Coloring tournaments: from local to global

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Abstract

The chromatic number of a directed graph $D$ is the minimum number of colors needed to color the vertices of $D$ such that each color class of $D$ induces an acyclic subdigraph. Thus, the chromatic number of a tournament $T$ is the minimum number of transitive subtournaments which cover the vertex set of $T$. We show in this paper that tournaments are significantly simpler than graphs with respect to coloring. Indeed, while undirected graphs can be altogether “locally simple” (every neighborhood is a stable set) and have large chromatic number, we show that locally simple tournaments are indeed simple. In particular, there is a function $f$ such that if the out-neighborhood of every vertex in a tournament $T$ has chromatic number at most $c$, then $T$ has chromatic number at most $f(c)$. This answers a question of Berger et al.

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1 Introduction

A directed graph is said to be acyclic if it does not contain any directed cycles. Given a loopless digraph $D$, a $k$-coloring of $D$ is a coloring of each of the vertices of $D$ with one of the colors from the set $\{1, \ldots, k\}$ such that each color class induces an acyclic subdigraph. The chromatic number $\chi(D)$ of $D$ is the smallest number $k$ for which $D$ admits a $k$-coloring. This digraph invariant was introduced by Neumann-Lara [13], and naturally generalizes many results on the graph chromatic number (see, for example, [4], [9], [10], [11], [12]). In this paper, we study the chromatic number of a class of tournaments where the out-neighborhood of every vertex has bounded chromatic number.

A tournament is a loopless digraph such that for every pair of distinct vertices $u, v$, exactly one of $uv, vu$ is an arc. Given a tournament $T$, a subset $X$ of $V(T)$ is transitive if the subtournament of $T$ induced by $X$ contains no directed cycle. Thus, $\chi(T)$ is the minimum $k$ such that $V(T)$ can be colored with $k$ colors where each color class is a transitive set. The coloring of tournaments has close relationship with the celebrated Erdős–Hajnal conjecture (cf. [1, 8]) and has been studied in [3, 5, 6, 2, 7].

Given $t \geq 1$, a tournament $T$ is $t$-local if for every vertex $v$, the subtournament of $T$ induced by the set of out-neighbors of $v$ has chromatic number at most $t$. The following conjecture was raised in [3] (Conjecture 2.6) and settled for $t = 2$ in [7].

**Conjecture 1.** There is a function $f$ such that every $t$-local tournament $T$ satisfies $\chi(T) \leq f(t)$.

The goal of this note is to provide a proof of Conjecture 1 for all $t$.

Given a set $S \subset V(T)$, we say that $S$ is a dominating set of $T$ if every vertex in $V \setminus S$ has an in-neighbor in $S$. The dominating number $\gamma(T)$ of a tournament $T$ is the smallest number $k$ such that $T$ has a dominating set of size $k$. The main tool to prove Conjecture 1 is the following theorem, which seems more interesting than our original goal.

**Theorem 2.** For every integer $k \geq 1$, there exist integers $K$ and $\ell$ such that every tournament $T$ with dominating number at least $K$ contains a subtournament on $\ell$ vertices and chromatic number at least $k$.

Roughly speaking, Theorem 2 asserts that if the dominating number of a tournament is sufficiently large, then it contains a bounded-size subtournament with large chromatic number. One may ask whether high dominating number is enough to force an induced copy of a specific (high chromatic number) subtournament. The following tournaments may be potential candidates. Let $S_1$ be the tournament with a single vertex. For every $i > 1$,
let $S_i$ be the tournament (with $2^i - 1$ vertices) obtained by blowing up two vertices of an oriented triangle into two copies of $S_{i-1}$. It is easy to check that $\chi(S_i) \geq i$. The following problem is trivial for $i \leq 2$ and verified for $i = 3$ in [7], while still open for all $i \geq 4$.

**Problem 3.** For every integer $i \geq 1$, there exist $f(i)$ such that every tournament $T$ with dominating number at least $f(i)$ contains an isomorphic copy of $S_i$.

On another note, it is natural to ask whether Theorem 2 still holds with a weaker hypothesis. In particular, is it true that for every $k$, if the chromatic number of a tournament is huge, then it contains a bounded-size subtournament with chromatic number at least $k$? Unfortunately, the answer is negative for any $k \geq 3$. It is well-known that for any $\ell$, there is an undirected simple graph $G$ with arbitrarily high chromatic number and girth at least $\ell + 1$. We fix an arbitrary enumeration of vertices of $G$ and create a tournament $T$ as follows: If $ij$ with $i < j$ is an edge of $G$ then $ij$ is an arc of $T$; otherwise, $ji$ is an arc of $T$. Then $T$ has arbitrarily high chromatic number while every subtournament of $T$ of size $\ell$ has chromatic number at most 2. However, a similar question for dominating number is still open.

**Problem 4.** For every integer $k \geq 1$, there exist integers $K$ and $\ell$ such that every tournament $T$ with dominating number at least $K$ contains a subtournament with $\ell$ vertices and dominating number at least $k$.

## 2 Proof of Conjecture 1

For every vertex $v$ in a tournament $T$, we denote by $N^+_T(v)$ the set of out-neighbors of $v$ in $T$. Given a subset $X$ of $V(T)$, let $N^+_T(X)$ denote the union of all $N^+_T(v)$, for $v \in X$, and denote by $N^+_T[X] := X \cup N^+_T(X)$. For every subset $X$ of $V(T)$, let $\chi_T(X)$ denote the chromatic number of the subtournament of $T$ induced by $X$.

Given a tournament $T$ and a subset $X$ of $V(T)$, we say a set $R \subseteq V(T)$ (not necessary disjoint from $X$) is a dominating set of $X$ in $T$ if every vertex in $X \setminus R$ has an in-neighbor in $R$. The dominating number $\gamma_T(X)$ of $X$ in $T$ is the smallest number $k$ such that $X$ has a dominating set of size $k$. When it is clear in the context, we omit the subscript $T$ in the notation.

Let $T$ be a tournament and $X, Y \subseteq V(T)$. The following inequalities are straightforward:

\[ \gamma_T(N^+[X]) \leq |X|, \]  \hspace{1cm} (1)

and

\[ \gamma_T(Y) \leq \gamma_T(X) + \gamma_T(Y \setminus X). \]  \hspace{1cm} (2)
Let us restate Theorem 2.

**Theorem 5.** For every integer \( k \geq 1 \), there exist integers \( K \) and \( \ell \) such that every tournament \( T \) with \( \gamma(T) \geq K \) contains a subtournament \( A \) on \( \ell \) vertices and \( \overrightarrow{\chi}(A) \geq k \).

**Proof.** We proceed by induction on \( k \). The claim is trivial for \( k = 1 \). For \( k = 2 \), we can choose \( K = 2 \) and \( \ell = 3 \). Indeed, if a tournament \( T \) satisfies \( \gamma(T) \geq 2 \), then \( T \) is not transitive and thus it contains an oriented triangle \( A \) of size \( \ell = 3 \) and \( \overrightarrow{\chi}(A) \geq 2 \).

Assuming now that \( (K, \ell) \) exists for \( k \), we want to find \( (K', \ell') \) for \( k + 1 \). For this, we set \( K' := k(K + \ell + 1) + K \), and fix \( \ell ' \) later. Let \( T \) be a tournament such that \( \gamma(T) \geq K' \). Let \( D \) be a dominating set of \( T \) of minimum size. Consider a subset \( W \) of \( D \) of size \( k(K + \ell + 1) \). From (1) and (2) we have

\[
\gamma(V \setminus N^+[W]) \geq \gamma(T) - \gamma(N^+[W]) \geq K' - |W| \geq K,
\]

where \( V \) is the vertex set of \( T \). Thus by induction hypothesis applied to \( k \), one can find a set \( A \subseteq V \setminus N^+[W] \) such that \( A \) has \( \ell \) vertices and \( \overrightarrow{\chi}(A) \geq k \). Note that by construction, \( A \cap W = \emptyset \) and all arcs between \( A \) and \( W \) are directed from \( A \) to \( W \).

Consider now a subset \( S \) of \( W \) of size \( K + \ell + 1 \). We claim that \( \gamma(N^+(S)) \geq K + \ell \). If not, we can choose a dominating set \( S' \) of \( N^+(S) \) of size at most \( K + \ell - 1 \). Note that \( x \) dominates \( S \) for any \( x \in A \), and so \( S' \cup \{x\} \) dominates \( N^+[S] \). Hence \( (D \setminus S) \cup S' \cup \{x\} \) would be a dominating set of \( T \) of size less than \( |D| \), which contradicts the minimality of \( |D| \). Therefore \( \gamma(N^+(S)) \geq K + \ell \).

Let \( N' \) be the set of vertices \( N^+(S) \setminus N^+(A) \). From (1) and (2) we have

\[
\gamma(N') \geq \gamma(N^+(S)) - \gamma(N^+(A)) \geq K + \ell - |A| = K.
\]

Thus by induction hypothesis applied to \( k \), there is a subset \( A_S \) of \( N' \) such that \( |A_S| = \ell \) and \( \overrightarrow{\chi}(A_S) \geq k \). Note that by construction, \( A_S \cap A = \emptyset \) and all arcs between \( A_S \) and \( A \) are directed from \( A_S \) to \( A \).

We now construct our subtournament of \( T \) with chromatic number at least \( k + 1 \). For this we consider the set of vertices \( A \cup W \) to which we add the collection of \( A_S \), for all subsets \( S \subseteq W \) of size \( K + \ell + 1 \). Call \( A' \) this new tournament and observe that its number of vertices is at most

\[
\ell' := \ell + k(K + \ell + 1) + \ell \left( \frac{k(K + \ell + 1)}{K + \ell + 1} \right).
\]

To conclude, it is sufficient to show that \( \overrightarrow{\chi}(A') \geq k + 1 \). Suppose not, and for contradiction, take a \( k \)-coloring of \( A' \). Since \( |W| = k(K + \ell + 1) \) there
is a monochromatic set \( S \) in \( W \) of size \( K + \ell + 1 \) (say, colored 1). Recall that we have all arcs from \( A_S \) to \( A \) and all arcs from \( A \) to \( S \), and note that since \( \chi(A) \geq k \) and \( \chi(A_S) \geq k \), both \( A \) and \( A_S \) have a vertex of each of the \( k \) colors. Hence there are \( u \in A \) and \( v \in A_S \) colored 1. Since \( A_S \subseteq N^+(S) \), there is \( v \in S \) such that \( vw \) is an arc. We then obtain the monochromatic cycle \( uvw \) of color 1, a contradiction. Thus, \( \chi(A') \geq k + 1 \), completing the proof.

We now show that Conjecture 1 is true.

**Theorem 6.** There is a function \( f \) such that every \( t \)-local tournament \( T \) satisfies \( \chi(T) \leq f(t) \).

**Proof.** Let \((K, \ell)\) satisfy Theorem 5 for \( k := t + 1 \). Let \( T \) be a \( t \)-local tournament. Thus, if \( \gamma(T) \geq K \) then \( T \) contains a set \( A \) of \( \ell \) vertices and \( \chi(A) \geq t + 1 \). If a vertex \( v \in V(T) \setminus A \) does not have an in-neighbor in \( A \), then \( A \subseteq N^+(v) \), and so \( t + 1 \leq \chi(A) \leq \chi(N^+(v)) \leq t \), a contradiction. Hence, \( A \) is a dominating set of \( T \). Note that

\[
\chi(N^+[v]) \leq \chi(N^+(v)) + \chi(\{v\}) \leq t + 1
\]

for every \( v \in V(T) \). Thus

\[
\chi(T) = \chi(N^+[A]) \leq \sum_{v \in A} \chi(N^+[v]) \leq (t + 1)|A| = (t + 1)\ell.
\]

Otherwise, \( \gamma(T) < K \). Let \( D \) be a dominating set of \( T \) with minimum size. Then

\[
\chi(T) = \chi(N^+[D]) \leq \sum_{v \in D} \chi(N^+[v]) \leq (t + 1)|D| < (t + 1)K.
\]

Consequently, \( t \)-local tournaments have chromatic number at most \( f(t) := \max((t + 1)K, (t + 1)\ell) \).

The implication of our result is that we are possibly missing a key-definition of what is a “large” (or “dense”) hypergraph (i.e., a set of subsets). It could be that for a suitable definition of “large” (for which “large” intersecting “large” would be “large”), we would obtain that for any tournament \( T \) on vertex set \( V \), the set of out-neighborhoods of vertices of \( T \) is “large”, and in addition the set of subsets of vertices of a \( K \)-chromatic tournament inducing at least chromatic number \( k \) is also “large”. Hence, if two large sets are intersecting in a non-empty way, one could find an out-neighborhood with chromatic number \( k \).
If such a notion would exist, it should decorrelate the two large sets (out-neighborhoods and $k$-chromatic), and thus imply the following: If $T_1, T_2$ are tournaments on the same set of vertices and $\chi(T_1)$ is huge, then there is a vertex $v$ such that $T_1$ induces on $N_{T_2}^+(v)$ a subtournament of large chromatic number. A very similar conjecture was proposed by Alex Scott and Paul Seymour.

**Conjecture 7.** [14] For every $k$, there exists $K$ such that if $T$ and $G$ are respectively a tournament and a graph on the same set of vertices with $G$ of chromatic number at least $K$, then there is a vertex $v$ such that $G$ induces on $N_T^+(v)$ a subgraph of $G$ of chromatic number at least $k$.

**References**


