\tilde{q}_4 - \frac{\nu}{2\sqrt{s}}(1+\delta^2)\tilde{q}_2 + \frac{D_{2,0}\tilde{q}_0}{\sqrt{s}} + \frac{D_{2,2}\tilde{q}_2}{\sqrt{s}} \quad (145)$$

$$+ \left(C_{2,2} - \frac{\delta\nu}{2}\right)\frac{q_2}{\sqrt{s}} + \Theta_{2,0}^*\frac{c_2}{\kappa}\frac{\tilde{q}_2}{\sqrt{s}} + \frac{\Theta_{2,0}^* R_{0,1}^*}{\kappa}\frac{1}{s^{\frac{3}{2}}} + O\left(\frac{1}{s^{\frac{7}{4}}}\right), \quad (146)$$

$$q_3' = -\frac{3}{2}q_3 + O\left(\frac{A^2}{s^{\frac{3}{2}}}\right), \quad (147)$$

$$\tilde{q}_3' = -\frac{1}{2}q_3 + O\left(\frac{A^2}{s^{\frac{3}{2}}}\right), \quad (148)$$

and

$$q_4' = -2q_4 + C_{4,2}\frac{q_2}{\sqrt{s}} + D_{4,2}\frac{\tilde{q}_2}{\sqrt{s}} + \frac{R_{4,2}^*}{s^{\frac{3}{2}}} + O\left(\frac{A^6}{s^{\frac{7}{4}}}\right),$$

$$\tilde{q}_4' = -\tilde{q}_4 + \tilde{C}_{4,2}\frac{q_2}{\sqrt{s}} + \tilde{D}_{4,2}\frac{\tilde{q}_2}{\sqrt{s}} + \frac{\tilde{R}_{4,2}^*}{s^{\frac{3}{2}}} + O\left(\frac{1}{s^{\frac{7}{4}}}\right).$$

Let us recall the definition of $\tilde{Q}_0, Q_4$ and $\tilde{Q}_4$ defined as in Definition 4.1 and using the odes in (149), and (138) and the fact that $q \in V_A(s)$, we can derive the following odes:

$$\left|Q_2'(s) + Q_2(s)\right| \leq \frac{C}{s^{\frac{7}{4}}}, \quad (149)$$

$$\left|Q_4'(s) + 2Q_4(s)\right| \leq \frac{CA^6}{s^{\frac{7}{4}}}, \quad (150)$$

$$\left|\tilde{Q}_4'(s) + Q_4(s)\right| \leq \frac{C}{s^{\frac{7}{4}}}, \quad (151)$$

and

$$\left|q_3' + \frac{3}{2}q_3\right| \leq \frac{CA^2}{s^{\frac{3}{2}}},$$

$$\left|\tilde{q}_3' + \frac{1}{2}\tilde{q}_3\right| \leq \frac{CA^2}{s^{\frac{3}{2}}}.$$

It is similar to the proof in Nouaili and Zaag, [NZ18] for the case $\beta = 0$. We use these estimates to conclude the result.

Finally, this also finish the proof of Proposition 4.10. ■

4.3.2 The infinite dimensional part: q_-

We proceed as section 5.2 form [MZ08] We have 2 parts:

- In Part 1, we project equation (48) to get equations satisfied by q_-.
- In Part 2, we prove the estimate on q_-.

Part 1: Projection of equation (48) **using the projector** P_- In the following, we will project equation (48) term by term.

First term: $\frac{\partial q}{\partial s}$
From (63), its projection is

$$P_-(\frac{\partial q}{\partial s}) = \frac{\partial q_-}{\partial s} \tag{152}$$

Second term: $\tilde{\mathcal{L}}_{\beta,\delta} q$
From (48), we have the following,

$$P_-(\mathcal{L}_{\beta,\delta} q) = \mathcal{L}_\beta q_- + P_-[(1+i\delta)\Re q_-].$$

Third term: $-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q$
Since P_- commutes with the multiplication by i, we deduce that

$$P_-[-i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q] = -i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right) q_-.$$

Fourth term: $V_1 q$ **and** $V_2\bar{q}$
We have the following:

Lemma 4.21 (Projection of $V_1 q$ and $V_2\bar{q}$) *The projection of $V_1 q$ and $V_2\bar{q}$ satisfy for all $s \geq 1$,*

$$\left\|\frac{P_-(V_1 q)}{1+|y|^{M+1}}\right\|_{L^\infty} \leq \left(\|V_1\|_{L^\infty} + \frac{C}{s^{1/2}}\right)\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} + \sum_{n=0}^{M} \frac{C}{s^{\frac{M+1-n}{4}}}(|\hat{q}_n| + |\tilde{q}_n|), \tag{153}$$

and the same holds for $V_2\bar{q}$.

Using the fact that $q(s) \in \mathcal{V}_A(s)$, we get the following

Corollary 4.5 *For all $A \geq 1$, there exists $s_{14}(A)$ such that for all $s \geq s_{14}$, if $q(s) \in \mathcal{V}_A(s)$ then,*

$$\left\|\frac{P_-(V_1 q)}{1+|y|^{M+1}}\right\|_{L^\infty} \leq \|V_1\|_{L^\infty}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} + C\frac{A^M}{s^{\frac{M+2}{4}}},$$

and the same holds for $V_2 q$.

Proof of Lemma 4.21: We just give the proof for $V_1 q$ since the proof for $V_2\bar{q}$ is similar. From Subsection 3.5, we write $q = q_+ + q_-$ and

$$P_-(V_1 q) = V_1 q - P_+(V_1 q_-) + P_-(V_1 q_+).$$

Moreover, we claim that the following estimates hold

$$\begin{array}{rcl}\left\|\frac{V_1 q_-}{1+|y|^{M+1}}\right\|_{L^\infty} & \leq & \|V_1\|_{L^\infty}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}\\ \left\|\frac{P_+(V_1 q_-)}{1+|y|^{M+1}}\right\|_{L^\infty} & \leq & \frac{C}{s^{1/2}}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}.\end{array}$$

Indeed, the first one is obvious. To prove the second one, we use (98) to show that

$$|P_{n,M}(V_1 q_-)| + |\tilde{P}_{n,M}(V_1 q_-)| \leq \frac{C}{s^{1/2}}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty}.$$

To control $P_-(V_1 q_+) = \sum_{n\leq M} P_-(V_1(q_n h_n + \tilde{q}_n\tilde{h}_n))$, we argue as follows.
If $M-n$ is odd, we take $k = \frac{M-1-n}{2}$ in (99), hence

$$P_-(V_1(q_n h_n + \tilde{q}_n\tilde{h}_n)) = \sum_{j=1}^{k}\frac{1}{s^{\frac{j}{2}}}P_-\left(W_{1,j}(q_n h_n + \tilde{q}_n\tilde{h}_n)\right) + P_-\left((q_n h_n + \tilde{q}_n\tilde{h}_n)\tilde{W}_{1,k}\right)$$

Since $2k+n \leq M$, we deduce that $P_-\left(W_{1,j}(q_n h_n + \tilde{q}_n\tilde{h}_n)\right) = 0$ for all $0 \leq j \leq k$. Moreover, using that

$$|\tilde{W}_{1,k}| \leq C\frac{(1+|y|^{2k+2})}{s^{\frac{k+1}{2}}}$$

and applying Lemma A.3, we deduce that

$$\left\|\frac{P_-(V_1(q_n h_n + \tilde{q}_n\tilde{h}_n))}{1+|y|^{M+1}}\right\|_{L^\infty} \leq C\frac{|q_n| + |\tilde{q}_n|}{s^{\frac{M+1-n}{4}}}. \tag{154}$$

If $M-n$ is even, we take $k = \frac{M-n}{2}$ in (99) and use that

$$|\tilde{W}_{1,k}| \leq C\frac{1+|y|^{2k+1}}{s^{\frac{k}{2}+\frac{1}{4}}},$$

to deduce that (154) holds. This ends the proof of Lemma 4.21.■

Fifth term: $B(q,y,s)$**.**
Using (50), we have the following estimate from Lemmas A.3 and 4.16.

Lemma 4.22 *For all* $K \geq 1$ *and* $A \geq 1$*, there exists* $s_{15}(K,A)$ *such that for all* $s \geq s_{15}$*, if* $q(s) \in \mathcal{V}_A(s)$*, then*

$$\left\|\frac{P_-(B(q,y,s))}{1+|y|^{M+1}}\right\|_{L^\infty} \leq C(M)\left[\left(\frac{A^{M+2}}{s^{\frac{1}{4}}}\right)^{\bar{p}} + \frac{A^5}{s^{\frac{1}{2}}}\right]\frac{1}{s^{\frac{M+1}{4}}}, \tag{155}$$

where $\bar{p} = \min(p,2)$*.*

Proof: The proof is very similar to the proof of the previous lemma. From Lemma 4.16, we deduce that for all s there exists a polynomial B_M of degree M in y such that for all y and s, we have

$$|B - B_M(y)| \leq C\left[\left(\frac{A^{M+2}}{s^{\frac{1}{4}}}\right)^{\bar{p}} + \frac{A^{[5+(M+1)^2]}}{s^{\frac{1}{2}}}\right]\frac{(1+|y|^{M+1})}{s^{\frac{M+1}{4}}}. \tag{156}$$

Indeed, we can take B_M to be the polynomial

$$B_M = P_{+,M}\left[\sum_{l=0}^{M}\sum_{\substack{0\le j,k\le M+1\\ 2\le j+k\le M+1}}\frac{1}{s^{l/2}}\left[B^l_{j,k}(\frac{y}{s^{\frac14}})q_+^j\bar{q}_+^k\right]\right].$$

Then the fact that $B-B_M(y)$ is controlled by the right hand side of (156) is a consequence of the following estimates in the outer region and in the inner region.
First, in the region $|y|\ge s^{\frac14}$, we have from Lemma 4.16,

$$|B|\le C|q|^{\bar{p}}\le C\left(\frac{A^{M+2}}{s^{\frac14}}\right)^{\bar{p}}$$

and from the proof of Lemma 4.17, we know that for $0\le n\le M$,

$$|\tilde{P}_n(B_M(q,y,s))|+|P_{n,M}(B_M(q,y,s))|\le C\frac{A^n}{s^{\frac{n+2}{4}}}.$$

Beside, in the region $|y|\le s^{\frac14}$, we can use the same argument as in the proof of Lemma 4.16 to deduce that the coefficients of degree $k\ge M+1$ of the polynomial

$$\sum_{l=0}^{M}\sum_{\substack{0\le j,k\le M+1\\ 2\le j+k\le M+1}}\frac{1}{s^{l/2}}\left[B^l_{j,k}(\frac{y}{s^{\frac14}})q_+^j\bar{q}_+^k\right]-B_M,$$

is controlled by $C\frac{A^k}{s^{\frac{k}{4}+\frac12}}$ and hence

$$\left|\sum_{l=0}^{M}\sum_{\substack{0\le j,k\le M+1\\ 2\le j+k\le M+1}}\frac{1}{s^{l/2}}\left[B^l_{j,k}(\frac{y}{s^{\frac14}})q_+^j\bar{q}_+^k\right]-B_M\right|\le C\frac{A^{2M+2}}{s^{\frac{M+3}{4}}}(1+|y|^{M+1}),$$

in the region $|y|\le s^{\frac14}$.
Moreover, using that $|q|\le C\frac{A^{M+1}}{s^{\frac14}}$ in the region $|y|\le s^{\frac14}$, we deduce that for all $s\ge 2A^2$, we have

$$\left|\sum_{l=0}^{M}\sum_{\substack{0\le j,k\le M+1\\ 2\le j+k\le M+1}}\frac{1}{s^{l/2}}\left[B^l_{j,k}(\frac{y}{s^{\frac14}})q_+^j\bar{q}_+^k\right]\right|\le C\frac{A^{2M+2}}{s^{\frac{M+3}{4}}}(1+|y|^{M+1}).$$

Finally, to control the term $|q|^{M+2}$, we use the fact that in the region $|y| \le s^{\frac14}$, we have the following two estimates $|q| \le C\frac{A^{M+1}}{s^{\frac14}}$ and $|q| \le A^5 \frac{1}{s^{\frac34}}(1+|y|^{M+1})$ if $\sqrt{s} \ge 2A^2$. Hence

$$|q|^{M+2} \le C\frac{A^5}{s^{\frac34}}\left(\frac{A^{M+1}}{s^{\frac14}}\right)^{M+1}(1+|y|^{M+1}).$$

This ends the proof of estimate (156) and conclude the proof of (155) by applying Lemma A.3.■

Sixth term: $R^*(\theta', y, s)$.
We claim the following:

Lemma 4.23 *If $|\theta'(s)| \le \frac{CA^5}{s^{3/2}}$, then the following holds*

$$\left\|\frac{P_-(R^*(\theta',y,s))}{1+|y|^{M+1}}\right\| \le C\frac{1}{s^{\frac{M+3}{4}}}$$

Proof: Taking $n = \frac{M}{2}+1$ (remember M is even), we write from Lemma 4.18 $R^*(\theta', y, s) = \Pi_n(\theta', y, s) + \tilde{\Pi}_n(\theta', y, s)$. Since $2n-2 = M$, we see from subsection 3.5 that

$$|\tilde{\Pi}_n(\theta', y, s)| \le C\frac{1+|y|^{2n-2}}{s^{\frac{n+1}{2}}} \le C\frac{1+|y|^{M+1}}{s^{\frac{M+3}{4}}} \tag{157}$$

in the region $|y| < s^{1/4}$. It is easy to see using 52 and the definition of Π_n that (157) holds for all $y \in \mathbb{R}$ and $s \ge 1$. Then applying Lemma A.3, we conclude easily.■

Part 2: Proof of the last but one identity in (iv) of Proposition 4.10 (estimate on q_-)
If we apply the projection P_- to the equation (48) satisfied by q, we see that q_- satisfies the following equation:

$$\frac{\partial q_-}{\partial s} = \mathcal{L}_{\beta,\delta}q_- + P_-[(1+i\delta)\Re q_-] + P_-[-i(\frac{\mu}{s}+\theta'(s))q + V_1 q + V_2\bar{q} + B(q,y,s) + R^*(\theta',y,s)].$$

Here, we have used the important fact that $P_-[(1+i\delta)\Re q_+] = 0$. As it was mentioned in [MZ08], in this part we use the operator $\mathcal{L}_\beta$, unlike the case for q_n and $\tilde{q}_n$, where we used the properties of $\mathcal{L}_{\beta,\delta}$. The fact that M is large as fixed in (60) is crucial in the proof.

Using the kernel of the semigroup generated by $\mathcal{L}_\beta$, we get for all $s \in [\tau, s_1]$,

$$\begin{aligned} q_-(s) &= e^{(s-\tau)\mathcal{L}_\beta} q_-(\tau) \\ &+ \int_\tau^s e^{(s-s')\mathcal{L}_\beta} P_-[(1+i\delta)\Re q_-] ds' \\ &+ \int_\tau^s e^{(s-s')\mathcal{L}_\beta} P_-\left[-i(\frac{\mu}{s}+\theta'(s'))q + V_1 q + V_2\bar{q} + B(q,y,s') + R^*(\theta',y,s')\right] ds'. \end{aligned}$$

Using Lemma A.2, we get

$$\begin{aligned} &\left\|\frac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} \le e^{-\frac{M+1}{2}(s-\tau)}\left\|\frac{q_-(\tau)}{1+|y|^{M+1}}\right\|_{L^\infty} \\ &+ \int_\tau^s e^{-\frac{M+1}{2}(s-s')}\sqrt{1+\delta^2}\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} ds' \\ &+ \int_\tau^s e^{-\frac{M+1}{2}(s-s')}\left\|\frac{P_-\left[-i(\frac{\mu}{s}+\theta'(s'))q + V_1 q + V_2\bar{q} + B(q,y,s') + R^*(\theta',y,s')\right]}{1+|y|^{M+1}}\right\|_{L^\infty} ds' \end{aligned}$$

Assuming that $q(s') \in \mathcal{V}_A(s')$, the results from Part 1 yields (use (i) of Proposition 4.10 to bound $\theta'(s)$)

$$\begin{aligned}
&\left\|\frac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} \le e^{-\frac{M+1}{2}(s-\tau)}\left\|\frac{q_-(\tau)}{1+|y|^{M+1}}\right\|_{L^\infty}\\
&+\int_\tau^s e^{-\frac{M+1}{2}(s-s')}\left(\sqrt{1+\delta^2}+\||V_1|+|V_2|\|_{L^\infty}\right)\left\|\frac{q_-}{1+|y|^{M+1}}\right\|_{L^\infty} ds'\\
&+C(M)\int_\tau^s e^{-\frac{M+1}{2}(s-s')}\left[\frac{A^{(M+1)^2+5}}{(s')^{\frac{M+3}{4}}}+\frac{A^{(M+2)\bar p}}{(s')^{\frac{\bar p-1}{2}}}\frac{1}{(s')^{\frac{M+2}{2}}}+\frac{A^M}{(s')^{\frac{M+2}{2}}}\right]ds'.
\end{aligned}$$

Since we have already fixed M in (60) such that

$$M \ge 4\left(\sqrt{1+\delta^2}+1+2\max_{i=1,2,y\in\mathbb{R},s\ge1}|V_i(y,s)|\right),$$

using Gronwall's lemma or Maximum principle and (90), we deduce that

$$\begin{aligned}
e^{\frac{M+1}{2}s}\left\|\frac{q_-(s)}{1+|y|^{M+1}}\right\|_{L^\infty} &\le e^{\frac{M+1}{4}(s-\tau)}e^{\frac{M+1}{2}\tau}\left\|\frac{q_-(\tau)}{1+|y|^{M+1}}\right\|_{L^\infty}\\
&+e^{\frac{M+1}{2}s}2^{\frac{M+3}{4}}\left[\frac{A^{(M+1)^2+5}}{s^{\frac{M+3}{4}}}+\frac{A^{(M+2)\bar p}}{s^{\frac{\bar p-1}{2}}}\frac{1}{(s')^{\frac{M+2}{2}}}+\frac{A^M}{s^{\frac{M+2}{2}}}\right]
\end{aligned}$$

which concludes the proof of the last but one identity in (iv) of Proposition 4.10.

4.3.3 The outer region: q_e

Here, we finish the proof of Proposition 4.10 by proving the last inequality in (iv). Since $q(s) \in \mathcal{V}_A(s)$ for all $s \in [\tau, s_1]$, it holds from Claim 4.8 and Proposition 4.10 that

$$\|q(s)\|_{L^\infty(|y|<2Ks^{1/4})} \le C\frac{A^{M+1}}{s^{1/4}} \text{ and } |\theta'(s)| \le \frac{CA^5}{s^{3/2}}. \tag{158}$$

Then, we derive from (44) an equation satisfied by q_e, where q_e is defined by (47):

$$\begin{aligned}
\frac{\partial q_e}{\partial s} &= \mathcal{L}_\beta q_e - \tfrac{1}{p-1}q_e + (1-\chi)e^{\frac{i\delta}{p-1}s}\left\{L(q,\theta',y,s)+R^*(\theta',y,s)\right\}\\
&-e^{\frac{i\delta}{p-1}s}q(s)\left(\partial_s\chi+(1+i\beta)\Delta\chi+\frac12 y\cdot\nabla\chi\right)+2e^{\frac{i\delta}{p-1}s}(1+i\beta)div(q(s)\nabla\chi).
\end{aligned}\tag{159}$$

Writing this equation in its integral form and using the maximum principle satisfied by $e^{\tau\mathcal{L}_\beta}$ (see Lemma A.1, see Appendix below), we write

$$\begin{aligned}
\|q_e(s)\|_{L^\infty} &\le e^{-\frac{s-\tau}{p-1}}\|q_e(\tau)\|_{L^\infty},\\
&+\int_\tau^s e^{-\frac{s-s'}{p-1}}\left(\|(1-\chi)L(q,\theta',y,s')\|_{L^\infty}+\|(1-\chi)R^*(\theta',y,s')\|_{L^\infty}\right)ds'\\
&+\int_\tau^s e^{-\frac{s-s'}{p-1}}\left\|q(s')\left(\partial_s\chi+(1+i\beta)\Delta\chi+\frac12 y\cdot\nabla\chi\right)\right\|_{L^\infty}ds'\\
&+\int_\tau^s e^{-\frac{s-s'}{p-1}}\frac{1}{\sqrt{1-e^{-(s-s')}}}\|q(s')\nabla\chi\|_{L^\infty}ds'.
\end{aligned}$$

Let us bounds the norms in the three last lines of this inequality.
First from (46) and (158)

$$\begin{aligned}\left\|q(s')\left(\partial_s\chi+(1+i\beta)\Delta\chi+\tfrac{1}{2}y\cdot\nabla\chi\right)\right\|_{L^\infty} &\le C(1+\tfrac{1}{K^2s'})\|q(s')\|_{L^\infty(|y|<2Ks'^{1/4})}\\ &\le C\frac{A^{M+1}}{(s')^{1/4}},\end{aligned}\tag{160}$$

$$\|q(s')\nabla\chi\|_{L^\infty}\le\frac{C}{K(s')^{1/4}}\|q(s')\|_{L^\infty(|y|<2K(s')^{1/4})}\le C\frac{A^{M+1}}{\sqrt{s'}},\tag{161}$$

for s' large enough.
Second note that the residual term $(1-\chi)R^*$ is small as well. Indeed, recalling the bound (51) on R, we write from the definition of R^* (45) and (158):

$$\|(1-\chi)R^*(\theta',y,s')\|_{L^\infty}\le\frac{C}{(s')^{1/4}}+|\theta'(s')|\le\frac{C}{(s')^{1/4}}\tag{162}$$

for s' large enough.
Third, the term $(1-\chi)L(q,\theta',y,s')$ given in (45) is less than $\epsilon|q_e|$ with $\epsilon=\frac{1}{2(p-1)}$. Indeed, it holds from (158) that:

$$\begin{aligned}&\|(1-\chi)L(q,\theta',y,s')\|_{L^\infty}\\ &\le C\|q_e(s')\|_{L^\infty}\left(\|\varphi(s')\|^{p-1}_{L^\infty(|y|\ge Ks'^{1/4})}+\|q(s')\|^{p-1}_{L^\infty(|y|\ge Ks'^{1/4})}+\frac{1}{s'}+|\theta'(s)|\right),\\ &\le\tfrac{1}{2(p-1)}\|q_e(s')\|_{L^\infty},\end{aligned}\tag{163}$$

whenever K and s' are large (in order to ensure that $\|\varphi(s')\|_{L^\infty(|y|\ge Ks'^{1/4})}$ is small). Notice that it is only here that we need the fact that K is big enough. Using estimates (158), (160), (161), (162) and (163), we write

$$\begin{aligned}\|q_e(s)\|_{L^\infty}&\le e^{-\frac{s-\tau}{p-1}}\|q_e(\tau)\|_{L^\infty}\\ &+\int_\tau^s e^{-\frac{s-s'}{p-1}}\left(\frac{1}{2(p-1)}\|q_e(s')\|_{L^\infty}+C\frac{A^{M+1}}{(s')^{\frac14}}+C\frac{A^{M+1}}{(s')^{\frac12}}\frac{1}{\sqrt{1-e^{-(s-s')}}}\right)ds'.\end{aligned}$$

Using Gronwall's inequality or Maximum principle, we end-up with

$$\|q_e(s)\|_{L^\infty}\le e^{-\frac{(s-\tau)}{2(p-1)}}\|q_e(\tau)\|_{L^\infty}+\frac{CA^{M+1}}{\tau^{\frac14}}(s-\tau+\sqrt{s-\tau}),$$

which concludes the proof of Proposition 4.10.

5 Single point blow-up and final profile

In this section, we prove Theorem 1. Here, we use the solution of problem (48)-(79) constructed in the last section to exhibit a blow-up solution of equation (1) and prove Theorem 1.
(i) Consider $(q(s),\theta(s))$ constructed in Section 4 such that (65) holds. From (65) and the

properties of the shrinking set given in Claim 4.8, we see that $\theta(s) \to \theta_0$ as $s \to \infty$ such that

$$|\theta(s) - \theta_0| \le CA^{10} \int_s^\infty \frac{1}{\tau^{\frac{5}{4}}} d\tau \le \frac{CA^{10}}{s^{\frac{1}{4}}} \text{ and } \|q(s)\|_{L^\infty(\mathbb{R})} \le \frac{C_0(K,A)}{\sqrt{s}}. \tag{164}$$

Introducing $w(y,s) = e^{i(\nu\sqrt{s}+\mu\log s+\theta(s))}(\varphi(y,s) + q(y,s))$, we see that w is a solution of equation (19) that satisfies for all $s \ge \log T$ and $y \in \mathbb{R}$,

$$|w(y,s) - e^{i\theta_0 + i\nu\sqrt{s} + i\mu\log s}\varphi(y,s)| \le C\|q(s)\|_{L^\infty} + C|\theta(s) - \theta_0| \le \frac{C_0}{s^{\frac{1}{4}}}.$$

Introducing

$$u(x,t) = e^{-i\theta_0}\kappa^{i\delta}(T-t)^{\frac{1+i\delta}{p-1}} w\left(\frac{y}{\sqrt{T-t}}, -\log(T-t)\right),$$

we see from (18) and the definition of φ (41) that u is a solution of equation (1) defined for all $(x,T) \in \mathbb{R} \times [0,T)$ which satisfies (11).
If $x_0 = 0$. It remains to prove that when $x_0 \ne 0$, x_0 is not a blow-up point. The following result from Giga and Kohn [GK89] allows us to conclude:

Proposition 5.1 (Giga and Kohn - No blow-up under the ODE threshold) *For all $C_0 > 0$, there is $\eta_0 > 0$ such that if $v(\xi,\tau)$ solves*

$$|v_t - \Delta v| \le C_0(1 + |v|^p)$$

and satisfies

$$|v(\xi,\tau)| \le \eta_0(T-t)^{-1/(p-1)}$$

for all $(\xi,\tau) \in B(a,r) \times [T-r^2, T)$ for some $a \in \mathbb{R}$ and $r > 0$, then v does not blow up at (a,T).

Proof: See Theorem 2.1 page 850 in [GK89]. ■

Indeed, we see from (11) and (41) that

$$\sup_{|x-x_0|\le\frac{|x_0|}{2}} (T-t)^{\frac{1}{p-1}}|u(x,t)| \le \left|\varphi_0\left(\frac{|x_0|/2}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| + \frac{C}{|\log(T-t)|^{\frac{1}{4}}} \to 0$$

as $t \to T$, x_0 is not a blow-up point of u from Proposition 5.1. This concludes the proof of (i) of Theorem 1.
(ii) Arguing as Merle did in [Mer92], we derive the existence of a blow-up profile $u^* \in C^2(\mathbb{R}^*)$ such that $u(x,t) \to u^*(x)$ as $t \to T$, uniformly on compact sets of $\mathbb{R}^*$. The profile $u^*(x)$ is not defined at the origin. In the following, we would like to find its equivalent as $x \to 0$ and show that it is in fact singular at the origin. We argue as in Masmoudi and Zaag [MZ08]. Consider $K_0 > 0$ to be fixed large enough later. If $x_0 \ne 0$ is small enough, we introduce for all $(\xi,\tau) \in \mathbb{R} \times [-\frac{t_0(x_0)}{T-t_0(x_0)}, 1)$,

$$v(x_0,\xi,\tau) = (T - t_0(x_0))^{\frac{1+i\delta}{p-1}} v(x,t), \tag{165}$$

$$\text{where, } x = x_0 + \xi\sqrt{T-t_0(x_0)},\ t = t_0(x_0) + \tau(T - t_0(x_0)), \tag{166}$$

and $t_0(x_0)$ is uniquely determined by

$$|x_0| = K_0\sqrt{(T-t_0(x_0))|\log(T-t_0(x_0))|^{\frac{1}{2}}}. \tag{167}$$

From the invariance of problem (1) under dilation, $v(x_0,\xi,\tau)$ is also a solution of (1) on its domain. From (166), (167), (41), we have

$$\sup_{|\xi|<2|\log(T-t_0(x_0))|^{1/8}} |v(x_0,\xi,0)-\varphi_0(K_0)| \le \frac{C}{|\log(T-t_0(x_0))|^{\frac{1}{8}}} \to 0 \text{ as } x_0\to 0.$$

Using the continuity with respect to initial data for problem (1) associated to a space-localization in the ball $B(0,|\xi|<|\log(T-t_0(x_0))|^{1/8})$, we show as in Section 4 of [Zaa98] that

$$\sup_{|\xi|\le|\log(T-t_0(x_0))|^{1/8},\,0\le\tau<1} |v(x_0,\xi,\tau)-U_{K_0}(\tau)| \le \epsilon(x_0) \text{ as } x_0\to 0,$$

where $U_{K_0}(\tau) = ((p-1)(1-\tau)+bK_0^2)^{-\frac{1+i\delta}{p-1}}$ is the solution of the PDE (1) with constant initial data $\varphi_0(K_0)$. Making $\tau\to 1$ and using (166), we see that

$$\begin{aligned} u^*(x_0)=\lim_{t\to T} v(x,t) &= (T-t_0(x_0))^{-\frac{1+i\delta}{p-1}} e^{i\nu\sqrt{|\log(T-t_0(x_0))|}}|\log(T-t_0(x_0))|^{i\mu}\lim_{\tau\to 1} v(x_0,0,\tau) \\ &\sim (T-t_0(x_0))^{-\frac{1+i\delta}{p-1}} e^{i\nu\sqrt{|\log(T-t_0(x_0))|}}|\log(T-t_0(x_0))|^{i\mu}U_{K_0}(1) \end{aligned}$$

as $x_0\to 0$. Since we have from (167)

$$\log(T-t_0(x_0)) \sim 2\log|x_0| \text{ and } T-t_0(x_0) \sim \frac{|x_0|^2}{\sqrt{2}K_0^2\sqrt{|\log|x_0||}},$$

as $x_0\to 0$, this yields (ii) of Theorem 1 and concludes the proof of Theorem 1.

Proof of Theorem 2, we proceed as in the proof of (i) of Theorem 1 and using results established in Section 4 we end the proof.

A Spectral properties of $\mathcal{L}_\beta$

In this Appendix, we recall from Appendix A of [MZ08] some properties associated to the operator $\mathcal{L}_\beta$, defined in (45). We recall that:

$$\mathcal{L}_\beta v = (1+i\beta)\Delta v - \frac{1}{2}y\cdot\nabla v = \frac{1}{\rho_\beta}div(\rho_\beta\nabla w).$$

where

$$\rho_\beta(y) = \frac{e^{-\frac{|y|^2}{4(1+i\beta)}}}{(4\pi(1+i\beta))^{N/2}}.$$

Moreover, the operator $\mathcal{L}_\beta$ is self adjoint with respect to the weight ρ_β in the sense that

$$\int_{\mathbb{R}^N} u(y)\mathcal{L}_\beta w(y)\rho_\beta(y)dy = \int_{\mathbb{R}^N} w(y)\mathcal{L}_\beta u(y)\rho_\beta(y)dy. \tag{168}$$

In one space dimension ($N=1$), the eigenfunction f_n of $\mathcal{L}_\beta$ are dilations of standard Hermite polynomials $H_n(y)$:

$$f_n(y) = H_n(\frac{y}{2\sqrt{1+i\beta}}),\text{where } \mathcal{L}_\beta H_n = -\frac{n}{2}H_n.$$

If $N \geq 2$, its eigenfunction $f_\alpha(y_1, ..., y_N)$ where $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ is a multi-indic are given by

$$f_\alpha(y) = \Pi_{i=1}^N f_{\alpha_i}(y_i) = \Pi_{i=1}^N H_{\alpha_i}(\frac{y_i}{2\sqrt{1+i\beta}}).$$

The family f_α is orthogonal in the sense that for all α and $\xi \in \mathbb{N}^N$,

$$\int f_\alpha f_\xi \rho_\beta dy = \delta_{\alpha,\xi}\int f_\alpha^2 \rho_\beta dy.$$

The semigroup generated by $\mathcal{L}_\beta$ is well defined and has the following kernel:

$$e^{s\mathcal{L}_\beta}(y,x) = \frac{1}{[4\pi(1+i\beta)(1-e^{-s})]^{N/2}}\text{exp}\left[-\frac{|x - ye^{-\frac{s}{2}}|^2}{4(1+i\beta)(1-e^{-s})}\right]. \qquad (169)$$

In the following, we give some properties associated to the kernel.

Lemma A.1 *a) The semigroup associated to $\mathcal{L}_\beta$ satisfies the maximum principle:*

$$\|e^{s\mathcal{L}_\beta}\varphi\|_{L^\infty} \leq \|\varphi\|_{L^\infty}.$$

b) Moreover, we have

$$\|e^{s\mathcal{L}_\beta}\text{div}\,(\varphi)\|_{L^\infty} \leq \frac{C}{\sqrt{1-e^{-s}}}\|\varphi\|_{L^\infty},$$

where C only depends on β.

Proof: a) It follows directly by part, this also follows from the definition of the semigroup (169).
b) Using an integration by part, this also follows from the definition of the semigroup (169).

Lemma A.2 *There exists a constant C such that if ϕ satisfies*

$$\forall x \in \mathbb{R} \;\; |\phi(x)| \leq (1 + |x|^{M+1})$$

then for all $y \in \mathbb{R}$, we have

$$|e^{s\mathcal{L}_\beta}P_-(\phi(y))| \leq Ce^{-\frac{M+1}{2}s}(1 + |y|^{M+1})$$

Proof: This also follows directly from the semigroup's definition, through an integration by part, for a similar case see page 556-558 from [BK94].

Moreover, we have the following useful lemma about P_-.

Lemma A.3 *For all $k \geq 0$, we have*

$$\left\|\frac{P_-(\phi)}{1+|y|^{M+k}}\right\|_{L^\infty} \leq C\left\|\frac{\phi}{1+|y|^{M+k}}\right\|.$$

Proof: Using (61), we have

$$|\phi_n| \le C \left\| \frac{\phi}{1+|y|^{M+k}} \right\|_{L^\infty}.$$

Since for all $m \le M$, $|h_m(y)| \le C(1+|y|^{m+k})$ and

$$|\phi| \le C \left\| \frac{\phi}{1+|y|^{M+k}} \right\|_{L^\infty} (1+|y|^{m+k}),$$

the result follows from definition (59) of ϕ.

B Details of expansions of the potential terms: V_1 and V_2

In this section, we aim at giving some expansions of V_1 and V_2 in order to give the conclusion of item (i) from Lemma 4.15 and some related constants. Indeed, we recall the definition of V_1 and V_2

$$\begin{aligned} V_1(y,s) &= (1+i\delta)\frac{p+1}{2}\left(|\varphi|^{p-1} - \frac{1}{p-1}\right), \\ V_2(y,s) &= (1+i\delta)\frac{p-1}{2}\left(|\varphi|^{p-3}\varphi^2 - \frac{1}{p-1}\right), \end{aligned}$$

where

$$\varphi(y,s) = \varphi_0(y,s) + \frac{(1+i\delta)a}{\sqrt{s}} = \kappa\left(1 + \frac{b}{p-1}\frac{|y|^2}{\sqrt{s}}\right)^{-\frac{1}{p-1}} + \frac{(1+i\delta)a}{\sqrt{s}},$$

and

$$a = \frac{2\kappa b(1-\delta\beta)}{(p-1)^2}.$$

Then, using Taylor expansion, we claim to the following asymptotic behaviors

$$V_1(y,s) = \frac{1}{\sqrt{s}}W_{1,1}(y) + \frac{1}{s}W_{1,2} + O\left(\frac{1+|y|^6}{s^{\frac{3}{2}}}\right), \tag{170}$$

and

$$V_2(y,s) = \frac{1}{\sqrt{s}}W_{2,1}(y) + \frac{1}{s}W_{2,2}(y) + O(\frac{1+|y|^6}{s^{\frac{3}{2}}}), \tag{171}$$

where

$$\begin{aligned} W_{1,1}(y) &= (1+i\delta)\frac{(p+1)}{2}\frac{b}{(p-1)^2}(-y^2 + 2(1-\delta\beta)), \\ W_{1,2}(y) &= (1+i\delta)\frac{(p+1)}{2}\frac{b^2}{(p-1)^3}\left\{y^4 - \frac{(2(1-\delta\beta)(p-2+\delta^2))}{p-1}y^2\right. \\ &+ \left.\frac{(p-1)(1+\delta^2)(1-\delta\beta)^2 + (p-3)(1-\delta^2)(1-\delta\beta)^2}{p-1}\right\} \\ &= (1+i\delta)\frac{(p+1)}{2}\frac{b^2}{(p-1)^4}\left\{(p-1)y^4 - [2(1-\delta\beta)(p-2+\delta^2)]y^2\right. \\ &+ \left. 2(p-2+\delta^2)(1-\delta\beta)^2\right\} \end{aligned}$$

and

$$
\begin{aligned}
W_{2,1}(y) &= (1+i\delta)\frac{b}{2(p-1)^2}\{(p-1+2i\delta)(-y^2+2(1-\delta\beta))\},\\
W_{2,2}(y) &= (1+i\delta)\frac{b^2}{2(p-1)^4}\left\{(p-1+2i\delta)(p-1+i\delta)y^4\right.\\
- &\quad (2(p-1)(p-2)+(2p-10)\delta^2+(8p-16)\delta i)(1-\delta\beta)y^2\\
+ &\quad \left.(1-\delta\beta)^2\left[\frac{(p+1)(p-1)(1+i\delta)^2}{2}+(p+1)(p-3)(1+\delta^2)+\frac{(p-3)(p-5)(1-i\delta)^2}{2}\right]\right\}.
\end{aligned}
$$

For the proof of (170) and (171). We give only the second one because the first one can be obtained in the same way. Let us start the proof (171): We first write

$$
\varphi = \varphi_0(z) + \frac{a(1+i\delta)}{\sqrt{s}} \equiv \varphi_0 + A.
$$

Using Taylor expansion of the function $|\varphi_0+z|^{p-3}(\varphi_0+z)^2$ at $z=0$, applying at $z=A$, we derive

$$
\begin{aligned}
|\varphi|^{p-3}\varphi^2 &= |\varphi_0|^{p-3}\varphi_0^2 + \frac{p+1}{2}|\varphi_0|^{p-3}\varphi_0 A + \frac{(p-3)}{2}|\varphi_0|^{p-5}\varphi_0^3\bar{A} + \frac{(p+1)(p-3)}{8}|\varphi_0|^{p-3}A^2\\
&+ \frac{(p+1)(p-3)}{4}|\varphi_0|^{p-5}\varphi_0^2|A|^2 + \frac{(p-3)(p-5)}{8}|\varphi_0|^{p-7}\varphi_0^4\bar{A}^2 + O\left(\frac{1}{s^{\frac{3}{2}}}\right).
\end{aligned}
$$

Besides that, we also have

$$
\begin{aligned}
|\varphi_0|^{p-3}\varphi_0^2 &= \frac{1}{p-1} - \frac{(p-1+2i\delta)by^2}{(p-1)^3\sqrt{s}} + \frac{(p-1+2i\delta)(p-1+i\delta)b^2y^4}{(p-1)^5 s}\\
&+ O(\frac{1+|y|^6}{s^{\frac{3}{2}}}),\\
\frac{(p+1)}{2}|\varphi_0|^{p-3}\varphi_0 A &= \frac{(p+1)(1+i\delta)a}{2\kappa(p-1)\sqrt{s}} - \frac{(p+1)(1+i\delta)(p-2+i\delta)aby^2}{2\kappa(p-1)^3 s}\\
&+ O\left(\frac{1+|y|^6}{s^{\frac{3}{2}}}\right),\\
\frac{(p-3)}{2}|\varphi_0|^{p-5}\varphi_0^3\bar{A} &= \frac{(p-3)(1-i\delta)a}{2\kappa(p-1)\sqrt{s}} - \frac{(p-3)(1-i\delta)(p-2+3i\delta)aby^2}{2\kappa(p-1)^3 s}\\
&+ O\left(\frac{1+|y|^6}{s^{\frac{3}{2}}}\right),\\
\frac{(p+1)(p-3)}{8}|\varphi_0|^{p-3}A^2 &= \frac{(p+1)(p-3)}{8}\frac{1}{\kappa^2(p-1)}\frac{(1+i\delta)^2a^2}{s} + O\left(\frac{1+|y|^2}{s^{\frac{3}{2}}}\right),\\
\frac{(p+1)(p-3)}{4}|\varphi_0|^{p-5}\varphi_0^2|A|^2 &= \frac{(p+1)(p-3)}{4}\frac{1}{\kappa^2(p-1)}\frac{(1+\delta^2)a^2}{s} + O\left(\frac{1+|y|^2}{s^{\frac{3}{2}}}\right),\\
\frac{(p-3)(p-5)}{8}|\varphi_0|^{p-7}\varphi_0^4\bar{A}^2 &= \frac{(p-3)(p-5)}{8}\frac{1}{\kappa^2(p-1)}\frac{(1-i\delta)^2a^2}{s} + O\left(\frac{1+|y|^2}{s^{\frac{3}{2}}}\right).
\end{aligned}
$$

Thus, plugging these asymptotic in the formula of V_2, we can derive (171).

In addition to that, we aim at determining the constants given in item (ii) of Lemma 4.15:

$$
\begin{aligned}
\tilde{D}_{4,2} &= \tilde{P}_{4,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) = \frac{b(\delta^2-p)}{(p-1)^2},\\
D_{2,2} &= P_{2,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2)\\
&= -\frac{b}{2(p-1)^2}\{-24p\delta + 56\delta^3 + 64\delta^2\beta + 32\delta + 24p\delta^2\beta + 40\delta^4\beta\},\\
\tilde{D}_{2,2} &= \tilde{P}_{2,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) = \frac{4b}{p-1}\delta\beta(1+\delta^2),\\
\tilde{L}_{2,4} &= \tilde{P}_{2,M}(i\tilde{h}_4) = 6\delta^2\beta - 12\delta - 6\beta,\\
D_{4,2} &= P_{4,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) = \frac{b}{(p-1)^2}\left\{-2\delta(1+\delta^2)\right\}\\
\tilde{D}_{2,0} &= \tilde{P}_{2,M}(W_{1,1}\tilde{h}_0 + W_{2,1}\bar{\tilde{h}}_0) = -\frac{b}{2(p-1)^2}(2p - 2\delta^2)\\
\tilde{L}_{0,2} &= \tilde{P}_{0,M}(i\tilde{h}_2) = -2\delta + \delta^2\beta - \beta,\\
\tilde{D}_{0,2} &= \tilde{P}_{0,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2)\\
&= -\frac{b}{2(p-1)^2}\{-32\delta\beta - 12p\beta^2 + 12\delta^2\beta^2 - 16\delta^2 + 16p - 4\delta^4\beta^2 + 4p\delta^2\beta^2 - 32p\delta\beta\},\\
\tilde{C}_{2,2} &= \tilde{P}_{2,M}(W_{1,1}h_2 + W_{2,1}\bar{h}_2)\\
&= -\frac{b}{2(p-1)^2}\{-14\delta^2\beta + 2p\beta - 12\beta\},\\
\tilde{C}_{2,4} &= \tilde{P}_{2,M}(W_{1,1}h_4 + W_{2,1}\bar{h}_4)\\
&= -\frac{b}{2(p-1)^2}\{96p\beta + 224\delta^3\beta^2 - 288\delta^2\beta - 128p\delta\beta^2 - 192\beta + 96\delta\beta^2\}\\
\tilde{D}_{2,4} &= \tilde{P}_{2,M}(W_{1,1}\tilde{h}_4 + W_{2,1}\bar{\tilde{h}}_4)\\
= -\frac{b}{2(p-1)^2}&\{-96p\delta^2\beta^2 - 168p\delta\beta + 96p - 528\delta\beta - 96\delta^2 + 216\delta^2\beta^2 - 168p\beta^2 + 144\delta^4\beta^2 - 360\delta^3\beta\}\\
\tilde{F}_{2,2} &= \tilde{P}_{2,M}(W_{1,2}\tilde{h}_2 + W_{2,2}\bar{\tilde{h}}_2)\\
&= \frac{b^2}{2(p-1)^4}\left\{-240p + 276p^2 - 312p\delta^2 - 204\delta^4 + (-288p - 552p^2 + 696)\delta\beta\right.\\
&+ \left.(432-144p)\delta^3\beta + 144\delta^5\beta + (180p - 180p^2)\beta^2 + (96p^2 + 288p - 96)\delta^2\beta^2 + (108+36p)\delta^4\beta^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
D_{0,2} &= P_{0,M}(W_{1,1}\tilde{h}_2 + W_{1,2}\bar{\tilde{h}}_2)\\
&= -\frac{b}{2(p-1)^2}\{32\delta + 24\delta^5\beta^2 + 64\delta^2\beta + 48\delta^3\beta^2 + 64\delta^4\beta + 32\delta^3 + 24\delta\beta^2 + 96p\delta^3\beta^2 + 96p\delta\beta^2\},\\
L_{0,2} &= P_{0,M}(i\tilde{h}_2) = 4\delta\beta + 4\delta^3\beta.
\end{aligned}
$$

We would like to explain a little bit how we obtain these constants. For example,

$\tilde{D}_{4,2} = \tilde{P}_{4,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2)$. Indeed, we first write

$$W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2 = \sum_{j=0}^{k} c_j y^j,$$

where $c_j \in \mathbb{C}$. Then, we the definition of h_j and $\tilde{h}_j$ to write

$$W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2 = \sum_{j=0}^{k} c_k y^j = \sum_{j=0}^{k}(a_j h_j + \tilde{a}_j \tilde{h}_j).$$

Thus, we obtain

$$P_{j,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) = a_j \text{ and } \tilde{P}_{j,M}(W_{1,1}\tilde{h}_2 + W_{2,1}\bar{\tilde{h}}_2) = \tilde{a}_j.$$

C Details of some expansions of $B(q)$

In this section, we will give the supplement of the proof of Lemma 4.17 and some constants relating. Indeed, let us recall from (49) that:

$$B(q,y,s) = (1+i\delta)\left(|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi - |\varphi|^{p-1}q - \frac{p-1}{2}|\varphi|^{p-3}\varphi(\varphi\bar{q} + \bar{\varphi}q)\right).$$

We first use the expansion in (173), we first derive

$$\begin{aligned}
B(q) &= \frac{(1+i\delta)}{8\kappa}\{(p-3)\bar{q}^2 + 2(p+1)\bar{q}q + (p+1)q^2\}\\
&= \frac{(1+i\delta)}{8\kappa}\left\{q_2^2[(p-3)\bar{h}_2^2 + 2(p+1)\bar{h}_2 h_2 + (p+1)h_2^2]\right.\\
&+ \tilde{q}_2^2[(p-3)\bar{\tilde{h}}_2^2 + 2(p+1)\tilde{h}_2\bar{\tilde{h}}_2 + (p+1)\tilde{h}_2^2]\\
&+ \tilde{q}_0 q_2[2(p-3)\bar{\tilde{h}}_0\bar{h}_2 + 2(p+1)(\bar{\tilde{h}}_0 h_2 + \tilde{h}_0\bar{h}_2) + 2(p+1)\tilde{h}_0 h_2]\\
&+ \tilde{q}_0\tilde{q}_2[2(p-3)\bar{\tilde{h}}_0\bar{\tilde{h}}_2 + 2(p+1)(\bar{\tilde{h}}_0\tilde{h}_2 + \tilde{h}_0\bar{\tilde{h}}_2) + 2(p+1)\tilde{h}_0\tilde{h}_2]\\
&+ \left. q_2\tilde{q}_2[2(p-3)\bar{\tilde{h}}_2\bar{h}_2 + 2(p+1)(\bar{\tilde{h}}_2 h_2 + \tilde{h}_2\bar{h}_2) + 2(p+1)h_2\tilde{h}_2]\right\} + O(|q|^3).
\end{aligned}$$

Then, from then fact that $q \in \mathcal{V}_A(s)$, estimates (116) and (117) directly follow. In addition to that, we also derive the following

$$\tilde{P}_{2,M}(B(q)) = q_2^2\tilde{B}_2(q_2^2) + \tilde{q}_2^2\tilde{B}_2(\tilde{q}_2^2) + \tilde{q}_0 q_2\tilde{B}_2(\tilde{q}_0 q_2) + \tilde{q}_0\tilde{q}_2\tilde{B}_2(\tilde{q}_0\tilde{q}_2) + q_2\tilde{q}_2\tilde{B}_2(q_2\tilde{q}_2) + O\left(\frac{1}{s^3}\right).$$

where

$$\begin{aligned}
\tilde{B}_2(q_2^2) &= \frac{1}{8\kappa}(32 - 64\delta\beta),\\
\tilde{B}_2(\tilde{q}_2^2) &= \frac{1}{\kappa}[4(p-\delta^2) - \delta\beta(4p + 2\delta^2 + 6)],\\
\tilde{B}_2(\tilde{q}_0 q_2) &= 0,\\
\tilde{B}_2(\tilde{q}_0\tilde{q}_2) &= \frac{1}{8\kappa}[8(p-\delta^2)] = \frac{1}{\kappa}[p-\delta^2],\\
\tilde{B}_2(q_2\tilde{q}_2) &= \frac{1}{8\kappa}[-56\delta^2\beta + 8p\beta - 48\beta] = \frac{1}{\kappa}[-7\delta^2\beta + p\beta - 6\beta].
\end{aligned}$$

Using the fact that $q \in \mathcal{V}_A(s)$, we get

$$\tilde{P}_{2,M}(B) = \tilde{B}_2 \tilde{q}_2^2 + B_1 \frac{\tilde{q}_2}{s} + \frac{B_2}{s^2} + O\left(\frac{A^6}{s^{9/4}}\right), \tag{172}$$

where

$$\begin{aligned}
\tilde{B}_2 &= \frac{1}{\kappa}\left\{4(p-\delta^2) - \delta\beta(6+4p+2\delta^2)\right\},\\
B_1 &= \frac{1}{\kappa}\left\{(-7\delta^2\beta + p\beta - 6\beta)R^*_{2,1} - (p-\delta^2)\tilde{R}^*_{0,1}\right\},\\
B_2 &= \frac{(R^*_{2,1})^2}{s^2}\frac{(32-64\delta\beta)}{8\kappa}.
\end{aligned}$$

D Details of expansions of $R^*(y, s, \theta'(s))$

Using the definition of φ, the fact that φ_0 satisfies (43) and (48), we see that R^* is in fact a function of θ', $z = \frac{y}{s^{1/4}}$ and s that can be written as

$$\begin{aligned}
R^* &= \frac{(1+i\beta)}{s^{\frac{1}{2}}}\Delta_z\varphi_0(z) - \frac{1}{2}z\cdot\nabla\varphi_0(z) - \frac{(1+i\delta)}{p-1}\varphi_0(z) - \frac{(1+i\delta)^2}{p-1}\frac{a}{\sqrt{s}}\\
&+ (1+i\delta)F\left(\varphi_0(z) + \frac{a}{\sqrt{s}}(1+i\delta)\right) + \frac{1}{4s}z\cdot\nabla\varphi_0 + \frac{a}{2s^{\frac{3}{2}}}(1+i\delta)\\
&- i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{\sqrt{s}} + \theta'(s)\right)\varphi\\
&= \frac{(1+i\beta)}{s^{\frac{1}{2}}}\Delta_z\varphi_0(z) + \frac{1}{4s}z\cdot\nabla\varphi_0 + \frac{a}{2s^{\frac{3}{2}}}(1+i\delta) - \frac{(1+i\delta)^2 a}{(p-1)\sqrt{s}}\\
&+ (1+i\delta)\left(F\left(\varphi_0(z) + \frac{a}{\sqrt{s}}(1+i\delta)\right) - F(\varphi_0(z))\right)\\
&- i\left(\frac{\nu}{2\sqrt{s}} + \frac{\mu}{s} + \theta'(s)\right)\left(\varphi_0(z) + \frac{a}{\sqrt{s}}(1+i\delta)\right),\\
&= R_1^*(y,s) + \theta'(s)\Theta(y,s),
\end{aligned}$$

where $F(w) = |w|^{p-1}w$ and $\Theta(y,s) = -i\left(\varphi_0(y,s) + \frac{a(1+i\delta)}{\sqrt{s}}\right)$.

Expansion of $R_1^*(y,s)$ in terms oh h_j and $\tilde{h}_j$

In fact, we firstly try to expand each term of $R_1^*(y,s)$ as a power series in $\frac{1}{s}$ and y as $s\to\infty$, uniformly for $|y| \le Cs^{\frac{1}{4}}$, for some C, a positive constant.

As a matter of fact, we have the following

$$
\begin{aligned}
&|a_0+u|^{p-1}(a_0+u) = |a_0|^{p-1}a_0 + \frac{p+1}{2}|a_0|^{p-1}u + \frac{p-1}{2}|a_0|^{p-3}a_0^2\bar{u} \\
+ \quad & \frac{(p-1)(p-3)}{8}|a_0|^{p-5}a_0^3\bar{u}^2 + \frac{(p+1)(p-1)}{4}|a_0|^{p-3}a_0u\bar{u} + \frac{(p+1)(p-1)}{8}|a_0|^{p-3}\bar{a}_0u^2 \\
+ \quad & \frac{(p-1)(p-3)(p-5)}{48}|a_0|^{p-7}a_0^4\bar{u}^3 + \frac{(p+1)(p-1)(p-3)}{16}|a_0|^{p-5}a_0^2u\bar{u}^2, \\
+ \quad & \frac{(p+1)(p-1)^2}{16}|a_0|^{p-3}u^2\bar{u} + \frac{(p+1)(p-1)(p-3)}{48}|a_0|^{p-5}\bar{a}_0^2u^3 + O(u^4), \qquad (173)
\end{aligned}
$$

as $u \to 0$.

Let us consider $a_0 = \varphi_0(z) = \kappa(1+\frac{b}{p-1}z^2)^{-\frac{1+i\delta}{p-1}}$, where $z = \frac{y}{\sqrt{s}}$ and $u = \frac{a}{\sqrt{s}}(1+i\delta)$, then we have in addition

$$
\begin{aligned}
\frac{p+1}{2}|a_0|^{p-1}u &= \frac{(p+1)a(1+i\delta)}{2(p-1)\sqrt{s}}\left(1+\frac{b}{p-1}z^2\right)^{-1} \\
&= \frac{(p+1)a(1+i\delta)}{2(p-1)\sqrt{s}}\left(1-\frac{b}{p-1}z^2+\frac{b^2z^4}{(p-1)^2}+O(z^6)\right), \\
\frac{p-1}{2}|a_0|^{p-3}a_0^2\bar{u} &= \frac{a(1-i\delta)}{2\sqrt{s}}\left(1+\frac{b}{p-1}z^2\right)^{-\frac{(p-1+2i\delta)}{p-1}} \\
&= \frac{a(1-i\delta)}{2\sqrt{s}}\left(1-\frac{(p-1+2i\delta)b}{(p-1)^2}z^2+\frac{(p-1+2i\delta)(p-1+i\delta)b^2}{(p-1)^4}z^4\right) \\
&+ \quad O(z^6)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{(p-1)(p-3)}{8}|a_0|^{p-5}a_0^3\bar{u}^2 &= \frac{(p-3)(1-i\delta)^2a^2}{8\kappa s}\left(1+\frac{b}{p-1}z^2\right)^{-\frac{(p-2+3i\delta)}{p-1}} \\
&= \frac{(p-3)(1-i\delta)^2a^2}{8\kappa s}\left(1-\frac{(p-2+3i\delta)bz^2}{(p-1)^2}+O(z^4)\right), \\
\frac{(p+1)(p-1)}{4}|a_0|^{p-3}a_0u\bar{u} &= \frac{(p+1)(1+\delta^2)a^2}{4\kappa s}\left(1+\frac{b}{p-1}z^2\right)^{-\frac{(p-2+i\delta)}{p-1}} \\
&= \frac{(p+1)(1+\delta^2)a^2}{4\kappa s}\left(1-\frac{(p-2+i\delta)bz^2}{(p-1)^2}+O(z^4)\right), \\
\frac{(p+1)(p-1)}{8}|a_0|^{p-3}\bar{a}_0u^2 &= \frac{(p+1)(1+i\delta)^2a^2}{8\kappa s}\left(1+\frac{b}{p-1}z^2\right)^{-\frac{(p-2-i\delta)}{p-1}} \\
&= \frac{(p+1)(1+i\delta)^2a^2}{8\kappa s}\left(1-\frac{(p-2-i\delta)bz^2}{(p-1)^2}+O(z^4)\right),
\end{aligned}
$$

Besides that, we also get the following

$$\begin{aligned}
&\frac{1}{4s}z\cdot\nabla\varphi_0(z)\\
=\;& -\frac{\kappa(1+i\delta)bz^2}{2(p-1)^2s}+O\left(\frac{z^4}{s}\right),\\
&\frac{(1+i\beta)}{s^{\frac12}}\Delta_z\varphi_0(z)\\
=\;& -\frac{2\kappa(1+i\delta)(1+i\beta)b}{(p-1)^2\sqrt{s}}\left(1-\frac{(p+i\delta)bz^2}{(p-1)^2}+\frac{(p+i\delta)(2p-1+i\delta)b^2z^4}{2(p-1)^4}+O(z^6)\right)\\
+\;& \frac{4\kappa(1+i\delta)(1+i\beta)(p+i\delta)b^2z^2}{(p-1)^4\sqrt{s}}\left(1-\frac{(2p-1+i\delta)bz^2}{(p-1)^2}+O(z^4)\right)\\
=\;& -\frac{2\kappa(1+i\delta)(1+i\beta)b}{(p-1)^2\sqrt{s}}+i\frac{6\kappa(p+1)\delta(1+\beta^2)b^2y^2}{(p-1)^4s}\\
+\;& \frac{5\kappa(p+1)\delta(1+\beta^2)(\delta-i(2p-1))b^3y^4}{(p-1)^6s^{\frac32}}+O\left(\frac{1+|y|^6}{s^2}\right),\\
&\varphi_0(z)+(1+i\delta)\frac{a}{s^{\frac12}}\\
=\;& \kappa-\frac{\kappa(1+i\delta)bz^2}{(p-1)^2}+\frac{\kappa(p+1)\delta(\beta+i)b^2z^4}{2(p-1)^4}+(1+i\delta)\frac{a}{\sqrt{s}}+O(z^6).
\end{aligned}$$

So, in expansion of R_1^* in series of $\frac{1}{s^j}$, we have

+ Order $\frac{1}{\sqrt{s}}$:

$$\begin{aligned}
& -\frac{2\kappa(1+i\delta)(1+i\beta)b}{(p-1)^2}-\frac{(1+i\delta)^2a}{p-1}+(1+i\delta)\left(\frac{(p+1)(1+i\delta)a}{2(p-1)}+\frac{(1-i\delta)a}{2}\right)-i\frac{\kappa\nu}{2}\\
=\;& -i\frac{\kappa\nu}{2}-\frac{2\kappa(1+i\delta)(1+i\beta)b}{(p-1)^2}+(1+i\delta)a,
\end{aligned}$$

Then, we can write R_1^* as follows

$$R_1^*(y,s)=\frac{1}{\sqrt{s}}\mathcal{R}_0(y)+\frac{1}{s}\mathcal{R}_1(y)+\frac{1}{s^{\frac32}}\mathcal{R}_2(y)+\frac{1}{s^2}\mathcal{R}_3(y)+\tilde{\mathcal{R}}(y,s),$$

where $\tilde{\mathcal{R}}$ satisfies

$$|\tilde{\mathcal{R}}(y,s)|\le\frac{C(1+|y|^8)}{s^{\frac52}},$$

which implies that

$$|P_{j,M}(\tilde{\mathcal{R}})|+|\tilde{P}_{j,M}(\tilde{\mathcal{R}})|\le\frac{C}{s^{\frac52}}.$$

In addition to that, by using a explicit calculation, we can show that $\mathcal{R}_j(y)$ is a polynomial of order $2j$. Besides that, these polynomials don't contain any odd order of y.

Moreover, we can write $\mathcal{R}_j$ in terms of h_k, and $\tilde{h}_k$ as follows

$$\mathcal{R}_j(y) = \sum_{k=0}^{2j} (R^*_{k,j} h_k + \tilde{R}^*_{k,j} \tilde{h}_k)$$

More precisely, we have

$$\begin{aligned}
\tilde{R}^*_{0,0} &= a - 2(1-\beta\delta)\frac{b}{(p-1)^2}), \\
R^*_{0,0} &= \kappa\frac{\nu}{2} - 2\kappa\beta(1+\delta^2)\frac{b}{(p-1)^2}, \\
\tilde{R}^*_{2,1} &= \left(\frac{2(\delta^2-p)ab - \kappa\delta\nu b}{2(p-1)^2}\right) \\
R^*_{2,1} &= \left(\frac{(1+\delta^2)}{2(p-1)^2}(\kappa\nu b - 4\delta ab) + \frac{6\kappa(p+1)\delta(1+\beta^2)b^2}{(p-1)^4}\right) \\
\tilde{R}^*_{0,1} &= \frac{a^2(p-\delta^2)+\kappa\delta\nu a}{2\kappa} + \frac{(1-\delta\beta)(2(\delta^2-p)ab-\kappa\delta\nu b)}{(p-1)^2} \\
&- \frac{(1+\delta^2)}{2(p-1)^2}\beta(\kappa\nu b - 4\delta ab) - \frac{6\kappa(p+1)\delta\beta(1+\beta^2)b^2}{(p-1)^4} \\
R^*_{0,1} &= \frac{(1+\delta\beta)(1+\delta^2)(\kappa\nu b - 4\delta ab)}{(p-1)^2} + \frac{12\kappa(p+1)\delta(1+\delta\beta)(1+\beta^2)b^2}{(p-1)^4} \\
&+ \frac{(1+\delta^2)(8\delta a^2 - 4\kappa\nu a)}{8\kappa} - \kappa\mu, \\
\tilde{R}^*_{2,2} &= \frac{5\kappa(p+1)\delta(1+\beta^2)b^3}{(p-1)^6}\left[12\delta - 6\delta^2\beta + 6(2p-1)\beta\right] \\
&+ \frac{\nu\kappa(p+1)\delta b^2}{4(p-1)^4}[12 - 6\delta\beta + 6\beta^2] \\
&+ \frac{ab^2}{2(p-1)^4}[24p^2 - 24p + (30-6p-24p^2)\delta\beta - 24p\delta^2 - 24\delta^4 + (18-6p)\delta^3\beta + 12\delta^5\beta] \\
&- \frac{\kappa b}{2(p-1)^2} - \frac{\mu\kappa\delta b}{(p-1)^2} - \frac{a^2 b}{8\kappa(p-1)^2}\left\{4p^2 - 8p - 12\delta^4\right\}.
\end{aligned}$$

Besides that, the other constants which will not give in this paper, because we need only the existence of μ. In addition to that, in order to make more clearly our computation, we also give below the calculations of polynomial $\mathcal{R}_j$. Then, the readers can check the accuracy of our constant.

+ Formula of $\mathcal{R}_0(y)$: As we mentioned above, $\mathcal{R}_0(y)$ is a polynomial of order 0 in y,

$$\begin{aligned}
&\mathcal{R}_0(y) \\
&= -\frac{2\kappa(1+i\delta)(1+i\beta)b}{(p-1)^2} - \frac{(1+i\delta)^2 a}{p-1} + (1+i\delta)\left(\frac{(p+1)(1+i\delta)a}{2(p-1)} + \frac{(1-i\delta)a}{2}\right) - i\frac{\kappa\nu}{2} \\
&= -i\frac{\kappa\nu}{2} - \frac{2\kappa(1+i\delta)(1+i\beta)b}{(p-1)^2} + (1+i\delta)a,
\end{aligned}$$

+ Formula of $\mathcal{R}_1(y)$: As we mentioned, $\mathcal{R}_1(y)$ is a polynomial of order 2

$$\mathcal{R}_1(y) = \mathcal{R}_{1,0} + \mathcal{R}_{1,2} y^2,$$

where

$$\begin{aligned}
& \mathcal{R}_{1,0} \\
= \;& (1+i\delta)\left(\frac{(p-3)(1-i\delta)^2a^2}{8\kappa} + \frac{(p+1)(1+\delta^2)a^2}{4\kappa} + \frac{(p+1)(1+i\delta)^2a^2}{8\kappa} - i\frac{\nu a}{2}\right) - i\mu\kappa \\
= \;& (1+i\delta)\left(a^2\frac{4p+8\delta i+4\delta^2}{8\kappa} - i\frac{4\kappa\nu a}{8\kappa}\right) - i\mu\kappa \\
= \;& (1+i\delta)\left(\frac{a^2(p-\delta^2)+\kappa\delta\nu a}{2\kappa}\right) \\
+ \;& i\left(\frac{(1+\delta^2)(8\delta a^2 - 4\kappa\nu a)}{8\kappa} - \kappa\mu\right) \\
& \mathcal{R}_{1,2} = i\frac{\kappa(1+i\delta)\nu b}{2(p-1)^2} + i\frac{6\kappa(p+1)\delta(1+\beta^2)b^2}{(p-1)^4} \\
+ \;& (1+i\delta)\left(\frac{(p+1)(1+i\delta)a}{2(p-1)}\left(-\frac{b}{p-1}\right)\right) + \frac{(1-i\delta)a}{2}\left(-\frac{(p-1+2i\delta)b}{(p-1)^2}\right)
\end{aligned}$$

+ Formula of $\mathcal{R}_2(y)$: In fact, this polynomial is of order 4 that we will give in the following

$$\mathcal{R}_2(y) = \mathcal{R}_{2,0} + \mathcal{R}_{2,1}y^2 + \mathcal{R}_{2,2}y^4,$$

where

$$\begin{aligned}
\mathcal{R}_{2,1} \;=\;& -\frac{\kappa(1+i\delta)b}{2(p-1)^2} \\
+\;& (1+i\delta)\left\{-\frac{(p-3)(1-i\delta)^2a^2}{8\kappa}\frac{(p-2+3i\delta)b}{(p-1)^2}\right. \\
-\;& \left.\frac{(p+1)(1+\delta^2)a^2}{4\kappa}\frac{(p-2+i\delta)b}{(p-1)^2} - \frac{(p+1)(1+i\delta)^2a^2}{8\kappa}\frac{(p-2-i\delta)b}{(p-1)^2}\right\} \\
+\;& \frac{i\mu\kappa(1+i\delta)b}{(p-1)^2} \\
=\;& -\frac{\kappa(1+i\delta)b}{2(p-1)^2} + \frac{i\mu\kappa(1+i\delta)b}{(p-1)^2} \\
+\;& (1+i\delta)\frac{a^2b}{8\kappa(p-1)^2}\left\{-(p-3)(1-i\delta)^2(p-2+3i\delta)\right. \\
-\;& \left. 2(p+1)(1+\delta^2)(p-2+i\delta) - (p+1)(1+i\delta)^2(p-2-i\delta)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_{2,2} &= \frac{5\kappa(p+1)\delta(1+\beta^2)(\delta-i(2p-1))b^3}{(p-1)^6} \\
&+ \frac{\nu\kappa(p+1)\delta(1-i\beta)b^2}{4(p-1)^4} \\
&+ \frac{ab^2}{2(p-1)^4}\left[(p^2-1)(1+i\delta)^2+(1+\delta^2)(p-1+2i\delta)(p-1+i\delta)\right] \\
&= \frac{5\kappa(p+1)\delta(1+\beta^2)b^3}{(p-1)^6}(\delta-i(2p-1)) \\
&+ \frac{\nu\kappa(p+1)\delta b^2}{4(p-1)^4}(1-i\beta) \\
&+ \frac{ab^2}{2(p-1)^4}\left\{2p^2-2p-2p\delta^2-2\delta^4+i[(2p^2+3p-5)\delta+(3p-3)\delta^3]\right\},
\end{aligned}$$

and $\mathcal{R}_{2,0}$ is some constant in $\mathbb{C}$ which we don't need to calculate explicitly. Moreover, we will not need the exact formula of $\mathcal{R}_3(y)$, we just need to know that it is a polynomial of order 6. Finally, in order to finish the calculation on $R^*(\theta'(s))$, it remains to expand the second term $\theta'(s)\Theta(y,s)$.

Expansion of $\theta'(s)\Theta(y)$

We introduce

$$\Theta(y,s) = -i\left(\varphi_0(y,s)+\frac{a(1+i\delta)}{\sqrt{s}}\right),$$

where φ_0 and a defined as in (41) and (42), respectively. Using Taylor expansions, we write

$$\begin{aligned}
\Theta(y,s) &= -i\kappa-\kappa(\delta-i)\frac{y^2}{\sqrt{s}}\frac{b}{(p-1)^2}+a(\delta-i)\frac{1}{\sqrt{s}} \\
&+ \kappa(1-i\beta)\delta(p+1)\frac{y^4}{s}\frac{b^2}{2(p-1)^4}+\tilde{\Theta}(y,s),
\end{aligned}$$

where $\tilde{\Theta}(y,s)$ satisfies the following

$$\left|\tilde{\Theta}(y,s)\right| \le \frac{C(1+|y|^6)}{s^{\frac{3}{2}}},$$

which yields

$$|P_{j,M}(\tilde{\Theta})|+|\tilde{P}_{j,M}(\tilde{\Theta})| \le \frac{C}{s^{\frac{3}{2}}}.$$

and

$$-i\kappa-\kappa(\delta-i)\frac{y^2}{\sqrt{s}}\frac{b}{(p-1)^2}+a(\delta-i)\frac{1}{\sqrt{s}}+\kappa(1-i\beta)\delta(p+1)\frac{y^4}{s}\frac{b^2}{2(p-1)^4} \quad (174)$$

$$= \left(-\kappa+\frac{\Theta^*_{0,0}}{\sqrt{s}}\right)h_0+\frac{\tilde{\Theta}^*_{0,0}}{\sqrt{s}}\tilde{h}_0+\frac{\Theta^*_{2,0}}{\sqrt{s}}h_2+\frac{\tilde{\Theta}^*_{2,0}}{\sqrt{s}}\tilde{h}_2+\frac{\tilde{\Theta}^*_{2,1}}{s}\tilde{h}_2. \quad (175)$$

In addition to that, we can calculate these constants and we obtain

$$\begin{aligned}
\Theta^*_{0,0} &= 4(1+\delta^2)\delta\beta\frac{\kappa b}{(p-1)^2},\\
\tilde{\Theta}^*_{0,0} &= -\beta(1+\delta^2)\frac{\kappa b}{(p-1)^2},\\
\tilde{\Theta}^*_{2,0} &= -\delta\frac{\kappa b}{(p-1)^2},\\
\Theta^*_{2,0} &= (1+\delta^2)\frac{\kappa b}{(p-1)^2},\\
\tilde{\Theta}^*_{2,1} &= -3\delta(p+1)(-\beta^2+\beta\delta-2)\frac{\kappa b^2}{(p-1)^4}.
\end{aligned}$$

E Formal derivation of constant b and μ

We begin by recalling that we need the following equation, for the determination of b

$$\begin{aligned}
&-\tfrac{1}{2}R_2'r - \tfrac{R_2}{p-1} + p|R_0|^{p-1}R_2 + R_1'' - 2R_0\varphi_0'\varphi_1' - R_1\varphi_0'^2 - 2\beta R_0'\varphi_1' - 2\beta R_1'\varphi_0'\\
&-\beta R_0\varphi_1'' - \beta R_1\varphi_0'' + \tfrac{1}{4}R_0'r + \tfrac{p(p-1)}{2}|R_0|^{p-3}R_0R_1^2 = 0
\end{aligned} \tag{176}$$

Let us introduce the following

$$\begin{aligned}
R_0 &= (p-1+br^2)^{-\frac{1}{p-1}},\\
R_1 &= (A_1+B_1r^2)(p-1+br^2)^{-\frac{p}{p-1}},\\
\varphi_0' &= \tilde{A}_0r(p-1+br^2)^{-1},\\
\varphi_0'' &= \tilde{A}_0(p-1+br^2)^{-1} - 2\tilde{A}_0br^2(p-1+br^2)^{-2}\\
\varphi_1' &= \tilde{A}_1r(p-1+br^2)^{-1} + \tilde{B}_1r(p-1+br^2)^{-2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
R_1' &= 2B_1r(p-1+br^2)^{-\frac{p}{p-1}} - \frac{2pA_1br}{p-1}(p-1+br^2)^{-\frac{2p-1}{p-1}}\\
&- \frac{2pB_1br^3}{p-1}(p-1+br^2)^{-\frac{(2p-1)}{p-1}},\\
R_1'' &= 2B_1(p-1+br^2)^{-\frac{p}{p-1}} - \frac{8pB_1br^2}{p-1}(p-1+br^2)^{-\frac{2p-1}{p-1}}\\
&+ (A_1+B_1r^2)\left(-\frac{2pb}{p-1}(p-1+br^2)^{-\frac{(2p-1)}{p-1}} + \frac{4p(2p-1)b^2r^2}{(p-1)^2}(p-1+br^2)^{-\frac{(3p-2)}{p-1}}\right)\\
\varphi_1'' &= \tilde{A}_1(p-1+br^2)^{-1} + \tilde{B}_1(p-1+br^2)^{-2}\\
&- 2\tilde{A}_1br^2(p-1+br^2)^{-2} - 4\tilde{B}_1br^2(p-1+br^2)^{-3}.
\end{aligned}$$

Next, we have to compute

$$\begin{aligned}
F_3 &= R_1'' - 2R_0\varphi_0'\varphi_1' - R_1\varphi_0'^2 - 2\beta R_0'\varphi_1' - 2\beta R_1'\varphi_0' - \beta R_0\varphi_1'' - \beta R_1\varphi_0''\\
&+ \frac{1}{4}R_0'r + \frac{p(p-1)}{2}|R_0|^{p-2}R_1^2,
\end{aligned}$$

where

$$
\begin{aligned}
R_1'' &= 2B_1(p-1+br^2)^{-\frac{p}{p-1}} - \frac{8pB_1br^2}{p-1}(p-1+br^2)^{-\frac{2p-1}{p-1}} \\
&+ (A_1+B_1r^2)\left(-\frac{2pb}{p-1}(p-1+br^2)^{-\frac{(2p-1)}{p-1}} + \frac{4p(2p-1)b^2r^2}{(p-1)^2}(p-1+br^2)^{-\frac{(3p-2)}{p-1}}\right) \\
2R_0\varphi_0'\varphi_1' &= 2\tilde{A}_0r(p-1+br^2)^{-\frac{p}{p-1}}\left(\tilde{A}_1r(p-1+br^2)^{-1} + \tilde{B}_1r(p-1+br^2)^{-2}\right) \\
&= 2\tilde{A}_0\tilde{A}_1r^2(p-1+br^2)^{-\frac{(2p-1)}{p-1}} + 2\tilde{A}_0\tilde{B}_1r^2(p-1+br^2)^{-\frac{(3p-2)}{p-1}}. \\
R_1\varphi_0'^2 &= (A_1+B_1r^2)(p-1+br^2)^{-\frac{p}{p-1}}(\tilde{A}_0r(p-1+br^2)^{-1})^2 \\
&= (\tilde{A}_0^2A_1r^2 + \tilde{A}_0^2B_1r^4)(p-1+br^2)^{-\frac{(3p-2)}{p-1}}. \\
2\beta R_0'\varphi_1' &= 2\beta\left(-\frac{2br}{p-1}(p-1+br^2)^{-\frac{p}{p-1}}\right)(\tilde{A}_1r(p-1+br^2)^{-1} + \tilde{B}_1r(p-1+br^2)^{-2}) \\
&= -\frac{4\beta\tilde{A}_1br^2}{p-1}(p-1+br^2)^{-\frac{(2p-1)}{p-1}} - \frac{4\beta\tilde{B}_1br^2}{p-1}(p-1+br^2)^{-\frac{(3p-2)}{p-1}}.
\end{aligned}
$$

$$
\begin{aligned}
2\beta R_1'\varphi_0' &= 2\beta\left(2B_1r(p-1+br^2)^{-\frac{p}{p-1}} - \frac{2pA_1br}{p-1}(p-1+br^2)^{-\frac{(2p-1)}{p-1}} - \frac{2pB_1br^3}{p-1}(p-1+br^2)^{-\frac{(2p-1)}{p-1}}\right) \\
&\times (\tilde{A}_0r(p-1+br^2)^{-1}) \\
&= 4\beta\tilde{A}_0B_1r^2(p-1+br^2)^{-\frac{(2p-1)}{p-1}} - \frac{4p\beta\tilde{A}_0A_1br^2}{p-1}(p-1+br^2)^{-\frac{3p-2}{p-1}} \\
&- \frac{4p\beta\tilde{A}_0B_1br^4}{p-1}(p-1+br^2)^{-\frac{3p-2}{p-1}} \\
\beta R_0\varphi_1'' &= \beta(p-1+br^2)^{-\frac{1}{p-1}}\left(\tilde{A}_1(p-1+br^2)^{-1} + \tilde{B}_1(p-1+br^2)^{-2}\right) \\
&+ \beta(p-1+br^2)^{-\frac{1}{p-1}}\left(-2\tilde{A}_1br^2(p-1+br^2)^{-\frac{(2p-1)}{p-1}} - 4\tilde{B}_1br^2(p-1+br^2)^{-\frac{3p-2}{p-1}}\right) \\
&= \tilde{A}_1\beta(p-1+br^2)^{-\frac{p}{p-1}} + \tilde{B}_1\beta(p-1+br^2)^{-\frac{(2p-1)}{p-1}} \\
&- 2\beta\tilde{A}_1br^2(p-1+br^2)^{-\frac{(2p-1)}{p-1}} - 4\beta\tilde{B}_1br^2(p-1+br^2)^{-\frac{(3p-2)}{p-1}} \\
\beta R_1\varphi_0'' &= \beta(A_1+B_1r^2)(p-1+br^2)^{-\frac{p}{p-1}}\left(\tilde{A}_0(p-1+br^2)^{-1} - 2\tilde{A}_0br^2(p-1+br^2)^{-2}\right) \\
&= (\beta\tilde{A}_0A_1 + \beta\tilde{A}_0B_1r^2)(p-1+br^2)^{-\frac{(2p-1)}{p-1}} - (2\beta\tilde{A}_0A_1br^2 + 2\beta\tilde{A}_0B_1br^4)(p-1+br^2)^{-\frac{(3p-2)}{p-1}} \\
\frac{1}{4}R_0'r &= -\frac{br^2}{2(p-1)}(p-1+br^2)^{-\frac{p}{p-1}}.
\end{aligned}
$$

$$
\frac{p(p-1)}{2}|R_0|^{p-3}R_0R_1^2 = \frac{p(p-1)}{2}(A_1+B_1r^2)^2(p-1+br^2)^{-\frac{(3p-2)}{p-1}}.
$$

In the following, we decompse F_3 as:

$$
F_3 = Q_1(r) + Q_2(r) + Q_3(r),
$$

Where

$$\begin{aligned}
Q_1 &= (p-1+br^2)^{-\frac{p}{p-1}}\left(2B_1-\tilde{A}_1\beta-\frac{br^2}{2(p-1)}\right)\\
&= Q_{1,1}(p-1+br^2)^{-\frac{p}{p-1}}+Q_{1,2}r^2(p-1+br^2)^{-\frac{p}{p-1}}.\\
Q_2 &= (p-1+br^2)^{-\frac{(2p-1)}{p-1}}\left(-\frac{2pbA_1}{p-1}-\beta\tilde{B}_1-\beta\tilde{A}_0A_1\right)\\
&+ (p-1+br^2)^{-\frac{(2p-1)}{p-1}}\left(-\frac{10pB_1b}{p-1}-2\tilde{A}_0\tilde{A}_1+\frac{4\beta\tilde{A}_1b}{p-1}-4\beta\tilde{A}_0B_1+2\beta\tilde{A}_1b-\beta\tilde{A}_0B_1\right)r^2\\
&= Q_{2,1}(p-1+br^2)^{-\frac{(2p-1)}{p-1}}+Q_{2,2}r^2(p-1+br^2)^{-\frac{(2p-1)}{p-1}}.\\
Q_3 &= (p-1+br^2)^{-\frac{3p-2}{p-1}}\left(\frac{p(p-1)}{2}A_1^2\right)\\
&+ (p-1+br^2)^{-\frac{3p-2}{p-1}}r^2\left(\frac{4p(2p-1)A_1b^2}{(p-1)^2}-2\tilde{A}_0\tilde{B}_1-\tilde{A}_0^2A_1\right)\\
&+ (p-1+br^2)^{-\frac{3p-2}{p-1}}r^2\left(\frac{4p\beta\tilde{B}_1b}{p-1}+\frac{2(3p-1)\beta\tilde{A}_0A_1b}{p-1}+p(p-1)A_1B_1\right)\\
&+ (p-1+br^2)^{-\frac{3p-2}{p-1}}r^4\\
&= Q_{3,1}(p-1+br^2)^{-\frac{3p-2}{p-1}}+Q_{3,2}r^2(p-1+br^2)^{-\frac{3p-2}{p-1}}+Q_{3,3}r^4(p-1+br^2)^{-\frac{3p-2}{p-1}}.
\end{aligned}$$

By using variation of constant, we deduce that

$$R_2 = H^{-1}(r)\left(\int F_3\frac{2H}{r}H dr\right),$$

where

$$H(r)=\frac{(p-1+br^2)^{\frac{p}{p-1}}}{r^2}.$$

We see that the

$$\frac{2H}{r}F_3 = 2\left(Q_{1,2}-\frac{bQ_{2,1}}{(p-1)^2}+\frac{Q_{2,2}}{p-1}-\frac{2bQ_{3,1}}{(p-1)^3}+\frac{Q_{3,2}}{(p-1)^2}\right)\frac{1}{r}+\text{ integrable term }.$$

As a matter of fact, we always find R_2 such that ΔR_2 is continuous at 0. This implies that R_2 may not contain terms $\log|r|$, consequently, we obtain,

$$P = Q_{1,2}-\frac{bQ_{2,1}}{(p-1)^2}+\frac{Q_{2,2}}{p-1}-\frac{2bQ_{3,1}}{(p-1)^3}+\frac{Q_{3,2}}{(p-1)^2}=0. \tag{177}$$

We recall that

$$\begin{aligned}
A_1 &= -\frac{2b(\delta\beta-1)}{p-1} \text{ and } B_1=\mathcal{C},\\
\tilde{A}_0 &= -\frac{2\delta b}{p-1},\\
\tilde{A}_1 &= \frac{4\beta(1+\delta^2)b^2}{(p-1)^2} \text{ and } \tilde{B}_1=\frac{4b^2}{(p-1)^2}\big((p+3)\delta+\beta(2p+\delta^2(p-3))\big)+2(p-1)\delta\mathcal{C}.
\end{aligned}$$

We also define $\mathcal{C}_1 = 2(p-1)\delta\mathcal{C}$. Then,

$$\tilde{B}_1 = \frac{4b^2}{(p-1)^2}\big((p+3)\delta + \beta(2p + \delta^2(p-3))\big) + \mathcal{C}_1$$

$$\begin{aligned}
Q_{1,2} &= -\frac{b}{2(p-1)}, \\
Q_{2,1} &= -\frac{2pbA_1}{p-1} - \beta\tilde{B}_1 - \beta\tilde{A}_0 A_1 \\
&= -\frac{2p}{p-1} b\left(-\frac{2b(\delta\beta-1)}{p-1}\right) - \beta\left(\frac{4b^2}{(p-1)^2}\big((p+3)\delta + \beta(2p+\delta^2(p-3))\big) + \mathcal{C}_1\right) \\
&- \beta\left(-\frac{2\delta b}{p-1}\right)\left(-\frac{2b(\delta\beta-1)}{p-1}\right) \\
&= \frac{4p(\delta\beta-1)b^2}{(p-1)^2} - \frac{4\delta\beta(\delta\beta-1)b^2}{(p-1)^2} - \frac{4b^2}{(p-1)^2}\big((p+3)\delta\beta + \beta^2(2p+(p-3)\delta^2)\big) - \beta\mathcal{C}_1 \\
&= \frac{4b^2}{(p-1)^2}\big(p(\delta\beta-1) - \delta\beta(\delta\beta-1) - (p+3)\delta\beta - \beta^2(2p+(p-3)\delta^2)\big) - \beta\mathcal{C}_1 \\
&= \frac{4b^2}{(p-1)^2}\big((2-p)\delta^2\beta^2 - 2\delta\beta - p - 2p\beta^2\big) - 2(p-1)\delta\beta\mathcal{C}
\end{aligned}$$

and

$$\begin{aligned}
Q_{2,2} &= -\frac{10pB_1 b}{p-1} - 2\tilde{A}_0\tilde{A}_1 + \frac{4\beta\tilde{A}_1 b}{p-1} - 4\beta\tilde{A}_0 B_1 + 2\beta\tilde{A}_1 b - \beta\tilde{A}_0 B_1 \\
&= -\frac{10pbB_1}{p-1} + \frac{2(p+1)\beta\tilde{A}_1 b}{p-1} - 5\beta\tilde{A}_0 B_1 - 2\tilde{A}_0\tilde{A}_1 \\
&= -\frac{10pb}{p-1}\mathcal{C} + \frac{2(p+1)}{p-1}\beta\left(\frac{4\beta(1+\delta^2)b^2}{(p-1)^2}\right) b - 5\beta\left(-\frac{2\delta b}{p-1}\right)\mathcal{C} \\
&- 2\left(-\frac{2\delta b}{p-1}\right)\left(\frac{4\beta(1+\delta^2)b^2}{(p-1)^2}\right) \\
&= \frac{b^3}{(p-1)^3}\big(8(p+1)\beta^2(1+\delta^2) + 16\delta\beta(1+\delta^2)\big) + \left(\frac{10\delta\beta b}{p-1} - \frac{10pb}{p-1}\right)\mathcal{C} \\
&= \frac{b^3}{(p-1)^3}\big(16\delta^3\beta + 8(p+1)\delta^2\beta^2 + 8(p+1)\beta^2 + 16\delta\beta\big) + \left(\frac{10\delta\beta b}{p-1} - \frac{10pb}{p-1}\right)\mathcal{C} \\
Q_{3,1} &= \frac{2p(\delta\beta-1)^2 b^2}{p-1} = \frac{b^2}{p-1}(2p(\delta\beta-1)^2),
\end{aligned}$$

and

$$
\begin{aligned}
Q_{3,2} &= \frac{4p(2p-1)A_1b^2}{(p-1)^2} - 2\tilde{A}_0\tilde{B}_1 - \tilde{A}_0^2A_1 + \frac{4p\beta\tilde{B}_1b}{p-1} + \frac{2(3p-1)\beta\tilde{A}_0A_1b}{p-1} + p(p-1)A_1B_1 \\
&= \frac{4p(2p-1)}{(p-1)^2}\left(-\frac{2b(\delta\beta-1)}{p-1}\right)b^2 - 2\left(-\frac{2\delta b}{p-1}\right)\left\{\frac{4b^2}{(p-1)^2}((p+3)\delta+\beta(2p+(p-3)\delta)) + \mathcal{C}_1\right\} \\
&- \left(-\frac{2\delta b}{p-1}\right)^2\left(-\frac{2b(\delta\beta-1)}{p-1}\right) + \frac{4p\beta b}{p-1}\left\{\frac{4b^2}{(p-1)^2}\left((p+3)\delta+\beta(2p+(p-3)\delta^2)\right) + \mathcal{C}_1\right\} \\
&+ \frac{2(3p-1)\beta}{p-1}\left(-\frac{2\delta b}{p-1}\right)\left(-\frac{2b(\delta\beta-1)}{p-1}\right)b + p(p-1)\left(-\frac{2b(\delta\beta-1)}{p-1}\right)\mathcal{C} \\
&= -\frac{8p(2p-1)(\delta\beta-1)b^3}{(p-1)^3} + \frac{16\delta b^3}{(p-1)^3}\left((p+3)\delta+\beta(2p+(p-3)\delta^2)\right) + \frac{4\delta b}{p-1}\mathcal{C}_1 + \frac{8\delta^2(\delta\beta-1)b^3}{(p-1)^3} \\
&+ \frac{16p\beta b^3}{(p-1)^3}\left((p+3)\delta+\beta(2p+(p-3)\delta^2)\right) + \frac{4p\beta b}{p-1}\mathcal{C}_1 + \frac{8(3p-1)\delta\beta(\delta\beta-1)b^3}{(p-1)^3} - 2p(\delta\beta-1)b\mathcal{C} \\
&= \frac{b^3}{(p-1)^3}\big[-8p(2p-1)(\delta\beta-1) + 16(p+3)\delta^2 + 16\delta\beta(2p+(p-3)\delta^2) + 8\delta^2(\delta\beta-1) \\
&+ 16p(p+3)\delta\beta + 16p\beta^2(2p+(p-3)\delta^2) + 8(3p-1)\delta\beta(\delta\beta-1)\big] \\
&+ \frac{4\delta b}{p-1}\mathcal{C}_1 + \frac{4p\beta b}{p-1}\mathcal{C}_1 - 2p(\delta\beta-1)b\mathcal{C} \\
&= \frac{b^3}{(p-1)^3}\big[\delta^3\beta(16p-40) + \delta^2\beta^2(16p^2-24p-8) + \delta^2(16p+40) \\
&+ 32p^2\beta^2 + \delta\beta(64p+8) + 8p(2p-1)\big] \\
&+ b(8\delta^2 + 6p\delta\beta + 2p)\mathcal{C}
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
P &= Q_{1,2} - \frac{bQ_{2,1}}{(p-1)^2} + \frac{Q_{2,2}}{p-1} - \frac{2bQ_{3,1}}{(p-1)^3} + \frac{Q_{3,2}}{(p-1)^2} \\
&= -\frac{b}{2(p-1)} - \frac{b}{(p-1)^2}\left\{\frac{4b^2}{(p-1)^2}[(2-p)\delta^2\beta^2 - 2p\beta^2 - 2\delta\beta - p] - 2(p-1)\delta\beta\mathcal{C}\right\} \\
&+ \frac{1}{p-1}\left\{\frac{b^3}{(p-1)^3}[16\delta^3\beta + 8(p+1)\delta^2\beta^2 + 8(p+1)\beta^2 + 16\delta\beta] + \left(\frac{10\delta\beta b - 10pb}{p-1}\right)\mathcal{C}\right\} \\
&- \frac{2b}{(p-1)^3}\left\{\frac{b^2}{p-1}(2p\delta^2\beta^2 - 4p\delta\beta + 2p)\right\} \\
&+ \frac{1}{(p-1)^2}\big\{[(16p-40)\delta^3\beta + (16p^2-24p-8)\delta^2\beta^2 + (16p+40)\delta^2 + 32p^2\beta^2 + (64p+8)\delta\beta + 16p^2 - 8p] \\
&\times \frac{b^3}{(p-1)^3} + b(8\delta^2 + 6p\delta\beta + 2p)\mathcal{C}\big\}
\end{aligned}
$$

We conclude that $P=0$, gives us the constant b as follows;

$$\begin{aligned}
\frac{b}{2(p-1)} &= \frac{b^3}{(p-1)^5}\{-4(p-1)[(2-p)\delta^2\beta^2-2p\beta^2-2\delta\beta-p] \\
&+ (p-1)[16\delta^3\beta+8(p+1)\delta^2\beta^2+8(p+1)\beta^2+16\delta\beta] \\
&- 2(p-1)(2p\delta^2\beta^2-4p\delta\beta+2p) \\
&+ [(16p-40)\delta^3\beta+(16p^2-24p-8)\delta^2\beta^2+(16p+40)\delta^2+32p^2\beta^2+(64p+8)\delta\beta+16p^2-8p]\} \\
&+ \frac{\mathcal{C}}{(p-1)^2}\left(10\delta\beta b-10pb+2(p-1)\delta\beta b+b(8\delta^2+6p\delta\beta+2p)\right) \\
&= \frac{b^3}{(p-1)^5}\left[\delta^3\beta(32p-56)+\delta^2\beta^2(24p^2-32p-8)+\beta^2(48p^2-8p-8)+\delta^2(16p+40)\right. \\
&+ \left.\delta\beta(8p^2+80p-16)+16p^2-8p\right]..
\end{aligned}$$

By the critical condition we can write $\beta=\frac{p-\delta^2}{(p+1)\delta}$ and we obtain

$$\begin{aligned}
\frac{b}{2(p-1)} &= \frac{b^5}{(p-1)^5}\left[\delta^3\frac{p-\delta^2}{(p+1)\delta}(32p-56)+\delta^2\left(\frac{p-\delta^2}{(p+1)\delta}\right)^2(24p^2-32p-8)\right. \\
&+ \left(\frac{p-\delta^2}{(p+1)\delta}\right)^2(48p^2-8p-8) \\
&+ \left.\delta^2(16p+40)+\delta\frac{p-\delta^2}{(p+1)\delta}(8p^2+80p-16)+16p^2-8p\right] \\
&= \frac{b^5}{(p+1)^2\delta^2(p-1)^5}\left[\delta^6(-8p^2-8p+48)+\delta^4(-8p^3+72p^2-16p+48)\right. \\
&+ \left.\delta^2(48p^4-16p^3+72p^2-8p)+p^2(48p^2-8p-8)\right].
\end{aligned}$$

From the determination of μ

The equation $F_4(0)=0$ gives us the value of μ as follows which yields

$$\begin{aligned}
\mu &= \varphi_1''(0)-2\beta\varphi_0'(0)\varphi_1'(0)+2R_0^{-1}(0)R_0'(0)\varphi_1'(0)+2R_0^{-1}(0)R_1'(0)\varphi_0'(0) \\
&- 2R_0^{-2}(0)R_0'(0)\varphi_0'(0)R_1(0)+\beta R_0^{-1}(0)R_1''(0)-\beta R_0^{-2}R_0''(0)R_1(0) \\
&+ \delta(p-1)R_0^{p-2}(0)R_2(0)+\frac{\delta(p-1)(p-2)}{2}R_0^{p-3}(0)R_1^2(0).
\end{aligned}$$

+ The calculation of $\varphi_1''(0)$:

$$\begin{aligned}
\varphi_1''(0) &= \frac{4\beta(1+\delta^2)b^2}{(p-1)^3}+\frac{4b^2}{(p-1)^4}\{(p+3)\delta+\beta[2p+\delta^2(p-3)]\}+\frac{2\delta\mathcal{C}}{p-1} \\
&= \frac{4b^2}{(p-1)^4}\{(p+3)\delta+\beta[3p-1+\delta^2(2p-4)]\}+\frac{2\delta\mathcal{C}}{p-1}.
\end{aligned}$$

+ The calculation of $-2\beta\varphi_0'(0)\varphi_1'(0)$:

$$-2\beta\varphi_0'(0)\varphi_1'(0) = 0, \text{ because of } \varphi_0'(0)=0.$$

+ The calculation of $2R_0^{-1}(0)R_0'(0)\varphi_1'(0)$:

$$2R_0^{-1}(0)R_0'(0)\varphi_1'(0) = 0, \text{ because of } R_0'(0) = 0.$$

+ The calculation $2R_0^{-1}(0)R_1'(0)\varphi_0'(0)$:

$$2R_0^{-1}(0)R_1'(0)\varphi_0'(0) = 0 \text{ because of } \varphi_0'(0) = 0.$$

+ The calculation of $-2R_0^{-2}(0)R_0'(0)\varphi_0'(0)R_1(0)$

$$-2R_0^{-2}(0)R_0'(0)\varphi_0'(0)R_1(0) = 0 \text{ because of } \varphi_0'(0) = 0.$$

+ The calculation of $\beta R_0^{-1}(0)R_1''(0)$

$$\begin{aligned}\beta R_0^{-1}(0)R_1''(0) &= \beta(p-1)^{\frac{1}{p-1}}\left[2\mathcal{C}(p-1)^{-\frac{p}{p-1}} + \frac{4p(\delta\beta-1)b^2}{(p-1)^4}(p-1)^{-\frac{1}{p-1}}\right]\\ &= \frac{4p\beta(\delta\beta-1)b^2}{(p-1)^4} + \frac{2\beta\mathcal{C}}{p-1}.\end{aligned}$$

+ The calculation of $-\beta R_0^{-2}R_0''(0)R_1(0)$:

$$\begin{aligned}-\beta R_0^{-2}(0)R_0''(0)R_1(0) &= -\beta(p-1)^{\frac{2}{p-1}}\left(-\frac{2b}{p-1}\right)(p-1)^{-\frac{p}{p-1}}\left(-\frac{(\delta\beta-1)2b}{p-1}\right)(p-1)^{-\frac{p}{p-1}}\\ &= -\frac{4\beta(\delta\beta-1)b^2}{(p-1)^4}.\end{aligned}$$

+ The calculation of $\delta(p-1)R_0^{p-2}(0)R_2(0)$: The calculation is more difficult, we first determine $R_2(0)$ .We derive from (176) that

$$R_2(0) = -F_3(0), \tag{178}$$

where F_3 is defined in (36). We now find $F_3(0)$

$$\begin{aligned}F_3(0) &= R_1''(0) - 2R_0(0)\varphi_0'(0) - R_1(0)[\varphi_0'(0)]^2 - 2\beta R_0'(0)\varphi_1'(0) - 2\beta R_1'(0)\varphi(0) - \beta R_0(0)\varphi_1''(0)\\ &- \beta R_1(0)\varphi_0''(0) + \frac{p(p-1)}{2}R_0^{p-2}(0)R_1^2(0)\\ &= R_1''(0) - \beta R_0(0)\varphi_1''(0) - \beta R_1(0)\varphi_0''(0) + \frac{p(p-1)}{2}R_0^{p-2}(0)R_1^2(0).\end{aligned}$$

We have

$$
\begin{aligned}
& R_1''(0) = \frac{4p(\beta\delta-1)b^2}{(p-1)^4}(p-1)^{-\frac{1}{p-1}} + \frac{2\mathcal{C}}{p-1}(p-1)^{-\frac{1}{p-1}}, \\
& -\beta R_0(0)\varphi_1''(0) = -\left[\tilde{A}_1\beta(p-1)^{-\frac{p}{p-1}} + \tilde{B}_1\beta(p-1)^{-\frac{2p-1}{p-1}}\right] \\
= \; & -\left[\frac{4\beta^2(1+\delta^2)b^2}{(p-1)^3} + \frac{4b^2}{(p-1)^4}((p+3)\delta\beta + \beta^2[2p+\delta^2(p-3)]) + \frac{2\delta\beta\mathcal{C}}{p-1}\right](p-1)^{-\frac{1}{p-1}}, \\
& -\beta R_1(0)\varphi_0''(0) = -\left[\beta\tilde{A}_0 A_1(p-1)^{-\frac{2p-1}{p-1}}\right] \\
= \; & -\left[\frac{4\delta\beta(\delta\beta-1)b^2}{(p-1)^4}\right](p-1)^{-\frac{1}{p-1}}, \\
& \frac{p(p-1)}{2}R_0^{p-2}(0)R_1^2(0) = \frac{p(p-1)}{2}A_1^2(p-1)^{-\frac{3p-2}{p-1}} \\
= \; & \frac{2p(\delta\beta-1)^2}{(p-1)^4}(p-1)^{-\frac{1}{p-1}}.
\end{aligned}
$$

Then, we obtain the following

$$
\begin{aligned}
& F_3(0) = R_1''(0) - \beta R_0(0)\varphi_1''(0) - \beta R_1(0)\varphi_0''(0) + \frac{p(p-1)}{2}R_0^{p-2}(0)R_1^2(0) \\
= \; & \frac{4p(\delta\beta-1)b^2}{(p-1)^4}(p-1)^{-\frac{1}{p-1}} + \frac{2\mathcal{C}}{p-1}(p-1)^{-\frac{1}{p-1}} \\
- \; & \frac{4\beta^2(1+\delta^2)b^2}{(p-1)^3}(p-1)^{-\frac{1}{p-1}} - \frac{4b^2}{(p-1)^4}((p+3)\delta\beta + \beta^2[2P+\delta^2(p-3)])(p-1)^{-\frac{1}{p-1}} \\
- \; & \frac{2\delta\beta\mathcal{C}}{p-1}(p-1)^{-\frac{1}{p-1}} - \frac{4\delta\beta(\delta\beta-1)b^2}{(p-1)^4}(p-1)^{-\frac{1}{p-1}} + \frac{2p(\delta\beta-1)^2b^2}{(p-1)^4}(p-1)^{-\frac{1}{p-1}} \\
= \; & \left[\frac{2\mathcal{C}}{p-1} - \frac{2\delta\beta\mathcal{C}}{p-1}\right](p-1)^{-\frac{1}{p-1}} \\
+ \; & \left\{4p(\delta\beta-1) - 4(p-1)\beta^2(1+\delta^2) - 4((p+3)\delta\beta + \beta^2[2p+\delta^2(p-3)]) - 4\delta\beta(\delta\beta-1) + 2p(\delta\beta-1)^2\right\} \\
\times \; & \frac{b^2}{(p-1)^4}(p-1)^{-\frac{1}{p-1}} \\
= \; & \left[\frac{2\mathcal{C}}{p-1} - \frac{2\delta\beta\mathcal{C}}{p-1}\right](p-1)^{-\frac{1}{p-1}} \\
+ \; & \frac{b^2}{(p-1)^4}((\delta\beta-1)(2p+(2p-4)\delta\beta) - 4(p+3)\delta\beta - \beta^2[12p-4+\delta^2(8p-16)])(p-1)^{-\frac{1}{p-1}}.
\end{aligned}
$$

We then get from equation $F_4(0) = 0$:

$$
\begin{aligned}
R_2(0) \quad = \quad & -F_3(0) = \left[\frac{2\delta\beta\mathcal{C}}{p-1} - \frac{2\mathcal{C}}{p-1}\right](p-1)^{-\frac{1}{p-1}} \\
+ \quad & \frac{b^2}{(p-1)^4}\{4(p+3)\delta\beta + \beta^2[12p-4+\delta^2(8p-16)] - (\delta\beta-1)(2p+(2p-4)\delta\beta)\}(p-1)^{-\frac{1}{p-1}}
\end{aligned}
$$

Therefore, we get the following

$$
\begin{aligned}
&\delta(p-1)R_0^{p-2}(0)R_2(0) = \frac{2\delta^2\beta\mathcal{C}}{p-1} - \frac{2\delta\mathcal{C}}{p-1} \\
+ \quad & \frac{b^2}{(p-1)^4}\{4(p+3)\delta^2\beta + \delta\beta^2[12p-4+\delta^2(8p-16)] - (\delta\beta-1)(2p\delta+(2p-4)\delta^2\beta)\}.
\end{aligned}
$$

+ The calculation of $\frac{\delta(p-1)(p-2)}{2}R_0^{p-3}(0)R_1^2(0)$. We have

$$
\begin{aligned}
\frac{\delta(p-1)(p-2)}{2}R_0^{p-3}(0)R_1^2(0) &= \frac{\delta(p-1)(p-2)}{2}(p-1)^{-\frac{p-3}{p-1}}\left(-\frac{2b(\delta\beta-1)}{p-1}(p-1)^{-\frac{p}{p-1}}\right)^2 \\
&= \frac{2(p-2)\delta(\delta\beta-1)^2b^2}{(p-1)^4}.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&\mu = \varphi_1''(0) + \beta R_0^{-1}R_1''(0) - \beta R_0'' R_0^{-2}R_1(0) + \delta(p-1)R_0^{p-2}R_2(0) + \frac{\delta(p-1)(p-2)}{2}R_0^{p-3}R_1^2(0) \\
= \quad & \frac{b^2}{(p-1)^4}\left\{4(p+3)\delta + 4(3p-1)\beta + 4(2p-4)\delta^2\beta + 4p\beta^2\delta - 4p\beta - 4\beta^2\delta + 4\beta + 4(p+3)\delta^2\beta\right. \\
+ \quad & (12p-4)\delta\beta^2 + \delta^3\beta^2(8p-16) - 2p\delta^2\beta - (2p-4)\delta^3\beta^2 + 2p\delta - (2p-4)\delta^2\beta + 2(p-2)\delta^3\beta^2 \\
- \quad & \left.4(p-2)\delta^2\beta + 2(p-2)\delta\right\} + \frac{2\beta(1+\delta^2)\mathcal{C}}{p-1} \\
= \quad & \frac{b^2}{(p-1)^4}\left\{8(p+1)\delta + 8p\beta + (4p+8)\delta^2\beta + (16p-8)\delta\beta^2 + (8p-16)\delta^3\beta^2\right\} + \frac{2\beta(1+\delta^2)\mathcal{C}}{p-1}.
\end{aligned}
$$

F Cancellation of some coefficient in the ODE of $\tilde{q}_2$

In this part, we aim at giving some details of the computation some qualities related to the construction $\tilde{q}_2$'s ODE.

a) The cancellation of (137): We now rewrite this one as follows

$$
\frac{\nu}{2}R^*_{2,1} + \tilde{C}_{2,2}R^*_{2,1} - \tilde{D}_{2,0}\tilde{R}^*_{0,1} - \frac{\delta b}{(p-1)^2}R^*_{0,1} + \tilde{R}^*_{2,2} = 0.
$$

Using the definitions of the constants in Appendices B and D we can derive the right hand side as follows

$$
\begin{aligned}
&\frac{\nu}{2}R^*_{2,1} + \tilde{C}_{2,2}R^*_{2,1} - \tilde{D}_{2,0}\tilde{R}^*_{0,1} - \frac{\delta b}{(p-1)^2}R^*_{0,1} + \tilde{R}^*_{2,2} \\
= \quad & -\frac{\kappa b}{2(p-1)^2} \\
- \quad & \frac{4b^3\kappa}{(p-1)^6}\left\{-12\beta\delta p - 9\beta^3\delta p + 3\beta^3\delta^3 p\right. \\
& -3p\beta^2\delta^4 - 2p^2\beta\delta - 12p^2\beta^3\delta - 5\delta^2\beta^2p^2 - 16\delta^2p\beta^2 + 2\beta^2 - 5p^2 \\
+ \quad & \left.4\delta^4 - 13\delta^2 + 5p + \delta\beta + 8\beta\delta^3 + 3\beta^3\delta - 5\delta^2\beta^2 - 3\delta^5\beta + 3\beta^3\delta^3 - p\beta^2 - 7p\delta^2\right\}.
\end{aligned}
$$

In addition to that, using the critical condition

$$p-(p+1)\delta\beta-\delta^2=0,$$

then, we derive the following

$$\begin{aligned}
0 &= \frac{\nu}{2}R^*_{2,1}+\tilde{C}_{2,2}R^*_{2,1}-\tilde{D}_{2,0}\tilde{R}^*_{0,1}-\frac{\delta b}{(p-1)^2}R^*_{0,1}+\tilde{R}^*_{2,2}\\
&= -\frac{\kappa b}{2(p-1)^2}\\
&- \frac{8b^3\kappa}{(p-1)^6(p+1)^2\delta^2}(1+\delta^2)((p^2+p-6)\delta^4+(p^3-10p^2+p)\delta^2+p^3-6p^4+p^2).
\end{aligned}$$

This yields

$$b^2 = \frac{(p-1)^4(p+1)^2\delta^2}{-16(1+\delta^2)L},$$

where

$$L = (p^2+p-6)\delta^4+p(p^2-10p+1)\delta^2+p^2(-6p^2+p+1).$$

b/ Explain the decomposition $\tilde{\mathcal{A}}_2(\tilde{H}_1+1)+\tilde{H}_2$. Indeed, it is enough to prove that there no μ in $\tilde{H}_2$. Indeed, we recall formula of $\tilde{H}_2$ as follows:

$$\begin{aligned}
\tilde{H}_2 &= \frac{\nu}{2}\left[X_2+c_4[\tilde{C}_{4,2}R^*_{2,1}+\tilde{R}^*_{4,2}]-D_{2,0}\tilde{R}_{0,1}\right]-\frac{\nu}{2}\left[\tilde{K}_{2,4}\left(\frac{C_{4,2}R^*_{2,1}}{2}+\frac{R^*_{4,2}}{2}\right)\right]\\
&- \frac{\nu}{2}\left[\tilde{L}_{2,4}\left(\tilde{C}_{4,2}R^*_{2,1}+\tilde{R}^*_{4,2}\right)\right]+\mu R^*_{2,1}+\frac{R^*_{0,1}R^*_{2,1}}{\kappa}\\
&+ \tilde{D}_{2,0}\left[-\tilde{X}_0+\frac{\nu\tilde{K}_{0,2}R^*_{2,1}}{2}-\tilde{C}_{0,2}R^*_{2,1}\right]+\tilde{C}_{2,2}\left[X_2+c_4(\tilde{C}_{4,2}R^*_{2,1}+\tilde{R}_{4,2})-D_{2,0}\tilde{R}^*_{0,1}\right]\\
&+ \tilde{C}_{2,4}\left[\frac{C_{4,2}R^*_{2,1}}{2}+\frac{R^*_{4,2}}{2}\right]+\tilde{D}_{2,4}(\tilde{C}_{4,2}R^*_{2,1}+\tilde{R}^*_{4,2})+\tilde{E}_{2,2}R^*_{2,1}-\tilde{F}_{2,0}\tilde{R}^*_{0,1}\\
&+ \frac{1}{8\kappa}(32-64\delta\beta)(R^*_{2,1})^2\\
&- \frac{\delta b}{(p-1)^2}\left[\frac{\nu}{2}(1+\delta^2)\tilde{R}^*_{0,1}-\frac{\nu}{2}K_{0,2}R^*_{2,1}-D_{0,0}\tilde{R}^*_{0,1}+C_{0,2}R^*_{2,1}+R^*_{0,2}+\frac{\Theta^*_{0,0}R^*_{0,1}}{\kappa}\right]\\
&+ \tilde{R}^*_{2,3}+\frac{(p+1)\delta[12-6\delta\beta+6\beta^2]R^*_{0,1}b^2}{2(p-1)^4}.
\end{aligned}$$

Using the definitions of constants in $\tilde{H}_2$, we can write

$$\begin{aligned}
\tilde{H}_2 = &\frac{\nu}{2}X_2+\mu R^*_{2,1}+\frac{R^*_{0,1}R^*_{2,1}}{\kappa}-\tilde{D}_{2,0}\tilde{X}_0+\tilde{C}_{2,2}X_2-\frac{\delta b}{(p-1)^2}R^*_{0,2}-\frac{\delta b}{(p-1)^2}\frac{\Theta^*_{0,0}R^*_{0,1}}{\kappa}\\
&+ \tilde{R}^*_{2,3}+\frac{(p+1)\delta[12-6\delta\beta+6\beta^2]R^*_{0,1}b^2}{2(p-1)^4}+B(p,\delta,\beta),
\end{aligned}$$

where B doesn't contain μ, and the first terms contain μ. We repeat the process of removing terms that do not contain, and we can get

$$\begin{aligned}
\tilde{H}_2 &= \frac{\nu}{2}\left[R^*_{2,2} - \mu\Theta^*_{2,0}\right] - \tilde{D}_{2,0}\left[\tilde{R}^*_{0,2} - \mu\tilde{\Theta}^*_{0,0}\right] + \tilde{C}_{2,2}\left[R^*_{2,2} - \mu\Theta^*_{2,0}\right] - \frac{\delta b}{(p-1)^2}R^*_{0,2} \\
&+ \frac{\delta b}{(p-1)^2}\mu\Theta^*_{0,0} + \tilde{R}^*_{2,3} - \frac{\mu\kappa b^2(p+1)\delta[12 - 6\delta\beta + 6\beta^2]}{2(p-1)^2} + B_1,
\end{aligned}$$

where B_1 doesn't contain μ. In addition to that, we have

$$\begin{aligned}
R^*_{2,2} &= = \mu\Theta^*_{2,0} + \text{`term no } \mu\text{'}, \\
\tilde{R}^*_{0,2} &= \mu\tilde{\Theta}^*_{0,0} + \text{`term no } \mu\text{'}, \\
R^*_{2,2} &= = \mu\Theta^*_{2,0} + \text{`term no } \mu\text{'}, \\
R^*_{0,2} &= \mu\Theta^*_{0,0} + \text{`term no } \mu\text{'}, \\
R^*_{2,3} &= \mu\tilde{\Theta}^*_{2,1} + \text{`term no } \mu\text{'}.
\end{aligned}$$

This implies the fact that $\tilde{H}_2$ doesn't contain μ. Besides that, for the critical condition, we have $\tilde{H}_1 \leq -\frac{3}{2}$. Thus, we can write

$$\tilde{\mathcal{A}}_2(\tilde{H}_1 + 1) + \tilde{H}_2 = a_0(p,\delta)\mu + a_1(p,\delta,\beta),$$

with $a_0 \neq 0$.

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