Fractional diffusion limit for a kinetic equation with an interface*

Tomasz Komorowski† Stefano Olla‡ Lenya Ryzhik§

May 25, 2019

Abstract

We consider the limit of a linear kinetic equation, with reflection-transmission-absorption at an interface, with a degenerate scattering kernel. The equation arise from a microscopic chain of oscillators in contact with a heat bath. In the absence of the interface, the solutions exhibit a superdiffusive behavior in the long time limit. With the interface, the long time limit is the unique solution of a version of the fractional in space heat equation, with reflection-transmission-absorption at the interface. The limit problem corresponds to a certain stable process that is either absorbed, reflected, or transmitted upon crossing the interface.

Keywords: Diffusion Limits from Kinetic Equations, Fractional Laplacian, Stable Processes, Boundary Conditions at Interface

1 Introduction

We consider a linear phonon Boltzmann equation with an interface. This equation describes the evolution of the energy density $W(t,y,k)$ of phonons at time $t \geq 0$, spatial position $y \in \mathbb{R}$ and the frequency $k \in \mathbb{T} = [-1/2, 1/2]$ with identified endpoints. Outside the interface, located at $y = 0$, the density satisfies the kinetic equation

$$\begin{aligned} &\partial_t W(t,y,k) + \bar{\omega}'(k)\partial_y W(t,y,k) = \gamma_0 L_k W(t,y,k), \\ &W(0,y,k) = W_0(y,k). \end{aligned} \tag{1.1}$$

***Acknowledgement.** TK acknowledges the support of the National Science Centre under the NCN grant 2016/23/B/ST1/00492. LR was partially supported by the NSF grant DMS-1613603 and ONR grant N00014-17-1-2145. SO was partially supported by ANR-15-CE40-0020-01 grant LSD. TK expresses his gratitude to Prof. K. Bogdan for elucidating conversations on the subject of the paper and to the Department of Mathematics of the Technical University of Wrocław for its hospitability.

†Institute of Mathematics, Polish Academy Of Sciences, ul. Śniadeckich 8, 00-956 Warsaw, Poland, e-mail: `komorow@hektor.umcs.lublin.pl`

‡CEREMADE, UMR-CNRS, Université de Paris Dauphine, PSL Research University Place du Maréchal De Lattre De Tassigny, 75016 Paris, France, e-mail: `olla@ceremade.dauphine.fr`

§ Mathematics Department, Stanford University, Stanford, CA 94305, USA , email: `ryzhik@stanford.edu`

We denote by $\omega : \mathbb{T} \to [0, +\infty)$ the dispersion relation, and set the group velocity of the phonon $\bar{\omega}'(k) := \omega'(k)/(2\pi)$, $k \in \mathbb{T}$. The parameter $\gamma_0 > 0$ represents the phonon scattering rate, and the scattering operator L_k, acting only on the k-variable, is given by

$$L_k F(k) := \int_{\mathbb{T}} R(k, k') \left[F(k') - F(k)\right] dk', \quad k \in \mathbb{T}, \tag{1.2}$$

for a bounded and measurable function F.

When there is no interface, this is the Kolmogorov equation for a classical jump process. The interface conditions prescribe the outgoing phonon density in terms of what comes to the interface:

$$W(t, 0^+, k) = p_-(k) W(t, 0^+, -k) + p_+(k) W(t, 0^-, k) + T\mathfrak{g}(k), \text{ for } 0 < k \le 1/2, \tag{1.3}$$

and

$$W(t, 0^-, k) = p_-(k) W(t, 0^-, -k) + p_+(k) W(t, 0^+, k) + T\mathfrak{g}(k), \quad \text{for } -1/2 < k < 0, \tag{1.4}$$

with the energy balance

$$p_+(k) + p_-(k) + \mathfrak{g}(k) = 1. \tag{1.5}$$

Here, $p_-(k)$, $p_+(k)$ and $\mathfrak{g}(k)$ are, respectively, the probabilities of the phonon being reflected, transmitted or absorbed, while $T\mathfrak{g}(k)$ is the phonon production rate at the interface.

We assume that the absorption coefficient $\mathfrak{g}(k)$ and the reflection-transmission coefficients $p_\pm(k)$ are positive, continuous, even functions, satisfying (1.5) and such that

$$\lim_{k \to 0+} \mathfrak{g}(k) = \mathfrak{g}_0 > 0, \quad \lim_{k \to 0+} p_\pm(k) = p_\pm, \tag{1.6}$$

and there exist $C_0, \gamma > 0$ such that

$$|p_\pm(k) - p_\pm| \le C_0 |k|^\gamma, \quad k \in \mathbb{T}. \tag{1.7}$$

The large scale limit of the kinetic equation without an interface has been considered in [5, 13, 21]. The corresponding rescaled problem, with $N \to +\infty$, is

$$\begin{aligned} &\frac{1}{N^\alpha} \partial_t W_N(t, y, k) + \frac{1}{N} \bar{\omega}'(k) \partial_y W_N(t, y, k) = \gamma_0 L_k W_N(t, y, k), \\ &W_N(0, y, k) = W_0(y, k), \end{aligned} \tag{1.8}$$

with an appropriate exponent $\alpha > 0$, with $\alpha = 2$ corresponding to the classical diffusive scaling. An important feature of the phonon scattering is that the total scattering kernel

$$R(k) := \int_{\mathbb{T}} R(k, k') dk' \tag{1.9}$$

degenerates at $k = 0$ – phonons at a low frequency scatter much less. The correct choice of the time rescaling exponent α depends then on the properties of the dispersion relation. In the optical case, when $\bar{\omega}'(k) \sim k$, $|k| \ll 1$, so that the low frequency phonons not only scatter less but also travel slower, the scaling in (1.8) is diffusive, so that $\alpha = 2$ and $W_N(t, y, k)$ converges as $N \to +\infty$ to the solution to a heat equation

$$\partial_t W(t, y) = \hat{c}\partial_{yy}^2 W(t, y), \tag{1.10}$$

with the initial condition

$$W(0, y) = \int_{\mathbb{T}} W_0(y, k)dk,$$

and an appropriate diffusion coefficient $\hat{c} > 0$.

When, on the other hand, the dispersion relation is acoustic, so that $\bar{\omega}'(k) \sim \operatorname{sign} k$, for $|k| \ll 1$, and the phonons at low frequency scatter less but move as fast as other phonons, then the scaling is super-diffusive, with $\alpha = 3/2$ and the limit of $W_N(t, y, k)$ as $N \to +\infty$ satisfies the fractional heat equation

$$\begin{aligned} \partial_t W(t, y) &= -\hat{c}|\partial_{yy}^2|^{3/4} W(t, y), \\ W(0, y) &= \int_{\mathbb{T}} W_0(y, k)dk, \end{aligned} \tag{1.11}$$

with an appropriate fractional diffusion coefficient $\hat{c} > 0$. In both cases the limit $W(t, y)$ does not depend on the frequency k. Results of this type under various assumptions on the scattering kernel (but without an interface present) have been proved in [5, 6, 10, 13, 21].

Our interest here is to understand the long time behavior of the solutions to the kinetic equation with an acoustic dispersion relation in the presence of the interface, so that (1.8) holds away from $y = 0$, and the interface conditions (1.3)-(1.4) for W_N hold at $y = 0$. We allow the total scattering rate to degenerate as $R(k) \sim |\sin(\pi k)|^\beta$ for some $\beta > 1$. The case $\beta \in (0, 1)$ has been considered in [4], with the initial condition that is a local perturbation of the the equilibrium solution $W(t, y, k) \equiv T$. It leads to a diffusive scaling and the limit described by a heat equation (1.11), with a pure absorption interface condition $W(t, 0) = T$. In that situation, the degeneracy of scattering at low frequencies is not strong enough to prevent the diffusive behavior.

In order to formulate our main result, let us make some assumptions on the scattering kernel, reflection-transmission-absorption coefficients and the initial condition. We assume that the scattering kernel is symmetric

$$R(k, k') = R(k', k), \tag{1.12}$$

positive, except possibly at $k = 0$:

$$R(k, k') > 0, \quad k, k' \neq 0, \tag{1.13}$$

and the total scattering kernel has the asymptotics

$$R(k) := \int_{\mathbb{T}} R(k, k')dk' \sim R_0|\sin(\pi k)|^\beta, \quad |k| \ll 1, \tag{1.14}$$

with some $\beta \geq 0$ and $R_0 > 0$. We also assume that the normalized cross-section

$$p(k, k') := \frac{R(k, k')}{R(k)}, \quad k, k' \neq 0, \tag{1.15}$$

extends to a C^∞ function on $\mathbb{T}^2$. Note that

$$\int_{\mathbb{T}} p(k,k')dk' = 1, \text{ for all } k \neq 0. \tag{1.16}$$

For the dispersion relation, we assume that it is acoustic, that is,

$$\omega(k) \sim 2\omega_0'|\sin(\pi k)|, \quad |k| \ll 1, \tag{1.17}$$

with some $\omega_0' > 0$, and that $\omega(k)$ is even in k.

To make the precise assumptions on $W_0(y,k)$, we will use the notation $\mathbb{R}_+ = (0,+\infty)$, $\mathbb{R}_- = (-\infty,0)$, $\mathbb{R}_* = \mathbb{R}\setminus\{0\}$, $\bar{\mathbb{R}}_+ = [0,+\infty)$, $\bar{\mathbb{R}}_- = (-\infty,0]$, as well as $\mathbb{T}_* = \mathbb{T}\setminus\{0\}$, and $\mathbb{T}_\pm = [k:\ 0 < \pm k < 1/2]$. Given T, we let $\mathcal{C}_T$ be a subclass of $C_b(\mathbb{R}_*\times\mathbb{T}_*)$ of functions F that can be continuously extended to $\bar{\mathbb{R}}_\pm \times \mathbb{T}_*$ and satisfy the interface conditions

$$\begin{aligned} F(0^+,k) &= p_-(k)F(0^+,-k) + p_+(k)F(0^-,k) + \mathfrak{g}(k)T, \qquad &&\text{for } 0 < k \le 1/2, \\ F(0^-,k) &= p_-(k)F(0^-,-k) + p_+(k)F(0^+,k) + \mathfrak{g}(k)T, &&\text{for } -1/2 < k < 0. \end{aligned} \tag{1.18}$$

Note that $F \in \mathcal{C}_T$ if and only if $F - T' \in \mathcal{C}_{T-T'}$ for some T', because of (1.5).

In the presence of the interface, the fractional diffusion equation (1.11) is replaced by the following non-local equation

$$\begin{aligned} \partial_t W(t,y) = \hat{c}\int_{yy'>0} q_\beta(y-y')[W(t,y') - W(t,y)]dy' + \hat{c}\mathfrak{g}_0 \int_{yy'<0} q_\beta(y-y')[T - W(t,y)]dy' \\ + \hat{c}p_- \int_{yy'<0} q_\beta(y-y')[W(t,-y') - W(t,y)]dy' + \hat{c}p_+ \int_{yy'<0} q_\beta(y-y')[W(t,y') - W(t,y)]dy'. \end{aligned} \tag{1.19}$$

Here, $\hat{c}$ is a fractional diffusion coefficient given by (1.23) below, $p_\pm$ and $\mathfrak{g}_0$ are as in (1.6), and

$$q_\beta(y) = \frac{c_\beta}{|y|^{2+1/\beta}}, \quad c_\beta := \frac{2^{1+1/\beta}\Gamma\left(1+1/(2\beta)\right)}{\sqrt{\pi}\Gamma\left(-1/2+1/(2\beta)\right)}. \tag{1.20}$$

As we explain below, equation (1.19) automatically incorporates the interface conditions. Our main result is as follows.

Theorem 1.1 *In addition to the above assumptions about the scattering kernel $R(\cdot,\cdot)$ and the dispersion relation $\omega(\cdot)$, suppose that $\beta > 1$ and $W_0 \in \mathcal{C}_T$, and let $W_N(t,y,k)$ be the solution to (1.8) with $\alpha = 1 + 1/\beta$. Then, we have*

$$\lim_{N\to+\infty} \int_{\mathbb{R}\times\mathbb{T}} W_N(t,y,k)G(y,k)dydk = \int_{\mathbb{R}\times\mathbb{T}} W(t,y)G(y,k)dydk, \tag{1.21}$$

for any $t > 0$, and any test function $G \in C_0^\infty(\mathbb{R}\times\mathbb{T})$. The limit $W(t,y)$ is a weak solution of equation (1.19)*, in the sense of Definition* 2.3*, with the initial condition*

$$\bar{W}_0(y) := \int_{\mathbb{T}} W_0(y,k)dk \tag{1.22}$$

and the fractional diffusion coefficient

$$\hat{c} := \frac{2\pi^\beta(\omega_0')^{1+1/\beta}}{\beta(\gamma_0 R_0)^{1/\beta}} \text{ p.v.} \int_{\mathbb{R}} \frac{(e^{i\lambda}-1)d\lambda}{|\lambda|^{2+1/\beta}}. \tag{1.23}$$

The proof of this theorem proceeds as follows: as we have mentioned, the kinetic equation (1.8) is the Kolmogorov equation for a Markov process $(Z_N(t), K_N(t))$, where the frequency $K_N(t)$ is a certain jump process and the spatial component $Z_N(t)$ is the time integral of $\bar{\omega}'(K_N(t))$. This process can be generalized to incorporate the reflection-transmission-absorption at the interface. Similarly, we show that (1.19) is a Kolmogorov equation for a certain stable process $\zeta(t)$ that undergoes reflection-transmission-absorption at the interface. We prove that $Z_N(t,y)$ converges to $\zeta(t)$ in law. This shows that $W_N(t,y,k)$ converges to a weak solution $W(t,y)$ of (1.19), such that $W(t,y) = \mathbb{E}[\bar{W}_0(\zeta(t))|\zeta(0) = y]$.

Theorem 1.1 identifies the limit as a weak solution only in the sense of Definition 2.3 below, that does not characterize its behaviour at the interface. In order to obtain this information we need to prove that the limit belongs to a class of functions that satisfy a certain regularity condition at the interface (see (2.7) and (2.8)). When it is imposed the solution is unique.

In Theorem 2.5 we prove that the weak solution obtained in Theorem 1.1 belongs to this regularity class, under the further assumption that the initial condition $\bar{W}_0(y)$ belongs to L^1 add to an additive constant. To this end, we construct another approximation $\zeta_a(t)$ of $\zeta(t)$ that converges in law to $\zeta(t)$ as $a \to 0^+$ and

$$W_a(t,y) = \mathbb{E}\left[\bar{W}_0(\zeta_a(t))|\zeta_a(0) = y\right] \to W(t,y) = \mathbb{E}[\bar{W}_0(\zeta(t))|\zeta(0) = y].$$

However, we ensure that $W_a(t,y)$ also satisfies an energy estimate, see (7.10) below, thus so does $W(t,y)$ in the limit.

A kinetic problem with similar conditions at the interface appears as the macroscopic limit of a system of oscillators driven by a random noise that conserves energy, momentum and volume [3]. This microscopic model has been recently considered in [14], with a thermostat at a fixed temperature $T \geq 0$ acting on one particle, so that the phonons may be emitted, reflected or transmitted, and the corresponding macroscopic interface conditions have been obtained, in the absence of the bulk scattering, corresponding to $\gamma_0 = 0$ in (1.1). It is believed that the above macroscopic interface conditions also hold in the presence of interior microscopic scattering when $\gamma_0 > 0$. However, for the absorbing probability arising from this microscopic dynamics, we have $\mathfrak{g}_0 = 0$ (cf (1.6)). This generates a different interface condition for the macroscopic limit [15] from the one obtained here.

There seem to be few results on a fractional diffusion limit for kinetic equations in the presence of an interface. In [8], the case of absorbing, or reflecting boundary, but with the operator L that itself is a generator of a fractional diffusion, has been considered. Another situation, closer to ours, is a subject of [9], where the convergence of solutions to kinetic equations with the diffusive reflection conditions on the boundary is investigated. This condition is, however, different from our interface condition that concerns reflection-transmission-absorption. Also, in contrast to our situation, the results of [9], do not establish the uniqueness of the limit for solutions of the kinetic equation, stating only that it satisfies a certain fractional diffusive equation with a boundary condition. The question of the uniqueness of the solution for the limiting equation seems to be left open, see the remark after Theorem 1.2 in [9]. We mention here also a result of [2], where solutions of a stationary (time independent) linear kinetic equation are considered. The spatial domain is a half-space, with the absorbing-reflecting-emitting boundary, of a different type than in the present paper, and frequencies belong to a cylindrical domain. It has been shown that under an appropriate scaling the solutions converge to a harmonic function corresponding to a Neumann boundary, fractional Laplacian with exponent 1/2.

2 Some preliminaries

The classical solution of the kinetic interface problem

We start with the definition of a classical solution to the kinetic interface problem.

Definition 2.1 *We say that a function $W(t,x,k)$, $t \ge 0$, $x \in \mathbb{R}$, $k \in \mathbb{T}_*$, is a classical solution to equation* (1.1) *with the interface conditions* (1.3) *and* (1.4)*, if it is bounded and continuous on $\mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{T}_*$, and the following conditions hold:*

(1) The restrictions of W to $\mathbb{R}_+ \times \mathbb{R}_\iota \times \mathbb{T}_*$, $\iota \in \{-,+\}$, can be extended to bounded and continuous functions on $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_\iota \times \bar{\mathbb{T}}_{\iota'}$, $\iota' \in \{-,+\}$.

(2) For each $(t,y,k) \in \mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{T}_*$ fixed, the function $W(t+s, y+\bar{\omega}'(k)s, k)$ is of the C^1 class in the s-variable in a neighborhood of $s=0$, and the directional derivative

$$D_t W(t,y,k) = (\partial_t + \bar{\omega}'(k)\partial_y)\, W(t,y,k) := \frac{d}{ds}_{|s=0} W(t+s, y+\bar{\omega}'(k)s, k) \tag{2.1}$$

is bounded in $\mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{T}_*$ and satisfies

$$D_t W(t,y,k) = \gamma_0 L_k W(t,y,k), \qquad (t,y,k) \in \mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{T}_*, \tag{2.2}$$

together with (1.3) *and* (1.4) *and*

$$\lim_{t\to 0+} W(t,y,k) = W_0(y,k), \quad (y,k) \in \mathbb{R}_* \times \mathbb{T}_*. \tag{2.3}$$

The following result is standard.

Proposition 2.2 *Suppose that $W_0 \in \mathcal{C}_T$. Then, under the above hypotheses on the scattering kernel $R(k,k')$ and the dispersion relation $\omega(k)$, there exists a unique classical solution to equation* (1.1) *with the interface conditions* (1.3) *and* (1.4) *in the sense of Definition* 2.1.

The existence part is proved in Appendix A below, while uniqueness follows from Proposition 3.2, also below.

2.1 The fractional diffusion equation with an interface

Let us now discuss the weak solutions to the fractional diffusion equation with an interface that will arise as the long time asymptotics of the kinetic interface problem. For $\beta > 1$, we define the fractional Laplacian $\Lambda_\beta = (-\partial_y^2)^{(\beta+1)/(2\beta)}$ as the L^2 (self-adjoint) closure of the singular integral operator

$$\Lambda_\beta F(y) := \text{p.v.} \int_{\mathbb{R}} q_\beta(y-y')\, [F(y) - F(y')]\, dy', \quad F \in C_0^\infty(\mathbb{R}), \tag{2.4}$$

understood in the sense of the principal value, with $q_\beta(y)$ as in (1.20).

The operator $-\Lambda_\beta$ is the generator of a Lévy process. In order to introduce an interface, let us assume that if a particle tries to make a Lévy jump from y to y' such that y and y' have the same sign, then the jump happens almost surely. However, if y and y' have different signs, then with the probability p_+ the particle jumps to y', with probability p_- it jumps to $(-y')$ and with probability $\mathfrak{g}_0$ it is killed at the interface $y=0$, where a boundary condition $W(t,0)=T$ is prescribed. Recall that these probabilities satisfy (1.5). The corresponding Kolmogorov equation is then (1.19). Using relation (1.5), the right side of (1.19) can be re-written as

$$\partial_t W(t,y) = -\hat{c}\Lambda_\beta W(t,y) + \hat{c}\int\limits_{yy'<0} q_\beta(y-y')\big[\mathfrak{g}_0(T-W(t,y')) + p_-(W(t,-y')-W(t,y'))\big]dy'. \tag{2.5}$$

Definition 2.3 *A bounded function $W(t,y)$, $(t,y)\in\bar{\mathbb{R}}_+\times\mathbb{R}$, is a weak solution to equation* (2.5) *if for any $t_0>0$ and $G\in C_0^\infty([0,t_0]\times\mathbb{R}_*)$ we have*

$$\begin{aligned} 0 = &\int_0^{t_0} dt\int_{\mathbb{R}} \{\partial_t G(t,y) - \hat{c}\Lambda_\beta G(t,y)\} W(t,y)dy \\ &+ \hat{c}\int_0^{t_0} dt\int_{\mathbb{R}} G(t,y)dy\Big\{p_-\int_{[yy'<0]} q_\beta(y-y')[W(t,-y')-W(t,y')]dy' \\ &+ \mathfrak{g}_0\int_{[yy'<0]} q_\beta(y-y')[T-W(t,y')]dy'\Big\} - \int_{\mathbb{R}} G(t_0,y)W(t_0,y)dy + \int_{\mathbb{R}} G(0,y)W_0(y)dy. \end{aligned} \tag{2.6}$$

Notice that, since the support of the test functions G is bounded away from the interface, this weak formulation does not give information on the behaviour of the solution at the interface. In order to capture the behaviour of $W(t,y)$ for $y\to 0^\pm$ we need to consider solution in a certain regularity class. For this purpose we introduce the space $\mathcal{H}_0$ of functions that is the completion of $C_0^\infty(\mathbb{R}^d)$ in the norm

$$\begin{aligned} \|G\|_{\mathcal{H}_0}^2 := &\sum_{\iota=\pm}\int_{\mathbb{R}_\iota^2} q_\beta(y-y')[G(y)-G(y')]^2 dydy' + \mathfrak{g}_0\int_{\mathbb{R}_+\times\mathbb{R}_-} q_\beta(y-y')[G^2(y)+G^2(y')]dydy' \\ &+ \int_{\mathbb{R}_+\times\mathbb{R}_-} q_\beta(y-y')\left\{p_+[G(y)-G(y')]^2 + p_-[G(y)-G(-y')]^2\right\}dydy'. \end{aligned} \tag{2.7}$$

Note that the term in the last line above, with p_+ and p_-, is dominated by the term with the factor of $\mathfrak{g}_0$.

Since $q_\beta(y)\sim |y|^{-2-1/\beta}$, finitness of this norm forces $G(y)$ to decay to 0 at a certain rate, as $y\to 0$. Let us define the class of function

$$\mathfrak{H}_T := \{\exists\, T':\ W-T'\in C([0,+\infty),L^2(\mathbb{R})), W-T\in L^2_{\rm loc}([0,+\infty),\mathcal{H}_0)\} \tag{2.8}$$

Clearly if $W\in\mathfrak{H}_T$, then $W(t,y)\to T$, as $y\to 0$ for almost every $t\geq 0$.

Proposition 2.4 *A weak solution of* (2.6) *is unique in $\mathfrak{H}_T$.*

Proof. In fact, let $\bar{W}$ be the difference of two weak solutions in $\mathfrak{H}_T$. Then $\bar{W}(t,y)$ is in the space

$$C([0,+\infty),L^2(\mathbb{R}))\cap L^2_{\rm loc}([0,+\infty),\mathcal{H}_0),$$

and satisfies (2.6) for $T=0$. Approximating $\bar{W}$ by test functions G in (2.6) we obtain the identity

$$\frac{d}{dt}\|\bar{W}(t,\cdot)\|_2^2=-\hat{c}\|\bar{W}(t,\cdot)\|^2_{\mathcal{H}_0}. \tag{2.9}$$

Identity (2.9) immediately implies uniqueness of the solutions to (2.6) in the corresponding space. □

In Section 7 we prove the following.

Theorem 2.5 *Suppose that $W_0\in\mathcal{C}_T$ and there exists a constant T', so that $W_0-T'\in L^1(\mathbb{R}\times\mathbb{T})$. Let W be the limit of the solutions of the scaled kinetic equation described in Theorem 1.1. Then W belongs to $\mathfrak{H}_T$.*

3 Probabilistic representation for a solution to the kinetic equation with an interface

We now construct a probabilistic interpretation for the kinetic equation with reflection, transmission and absorption at an interface, as a generalization of the corresponding jump process without an interface. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and μ be a Borel measure on $\mathbb{T}$ given by

$$\mu(dk)=\frac{R(k)dk}{\bar{R}},\quad \bar{R}=\int_{\mathbb{T}}R(k)dk. \tag{3.1}$$

We denote by $(K_n^k)_{n\ge 0}$ a Markov chain such that $\mathbb{P}[K_0^k=k]=1$, with the transition operator

$$Pf(k)=\int_{\mathbb{T}}p(k,k')f(k')\mu(dk'),\quad k\in\mathbb{T},\ f\in L^\infty(\mathbb{T}). \tag{3.2}$$

Here, K_n^k are the particle momenta between the jump times. The measure μ is ergodic and invariant under P, and the operator P can be extended to $L^1(\mu)$ and

$$\|Pf\|_{L^\infty(\mu)}\le\|p\|_{L^\infty}\|f\|_{L^1(\pi)},\quad f\in L^1(\mu). \tag{3.3}$$

The transition operator is symmetric on $L^2(\mu)$. Since the transition probability density is strictly positive $\mu\otimes\mu$ a.s., the operator satisfies the spectral gap estimate

$$\sup[\|Pf\|_{L^2(\mu)}:\ f\perp 1,\ \|f\|_{L^2(\mu)}=1]<1. \tag{3.4}$$

We can easily conclude the following – see (1.2) for the definition of the operator L.

Proposition 3.1 *For any $F\in L^1(\mathbb{T})$ we have*

$$\lim_{t\to+\infty}\|e^{tL}F-\int_{\mathbb{T}}F(k)dk\|_{L^1(\mathbb{T})}=0. \tag{3.5}$$

Next, let τ_n, $n \geq 1$, be a sequence of i.i.d. exp(1) distributed random variables and $(\mathfrak{T}(t))_{t\geq 0}$ be a linear interpolation between the times

$$\mathfrak{T}(n) := \sum_{\ell=0}^{n} \bar{t}(K_\ell^k)\tau_\ell, \quad n = 0, 1, \ldots, \tag{3.6}$$

where

$$\bar{t}(k) := \frac{1}{\gamma_0 R(k)}. \tag{3.7}$$

That is, $\mathfrak{T}(n)$ is the time of the n-th jump, and the elapsed times between the consecutive jumps are $\bar{t}(K_\ell^k)\tau_\ell$. Between the jumps the particle moves with the constant speed $\omega'(K_\ell^k)$, and the corresponding spatial position $\tilde{Z}(t; y, k)$ is the linear interpolation between its locations at the jump times

$$Z_n(y, k) := y - \sum_{\ell=0}^{n} \bar{\omega}'(K_\ell^k)\bar{t}(K_\ell^k)\tau_\ell, \quad n = 0, 1, \ldots \tag{3.8}$$

Observe that there exists a constant $c_+ > 0$ such that

$$|\bar{\omega}'(k)\bar{t}(k)| \leq \frac{c_+\omega_0'}{\gamma_0 R_0 |k|^\beta}, \quad k \in \mathbb{T}. \tag{3.9}$$

Note that for each $n \geq 0$ and $(y, k) \in \mathbb{R} \times \mathbb{T}_*$ the law of $Z_n(y, k)$ is absolutely continuous with respect to the Lebesgue measure on the line.

We also note that

$$Y(t; y, k) := \tilde{Z}\left(\mathfrak{T}^{-1}(t); y, k\right) = y - \int_0^t \bar{\omega}'(K(s; k))ds, \quad K(t; k) := K^k_{[\mathfrak{T}^{-1}(t)]}, \tag{3.10}$$

and denote by $\mathcal{F}_t$ the natural filtration for the process $Y(t; y, k), K(t; k))$.

The jump process with reflection and transmission

We now add reflection and transmission to the jump process. Suppose that the starting point $y > 0$ and let

$$\mathfrak{s}_{y,1} := \inf[n > 0 : \; Z_n(y, k) < 0], \quad \mathfrak{s}_{y,2} := \inf[n > \mathfrak{s}_{y,1} : \; Z_n(y, k) > 0] \tag{3.11}$$

be the first times of the momenta jumps after the first crossing to the left and after the first crossing back to the right. Having defined $\mathfrak{s}_{y,2m-1}$, $\mathfrak{s}_{y,2m}$ for some $m \geq 1$ we let

$$\mathfrak{s}_{y,2m+1} := \inf[n > \mathfrak{s}_{y,2m} : \; Z_n(y, k) < 0], \quad \mathfrak{s}_{y,2m+2} := \inf[n > \mathfrak{s}_{y,2m+1} : \; Z_n(y, k) > 0]. \tag{3.12}$$

We define $\mathfrak{s}_{y,m}$ by symmetry also for $y < 0$. We also let

$$\tilde{\mathfrak{s}}_{y,1} := \inf[t > 0 : \; \tilde{Z}(t; y, k) < 0], \quad \tilde{\mathfrak{s}}_{y,2} := \inf[t > \mathfrak{s}_{y,1} : \; \tilde{Z}(t; y, k) > 0]$$

be the times when the trajectory crosses to the left and then crosses back to the right. Having defined $\tilde{\mathfrak{s}}_{y,2m-1}$, $\tilde{\mathfrak{s}}_{y,2m}$ for some $m \geq 1$, we set

$$\tilde{\mathfrak{s}}_{y,2m+1} := \inf[t > \tilde{\mathfrak{s}}_{y,2m} : \tilde{Z}(t;y,k) < 0], \quad \tilde{\mathfrak{s}}_{y,2m+2} := \inf[t > \tilde{\mathfrak{s}}_{y,2m+1} : \tilde{Z}(t;y,k) > 0]. \quad (3.13)$$

and. again, by symmetry we define $\tilde{\mathfrak{s}}_{y,m}$ for $y < 0$. Obviously, we have

$$\tilde{\mathfrak{s}}_{y,m} < \mathfrak{s}_{y,m} < \tilde{\mathfrak{s}}_{y,m+1}, \quad \text{a.s.}$$

We let $\hat{\sigma}_m^y$ be a $\{-1,0,1\}$-valued sequence of random variables that are independent, when conditioned on $(K_n^k)_{n\geq 0}$, such that

$$\mathbb{P}[\hat{\sigma}_m^y = 0|(K_n^k)_{n\geq 0}] = \mathfrak{g}(K^k_{\mathfrak{s}_{y,m}-1}), \quad \mathbb{P}[\hat{\sigma}_m^y = \pm 1|(K_n^k)_{n\geq 0}] = p_{\pm}(K^k_{\mathfrak{s}_{y,m}-1}).$$

These variables are responsible for whether the particle is reflected, transmitted or absorbed as it crosses the interface, and

$$\mathfrak{f} := \inf\big[m \geq 1 : \hat{\sigma}_m^y = 0\big] \quad (3.14)$$

is the crossing at which the particle is absorbed. For $m \geq 1$, we denote by $\mathfrak{F}_m$ the σ-algebra generated by $(Y(t;y,k), K(t;k))$, $0 \leq t \leq \tilde{\mathfrak{s}}_{y,m}$ and $\hat{\sigma}_\ell^y$, $\ell = 1,\dots,m$, with the convention that $\mathfrak{F}_0$ is the trivial σ-algebra. Recall that $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration corresponding to the process $(Y(t;y,k), K(t;k))_{t\geq 0}$.

We define the reflected-transmitted-absorbed process

$$\tilde{Z}^o(t;y,k) := \Big(\prod_{m=1}^{n-1} \hat{\sigma}_m^y\Big)\tilde{Z}(t;y,k), \quad t \in [\tilde{\mathfrak{s}}_{y,n-1}, \tilde{\mathfrak{s}}_{y,n}),$$

with the convention that the product over an empty set of indices equals 1 and the respective counterparts

$$Y^o(t;y,k) := \tilde{Z}^o(T^{-1}(t);y,k) = \Big(\prod_{m=1}^{n-1} \hat{\sigma}_m^y\Big)Y(t;y,k), \quad t \in [\tilde{\mathfrak{s}}_{y,n-1}, \tilde{\mathfrak{s}}_{y,n})$$

and

$$K^o(t;k) := \Big(\prod_{m=1}^{n-1} \hat{\sigma}_m^y\Big)K(t;k), \quad t \in [\tilde{\mathfrak{s}}_{y,n-1}, \tilde{\mathfrak{s}}_{y,n}).$$

In what follows, we assume the convention that $\bar{\omega}'(0) := 0$ even though $\omega(k)$ is not differentiable at $k = 0$. For $t \in (\tilde{\mathfrak{s}}_{y,n-1}, \tilde{\mathfrak{s}}_{y,n})$ we can write

$$\begin{aligned}\frac{dY^o(t;y,k)}{dt} &:= \Big(\prod_{m=1}^{n-1} \hat{\sigma}_m^y\Big)\frac{dY(t;y,k)}{dt} = -\Big(\prod_{m=1}^{n-1} \hat{\sigma}_m^y\Big)\omega'(K(t;k)) \\ &= -\omega'\Big(\Big(\prod_{m=1}^{n-1} \hat{\sigma}_m^y\Big)K(t;k)\Big) = -\omega'(K^o(t;k)). \end{aligned} \quad (3.15)$$

If $Y^o(t;y,k) \ge 0$ for $\tilde{\mathfrak{s}}_{y,m-1} < t < \tilde{\mathfrak{s}}_{y,m+1}$ – that is, the particle approached the interface from the right at the time $\tilde{\mathfrak{s}}_{y,m}$ and was reflected, then for $h > 0$ we define $\tilde{\mathfrak{s}}^h_{y,m} \in (\tilde{\mathfrak{s}}_{y,m-1}, \tilde{\mathfrak{s}}_{y,m})$ as the first exit time of $Y^o(t;y,k)$ from the half-line $[y > h]$ that happens after $\tilde{\mathfrak{s}}_{y,m-1}$, and $\tilde{\mathfrak{s}}^{h,e}_{y,m} \in (\tilde{\mathfrak{s}}_{y,m}, \tilde{\mathfrak{s}}_{y,m+1})$ as the first exit time of $Y^o(t;y,k)$ from the half-line $[y < h]$, after $\tilde{\mathfrak{s}}_{y,m}$. Note that both $\tilde{\mathfrak{s}}^h_{y,m}$ and $\tilde{\mathfrak{s}}^{h,e}_{y,m}$ are finite a.s. if $h > 0$ is sufficiently small, and we have

$$\lim_{h\to 0+} \tilde{\mathfrak{s}}^h_{y,m} = \lim_{h\to 0+} \tilde{\mathfrak{s}}^{h,e}_{y,m} = \tilde{\mathfrak{s}}_{y,m}, \quad \text{a.s.}$$

Analogous definitions can be introduced for all other configurations of the signs of $Y^o(t;y,k)$ in $\tilde{\mathfrak{s}}_{y,m-1} < t < \tilde{\mathfrak{s}}_{y,m+1}$.

A probabilistic representation for the kinetic equation

We will now prove the following.

Proposition 3.2 *If $W(t,y,k)$ is a solution to* (1.1)*, with the interface condition* (1.3)-(1.4)*, in the sense of Definition* 2.1*, such that $W(0,y,k) = W_0(y,k)$ and $W_0 \in \mathcal{C}_T$, then*

$$W(t,y,k) = \mathbb{E}\left[W_0\left(Y^o(t;y,k), K^o(t;k)\right), t < \tilde{\mathfrak{s}}_{y,\mathfrak{f}}\right] + T\mathbb{P}\left[t \ge \tilde{\mathfrak{s}}_{y,\mathfrak{f}}\right], \quad t \ge 0,\ y \in \mathbb{R}_*,\ k \in \mathbb{T}_*. \tag{3.16}$$

Proof. First, let $\widetilde{W}(t,y,k)$ be a solution to (1.1) as in Proposition 3.2 but with $T = 0$ in the interface conditions (1.3)-(1.4). We set

$$\begin{aligned}\mathcal{M}_m &:= \lim_{h\to 0+} 1_{[\mathfrak{f}>m-1]}\widetilde{W}\left(t - t\wedge\tilde{\mathfrak{s}}^h_{y,m+1}, Y^o(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};y,k), K^o(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};k)\right)\\ &+ 1_{[\mathfrak{f}\le m-1]}\widetilde{W}_0\left(Y^o(t;y,k), K^o(t;k)\right) - W(t,y,k)\end{aligned} \tag{3.17}$$

and consider the increments

$$\begin{aligned}&\mathcal{Z}_m := \lim_{h\to 0+}\Big\{\widetilde{W}\left(t - t\wedge\tilde{\mathfrak{s}}^h_{y,m+1}, Y^o(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};y,k), K^o(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};k)\right)\\ &-\widetilde{W}\left(t - t\wedge\tilde{\mathfrak{s}}^h_{y,m}, Y^o(t\wedge\tilde{\mathfrak{s}}^h_{y,m};y,k), K^o(t\wedge\tilde{\mathfrak{s}}^h_{y,m};k)\right)\Big\}, \quad m = 0,\ldots,\mathfrak{f}-1,\\ &\mathcal{Z}_{\mathfrak{f}} := -\lim_{h\to 0+}\widetilde{W}\left(t - \tilde{\mathfrak{s}}^h_{y,\mathfrak{f}}, Y^o(\tilde{\mathfrak{s}}^h_{y,\mathfrak{f}};y,k), K^o(\tilde{\mathfrak{s}}^h_{y,\mathfrak{f}};k)\right) \text{ on the event } [\mathfrak{s}_{y,\mathfrak{f}} \le t],\\ &\mathcal{Z}_{\mathfrak{f}} = 0, \text{ on the event } [\mathfrak{s}_{y,\mathfrak{f}} > t],\\ &\mathcal{Z}_m := 0, \quad m > \mathfrak{f},\end{aligned} \tag{3.18}$$

so that

$$\mathcal{M}_m := \sum_{j=0}^{m-1} \mathcal{Z}_j. \tag{3.19}$$

The key step in the proof of Proposition 3.2 is the following lemma.

Lemma 3.3 *We have*

$$\mathbb{E}[\mathcal{Z}_m|\mathfrak{F}_m] = 0, \quad m = 0,1,\ldots. \tag{3.20}$$

As an immediate corollary of Lemma 3.3, we know that the sequence $(\mathcal{M}_m)_{m\ge 1}$ is a zero mean martingale with respect to the filtration $(\mathfrak{F}_m)_{m\ge 1}$. Since $\mathfrak{f}+1$ is a stopping time with respect to the filtration $(\mathfrak{F}_m)_{m\ge 1}$, and the martingale $(\mathcal{M}_m)_{m\ge 1}$ is bounded, the optional stopping theorem implies that $\mathbb{E}\mathcal{M}_{\mathfrak{f}+1}=0$, which yields

$$\widetilde{W}(t,y,k)=\mathbb{E}\left[W_0\left(Y^o(t;y,k),K^o(t;k)\right),\,t<\tilde{\mathfrak{s}}_{y,\mathfrak{f}}\right],\quad t>0,\ y\in\mathbb{R}_*,\ k\in\mathbb{T}_*, \tag{3.21}$$

which is a special case of (3.16) with $T=0$.

In general, if $W(t,y,k)$ is as in Proposition 3.2, with $T\neq 0$, then $\widetilde{W}(t,x,k)=W(t,y,k)-T$ satisfies (1.1), with the interface condition given by (1.3) and (1.4) corresponding to $T=0$ and the initial condition $\widetilde{W}_0(y,k)=W_0(y,k)-T$. It follows from the above that

$$\begin{aligned}W(t,y,k)&=T+\widetilde{W}(t,y,k)=T+\mathbb{E}\left[\widetilde{W}_0\left(Y^o(t;y,k),K^o(t)\right),\,t<\tilde{\mathfrak{s}}_{y,\mathfrak{f}}\right]\\&=\mathbb{E}\left[W_0\left(Y^o(t;y,k),K^o(t)\right),\,t<\tilde{\mathfrak{s}}_{y,\mathfrak{f}}\right]+T\mathbb{P}\left[t\ge\tilde{\mathfrak{s}}_{y,\mathfrak{f}}\right]\end{aligned} \tag{3.22}$$

and (3.16) follows, finishing the proof of Proposition 3.2. □

Proof of Lemma 3.3. Let $\widetilde{W}_\pm(t,y,k)$ be the restrictions of $\widetilde{W}$ to $\{y>0\}$ and $\{y<0\}$, respectively. We extend them to the whole line in such a way that $D_t\widetilde{W}_\pm$ are well defined for all $(t,y,k)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{T}_*$ and they are bounded and measurable, and denote

$$F_\pm(t,y,k):=(L_k-\omega'(k)\partial_y-\partial_t)\widetilde{W}_\pm(t,y,k).$$

Note that $F_\pm(t,y,k)=0$ for $y\in\mathbb{R}_\pm$, respectively, and the processes

$$\mathcal{M}_\pm(u):=\widetilde{W}_\pm(t-u,Y(u;y,k),K(u;k))-\widetilde{W}_\pm(t,y,k)-\int_0^u F_\pm(t-s,Y(s;y,k),K(s;k))ds,$$

with $0\le u\le t$ are $\mathcal{F}_u$-martingales, so that

$$\begin{aligned}\mathbb{E}\Big\{\Big[&\widetilde{W}_{\sigma'}\left(t-t\wedge\tilde{\mathfrak{s}}^h_{y,m+1},\sigma Y(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};y,k),\sigma K(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};k))\right)\\&-\widetilde{W}_{\sigma'}\left(t-t\wedge\tilde{\mathfrak{s}}^{h,e}_{y,m},\sigma Y(t\wedge\tilde{\mathfrak{s}}^{h,e}_{y,m};y,k),\sigma K(t\wedge\tilde{\mathfrak{s}}^{h,e}_{y,m};k))\right)\Big]\Big|\mathcal{F}_{\tilde{\mathfrak{s}}_{y,m}}\Big\}=0,\end{aligned} \tag{3.23}$$

provided that

$$\sigma':=(-1)^m\sigma\mathrm{sign}\,y.$$

Note that since

$$(-1)^m\mathrm{sign}\,y=\mathrm{sign}\left(Y(t\wedge\tilde{\mathfrak{s}}^{h,e}_{y,m};y,k)\right)$$

we have

$$\sigma'=\mathrm{sign}\left(\sigma Y(t\wedge\tilde{\mathfrak{s}}^{h,e}_{y,m};y,k)\right).$$

The interface conditions (1.3) and (1.4) with $T=0$ can be written as

$$\begin{aligned}&p_+(\sigma K(\tilde{\mathfrak{s}}_{y,m};k))\widetilde{W}_{-\sigma'}\left(t-\tilde{\mathfrak{s}}_{y,m},0,\sigma K(\tilde{\mathfrak{s}}_{y,m};k)\right)\\&+p_-(\sigma K(\tilde{\mathfrak{s}}_{y,m};k))\widetilde{W}_{\sigma'}\left(t-\tilde{\mathfrak{s}}_{y,m},0,-\sigma K(\tilde{\mathfrak{s}}_{y,m};k)\right)=\widetilde{W}_{\sigma'}\left(t-\tilde{\mathfrak{s}}_{y,m},0,\sigma K(\tilde{\mathfrak{s}}_{y,m};k)\right),\end{aligned} \tag{3.24}$$

provided that

$$\sigma'\sigma\mathrm{sgn}\, K(\tilde{\mathfrak{s}}_{y,m};k) = -1. \tag{3.25}$$

We now need to replace the time $\tilde{\mathfrak{s}}^{h,e}_{y,m}$ in the second term in (3.23) by $\tilde{\mathfrak{s}}^{h}_{y,m}$, in order to convert the right side of (3.23) into a term of a telescoping sum, and to show that $\mathcal{M}_m$ is a martingale. To this end, suppose that $\Phi \in C_b((\mathbb{R}\times\mathbb{T})^{L+1}\times\{-1,0,1\}^{m-1})$ and consider the times $0 = t_0 < t_1 < \ldots < t_L$. Then, we have

$$\mathbb{E}\left[\mathcal{Z}_m\Phi_m\right] = \sum_{\varepsilon}\mathbb{E}\left[\mathcal{Z}_m\Phi_m,\ A_{\varepsilon_1,\ldots,\varepsilon_{m-1}}\right], \tag{3.26}$$

with

$$\Phi_m := \Phi\left(\left(Y(t_j\wedge\tilde{\mathfrak{s}}_{y,m};y,k), K(t_j\wedge\tilde{\mathfrak{s}}_{y,m};k)\right)_{0\le j\le L}, \hat{\sigma}^y_1,\ldots,\hat{\sigma}^y_{m-1}\right)$$

$$A_{\varepsilon_1,\ldots,\varepsilon_{m-1}} = [\hat{\sigma}^y_j = \varepsilon_j,\ j = 1,\ldots,m-1],$$

and the summation in (3.26) extending over all sequences $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_{m-1}) \in \{-1,0,1\}^{m-1}$. Suppose that some $\varepsilon_j = 0$. Then, we have

$$\mathfrak{f} = \min[j\in\{1,\ldots,m\}:\ \varepsilon_j = 0] \le m-1$$

and $\mathcal{Z}_m 1_{A_\varepsilon} = 0$ for the corresponding sequence $(\varepsilon_1,\ldots,\varepsilon_{m-1})$. On the other hand, if $\varepsilon_j \neq 0$ for all $j = 1,\ldots,m-1$, then

$$\mathbb{E}[\mathcal{Z}_m\Phi_m, A_\varepsilon] = \mathcal{I}_+ + \mathcal{I}_0 + \mathcal{I}_-,$$

where $\mathcal{I}_\iota$, $\iota\in\{-1,0,1\}$, correspond to the events $\hat{\sigma}^y_m = \iota$. Knowing the values $\hat{\sigma}^y_1,\ldots,\hat{\sigma}^y_{m-1}$ and the sign of y one can determine the sign σ in the equality

$$Y^o(t;y,k) = \sigma Y(t;y,k), \quad K^o(t;k) = \sigma K(t;k), \quad \tilde{\mathfrak{s}}^h_{y,m} \le t < \tilde{\mathfrak{s}}_{y,m},$$

hence

$$\widetilde{W}\left(t-\tilde{\mathfrak{s}}^h_{y,m}, Y^o(\tilde{\mathfrak{s}}^h_{y,m};y,k), K^o(\tilde{\mathfrak{s}}^h_{y,m};k)\right) = \widetilde{W}_{\sigma'}\left(t-\tilde{\mathfrak{s}}^h_{y,m}, \sigma Y(\tilde{\mathfrak{s}}^h_{y,m};y,k), \sigma K(\tilde{\mathfrak{s}}^h_{y,m};k)\right),$$

with $\sigma' := (-1)^m\sigma\mathrm{sign}\, y$. We have

$$\begin{aligned}
\mathcal{I}_\pm = &\lim_{h\to0+}\mathbb{E}\Big\{\Big\{\widetilde{W}_{\mp\sigma'}\left(t-t\wedge\tilde{\mathfrak{s}}^h_{y,m+1}, \pm\sigma Y(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};y,k), \pm\sigma K(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};k)\right)\\
&-\widetilde{W}_{\sigma'}\left(t-\tilde{\mathfrak{s}}^h_{y,m}, \sigma Y(\tilde{\mathfrak{s}}^h_{y,m};y,k), \sigma K(\tilde{\mathfrak{s}}^h_{y,m};k)\right)\Big\}\Phi_m, A_\varepsilon,\ \hat{\sigma}^y_m = \pm1,\ t\ge\tilde{\mathfrak{s}}^h_{y,m}\Big\}\\
= &\lim_{h\to0+}\mathbb{E}\Big\{\Big\{\widetilde{W}_{\mp\sigma'}\left(t-t\wedge\tilde{\mathfrak{s}}^h_{y,m+1}, \mp\sigma Y(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};y,k), \mp\sigma K(t\wedge\tilde{\mathfrak{s}}^h_{y,m+1};k)\right)p_\pm(\sigma K(\tilde{\mathfrak{s}}_{y,m};k))\\
&-\widetilde{W}_{\sigma'}\left(t-\tilde{\mathfrak{s}}_{y,m}, \sigma Y(\tilde{\mathfrak{s}}_{y,m};y,k), \sigma K(\tilde{\mathfrak{s}}_{y,m};k)\right)1_{[\hat{\sigma}^y_m=\pm1]}\Big\}\Phi_m, A_\varepsilon,\ t\ge\tilde{\mathfrak{s}}_{y,m}\Big\}.
\end{aligned}$$

Passing to the limit $h\to0+$ above, using the continuity of $\widetilde{W}_\pm$ up to the interface and the fact that $\mathfrak{s}^{h,e}_{y,m}\to\mathfrak{s}_{y,m}$ as $h\to0+$ a.s., and invoking (3.23), we conclude that

$$\begin{aligned}
\mathcal{I}_\pm = \mathbb{E}\Big\{\Big\{&\widetilde{W}_{\mp\sigma'}\left(t-\tilde{\mathfrak{s}}_{y,m}, \pm\sigma Y(\tilde{\mathfrak{s}}_{y,m};y,k), \pm\sigma K(\tilde{\mathfrak{s}}_{y,m};k)\right)p_\pm(K(\tilde{\mathfrak{s}}_{y,m};k))\\
&-\widetilde{W}_{\sigma'}\left(t-\tilde{\mathfrak{s}}_{y,m}, \sigma Y(\tilde{\mathfrak{s}}_{y,m};y,k), \sigma K(\tilde{\mathfrak{s}}_{y,m};k)\right)1_{[\hat{\sigma}^y_m=\pm1]}\Big\}\Phi_m, A_\varepsilon,\ t\ge\tilde{\mathfrak{s}}_{y,m}\Big\}.
\end{aligned}$$

On the event $[\hat{\sigma}^y_m = 0]$ we have $\mathfrak{f} = m$, therefore

$$\mathcal{I}_0 = -\mathbb{E}\left\{\widetilde{W}_{\sigma'}\left(t - \tilde{\mathfrak{s}}_{y,m}, \sigma Y(\tilde{\mathfrak{s}}_{y,m}; y, k), \sigma K(\tilde{\mathfrak{s}}_{y,m}; k)\right)\Phi_m, A_\varepsilon,\, \hat{\sigma}^y_m = 0,\, t \ge \tilde{\mathfrak{s}}_{y,m}\right\},$$

as follows from the condition $\mathcal{Z}_{\mathfrak{f}} = 0$ on the event $[\mathfrak{s}_{y,\mathfrak{f}} > t]$ in (3.18). Now, we conclude that from (3.24) that

$$\mathcal{I}_+ + \mathcal{I}_0 + \mathcal{I}_- = 0,$$

thus (3.20) follows. □

4 The scaled processes and their convergence

4.1 Convergence of processes without an interface

We consider the rescaled process

$$Z_N(t; y, k) := y - \frac{1}{N^{\beta/(1+\beta)}} \sum_{\ell=0}^{[Nt]} \bar{\omega}'(K^k_\ell)\bar{t}(K^k_\ell)\tau_\ell, \quad t \ge 0,$$

and $\tilde{Z}_N(t; y, k)$ be the linear interpolation in time between the points $Z_N(n/N; y, k)$, $n \ge 0$. Let also $(\mathfrak{T}_N(t; k))_{t\ge0}$ and $(Y_N(t; y, k))_{t\ge0}$ be the scaled versions of the processes defined by (3.6) and (3.10), respectively:

$$\mathfrak{T}_N(t; k) := \frac{1}{N} \sum_{\ell=0}^{m} \bar{t}(K^k_\ell)\tau_\ell, \quad t = \frac{m}{N} \tag{4.1}$$

and it is a linear interpolation otherwise, while

$$Y_N(t; y, k) := y - \frac{1}{N^{\beta/(1+\beta)}} \int_0^{Nt} \bar{\omega}'(K(s; k))ds = \tilde{Z}_N(\mathfrak{T}_N^{-1}(t, k), y, k). \tag{4.2}$$

To describe the limit, let $(\zeta(t))_{t\ge0}$ be the symmetric stable process with the Lévy exponent

$$\psi(\theta) = \frac{\hat{c}|\theta|^{1+1/\beta}}{\gamma_0 \bar{R}}, \quad \theta \in \mathbb{R},$$

and set

$$\zeta(t, y) := y + \zeta(t), \quad \tau(t) := t\bar{\tau}, \quad \eta(t, y) := \zeta(\tau^{-1}(t), y), \quad t \ge 0, \tag{4.3}$$

where

$$\bar{\tau} := \int_{\mathbb{T}} \bar{t}(k)\mu(dk) = \frac{1}{\gamma_0 \bar{R}}. \tag{4.4}$$

Proposition 4.1 *For any $t_0 > 0$ and $k \in \mathbb{T}$ we have*

$$\lim_{N\to+\infty} \mathbb{E}\Big\{ \sup_{t\in[0,t_0]} |\mathfrak{T}_N(t; k) - \tau(t)| \Big\} = 0. \tag{4.5}$$

The proof of the proposition is standard an we omit it.

The following result is a simpler version of Proposition 4.1 and Theorem 2.5(i) of [12], see also Theorem 2.4 of [13] and Theorem 3.2 of [5].

Proposition 4.2 *Suppose that $\beta > 1$ and $(y,k) \in \mathbb{R}_* \times \mathbb{T}_*$. Under the assumptions on the functions R and ω in Section 2, the joint law of $(Z_N(t,y,k), \mathfrak{T}_N(t;k))_{t\ge 0}$ converges in law, over $\mathcal{D}_2 := D([0,+\infty); \mathbb{R}\times\bar{\mathbb{R}}_+)$ equipped with the Skorokhod J_1-topology to $(\zeta(t,y),\tau(t))_{t\ge 0}$.*

The following result is an immediate consequence of the above theorem.

Corollary 4.3 *The process $(Y_N(t,y,k))_{t\ge 0}$ converge in law, as $N \to +\infty$, over $D[0,+\infty)$ equipped with the Skorokhod M_1-topology to $(\eta(t,y,k))_{t\ge 0}$.*

4.2 Joint convergence of processes and crossing times and positions

Using the analogues of (3.12)-(3.13) we can define crossing times $\mathfrak{s}^N_{y,m}$, $\tilde{\mathfrak{s}}^N_{y,m}$, $m, N = 1, 2, \dots$ for the scaled process $(Z_N(t,y,k))_{t\ge 0}$ and $(\tilde{Z}_N(t;y,k))_{t\ge 0}$, respectively. As a simple consequence of absolute continuity of the law of $Z_n(y,k)$ we conclude that for each $y \in \mathbb{R}$ there exists a strictly increasing sequence $(n^N_{y,m})_{m\ge 1}$ such that

$$\tilde{\mathfrak{s}}^N_{y,m} \le \mathfrak{s}^N_{y,m} = \frac{n^N_{y,m}}{N} < \tilde{\mathfrak{s}}^N_{y,m} + \frac{1}{N}, \qquad \text{a.s.} \tag{4.6}$$

Likewise, let $\mathfrak{u}_{y,m}$ be the consecutive times when the process $(\zeta(t,y))_{t\ge 0}$ crosses the level $z = 0$. The main result of this section is the following.

Theorem 4.4 *For any $(y,k) \in \mathbb{R} \times \mathbb{T}_*$ the random elements*

$$\Big((Z_N(t,y,k), \mathfrak{T}_N(t;k))_{t\ge 0}\,, (\mathfrak{s}^N_{y,m})_{m\ge 1}, (Z_N(\mathfrak{s}^N_{y,m}, y, k))_{m\ge 1} \Big)$$

converge in law, as $N \to +\infty$, over $\mathcal{D}_2 \times \bar{\mathbb{R}}^{\mathbb{N}}_+ \times \mathbb{R}^{\mathbb{N}}$ with the product of the J_1 and standard product topology on $(\mathbb{R}^{\mathbb{N}})^2$, to

$$\Big((\zeta(t,y), \tau(t))_{t\ge 0}, (\mathfrak{u}_{y,m})_{m\ge 1}, (\zeta(\mathfrak{u}_{y,m}, y, k)_{m\ge 1} \Big).$$

The proof of this result is contained in Appendix B.

We now formulate a property of the approximating process $(Z_N(t))_{t\ge 0}$ at the crossing times. We start with the following simple consequence of Corollary 2.2 of [20] and the strong Markov property of stable processes.

Lemma 4.5 *For each $y > 0$ we have*

$$\mathbb{P}\left[\zeta(\mathfrak{s}_{y,2m}, y) > 0 > \zeta(\mathfrak{s}_{y,2m-1}, y),\ m \ge 1\right] = 1. \tag{4.7}$$

If, on the other hand $y < 0$, then

$$\mathbb{P}\left[\zeta(\mathfrak{s}_{y,2m-1,y}, y) > 0 > \zeta(\mathfrak{s}_{y,2m}, y),\ m \ge 1\right] = 1. \tag{4.8}$$

As a consequence, we obtain the following estimate on the distance the particle travels upon a crossing, so that the jump is ”macroscopic”.

Corollary 4.6 *Suppose that $(y,k) \in \mathbb{R}_* \times \mathbb{T}_*$, $\varepsilon > 0$ and M is a positive integer. Then, there exist $C > 0$ that depends on ε and M such that*

$$\mathbb{P}\left[\min_{m=1,\dots,M} |\bar\omega'(K^k_{n^N_{y,m}})|\bar t(K^k_{n^N_{y,m}}) \le CN^{\beta/(1+\beta)}\right] < \varepsilon, \quad \text{for all } N \ge 1. \tag{4.9}$$

Proof. Suppose that $y > 0$. As a consequence of Theorem 4.4, for any M, the random vectors

$$(Z_N(\mathfrak{s}^N_{y,1}, y, k), \dots, Z_N(\mathfrak{s}^N_{y,M}, y, k)) \tag{4.10}$$

converge in law to $(\zeta(\mathfrak{u}_{y,1}, y), \dots, \zeta(\mathfrak{u}_{y,M}, y))$. Lemma 4.5 implies that given $\varepsilon > 0$, there exists $c > 0$ that depends on ε and M such that

$$\mathbb{P}\left[\zeta(\mathfrak{u}_{y,2m-1}, y) < -c,\, \zeta(\mathfrak{u}_{y,2m}, y) > c, \quad m = 1, \dots, M\right] > 1 - \varepsilon. \tag{4.11}$$

Let us set

$$A_N(c) := \left[Z_N(\mathfrak{s}^N_{y,2m-1}, y, k) < -c, Z_N(\mathfrak{s}^N_{y,2m}, y, k) > c, \quad m = 1, \dots, M\right].$$

The convergence in law of the vectors (4.10) and (4.11) imply that

$$\mathbb{P}[A_N(c_\varepsilon)] > 1 - \varepsilon \tag{4.12}$$

for all sufficiently large N. Decreasing $c > 0$ if necessary, we can claim that (4.12) holds for all $N \ge 1$, so that on $A_N(c)$ we have

$$y - \frac{1}{N^{\beta/(\beta+1)}} \sum_{n=0}^{n^N_{y,2m-1}} \bar\omega'(K^k_n)\bar t(K^k_n) < -c < 0 \le y - \frac{1}{N^{\beta/(\beta+1)}} \sum_{n=0}^{n^N_{y,2m-1}-1} \bar\omega'(K^k_n)\bar t(K^k_n)$$

and

$$y - \frac{1}{N^{\beta/(\beta+1)}} \sum_{n=0}^{n^N_{y,2m}} \bar\omega'(K^k_n)\bar t(K^k_n) > c > 0 \ge y - \frac{1}{N^{\beta/(\beta+1)}} \sum_{n=0}^{n^N_{y,2m}-1} \bar\omega'(K^k_n)\bar t(K^k_n),$$

both for all $m = 1, \dots, M$. Hence, on $A_N(c)$ we hve

$$|\bar\omega'(K^k_n)|\bar t(K^k_n) > cN^{\beta/(\beta+1)},\, m = 1, \dots, M,$$

which in turn yields (4.9). $\square$

4.3 Processes with reflection, transmission and killing

We now restore writing y, k in the notation of the processes, with $(y,k) \in \mathbb{R} \times \mathbb{T}_*$. To set the notation for the rescaled processes, let $(\hat\sigma^N_{y,m})$ be a $\{-1, 0, 1\}$-valued sequence of random variables that are independent, when conditioned on $(K^k_n)_{n\ge0}$, and set

$$\mathbb{P}[\hat\sigma^N_{y,m} = 0 | (K^k_n)_{n\ge0}] = \mathfrak{g}(K^k_{n^N_{y,m}}), \quad \mathbb{P}[\hat\sigma^N_{y,m} = \pm1 | (K^k_n)_{n\ge0}] = p_\pm(K^k_{n^N_{y,m}}),$$

as well as

$$\mathfrak{f}_N := \min[m : \hat{\sigma}^N_{y,m} = 0], \quad \tilde{\mathfrak{s}}^N_{\mathfrak{f}} := \tilde{\mathfrak{s}}^N_{y,\mathfrak{f}_N} \quad \mathfrak{s}^N_{\mathfrak{f}} := \mathfrak{s}^N_{y,\mathfrak{f}_N}. \tag{4.13}$$

The killed-reflected-transmitted process $(\tilde{Z}^o_N(t,y,k))_{t\ge 0}$ can be written as

$$\tilde{Z}^o_N(t,y,k) := \Big(\prod_{j=1}^m \hat{\sigma}^N_{y,j}\Big)\tilde{Z}_N(t,y,k), \quad t \in [\tilde{\mathfrak{s}}^N_{y,m}, \tilde{\mathfrak{s}}^N_{y,m+1}),\ m = 0,1,\ldots. \tag{4.14}$$

We adopt the convention that for $m = 0$ the product above equals 1 and $\tilde{\mathfrak{s}}^N_{y,0} := 0$.

For the limit killed-reflected-transmitted process, similarly, we let $(\hat{\sigma}_m)_{m\ge 1}$ be a sequence of i.i.d. $\{-1,0,1\}$ random variables, independent of $(\zeta(t,y))_{t\ge 0}$, with

$$\mathbb{P}[\hat{\sigma}_m = 0] = \mathfrak{g}_0, \quad \mathbb{P}[\hat{\sigma}_m = \pm 1] = p_\pm, \quad m = 0,1,\ldots, \tag{4.15}$$

Here, as we recall, $\mathfrak{g}_0 = \mathfrak{g}(0)$ and $p_\pm := p_\pm(0)$. We also set

$$\mathfrak{f} := \min[m \ge 1 : \hat{\sigma}_m = 0], \quad \mathfrak{u}_{\mathfrak{f}} := \mathfrak{u}_{y,\mathfrak{f}}, \quad \mathfrak{u}_{y,0} := 0. \tag{4.16}$$

The killed-reflected-transmitted stable process has a representation

$$\zeta^o(t,y) := \Big(\prod_{j=1}^m \hat{\sigma}_j\Big)\zeta(t,y), \quad t \in [\mathfrak{u}_{y,m}, \mathfrak{u}_{y,m+1}),\ m = 0,1,\ldots. \tag{4.17}$$

Note that the processes $Z_N(t)$ are discontinuous in t while $\tilde{Z}_N(t)$ are continuous in time. As the process $\zeta(t)$ is discontinuous, it would not be possible to prove convergence of $\tilde{Z}_N(t)$ to $\zeta(t)$ in the Skorokhod space $D[0,+\infty)$ equipped with the J_1-topology. Hence, we will need to use the M_1-topology that allows convergence of continuous processes to a discontinuous limit. Accordingly, we denote by $\mathcal{X}$ the space $D[0,+\infty) \times C[0,+\infty) \times \bar{\mathbb{R}}^{\mathbb{N}}_+$, equipped with the product of M_1 and uniform convergence on compacts topologies in the first two variables and the standard product topology on $\bar{\mathbb{R}}^{\mathbb{N}}_+$. We will use below the metric d_∞ that metrizes the M_1-topology on $D[0,+\infty)$, see Appendix B.1 for a brief review of the required definitions. Our main result in this section is the following.

Theorem 4.7 *The random elements* $\Big((\tilde{Z}^o_N(t,y,k))_{t\ge 0}, (\mathfrak{T}_N(t,k))_{t\ge 0}, (\tilde{\mathfrak{s}}^N_{y,m})_{m\ge 1}\Big)$ *converge in law over* $\mathcal{X}$ *to the random element* $\Big((\zeta^o(t,y))_{t\ge 0}, (\tau(t))_{t\ge 0}, (\mathfrak{u}_{y,m})_{m\ge 1}\Big)$.

Proof. Let us define the process

$$Z^o_N(t) := \Big(\prod_{j=1}^m \hat{\sigma}^N_{y,j}\Big)Z_N(t,y,k), \quad t \in [\tilde{\mathfrak{s}}^N_{y,m}, \tilde{\mathfrak{s}}^N_{y,m+1}),\ m = 0,1,\ldots. \tag{4.18}$$

It is straightforward to show that for any $\eta > 0$ we have

$$\lim_{N\to+\infty} \mathbb{P}\Big[\mathrm{d}_\infty(Z^o_N, \tilde{Z}^o_N) \ge \eta\Big] = 0. \tag{4.19}$$

Therefore, we may now pass from $\tilde{Z}^o_N$ to Z^o_N and prove convergence of the discontinuous processes Z^o_N to the discontinuous jump process. This can be done using the J_1-topology as both processes are discontinuous, and is simpler than working directly in the M_1-topology. Accordingly, $\mathcal{X}'$ be the space $\mathcal{X}$, equipped with the product topology, where on the first component we put the J_1 topology rather than M_1. We will prove that the random elements

$$\Big((Z_N^o(t,y,k))_{t\geq 0}, (\mathfrak{T}_N(t,k))_{t\geq 0}, (\mathfrak{s}_{y,m}^N)_{m\geq 1}\Big)$$

converge in law over $\mathcal{X}'$ to $\Big((\zeta^o(t,y))_{t\geq 0}, (\tau(t))_{t\geq 0}, (\mathfrak{u}_{y,m})_{m\geq 1}\Big)$. Thanks to (4.6) and (4.19) this will finish the proof of the theorem. Since we have already proved the convergence of the last two components, see Proposition 4.1 and Theorem 4.4, we focus only on proving the convergence in law of $(Z_N^o(t))_{t\geq 0}$ over $D[0,+\infty)$, equipped with the J_1-topology to $(\zeta^o(t,y))_{t\geq 0}$.

Lemma 4.8 *For any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $\mathbb{P}\left[\mathfrak{f}_N \geq M_\varepsilon\right] < \varepsilon$ for all $N \geq 1$.*

Proof. From the continuity of $\mathfrak{g}(\cdot)$ and its strict positivity we have

$$\delta := \inf_{k\in\mathbb{T}} \mathfrak{g}(k) > 0.$$

The definition of the sequence $(\hat{\sigma}_{y,m}^N)$ implies that

$$\mathbb{P}\left[\mathfrak{f}_N \geq M\right] = \mathbb{P}\left[\hat{\sigma}_{y,1}^N, \dots, \hat{\sigma}_{y,M}^N \in \{-1,1\}\right] \leq (1-\delta)^M$$

and the conclusion of the lemma follows, upon a choice of a sufficiently large M. $\square$

Let $(U_m)_{m\geq 1}$ be a sequence of i.i.d. random variables, uniformly distributed in $(0,1)$, independent of the sequence $(K_n^k)_{n\geq 0}$. Let us define

$$\hat{\sigma}_{y,m}^N = 1_{(0,\mathfrak{p}_{m,+}^N)}(U_m) - 1_{(1-\mathfrak{p}_{m,-}^N,1)}(U_m) \tag{4.20}$$

and

$$\hat{\sigma}_m := 1_{(0,p_+)}(U_m) - 1_{(1-p_-,1)}(U_m), \quad m \geq 1, \tag{4.21}$$

where $\mathfrak{p}_{m,\pm}^N := p_\pm(K_{n_{y,m}^N}^k)$.

Lemma 4.9 *For any integer $M > 0$ and $\varepsilon > 0$ we have*

$$\mathbb{P}\Big[\bigcup_{m=1}^M [\hat{\sigma}_{y,m}^N \neq \hat{\sigma}_m]\Big] < \Big(\frac{c_+\omega_0'}{C\gamma_0 R_0}\Big)^{\gamma/\beta} \frac{2C_0 M}{N^{\gamma/(\beta+1)}} + \varepsilon, \quad N \geq 1, \tag{4.22}$$

where $C > 0$ is as in Corollary 4.6, while $C_0, \gamma > 0$ are as in (1.7).

Proof. Consider the event

$$A_N := \Big[\min_{m=1,\dots,M} |\bar{\omega}'\bar{t}(K_{n_{y,m}^N}^k)| \leq CN^{\beta/(\beta+1)}\Big],$$

where C is as in the statement of Corollary 4.6, and write

$$\begin{aligned}\mathbb{P}\Big[\bigcup_{m=1}^{M}\left[\hat{\sigma}^{N}_{y,m}\neq\hat{\sigma}_m\right]\Big] &\le \mathbb{P}\Big[\bigcup_{m=1}^{M}\left[\hat{\sigma}^{N}_{y,m}\neq\hat{\sigma}_m\right], A_N^c\Big]+\mathbb{P}[A_N]\\ &\le \mathbb{P}\Big[\bigcup_{m=1}^{M}[\hat{\sigma}^{N}_{y,m}\neq\hat{\sigma}_m],\ \min_{m=1,\dots,M}|\bar{\omega}'\bar{t}(K^k_{n^N_{y,m}})|>CN^{\beta/(\beta+1)}\Big]+\varepsilon \qquad (4.23)\\ &\le \sum_{m=1}^{M}\mathbb{P}\left[\hat{\sigma}^{N}_{y,m}\neq\hat{\sigma}_m,\ |\bar{\omega}'\bar{t}(K^k_{n^N_{y,m}})|>CN^{\beta/(\beta+1)}\right]+\varepsilon.\end{aligned}$$

Note that, for all $m\ge 1$ we have

$$\begin{aligned}&\mathbb{P}\left[\hat{\sigma}^{N}_{y,m}\neq\hat{\sigma}_m,\ |\bar{\omega}'\bar{t}(K^k_{n^N_{y,m}})|>CN^{\beta/(\beta+1)}\right]\\ &\le \mathbb{E}\Big[|1_{(0,\mathfrak{p}^N_{m,+})}(U_m)-1_{(0,p_+)}(U_m)|+|1_{(1-\mathfrak{p}^N_{m,-},1)}(U_m)-1_{(1-p_-,1)}(U_m)|,\ |\bar{\omega}'\bar{t}(K^k_{n^N_{y,m}})|>CN^{\beta/(\beta+1)}\Big]\\ &=\sum_{\iota=\pm}\mathbb{E}\left[|p_\iota(K^k_{n^N_{y,m}})-p_\iota|,\ |\bar{\omega}'\bar{t}(K^k_{n^N_{y,m}})|>CN^{\beta/(\beta+1)}\right].\end{aligned}$$

By virtue of (1.7) and (3.9), the right side can be estimated by

$$\sum_{\iota=\pm}\mathbb{E}\Big[\Big|p_\iota(K^k_{n^N_{y,m}})-p_\iota\Big|,\ |K^k_{n^N_{y,m}}|<\Big(\frac{c_+\omega_0'}{C\gamma_0R_0}\Big)^{1/\beta}\frac{1}{N^{1/(1+\beta)}}\Big]\le 2C_0\Big(\frac{c_+\omega_0'}{C\gamma_0R_0}\Big)^{\gamma/\beta}\frac{1}{N^{\gamma/(\beta+1)}},$$

and (4.22) follows. □

Next, we define the process

$$\hat{Z}^o_N(t):=\Big(\prod_{j=1}^{m}\hat{\sigma}_j\Big)Z_N(t,y,k),\quad t\in[\tilde{\mathfrak{s}}^N_{y,m},\tilde{\mathfrak{s}}^N_{y,m+1}),\ m=0,1,\dots, \qquad (4.24)$$

with the random variables σ_m given by (4.21). Using Lemma 4.8 to choose M large enough, and then Lemma 4.9 to choose N large, we conclude the following.

Corollary 4.10 *Let $(Z^o_N(t))_{t\ge0}$ be defined by (4.18) with $\left(\hat{\sigma}^N_{y,m}\right)_{m\ge1}$ given by (4.20). Then, for any $\varepsilon>0$ there exists N_0 such that*

$$\mathbb{P}\left[Z^o_N\neq\hat{Z}^o_N\right]<\varepsilon,\quad N\ge N_0. \qquad (4.25)$$

Theorem 4.4 and Corollary 4.10 immediately imply Theorem 4.7. □

Given any $t_0\ge0$ the limiting process $(\zeta^o(t,y))_{t\ge0}$ is a.s. continuous at t_0, as a consequence of an analogous property of $(\zeta(t,y))_{t\ge0}$ mentioned earlier (see Proposition 1.2.7 p. 21 of [7]). It follows that the coordinate mapping is continuous on an event of probability one in the M_1 topology, see Theorem 12.5.1 part (v) of [25]. As a consequence we conclude the following.

Corollary 4.11 *The processes $(\tilde{Z}^o_N(t,y,k))_{t\ge0}$ converge in the sense of finite-dimensional distributions, as $N\to+\infty$, to the process $(\zeta^o(t,y))_{t\ge0}$.*

4.4 The re-scaled process for the kinetic equation

Let us now introduce the process corresponding to the kinetic equation (1.8) with reflection-transmission-killing at the interface:

$$Y_N^o(t) := \tilde{Z}_N^o(\mathfrak{T}_N^{-1}(t;k)), \quad t \geq 0, \tag{4.26}$$

where $\mathfrak{T}_N$ and $\tilde{Z}_N^o$ are given by (4.1) and (4.14), respectively. We set

$$\hat{\mathfrak{s}}_{y,m}^N := \mathfrak{T}_N(\tilde{\mathfrak{s}}_{y,m}^N; k) \quad \text{and} \quad \hat{\mathfrak{s}}_{\mathfrak{f}}^N := \mathfrak{T}_N(\tilde{\mathfrak{s}}_{y,\mathfrak{f}_N}^N; k), \tag{4.27}$$

where $\mathfrak{f}_N$ is as in (4.13). We let furthermore $\tilde{\sigma}^N(t) \equiv 1$ for $t \in [0, \tilde{\mathfrak{s}}_{y,1}^N)$ and

$$\tilde{\sigma}_N(t) := \prod_{j=1}^m \hat{\sigma}_{y,j}^N, \quad t \in [\tilde{\mathfrak{s}}_{y,m}^N, \tilde{\mathfrak{s}}_{y,m+1}^N),\ m \geq 1, \tag{4.28}$$

and

$$K_N^o(t,y) := \sigma_N^o(t) K_N(t,k), \tag{4.29}$$

with $\sigma_N^o(t) := \tilde{\sigma}_N(\mathfrak{T}_N^{-1}(t))$ and

$$K_N(t,k) := K^k_{[\mathfrak{T}_N^{-1}(t;k)]}. \tag{4.30}$$

As in (3.15), we have

$$\frac{dY_N^o(t;y,k)}{dt} = -\omega'(K_N^o(t;k)), \quad t \in [0, \hat{\mathfrak{s}}_{\mathfrak{f}}^N). \tag{4.31}$$

We also set $\hat{\mathfrak{u}}_{y,m} := \tau^{-1}(\mathfrak{u}_{y,m})$ and

$$\eta^o(t,y) := \zeta^o(\tau^{-1}(t), y), \tag{4.32}$$

with ζ^o and τ given by (4.17) and (4.3), respectively. The following is a direct corollary of Theorems 4.4 and 4.7.

Theorem 4.12 *The random elements $((Y_N^o(t))_{t\geq 0}, (\hat{\mathfrak{s}}_{y,m}^N)_{m\geq 1})$ converge in law to*

$$((\eta^o(t,y))_{t\geq 0}, (\hat{\mathfrak{u}}_{y,m})_{m\geq 1}),$$

over $D[0,+\infty) \times \bar{\mathbb{R}}_+^{\mathbb{N}}$, with the product of the M_1 and standard product topologies.

Proof. By Theorem 4.7 and the Skorochod embedding theorem we can find equivalent versions of $((\tilde{Z}_N^o(t,y,k))_{t\geq 0}, (\mathfrak{T}_N(t,k))_{t\geq 0})$ converging a.s. to $((\zeta^o(t,y))_{t\geq 0}, (\tau(t))_{t\geq 0})$. Hence, $\mathfrak{T}_N^{-1}$ converge a.s. in the uniform topology on compacts to τ^{-1}. Invoking Theorem 7.2.3 p. 164 of [26], we conclude convergence of the $(Y_N^o(t))_{t\geq 0}$ to $((\eta^o(t,y))_{t\geq 0}$. The convergence of the second components is a consequence of (4.27). □

5 The proof of convergence in Theorem 1.1

It suffices to prove the convergence statement for

$$\widetilde{W}_N(t,y,k) := W_N(t,y,k) - T. \tag{5.1}$$

It satisfies (1.8) with the initial condition $\widetilde{W}_0 := W_0 - T \in \mathcal{C}_0$, so that the interface conditions (1.3), (1.4) correspond to $T = 0$. We will show that

$$\lim_{N\to+\infty}\int_{\mathbb{R}\times\mathbb{T}}\widetilde{W}_N(t,y,k)G(y,k)dydk = \int_{\mathbb{R}\times\mathbb{T}}\widetilde{W}(t,y)G(y,k)dydk, \tag{5.2}$$

for any $G \in C_c^\infty(\mathbb{R}\times\mathbb{T})$, where

$$\widetilde{W}(t,y) = \mathbb{E}\left[\bar{W}_0\left(\eta^o(t;y)\right),\, t < \hat{\mathfrak{u}}_{y,\mathfrak{f}}\right] \tag{5.3}$$

and

$$\bar{W}_0(y) := \int_{\mathbb{T}}\widetilde{W}_0(y,k)dk. \tag{5.4}$$

This will imply that

$$\begin{aligned}
&\lim_{N\to+\infty}\int_{\mathbb{R}\times\mathbb{T}}W_N(t,y,k)G(y,k)dydk = T\int_{\mathbb{R}\times\mathbb{T}}G(y,k)dydk\\
&+\lim_{N\to+\infty}\int_{\mathbb{R}\times\mathbb{T}}\widetilde{W}_N(t,y,k)G(y,k)dydk = T\int_{\mathbb{R}\times\mathbb{T}}G(y,k)dydk\\
&+\lim_{N\to+\infty}\int_{\mathbb{R}\times\mathbb{T}}\widetilde{W}(t,y)G(y,k)dydk = \int_{\mathbb{R}\times\mathbb{T}}W(t,y)G(y,k)dydk,
\end{aligned} \tag{5.5}$$

where

$$W(t,y) = T + \widetilde{W}(t,y). \tag{5.6}$$

We now prove (5.2). Using Proposition 3.2, we write

$$\widetilde{W}_N(t,y,k) = \mathbb{E}\left[\widetilde{W}_0\left(Y_N^o(t;y,k), K_N^o(t,k)\right),\, t \le \tilde{\mathfrak{s}}_{y,\mathfrak{f}_N}^N\right]. \tag{5.7}$$

For a given test function $G \in C_c^\infty(\mathbb{R}\times\mathbb{T})$ let us set

$$I_N := \int_{\mathbb{R}\times\mathbb{T}}W_N(t,y,k)G(t,y,k)dydk. \tag{5.8}$$

Our goal is to show that

$$\limsup_{N\to+\infty}\left|I_N - \int_{\mathbb{R}\times\mathbb{T}}\mathbb{E}\left[\bar{W}_0\left(\eta^o(t;y)\right),\, t < \hat{\mathfrak{u}}_{y,\mathfrak{f}}\right]G(y,k)dydk\right| < \varepsilon, \tag{5.9}$$

where $\varepsilon > 0$ is arbitrary and $\bar{W}_0(y)$ is given by (5.4). Since $\widetilde{W}_0$ is continuous outside the interface $[y = 0]$, for any $\delta > 0$ we can write that

$$\widetilde{W}_0(y,k) = W_0^1(y,k) + W_0^2(y,k),$$

where $W_0^2 \in C_b(\mathbb{R}\times\mathbb{T})$, and

$$\operatorname{supp} W_0^1 \subset [|y|<2\delta]\times\mathbb{T}, \quad \operatorname{supp} W_0^2 \subset [|y|>\delta/2]\times\mathbb{T}, \text{ and } \|W_0^j\|_\infty \le \|\widetilde{W}_0\|_\infty,\ j=1,2. \tag{5.10}$$

We can decompose accordingly $I_N = I_N^1 + I_N^2$ and $\bar{W}_0 = \bar{W}_0^1 + \bar{W}_0^2$. Then, we have

$$\begin{aligned}
&\limsup_{N\to+\infty}\left| I_N^1 - \int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[\bar{W}_0^1\left(\eta^o(t;y)\right),\, t<\hat{\mathfrak{u}}_{y,\mathfrak{f}}\right] G(y,k)dydk\right| \qquad (5.11)\\
&\le \|\widetilde{W}_0\|_\infty \int_{\mathbb{R}\times\mathbb{T}} |G(y,k)| \left(\limsup_{N\to+\infty}\mathbb{P}\left[|Y_N(t;y,k)|<2\delta\right] + \mathbb{P}\left[|\eta(t;y)|<2\delta\right]\right) dydk < \frac{\varepsilon}{10},
\end{aligned}$$

provided that $\delta>0$ is sufficiently small.

Let us set

$$I_N^2(\delta) := \int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[W_0^2\left(Y_N^o(t-\delta;y,k), K_N^o(t,k)\right),\, t-\delta \le \tilde{\mathfrak{s}}^N_{y,\mathfrak{f}_N}\right] G(y,k)dydk. \tag{5.12}$$

By virtue of Lemma 4.8 and Theorem 4.12 we can write

$$\begin{aligned}
&\limsup_{N\to+\infty}\left|I_N^2 - I_N^2(\delta)\right|\\
&\le \limsup_{N\to+\infty}\int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[\sup_{k'}\left|W_0^2\left(Y_N^o(t;y,k),k'\right) - W_0^2\left(Y_N^o(t-\delta;y,k),k'\right)\right|\right] |G(y,k)|dydk\\
&+ \|\widetilde{W}_0\|_\infty \limsup_{N\to+\infty}\int_{\mathbb{R}\times\mathbb{T}} \mathbb{P}\left[t-\delta \le \tilde{\mathfrak{s}}^N_{y,\mathfrak{f}_N} < t\right] |G(y,k)|dydk < \frac{\varepsilon}{10} \qquad (5.13)
\end{aligned}$$

if $\delta>0$ is sufficiently small. We have used here the fact that, for each $m\ge 1$ the law of $(\hat{\mathfrak{u}}_{y,1},\dots,\hat{\mathfrak{u}}_{y,m})$ – the limit of the laws of $(\tilde{\mathfrak{s}}^N_{y,1},\dots,\tilde{\mathfrak{s}}^N_{y,m})$, as $N\to+\infty$, – is absolutely continuous with respect to the Lebesgue measure. This is a consequence of the strong Markov property of $(\eta(t,y))_{t\ge 0}$ and the fact that the joint law of $(\hat{\mathfrak{u}}_{y,1}, \eta(\hat{\mathfrak{u}}_{y,1},y))$ is absolutely continuous with respect to the Lebesgue measure, see e.g. Theorem 1, p. 93 of [11].

To conclude (5.9), it suffices to prove that we can choose a sufficiently small $\delta>0$ so that

$$\limsup_{N\to+\infty}\left| I_N^2(\delta) - \int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[\bar{W}_0^2\left(\eta^o(t;y)\right),\, t<\hat{\mathfrak{u}}_{y,\mathfrak{f}}\right] G(y,k)dydk\right| < \frac{\varepsilon}{10}. \tag{5.14}$$

To this end, we will assume, without loss of any generality, that $W_0^2 \in C_c^\infty(\mathbb{R}\times\mathbb{T})$. Indeed, for any $W_0^2 \in C_b(\mathbb{R}\times\mathbb{T})$ satisfying (5.10) and $R>0$, we can find $W_0^{2,s} \in C_c^\infty(\mathbb{R}\times\mathbb{T})$ and such that

$$\|W_0^{2,s}\|_\infty \le \|W_0^2\|_\infty + 1 \text{ and } \sup_{|y|\le R, k\in\mathbb{T}} |W_0^{2,s}(y,k) - W_0^2(y,k)| < \frac{\varepsilon}{100}.$$

Thanks to the already established tightness of the laws of $Y_N^o(t,y,k)$ we can easily see that, upon the choice of a sufficiently large $R>0$,

$$\limsup_{N\to+\infty}\left|I_N^2(\delta) - I_N^{2,s}(\delta)\right| < \frac{\varepsilon}{10},$$

where $I_N^{2,s}(\delta)$ is defined by (5.12), with $W_0^{2,s}$ replacing W_0^2. From here on, we will restrict our attention to $I_N^{2,s}(\delta)$.

Using Lemmas 4.8 and 4.9, together with the conclusion of Theorem 4.12 one can show that for a sufficiently small $\delta > 0$ we have

$$\limsup_{N\to+\infty} \left| I_N^2(\delta) - \tilde{I}_N^2(\delta) \right| < \frac{\varepsilon}{10}, \tag{5.15}$$

where

$$\tilde{I}_N^2(\delta) := \int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[W_0^2\left(Y_N^o(t-\delta; y,k), \hat{K}_N^o(t,k) \right), \, t-\delta \le \tilde{\mathfrak{s}}_{y,\mathfrak{f}_N}^N \right] G(y,k)dydk,$$

and

$$\hat{K}_N^o(t,y) := \hat{\sigma}_N^o(t-\delta) K_N(t,k), \tag{5.16}$$

where $\hat{\sigma}_N^o(t) := \hat{\sigma}_N(T_N^{-1}(t))$, $\hat{\sigma}_N(t) \equiv 1$ for $t \in [0, \tilde{\mathfrak{s}}_{y,1}^N)$, and

$$\hat{\sigma}_N(t) := \prod_{j=1}^m \hat{\sigma}_j, \quad t \in [\tilde{\mathfrak{s}}_{y,m}^N, \tilde{\mathfrak{s}}_{y,m+1}^N), \, m \ge 1. \tag{5.17}$$

Conditioning on $\mathcal{K}_{t-\delta}$, where $\left(\mathcal{K}_t^N\right)_{t\ge 0}$ is the natural filtration of $(K_N(t,k))_{t\ge0}$, we write

$$\tilde{I}_N^2(\delta) = \hat{I}_N^2(\delta) + \bar{I}_N^2(\delta),$$

where

$$\hat{I}_N^2(\delta) := \int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[\bar{W}_0^2(Y_N^o(t-\delta;y,k)), \, t-\delta \le \tilde{\mathfrak{s}}_{y,\mathfrak{f}_N}^N \right] G(y,k)dydk,$$
$$\bar{I}_N^2(\delta) := \int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[w_N(Y_N^o(t-\delta;y,k), \hat{\sigma}_N^o(t-\delta) K_N(t-\delta,k)), \, t-\delta \le \tilde{\mathfrak{s}}_{y,\mathfrak{f}_N}^N \right] G(y,k)dydk$$

and

$$\bar{W}_0^2(y) := \int_{\mathbb{T}} W_0^2(y,k)\mu(dk).$$

We have used above the notation

$$w_N(y,k) := e^{N\delta L} W_0^2\left(y,\cdot\right)(k) = \frac{1}{2\pi} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{N\delta L} \mathfrak{e}_\ell(k) \int_{\mathbb{R}} \widehat{W}_0^2\left(\xi,\ell\right) e^{i\xi y} d\xi,$$

with the generator L given by (1.2), $\mathfrak{e}_\ell(k) := \exp\left\{2\pi i k\ell\right\}$ and

$$\widehat{W}_0^2\left(\xi,\ell\right) = \int_{\mathbb{R}\times\mathbb{T}} W_0^2\left(y,k\right) e^{-i\xi y} \mathfrak{e}_\ell^\star(k) dydk.$$

The term $\bar{I}_N^2(\delta)$ we can estimated as follows:

$$\begin{aligned} \left|\bar{I}_N^2(\delta)\right| &\le \int_{\mathbb{R}} \sup_{k\in\mathbb{T}} |G(y,k)|dy \\ &\times \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}\times\mathbb{T}} |\widehat{W}_0^2\left(\xi,\ell\right)| \mathbb{E}\Big[|e^{N\delta L}\mathfrak{e}_\ell(K_N(t-\delta,k))| + |e^{N\delta L}\mathfrak{e}_\ell(-K_N(t-\delta,k))| \Big] d\xi dk \\ &= 2\int_{\mathbb{R}} \sup_{k\in\mathbb{T}} |G(y,k)|dy \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \|e^{N\delta L}\mathfrak{e}_\ell\|_{L^1(\mathbb{T})} \int_{\mathbb{R}} |\widehat{W}_0^2\left(\xi,\ell\right)| d\xi. \end{aligned} \tag{5.18}$$

Now, (3.5) implies that $\|e^{N\delta L}\mathfrak{e}_\ell\|_{L^1(\mathbb{T})} \to 0$, as $N \to +\infty$, for each $\ell \neq 0$. Therefore,

$$\lim_{N\to+\infty} \bar{I}_N^2(\delta) = 0.$$

It follows from Theorem 4.12 that

$$\limsup_{N\to+\infty} \left| \hat{I}_N^2(\delta) - \int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}\left[\bar{W}_0^2\left(\eta^o(t;y)\right), \, t < \hat{\mathfrak{u}}_{y,\mathfrak{f}}\right] G(y,k)dydk \right| < \frac{\varepsilon}{2}, \tag{5.19}$$

provided that $\delta > 0$ is sufficiently small. This ends the proof of (5.2).

6 Proof of Theorem 1.1: description of the limit

So far, we have shown the weak convergence of $W_N(t,y,k)$ to $W(t,y)$, in the sense of (5.2), with $W(t,y)$ defined in (5.6) and (5.3). We now identify $W(t,y)$ as a weak solution to (2.5) if $W_0 \in \mathcal{C}_T$. Thanks to (5.1) and (5.6) it suffices only to consider the case $T = 0$. Consider a regularized scattering kernel: take $a \in (0,1)$ and set

$$q_\beta^{(a)}(y) = \frac{c_\beta 1_{(a,+\infty)}(|y|)}{|y|^{2+1/\beta}}, \quad y \in \mathbb{R}_*.$$

Let $(\eta_a(t,y))_{t\ge0}$ be a Levy process starting at $y \in \mathbb{R}$, with the generator $-\hat{c}\Lambda_\beta^{(a)}$, where

$$\Lambda_\beta^{(a)} F(y) := \int_{\mathbb{R}} q_\beta^{(a)}(y-y')[F(y) - F(y')]dy', \quad F \in B_b(\mathbb{R}). \tag{6.1}$$

It is well known, see e.g. Section 2.5 of [16], that $(\eta_a(t,y))_{t\ge0}$ converge in law, as $a \to 0^+$, over $D[0,+\infty)$, with the topology of the uniform convergence on compacts, to $(\eta(t,y))_{t\ge0}$, the symmetric stable process with the generator $-\hat{c}\Lambda_\beta$, as in (2.4).

We define inductively the times of jumps over the interface

$$\hat{\mathfrak{u}}_{y,1}^a := \inf[t > 0 : \, \eta_a(t-,y)\eta_a(t,y) < 0], \quad \hat{\mathfrak{u}}_{y,m+1}^a := \inf[t > \hat{\mathfrak{u}}_{y,m}^a : \, \eta_a(t-,y)\eta_a(t,y) < 0].$$

To set up the reflected-transmitted-killed process, let $(\hat{\sigma}_m)_{m\ge1}$ be a sequence of i.i.d. $\{-1,0,1\}$ random variables, independent of $(\eta_a(t,y))_{t\ge0}$ distributed according to (4.15), and set

$$\eta_a^o(t,y) := \Big(\prod_{j=1}^m \hat{\sigma}_j\Big)\eta_a(t,y), \quad t \in [\hat{\mathfrak{u}}_{y,m}^a, \hat{\mathfrak{u}}_{y,m+1}^a), \, m \ge 0, \tag{6.2}$$

where $\hat{\mathfrak{u}}_{y,0}^a := 0$. Using Theorem B.3 together with the argument in Section 4.3 we easily conclude the following.

Theorem 6.1 *The random elements $((\eta_a^o(t,y))_{t\ge0}, (\hat{\mathfrak{u}}_{y,m}^a)_{m\ge1})$ converge in law over the product space $D[0,+\infty) \times \bar{\mathbb{R}}_+^{\mathbb{N}}$, equipped with the product of the topology of uniform convergence on compacts and the standard infinite product topology, to $((\eta^o(t,y))_{t\ge0}, (\hat{\mathfrak{u}}_{y,m})_{m\ge1})$, as $a \to 0$.*

As a direct corollary of the above theorem we conclude that

$$\lim_{a\to 0+}\widetilde{W}^{(a)}(t,y)=\widetilde{W}(t,y),\quad (t,y)\in\bar{\mathbb{R}}_+\times\mathbb{R}_*, \tag{6.3}$$

with $\widetilde{W}(t,y)$, the limit of $W_N(t,y,k)$, given by (5.3), and

$$\widetilde{W}^{(a)}(t,y)=\mathbb{E}\left[\bar{W}_0\left(\eta_a^o(t;y)\right),\, t<\hat{\mathfrak{u}}^a_{y,\mathfrak{f}}\right], \tag{6.4}$$

where $\bar{W}_0$ is given by (5.4), and $\mathfrak{f}$ by (4.16).

Note that $\widetilde{W}^{(a)}(t,y)$ satisfies

$$\partial_t\widetilde{W}^{(a)}(t,y)=\hat{c}\hat{L}_a\widetilde{W}^{(a)}(t,y),\quad (t,y)\in\mathbb{R}_+\times\mathbb{R}_*, \tag{6.5}$$

in the classical sense, where

$$\begin{aligned}\hat{L}_aF(y):=&-\Lambda_\beta^{(a)}F(y)+p_-\int_{[yy'<0]}q_\beta(y-y')[F(-y')-F(y')]dy'\\&-\mathfrak{g}_0\int_{[yy'<0]}q_\beta(y-y')F(y')dy',\quad F\in B_b(\mathbb{R}).\end{aligned}$$

Indeed, let

$$Q^{(a)}:=\hat{c}\int_{\mathbb{R}}q_\beta^{(a)}(y)dy,$$

and for $\Delta t\ll 1$ and $t>0$ write

$$\begin{aligned}\widetilde{W}^{(a)}(t+\Delta t,y)&=\mathbb{E}[\bar{W}_0(\eta_a^o(t+\Delta t,y)),\,t+\Delta t<\hat{\mathfrak{u}}^a_{y,\mathfrak{f}}]=\mathbb{E}\left[\bar{W}_0(\eta_a^o(t,\eta_a^o(\Delta t,y)),t<\hat{\mathfrak{u}}^a_{\eta_a^o(\Delta t,y),\mathfrak{f}}\right]\\&=e^{-Q^{(a)}\Delta t}\widetilde{W}^{(a)}(t,y)+\hat{c}\int_{[yy'>0]}q_\beta(y-y')\widetilde{W}^{(a)}(t,y')dy'\Delta t\\&+\underbrace{(1-p_--\mathfrak{g}_0)}_{=p_+}\hat{c}\int_{[yy'<0]}q_\beta(y-y')\widetilde{W}^{(a)}(t,y')dy'\Delta t\\&+p_-\hat{c}\int_{[yy'>0]}q_\beta(y+y')\widetilde{W}^{(a)}(t,y')dy'\Delta t+o(\Delta t).\end{aligned}$$

It follows that

$$\widetilde{W}^{(a)}(t+\Delta t,y)-\widetilde{W}^{(a)}(t,y)=\hat{c}\hat{L}_a\widetilde{W}^{(a)}(t,y)\Delta t+o(\Delta t),$$

which implies (6.5). Thanks to (6.3), we conclude that $\widetilde{W}$ satisfies Definition 2.3.

7 Proof of Theorem 2.5

By considering the kinetic equation with the initial data $W_0':=W_0-T'$ we may assume that $T'=0$ and $W_0\in L^1(\mathbb{R}\times\mathbb{T})$.

Assume first that $T = 0$. Let $\|\cdot\|_{\mathcal{H}_a}$ be defined by the analog to (2.7), with the kernel $q_\beta^{(a)}(\cdot)$ replacing $q_\beta(\cdot)$, and the Hilbert space $\mathcal{H}_a$ be the completion of $C_0^\infty(\mathbb{R}_*)$ in the respective norm. Obviously, we have

$$\|G\|_{\mathcal{H}_a} \le \|G\|_{\mathcal{H}_{a'}}, \quad G \in C_0^\infty(\mathbb{R}_*),\ a > a' \ge 0. \tag{7.1}$$

As with (2.9), we have

$$\frac{d}{dt}\|\widetilde{W}^{(a)}(t)\|^2_{L^2(\mathbb{R})} = -\hat{c}\|\widetilde{W}^{(a)}(t)\|^2_{\mathcal{H}_a}, \tag{7.2}$$

so that

$$\|\widetilde{W}^{(a)}(t)\|^2_{L^2(\mathbb{R})} + \hat{c}\int_0^t \|\widetilde{W}^{(a)}(s)\|^2_{\mathcal{H}_a} ds = \|\bar{W}_0\|^2_{L^2(\mathbb{R})}, \quad t \ge 0,\ a > 0. \tag{7.3}$$

Letting $a \to 0+$ we conclude, from (7.3) and (6.3) that

$$\|\widetilde{W}(t)\|^2_{L^2(\mathbb{R})} + \hat{c}\int_0^t \|\widetilde{W}(s)\|^2_{\mathcal{H}_0} ds \le \|\bar{W}_0\|^2_{L^2(\mathbb{R})}, \quad t \ge 0, \tag{7.4}$$

which implies part (i) of Definition 2.3.

When $T \ne 0$, let us set

$$W^{(a)}(t,y) := \mathbb{E}\left[\bar{W}_0\left(\eta_a^o(t;y)\right),\ t < \hat{\mathfrak{u}}^a_{y,\mathfrak{f}}\right] + T\mathbb{P}[t \ge \hat{\mathfrak{u}}^a_{y,\mathfrak{f}}], \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}_*, \tag{7.5}$$

where $\bar{W}_0$ is given by (5.4). It follows from (6.5) that $W^{(a)}$ satisfies

$$\partial_t W^{(a)}(t,y) = \hat{c}L_a W^{(a)}(t,y) + \hat{c}p_- \int_{[yy'<0]} q_\beta^{(a)}(y-y')[W^{(a)}(t,-y') - W^{(a)}(t,y')]dy'$$
$$+ \hat{c}\mathfrak{g}_0 \int_{[yy'<0]} q_\beta^{(a)}(y-y')[T - W^{(a)}(t,y')]dy', \tag{7.6}$$

while (5.6) and (6.3) imply

$$\lim_{a\to 0+} W^{(a)}(t,y) = W(t,y), \quad (t,y) \in \bar{\mathbb{R}}_+ \times \mathbb{R}_* \tag{7.7}$$

and

$$W(t,y) = \mathbb{E}\left[\bar{W}_0\left(\eta^o(t;y)\right),\ t < \hat{\mathfrak{u}}_{y,\mathfrak{f}}\right] + T\mathbb{P}[t \ge \hat{\mathfrak{u}}_{y,\mathfrak{f}}], \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}_*. \tag{7.8}$$

Part (i) of Definition 2.3 is a direct conclusion from the following.

Proposition 7.1 *If $\bar{W}_0 \in C_b(\mathbb{R}_*) \cap L^1(\mathbb{R})$, then $W \in L^\infty_{\rm loc}([0,+\infty), L^2(\mathbb{R}))$ and $W - T \in L^2_{\rm loc}([0,+\infty); \mathcal{H}_0)$.*

Proof. Let $\widetilde{W}^{(a)}(t) := W^{(a)}(t) - T$. Multiplying both sides of (7.6) by $\widetilde{W}^{(a)}(t,y)$ and integrating in the y variable we obtain

$$\|\bar{W}_0\|^2_{L^2(\mathbb{R})} = \|W^{(a)}(t)\|^2_{L^2(\mathbb{R})} + 2T\left(\int_{\mathbb{R}} W^{(a)}(t,y)dy - \int_{\mathbb{R}} \bar{W}_0(y)dy\right) + \hat{c}\int_0^t \|\widetilde{W}^{(a)}(s)\|^2_{\mathcal{H}_a} ds, \tag{7.9}$$

hence

$$\|W^{(a)}(t)\|^2_{L^2(\mathbb{R})}+\hat{c}\int_0^t\|\widetilde{W}^{(a)}(s)\|^2_{\mathcal{H}_a}ds\le\|\bar{W}_0\|^2_{L^2(\mathbb{R})}+2T\left(\|W^{(a)}(t)\|_{L^1(\mathbb{R})}+\|\bar{W}_0\|_{L^1(\mathbb{R})}\right). \tag{7.10}$$

To estimate the L^1-norm in the right side, note that (7.5) implies

$$|W^{(a)}(t,y)|\le\mathbb{E}\left[|\bar{W}_0\left(\eta_a(t;y)\right)|+|\bar{W}_0\left(-\eta_a(t;y)\right)|\right]+T\mathbb{P}[t\ge\hat{\mathfrak{u}}^a_{y,1}], \tag{7.11}$$

where $\eta_a(\cdot,y)$ is the Levy process with the generator (6.1) starting at y, thus

$$\|W^{(a)}(t)\|_{L^1(\mathbb{R})}\le2\|\bar{W}_0\|_{L^1(\mathbb{R})}+2T\int_0^{+\infty}\mathbb{P}[t\ge\hat{\mathfrak{u}}^a_{y,1}]dy\le2\|\bar{W}_0\|_{L^1(\mathbb{R})}+2T\mathbb{E}[\sup_{s\in[0,t]}\eta_a(s;0)]. \tag{7.12}$$

Since $(\eta_a(t,0))_{t\ge0}$ is a martingale, we may use the Doob maximal inequality $((\eta_a(t,0))_{t\ge0}$ to see that there exists $C>0$ such that

$$\left\{\mathbb{E}\Big[(\sup_{s\in[0,t]}\eta_a(s;0))^\kappa\Big]\right\}^{1/\kappa}\le C\left\{\mathbb{E}\left[|\eta_a(t;0)|^\kappa\right]\right\}^{1/\kappa}, \tag{7.13}$$

with $1<\kappa<1+1/\beta$. The argument in the proof of Lemma 5.25.7, p. 161 of [22] implies

$$\limsup_{a\to0+}\mathbb{E}\left[|\eta_a(t;0)|^\kappa\right]<+\infty,$$

so that, in particular, $\mathbb{E}[\sup_{s\in[0,t]}\eta(s;0)]<+\infty$. Letting $a\to0^+$ in (7.12), we obtain

$$\begin{aligned}\lim_{a\to0+}\|W^{(a)}(t)\|_{L^1(\mathbb{R})}&=\|W(t)\|_{L^1(\mathbb{R})}\le2\|\bar{W}_0\|_{L^1(\mathbb{R})}+2T\mathbb{E}[\sup_{s\in[0,t]}\eta(s;0)]\\&\le2\|\bar{W}_0\|_{L^1(\mathbb{R})}+2Tt^{\beta/(1+\beta)}\mathbb{E}[\sup_{s\in[0,1]}\eta(s;0)].\end{aligned} \tag{7.14}$$

The last inequality follows from the self-similarity of the stable process $(\eta(t;0))_{t\ge0}$. Now, we use (7.14) to bound the right side of (7.10), and pass to the limit $a\to0^+$ of that inequality to finish the proof. □

A Proof of the existence part of Proposition 2.2

We may assume without loss of generality that $T=0$ in (1.3) and (1.4), since if $W(t,y,k)$ is a solution of (1.1) in this case, with the respective interface conditions, then $W(t,y,k)+T$ solves the corresponding problem with a given temperature $T>0$. Consider a semigroup of bounded operators on $L^\infty(\mathbb{R}\times\mathbb{T}_*)$ defined by

$$\begin{aligned}S_tW_0\left(y,k\right)&=e^{-\gamma_0R(k)t}W_0\left(y-\bar{\omega}'(k)t,k\right)1_{[0,\bar{\omega}'(k)t]^c}(y)+p_+(k)e^{-\gamma_0R(k)t}\\&\times W_0\left(y-\bar{\omega}'(k)t,k\right)1_{[0,\bar{\omega}'(k)t]}(y)+p_-(k)e^{-\gamma_0R(k)t}W_0\left(-y+\bar{\omega}'(k)t,-k\right)1_{[0,\bar{\omega}'(k)t]}(y),\end{aligned} \tag{A.1}$$

with $W_0\in L^\infty(\mathbb{R}\times\mathbb{T}_*)$, $t\ge0$ and $(y,k)\in\mathbb{R}_*\times\mathbb{T}_*$. Note that if W_0 is continuous on $\mathbb{R}_*\times\mathbb{T}_*$, then $S_tW_0(y,k)$ satisfies the interface conditions (1.3) and (1.4) (with $T=0$) for all $t>0$, so that S_t maps $\mathcal{C}_0$ to $\mathcal{C}_0$, for any $t \geq 0$ fixed. In addition, $(S_t)_{t\geq 0}$ is a C_0-semigroup on $\mathcal{C}_0$, with the supremum norm, satisfying

$$D_t[S_t W_0(y,k)] = -\gamma_0 R(k) S_t W_0(y,k), \tag{A.2}$$

together with the interface condition (1.18) and the initial condition

$$\lim_{t\to 0} S_t W_0(y,k) = W_0(y,k), \quad (y,k) \in \mathbb{R}_* \times \mathbb{T}_*.$$

Using this semigroup, we can rewrite equation (1.1) in the mild formulation

$$W(t,y,k) = S_t W_0(y,k) + \gamma_0 \int_0^t S_{t-s} \mathcal{R} W(s,y,k) ds, \quad (t,y,k) \in \mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{T}_*, \tag{A.3}$$

with

$$\mathcal{R}F(y,k) := \int_{\mathbb{T}} R(k,k') F(y,k') dk', \quad F \in L^\infty(\mathbb{R} \times \mathbb{T}). \tag{A.4}$$

The solution of (A.3) with $W_0 \in \mathcal{C}_0$ can be written as the Duhamel series

$$W(t,y,k) = \sum_{n=0}^{+\infty} S^{(n)}(t,y,k), \quad (t,y,k) \in \mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{T}_*, \tag{A.5}$$

where

$$\begin{aligned} &S^{(0)}(t,y,k) := S_t W_0(y,k), \\ &S^{(n)}(t,y,k) := \gamma_0^n \int_{\Delta_n(t)} S_{t-s_1} \mathcal{R} S_{s_1-s_2} \ldots \mathcal{R} S_{s_{n-1}-s_n} \mathcal{R} S_{s_n} W_0(y,k) ds_{1,n}, \quad n \geq 1, \end{aligned}$$

and

$$\Delta_n(t) := [t \geq s_1 \geq \ldots \geq s_n \geq 0], \quad ds_{1,n} := ds_1 \ldots ds_n.$$

Since W_0 is bounded, the series is uniformly convergent on any $[0,t] \times \mathbb{R} \times \mathbb{T}_*$. Moreover, if $W_0 \in \mathcal{C}_0$, then $S_s W_0 \in \mathcal{C}_0$ for all $s \geq 0$ and the function $\mathcal{R} S_s W_0$ is bounded and continuous in $\mathbb{R}_* \times \mathbb{T}_*$, though it need not satisfy (1.18). On the other hand, the function $S_{t-s} \mathcal{R} S_s W_0$ satisfies the interface condition (1.18) for all $s \in [0,t]$, thus so does

$$S^{(1)}(t,y,k) = \int_0^t S_{t-s} \mathcal{R} S_s W_0(y,k) ds,$$

and $S^{(1)}(t,\cdot,\cdot) \in \mathcal{C}_0$ for each $t \geq 0$. A similar argument shows that $S^{(n)}(t,\cdot,\cdot) \in \mathcal{C}_0$ for all $t \geq 0$ and $n \geq 1$. Hence, $W(t,\cdot)$ defined by the series (A.3) belongs to $\mathcal{C}_0$ for each $t > 0$. One can also verify easily that both (2.2) and (2.3) hold. Thus, $W(t,y,k)$ is a solution of (1.1) in the sense of Definition 2.1, which ends the proof of the existence part of Proposition 2.2.□

B Proof of Theorem 4.4

B.1 Preliminaries on the Skorokhod space $D[0,+\infty)$

Let us denote by $D[0,+\infty)$ the space of the cadlag functions, see [1]. The J_1-topology on $D[0,+\infty)$ is induced by the metric

$$\rho_\infty(X_1,X_2) := \int_0^{+\infty} (\rho_T(X_1,X_2)\wedge 1)e^{-T}dT, \quad X_1,X_2\in D[0,+\infty),$$

where

$$\rho_T(X_1,X_2) := \inf_{\lambda\in\Lambda_T} \max\Big\{\sup_{t\in[0,T]}|\lambda(t)-t|, \sup_{t\in[0,T]}|X_1\circ\lambda(t)-X_2(t)|\Big\}, \quad X_1,X_2\in D[0,T]. \tag{B.1}$$

Here $D[0,T]$ is the space of cadlag functions on $[0,T]$ and Λ_T is the collection of homeomorphisms $\lambda:[0,T]\to[0,T]$ such that $\lambda(0)=0$, $\lambda(T)=T$. Theorem 16.1 of [1] says that for a sequence $(X_n)_{n\geq 1}$ of cadlag functions: $X_n\to_{\rho_\infty} X$ iff there exists a sequence of strictly increasing homeomorphisms $\lambda_n:[0,+\infty)\to[0,+\infty)$ such that for each $N>0$ we have

$$\lim_{n\to+\infty}\sup_{0\leq t\leq N}|t-\lambda_n(t)|=0 \quad\text{and}\quad \lim_{n\to+\infty}\sup_{0\leq t\leq N}|X_n(t)-X\circ\lambda_n(t)|=0. \tag{B.2}$$

The M_1-topology on $D[0,+\infty)$ is defined as follows. For a given $X\in D[0,T]$, let Γ_X be the graph of X:

$$\Gamma_X := [(t,z): t\in[0,T],\, z=cX(t-)+(1-c)X(t)],\ \text{for some } c\in[0,1]]. \tag{B.3}$$

We define an order on Γ_X by letting $(t_1,z_1)\leq(t_2,z_2)$ iff $t_1<t_2$, or $t_1=t_2$ and

$$|X(t_1-)-z_1|\leq|X(t_1-)-z_2|.$$

Denote by $\Pi(X)$ the set of all continuous mappings $\gamma=(\gamma^{(1)},\gamma^{(2)}):[0,1]\to\Gamma_X$ that are non-decreasing, i.e. $t_1\leq t_2$ implies that $\gamma(t_1)\leq\gamma(t_2)$. The metric $\mathrm{d}_T(\cdot,\cdot)$ is defined as follows:

$$\mathrm{d}_T(X_1,X_2) := \inf[\|\gamma_1^{(1)}-\gamma_2^{(1)}\|_\infty \vee \|\gamma_1^{(2)}-\gamma_2^{(2)}\|_\infty,\ \gamma_i=(\gamma_i^{(1)},\gamma_i^{(2)})\in\Pi(X_i),\ i=1,2].$$

This metric induces the M_1-topology in $D[0,T]$, see [25], Theorem 13.2.1. The corresponding topology in $D[0,+\infty)$ is defined by

$$\mathrm{d}_\infty(X_1,X_2) := \int_0^{+\infty}(\mathrm{d}_T(X_1,X_2)\wedge 1)e^{-T}dT, \quad X_1,X_2\in D[0,+\infty). \tag{B.4}$$

Then (see Theorem 6.3.2 of [26]), we have

$$\mathrm{d}_\infty(X_1,X_2)\leq\rho_\infty(X_1,X_2), \quad X_1,X_2\in D[0,+\infty).$$

Let $\bar{T}_y, \underline{T}_y : D[0,+\infty)\to[0,+\infty]$ be

$$\bar{T}_y(X) := \inf[t>0:\ X(t)>y], \quad \underline{T}_y(X) := \inf[t>0:\ X(t)<y], \quad X\in D[0,+\infty),$$

and for $a\geq 0$, let $\theta_a: D[0,+\infty)\to D[0,+\infty)$ be

$$\theta_a(X)(t) := X(t+a), \quad t\geq 0. \tag{B.5}$$

Finally, we use the notation $\mathcal{M},\mathcal{M}': D[0,+\infty)\to D[0,+\infty)$ for

$$\mathcal{M}(X)(t) := \sup_{0\leq s\leq t} X(s), \quad \mathcal{M}'(X) := -\mathcal{M}(-X) = \inf_{0\leq s\leq t} X(s). \tag{B.6}$$

Both of these mappings are J_1-continuous, see Theorem 7.4.1 of [26].

Joint convergence of $((Z_N(t), T_N(t))_{t\geq 0}, \mathfrak{t}_{y,1}^N)$

To simplify the notation, we suppress writing k and y in the notation of the processes, denoting them by $(Z_N(t))_{t\geq 0}$ and $(\zeta(t))_{t\geq 0}$, respectively. For $y > 0$, we introduce the consecutive times the trajectory $Z_N(t)$ crosses the level y: $\mathfrak{t}_{y,1}^N := \bar{T}_y(Z_N)$, and $\mathfrak{t}_{y,2}^N := \underline{T}_y \circ \theta_{\bar{T}_y(Z_N)}(Z_N)$. Having defined $\mathfrak{t}_{y,2m-1}^N$, $\mathfrak{t}_{y,2m}^N$ for some $m \geq 1$, we set

$$\mathfrak{t}_{y,2m+1}^N := \bar{T}_y \circ \theta_{\mathfrak{t}_{y,2m}^N}(Z_N), \quad \mathfrak{t}_{y,2m+2}^N := \underline{T}_y \circ \theta_{\mathfrak{t}_{y,2m+1}^{N,o}}(Z_N). \tag{B.7}$$

We introduce the crossing times for $y < 0$ similarly, as well as the crossing times $(\mathfrak{t}_{y,m})_{m\geq 1}$ for the process $(\zeta(t))_{t\geq 0}$. The conclusion of the theorem is equivalent to proving that

$$((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}, (\mathfrak{t}_{y,m}^N)_{m\geq 1}, (Z_N(\mathfrak{t}_{y,m}^N))_{m\geq 1})$$

converge in law, as $N \to +\infty$, to

$$((\zeta(t), \tau(t))_{t\geq 0}, (\mathfrak{t}_{y,m})_{m\geq 1}, (\zeta(\mathfrak{t}_{y,m}))_{m\geq 1}).$$

We prove this by an induction argument on m. Let us fix $y > 0$. Since $(Z_N(t))_{t\geq 0}$ converges in law to $(Z(t))_{t\geq 0}$ the processes

$$M_N(t) := \mathcal{M}(Z_N)(t), \quad t \geq 0. \tag{B.8}$$

are likewise convergent to $M(t) := \mathcal{M}(\zeta)(t)$, $t \geq 0$. Since $\mathbb{P}[\zeta(t) = \zeta(t-)] = 1$ for each $t \geq 0$ (see, for example, Proposition 1.2.7 of [7]) we have $\mathbb{P}[M(t) = M(t-)] = 1$, and, as a result, the finite-dimensional marginals of $(M_N(t))_{t\geq 0}$ converge to those of $(M(t))_{t\geq 0}$, see, for instance, Theorem 3.16.6 of [1]. Since, in addition the law of $M(t)$ is absolutely continuous, see e.g. Theorem 4.6 of [17], we have

$$\mathbb{P}[\mathfrak{t}_{y,1}^N \leq t] = \mathbb{P}[M_N(t) > y] \to \mathbb{P}[M(t) > y] = \mathbb{P}[\mathfrak{t}_{y,1} \leq t], \quad \text{as } N \to +\infty, \tag{B.9}$$

for any $y, t > 0$. Hence, both marginals of $((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}, \mathfrak{t}_{y,1}^N)$ converge in law towards the respective laws of the marginals of $((\zeta(t), \tau(t))_{t\geq 0}, \mathfrak{t}_{y,1})$. We need to show the joint convergence.

Let us recall that $\mathcal{D}_2 := D([0, +\infty); \mathbb{R} \times \bar{\mathbb{R}}_+)$, and let $F : \mathcal{D}_2 \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ be a bounded and continuous function. We need to show that, see Theorem 1.1.1(ii) of [23],

$$\lim_{N\to+\infty} \mathbb{E}F((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}, \mathfrak{t}_{y,1}^N) = \mathbb{E}F((\zeta(t), \tau(t))_{t\geq 0}, \mathfrak{t}_{y,1}). \tag{B.10}$$

It is straightforward to check that it is suffices to prove (B.10) only for functions of the form $F(\omega, t) = G(\omega)\psi(t)$, with a bounded continuous function $G : \mathcal{D}_2 \to \mathbb{R}$ and a compactly supported continuous function $\psi : \bar{\mathbb{R}}_+ \to \mathbb{R}$:

$$\lim_{N\to+\infty} \mathbb{E}\left[G((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0})\psi(\mathfrak{t}_{y,1}^N)\right] = \mathbb{E}\left[G((\zeta(t), \tau(t))_{t\geq 0})\psi(\mathfrak{t}_{y,1})\right]. \tag{B.11}$$

To this end, suppose that $t > 0$ and $y > 0$ are fixed, and consider the function

$$\mathfrak{F}_t(X, S) = G(X, S)1_{(y,+\infty)}(\pi_t \circ \mathcal{M}(X)),$$

where $\pi_t(X) := X(t)$. We claim that the set $\mathrm{Disc}(\mathfrak{F}_t)$ of discontinuities of the function $\mathfrak{F}_t$ has zero measure under the law of $(\zeta(t), \tau(t))_{t\geq 0}$. First, observe that $\mathrm{Disc}(\pi_t \circ \mathcal{M})$ is of zero measure. Indeed, if $X \in \mathrm{Disc}(\pi_t \circ \mathcal{M})$, then $\mathcal{M}(X) \in \mathrm{Disc}(\pi_t)$. Theorem 16.6(i) of [1] implies that then $\mathcal{M}(X)(t-) \neq \mathcal{M}(X)(t)$, which implies $X(t-) \neq X(t)$, and the latter set has zero measure under the law of $(\zeta(t))_{t\geq 0}$. On the other hand, if $X \in \mathrm{Disc}(1_{(y,+\infty)} \circ \pi_t \circ \mathcal{M})$ but $X \notin \mathrm{Disc}(\pi_t \circ \mathcal{M})$, it follows that $\pi_t \circ \mathcal{M}(X) = y$, which is a set of measure zero, by Theorem 4.6 of [17].

The above implies that the set of discontinuities of $\mathfrak{F}_t$ has measure zero. Hence, by Theorem 2.7 of [1], we have

$$\lim_{N\to+\infty} \mathbb{E}\left[\mathfrak{F}_t((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0})\right] = \mathbb{E}\left[\mathfrak{F}_t((\zeta(t), \tau(t))_{t\geq 0})\right], \tag{B.12}$$

or equivalently

$$\lim_{N\to+\infty} \mathbb{E}\left[G(Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}) 1_{[0,t]}(\mathfrak{t}_{y,1}^N)\right] = \mathbb{E}\left[G((\zeta(t), \tau(t))_{t\geq 0}) 1_{[0,t]}(\mathfrak{t}_{y,1})\right] \tag{B.13}$$

for any $t > 0$. The above implies

$$\lim_{N\to+\infty} \mathbb{E}\left[G(Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}) 1_{(s,t]}(\mathfrak{t}_{y,1}^N)\right] = \mathbb{E}\left[G((\zeta(t), \tau(t))_{t\geq 0}) 1_{(s,t]}(\mathfrak{t}_{y,1})\right] \tag{B.14}$$

for any $0 \leq s < t$. We can approximate (in the supremum norm) any compactly supported, continuous function ψ by step functions of the form $\sum_{i=1}^{I} c_i 1_{(s_i,t_i]}$, $s_i < t_i$. This ends the proof of (B.11).

Convergence of $((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}, \mathfrak{t}_{y,1}^N, Z_N(\mathfrak{t}_{y,1}^N))$

By the already proved part of the theorem we know that $((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}, \mathfrak{t}_{y,1}^N)$ converges in law to $((\zeta(t), \tau(t))_{t\geq 0}, \mathfrak{t}_{y,1})$. According to the Skorokhod embedding theorem, see e.g. Theorem I.6.7 of [1], we can assume that there exists a realization of the sequence of the processes $((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}, \mathfrak{t}_{y,1}^N)$, over a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that

$$\lim_{N\to+\infty} \rho_\infty((Z_N, \mathfrak{T}_N), (\zeta, \tau)) = 0 \quad \text{and} \quad \lim_{N\to+\infty} |\mathfrak{t}_{y,1}^N - \mathfrak{t}_{y,1}| = 0, \quad \text{a.s.} \tag{B.15}$$

and let

$$\mathfrak{Z}_m^N := Z_N(\mathfrak{t}_{y,m}^N), \quad \mathfrak{Z}_m := \zeta(\mathfrak{t}_{y,m}), \quad N, m \geq 1.$$

Lemma B.1 *For the above realization of the sequence $((Z_N(t), \mathfrak{T}_N(t))_{t\geq 0}, \mathfrak{t}_{y,1}^N)$, we have*

$$\lim_{N\to+\infty} |\mathfrak{Z}_1^N - \mathfrak{Z}_1| = 0, \quad \textit{a.s.} \tag{B.16}$$

Proof. Assume that $y > 0$. Thanks to (B.15), there exist a sequence λ_N of increasing homeomorphisms of $[0, +\infty)$ such that for any $T > 0$, we have, a.s:

$$\lim_{N\to+\infty} \sup_{t\in[0,T]} |Z_N(t) - \zeta \circ \lambda_N(t)| = 0, \quad \lim_{N\to+\infty} \sup_{t\in[0,T]} |t - \lambda_N(t)| = 0, \quad \lim_{N\to+\infty} \bar{T}_y(Z_N) = \bar{T}_y(\zeta), \tag{B.17}$$

hence

$$\lim_{N\to+\infty} \lambda_N \left(\bar{T}_y(Z_N)\right) = \bar{T}_y(\zeta), \quad \text{a.s.} \tag{B.18}$$

We claim that for $\mathbb{P}$ a.s. $\omega \in \Omega$ there exists $N_0(\omega)$ such that

$$\lambda_N \left(\bar{T}_y(Z_N(\omega))\right) = \bar{T}_y(\zeta(\omega)), \quad N \geq N_0. \tag{B.19}$$

Indeed, consider two cases.

Case (1) For a given $\omega \in \Omega$ there exists an infinite sequence N_k such that

$$\lambda_{N_k} \left(\bar{T}_y(Z_{N_k}(\omega))\right) > \bar{T}_y(\zeta(\omega)). \tag{B.20}$$

Then, there exists a (random) sequence (t_{N_k}) that satisfies

$$\bar{T}_y(Z_{N_k}(\omega)) > t_{N_k} > \lambda_{N_k}^{-1} \left(\bar{T}_y(\zeta(\omega))\right).$$

From (B.18), we conclude that

$$\lim_{k\to+\infty} t_{N_k} = \bar{T}_y(\zeta(\omega)),$$

therefore, by (B.17), we have

$$\lim_{k\to+\infty} \lambda_{N_k}(t_{N_k}) = \bar{T}_y(\zeta(\omega)).$$

From the right continuity of ζ and (B.20), we deduce that then

$$\lim_{k\to+\infty} \zeta\left(\lambda_{N_k}(t_{N_k})\right) = \zeta\left(\bar{T}_y(\zeta(\omega))\right). \tag{B.21}$$

From the first equality in (B.17) we infer that

$$\lim_{k\to+\infty} Z_{N_k}(t_{N_k}) = \zeta\left(\bar{T}_y(\zeta(\omega))\right). \tag{B.22}$$

However, since $\bar{T}_y(Z_{N_k}(\omega)) > t_{N_k}$ we have

$$Z_{N_k}(t_{N_k}) \leq y,$$

which would imply that

$$\zeta\left(\bar{T}_y(\zeta(\omega))\right) \leq y, \tag{B.23}$$

hence

$$\omega \in \mathcal{N}_0 = \{\omega : \ \zeta\left(\bar{T}_y(\zeta(\omega))\right) = y\}. \tag{B.24}$$

According to Corollary 2.2 of [20], the probability of $\mathcal{N}_0$ is zero.

Case (2). For a given $\omega \in \Omega$ there exist infinitely many N_k-s such that

$$\lambda_{N_k} \left(\bar{T}_y(Z_{N_k}(\omega))\right) < \bar{T}_y(\zeta(\omega)), \tag{B.25}$$

so that

$$\lim_{k\to+\infty} \zeta\left(\lambda_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right)\right) = \zeta\left(\bar{T}_y(\zeta(\omega))-\right) \leq y. \tag{B.26}$$

On the other hand we have

$$Z_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right) \geq y,$$

and, by (B.17),

$$\lim_{k\to+\infty}\left|\zeta\left(\lambda_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right)\right) - Z_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right)\right| = 0, \tag{B.27}$$

so that

$$\lim_{k\to+\infty}\zeta\left(\lambda_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right)\right) = \lim_{k\to+\infty} Z_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right) \geq y. \tag{B.28}$$

Comparing to (B.26), we see that

$$\lim_{k\to+\infty}\zeta\left(\lambda_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right)\right) = \lim_{k\to+\infty} Z_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right) = y. \tag{B.29}$$

Therefore, from (B.26) and (B.29) we get

$$\lim_{k\to+\infty}\zeta\left(\lambda_{N_k}\left(\bar{T}_y(Z_{N_k}(\omega))\right)\right) = \zeta\left(\bar{T}_y(\zeta(\omega))-\right) = y. \tag{B.30}$$

Hence, we either have

$$\omega \in \mathcal{N}_1 = [\omega :\ \zeta\left(\bar{T}_y(\zeta(\omega))-\right) = y,\ \zeta\left(\bar{T}_y(\zeta(\omega))\right) > y], \tag{B.31}$$

an event that has probability zero by Proposition, on p. 695 of [19], or $\omega \in \mathcal{N}_0$. We conclude that (B.19) holds. This however, obviously implies (B.16), as

$$\mathfrak{z}_1^N = Z_N\left(\bar{T}_y(Z_N(\omega))\right) \quad \text{and} \quad \mathfrak{z}_1 = \zeta\left(\bar{T}_y(\zeta(\omega))\right),$$

finishing the proof. □

Generalization to subsequent exit times – the end of the proof of Theorem 4.4

Corollary B.2 *Under the assumptions of Lemma B.1, for any $y \in \mathbb{R}$ we have*

$$\lim_{N\to+\infty}\rho_\infty(\theta_{\mathfrak{t}^N_{y,1}}(Z_N), \theta_{\mathfrak{t}_{y,1}}(\zeta)) = 0, \quad a.s. \tag{B.32}$$

Proof. Define the following increasing homeomorphism of $[0, +\infty)$:

$$\tilde{\lambda}_N(t) := \lambda_N\left(\bar{T}_y(Z_N(\omega)) + t\right) - \lambda_N(\bar{T}_y(Z_N(\omega))), \quad t \geq 0.$$

Thanks to the first two equalities in (B.17), for any $T > 0$ we have

$$\lim_{N\to+\infty}\sup_{t\in[0,T]}|Z_N(\bar{T}_y(Z_N(\omega)) + t) - \zeta(\tilde{\lambda}_N(t) + \lambda_N(\bar{T}_y(Z_N(\omega))))| = 0, \tag{B.33}$$

$$\lim_{N\to+\infty}\sup_{t\in[0,T]}|t - \tilde{\lambda}_N(t)| = 0.$$

It follows from the argument in the proof of Lemma B.1 that there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that for each $\omega \notin \mathcal{N}$ there exists N_0, for which (B.20) holds for all $N \ge N_0$. From this equality we conclude that

$$
\begin{aligned}
&\limsup_{N\to+\infty} \sup_{t\in[0,T]} |\theta_{\bar{T}_y(Z_N(\omega))}(Z_N)(t) - \theta_{\bar{T}_y(\zeta(\omega))}(\zeta)(\tilde{\lambda}_N(t))| \\
&= \limsup_{N\to+\infty} \sup_{t\in[0,T]} |Z_N(\bar{T}_y(Z_{,N}(\omega)) + t) - \zeta(\tilde{\lambda}_N(t) + \lambda_N(\bar{T}_y(Z_{\cdot,N}(\omega))))| \\
&= \limsup_{N\to+\infty} \sup_{t\in[0,T]} |Z_N(\bar{T}_y(Z_N(\omega)) + t) - \zeta(\lambda_N(t + \bar{T}_y(Z_N(\omega))))| = 0
\end{aligned}
$$

for any $T > 0$. We have shown therefore that (B.32) holds. $\square$

Let

$$
Z'_N(t;\omega) := \theta_{\mathfrak{t}^N_{y,1}}(Z_N)(t), \quad \zeta'(t) := \theta_{\mathfrak{t}_{y,1}}(\zeta)(t), \quad t \ge 0 \tag{B.34}
$$

and

$$
\tilde{\mathfrak{t}}^N_{1,y}(\omega) = \underline{\mathrm{T}}_y(Z'_N(\omega)), \quad \tilde{\mathfrak{t}}_{y,1}(\omega) = \underline{\mathrm{T}}_x(\zeta'(\omega)).
$$

Note that

$$
\mathfrak{t}^N_{y,2}(\omega) = \mathfrak{t}^N_{y,1}(\omega) + \tilde{\mathfrak{t}}^N_{y,1}(\omega), \quad \mathfrak{t}_{y,2}(\omega) = \mathfrak{t}_{y,1}(\omega) + \tilde{\mathfrak{t}}_{y,1}(\omega). \tag{B.35}
$$

Repeating the argument used in the proof of (B.9) we conclude that

$$
\mathbb{P}[\tilde{\mathfrak{t}}^N_{y,1} \le t] \to \mathbb{P}[\tilde{\mathfrak{t}}_{y,1} \le t], \qquad \text{as } N \to +\infty.
$$

This also proves the tightness of the random elements $((Z_N(t), \mathfrak{T}_N(t))_{t\ge0}, \mathfrak{t}^N_{y,1}, \mathfrak{t}^N_{y,2}, \mathfrak{Z}^N_1)$, $N \ge 1$. Using the same argument as in the proof of (B.10) we can reduce the proof of the convergence in law to showing that for any bounded and continuous functions $F : \mathcal{D}_2 \times \bar{\mathbb{R}}_+ \times \mathbb{R} \to \mathbb{R}$ and compactly supported continuous $\psi : \bar{\mathbb{R}}_+ \to \mathbb{R}$ we have

$$
\lim_{N\to+\infty} \mathbb{E}\left[F((Z_N(t), \mathfrak{T}_N(t))_{t\ge0}, \mathfrak{t}^N_{y,1}, \mathfrak{Z}^N_1)\psi(\tilde{\mathfrak{t}}^N_{y,1})\right] = \mathbb{E}\left[F(\zeta(t), \tau(t))_{t\ge0}, \mathfrak{t}_{y,1}, \mathfrak{Z}_1)\psi(\tilde{\mathfrak{t}}_{y,1})\right]. \tag{B.36}
$$

Suppose that $t > 0$ and consider the function

$$
\mathfrak{F}_t : (X, S, s, z) \mapsto F(X, S, s)1_{(y,+\infty)}(\pi_t \circ \mathcal{M}'(\theta_s(X)),
$$

where $(X, S, s, z) \in \mathcal{D}_2 \times \bar{\mathbb{R}}_+ \times \mathbb{R}$, F is as above and $\pi_t(X) := X(t)$, and let Q be the law of $((\zeta(t), \tau(t))_{t\ge0}, \mathfrak{t}_{y,1}, \mathfrak{Z}_1)$. We claim that the set $\mathrm{Disc}(\mathfrak{F}_t)$ of discontinuities of the function $\mathfrak{F}_t$ is Q-null. Indeed, first observe that the set D of discontinuities of

$$
(X, S, s, z) \mapsto \pi_t \circ \mathcal{M}'(\theta_s(X))
$$

is Q-null. If $(X, S, s, z) \in D$, then $\mathcal{M}'(\theta_s(X)) \in \mathrm{Disc}(\pi_t)$. According to Theorem 16.6(i) of [1], this is equivalent to $\mathcal{M}'(\theta_s(X))(t-) \ne \mathcal{M}'(\theta_s(X))(t)$. However, this set is contained in

$$
[(X, s) : \ X(s + t-) \ne X(s + t)].
$$

The Q-probability of the latter is

$$
\mathbb{P}\left[\zeta(\mathfrak{t}_{y,1} + t-) \ne \zeta(\mathfrak{t}_{y,1} + t)\right] = \mathbb{E}\left\{\mathbb{P}\left[\zeta(\mathfrak{t}_{y,1} + t-) \ne \zeta(\mathfrak{t}_{y,1} + t)\middle|\mathcal{F}_{\mathfrak{t}_{y,1}}\right]\right\}. \tag{B.37}
$$

The strong Markov property implies that the process $(\zeta(\mathfrak{t}_{y,1}+t)-\mathfrak{z}_1)_{t\ge 0}$ is independent of the σ-algebra $\mathcal{F}_{\mathfrak{t}_{xy1}}$ corresponding to the stopping time $\mathfrak{t}_{y,1}$, and the right side of (B.37) equals

$$\mathbb{P}\left[\zeta(t-)\neq\zeta(t)\right]=0. \tag{B.38}$$

Suppose now that $(X,s)\in D'$, the discontinuity set of

$$(X,S,s,z)\mapsto 1_{(y,+\infty)}\circ\pi_t\circ\mathcal{M}'(X\circ\theta_s)$$

and $(X,S,s,z)\notin D$, so that $\pi_t\circ\mathcal{M}'(\theta_s(X))=y$. Its probability equals

$$\mathbb{P}\Big[\inf_{\mathfrak{t}_{y,1}\le u\le \mathfrak{t}_{x,1}+t}\zeta(u)=y\Big]=\mathbb{E}\mathbb{P}[M'(t)=y-z]_{z=\mathfrak{z}_1},$$

where $M'(t)=\mathcal{M}'(\zeta)(t)$. By symmetry, the expression in the right equals

$$\mathbb{E}\mathbb{P}[M(t)=z-y]_{z=\mathfrak{z}_1}=0,$$

as the law of $M(t)$ is absolutely continuous. It follows that the set of discontinuities of $\mathfrak{F}_t$ is null. Hence, see Theorem 2.7 of [1], we have

$$\lim_{N\to+\infty}\mathbb{E}\left[\mathfrak{F}_t((Z_N(t),\mathfrak{T}_N(t))_{t\ge0},\mathfrak{t}^N_{y,1},\mathfrak{z}^N_1)\right]=\mathbb{E}\left[\mathfrak{F}_t(\zeta(t),\tau(t))_{t\ge0},\mathfrak{t}_{y,1},\mathfrak{z}_1)\right], \tag{B.39}$$

or equivalently

$$\lim_{N\to+\infty}\mathbb{E}\left[F((Z_N(t),\mathfrak{T}_N(t))_{t\ge0},\mathfrak{t}^N_{y,1},\mathfrak{z}^N_1)1_{[0,t]}(\mathfrak{t}^N_{x,2})\right]=\mathbb{E}\left[F((\zeta(t),\tau(t))_{t\ge0},\mathfrak{t}_{y,1},\mathfrak{z}_1)1_{[0,t]}(\mathfrak{t}_{y,2})\right] \tag{B.40}$$

for any $t>0$. The above implies that

$$\lim_{N\to+\infty}\mathbb{E}\left[F((Z_N(t),\mathfrak{T}_N(t))_{t\ge0},\mathfrak{t}^N_{y,1},\mathfrak{z}^N_1)1_{(s,t]}(\mathfrak{t}^N_{y,2})\right]=\mathbb{E}\left[F((\zeta(t),\tau(t))_{t\ge0},\mathfrak{t}_{y,1},\mathfrak{z}_1)1_{(s,t]}(\mathfrak{t}_{y,2})\right] \tag{B.41}$$

for any $0\le s<t$. We can approximate (in the supremum norm) any compactly supported function ψ by step functions of the form $\sum_{i=1}^I c_i 1_{(s_i,t_i]}$, $s_i<t_i$. This ends the proof of (B.36).

By the previous argument we already know that the random elements

$$\big((Z_N(t),\mathfrak{T}_N(t))_{t\ge0},\mathfrak{t}^N_{y,1},\mathfrak{t}^N_{y,2},\mathfrak{z}^N_1\big) \tag{B.42}$$

converge in law to

$$((\zeta(t),\tau(t))_{t\ge0},\mathfrak{t}_{y,1},\mathfrak{t}_{y,2},\mathfrak{z}_1)\,. \tag{B.43}$$

According to the Skorokhod embedding theorem, we can assume that there exist realizations of the random elements (B.42), (B.43) such that

$$\lim_{N\to+\infty}\rho_\infty((Z_N,\mathfrak{T}_N),(\zeta,\tau))=0\quad\text{and} \tag{B.44}$$

$$\sum_{i=1}^{2}\lim_{N\to+\infty}|\mathfrak{t}^N_{y,i}-\mathfrak{t}_{y,i}|=0,\quad \lim_{N\to+\infty}|\mathfrak{z}^N_1-\mathfrak{z}_1|=0,\quad\text{a.s.}$$

By Corollary B.2 we have

$$\lim_{N\to+\infty} \rho_\infty(Z'_N, \zeta') = 0, \tag{B.45}$$

where $\tilde Z_N$ and $\tilde\zeta$ are defined by (B.34). We can repeat the argument used in the proof of Lemma B.1 and conclude that the set of events ω, for which

$$\lim_{N\to+\infty} \mathfrak{z}_2^N \neq \mathfrak{z}_2$$

is contained in the set $\mathcal{N}$ of events ω such that

$$\zeta\left(\underline{\mathrm{T}}_y(\zeta(\omega))-\right) = y < \zeta\left(\underline{\mathrm{T}}_y(\zeta(\omega))\right), \quad \text{or} \quad \zeta\left(\underline{\mathrm{T}}_y(\zeta(\omega))\right) \le y, \tag{B.46}$$

which again by the same arguments as used there is of null probability.

The above argument, can be continued by induction and allows us to conclude the proof of Theorem 4.4. □

A further generalization

The argument of the present section, esentially without any modification, can be used to prove a slight generalization of Theorem 4.4, that we have used in the proof of Theorem 6.1. Suppose that $(\zeta_N(t,y))_{t\ge 0}$ is a sequence of processes that satisfy $\zeta_N(0,y) = y$ and converge in law, as $N \to +\infty$, in the J_1-topology over $D[0,+\infty)$ to $(\zeta(t,y))_{t\ge 0}$. We can define the consecutive crossing times $\mathfrak{s}_{y,m}^N$ $N, m = 1, 2, \dots$ for $(\zeta_N(t))_{t\ge 0}$ between the half-lines $\mathbb{R}_-$ and $\mathbb{R}_+$.

Theorem B.3 *For any $y \in \mathbb{R}_*$ the random elements*

$$((\zeta_N(t,y))_{t\ge 0}, (\mathfrak{s}_{y,m}^N)_{m\ge 1}, (\zeta_N(\mathfrak{s}_{y,m}^N, y))_{m\ge 1})$$

converge in law, as $N \to +\infty$, over $D[0,+\infty) \times \bar{\mathbb{R}}_+^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ with the product of the J_1 and standard product topology on $(\mathbb{R}^{\mathbb{N}})^2$, to $((\zeta(t,y))_{t\ge 0}, (\mathfrak{u}_{y,m})_{m\ge 1}, (\zeta(\mathfrak{u}_{y,m}, y, k)_{m\ge 1})$.

References

[1] P. Billingsley, *Probability and measure.* Second edition. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. xiv+622 pp. ISBN: 0-471-80478-9

[2] C. Bardos, F. Golse, and I. Moyano, Linear Boltzmann equation and fractional diffusion, 2018 KRM, 11, 1011–1036.

[3] G. Basile, S. Olla, H. Spohn, Wigner functions and stochastically perturbed lattice dynamics, Arch.Rat.Mech., **195**, 171–203, 2009.

[4] G. Basile, T. Komorowski, S. Olla, Diffusion limit for a kinetic equation with a thermostatted interface, to appear in KRM, arXiv:1903.02621.

[5] G. Basile and A. Bovier, Convergence of a kinetic equation to a fractional diffusion equation. Markov Process. Related Fields **16**, 2010, 15–44.

[6] N. Ben Abdallah, A. Mellet and M. Puel, Fractional diffusion limit for collisional kinetic equations: a Hilbert expansion approach, Kinet. Relat. Models **4**, 2011, 873–900.

[7] J. Bertoin, *Lévy processes.* Cambridge Tracts in Mathematics, 121. Cambridge University Press, Cambridge, 1996.

[8] L. Cesbron, Anomalous diffusion limit of kinetic equations in spatially bounded domains, Comm. Math. Phys. **364**, 233–286, 2018.

[9] L. Cesbron, A. Mellet, M. Puel, Fractional diffusion limit for a kinetic equation in the upper-half space with diffusive boundary conditions, arxiv:1805.04903v2.

[10] M. Frank, W. Sun, Fractional diffusion limits of non-classical transport equations, Kinet. Relat. Models **11**, 2018, 1503–1526.

[11] N. Ikeda, S. Watanabe, On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes, J. Math. Kyoto Univ. **2**, 1962, 79–95.

[12] Jara, M.; Komorowski, T. Limit theorems for some continuous-time random walks. Adv. Appl. Probab. **43**, 2011, 782–813.

[13] Jara, M., T. Komorowski, S. Olla, Limit theorems for additive functionals of a Markov chain, Ann. Appl. Prob. **19**, 2009, 2270–2300.

[14] T. Komorowski, S. Olla, L. Ryzhik, H. Spohn, High frequency limit for a chain of harmonic oscillators with a point Langevin thermostat., arXiv:1806.02089

[15] T. Komorowski, S. Olla and L. Ryzhik, in preparation.

[16] A. E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, 2006.

[17] M. Kwaśnicki, J. Małecki and M. Ryznar, Suprema of Lévy processes, Ann. Prob. **41**, 2013, 2047–2065.

[18] M. Kwaśnicki, Ten equivalent definitions of the fractional Laplace operator, Fract. Calc. Appl. Anal. **20**, 2017, 7–51.

[19] J. Lamperti, An invariance principle in renewal theory, Ann. Math. Stat. **33**, 1962, 685–696.

[20] P. W. Millar, Exit properties of stochastic processes with stationary independent increments. TAMS **178**, 1973, 459–479.

[21] A. Mellet, S. Mischler, C. Mouhot, Fractional diffusion limit for collisional kinetic equations, Arch. Rat. Mech. Anal., **199**, 2011, 493–525.

[22] K. I. Sato, *Levy processes and infinitely divisible distributions*, Cambridge Univ. Press, 1999.

[23] D.W. Stroock and S.R.S. Varadhan, *Multidimensional diffusion processes*, Grundlehren der Mathematischen Wissenschaften, 233. Springer-Verlag, Berlin-New York, 1979.

[24] V. A. Topchii, A. A. Vatutin Maximum of the critical Galton-Watson processes and left-continuous random walks, Theory Probab. Appl., **42**, 1998, 17–27.

[25] W. Whitt, *Stochastic Process Limits: An Introduction to Stochastic-Process Limits and their Application to Queues*, Springer, New York, 2002.

[26] W. Whitt, *Internet Supplement to. Stochastic-Process Limits.*, available at www.columbia.edu/ ww2040/supplementno.pdf