QUASI LINEAR PARABOLIC PDE IN A JUNCTION WITH NON LINEAR NEUMANN VERTEX CONDITION

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Abstract. The purpose of this article is to study quasi linear parabolic partial differential equations of second order, on a bounded junction, satisfying a nonlinear and non dynamical Neumann boundary condition at the junction point. We prove the existence and the uniqueness of a classical solution.

1. Introduction

In this paper, we study non degenerate quasi linear parabolic partial differential equations on a junction, satisfying a non linear Neumann boundary condition at the junction point $x = 0$:

$$\begin{cases} \partial_t u_i(t,x) - \sigma_i(x, \partial_x u_i(t,x))\partial^2_{x,x} u_i(t,x) + H_i(x, u_i(t,x), \partial_x u_i(t,x)) \quad = \quad 0, \\ \text{for all } x > 0, \text{ and for all } \ i \in \{1 \dots I\}, \\ F(u(t,0), \partial_x u(t,0)) \quad = \quad 0. \end{cases} \tag{1}$$

The well-known Kirchhoff law corresponds to the case where F is linear in $\partial_x u$ and independent of u.

Originally introduced by Nikol'skii [13] and Lumer [11, 12], the concept of ramified spaces and the analysis of partial differential equation on these spaces have attracted a lot of attention in the last 30 years. As explained in [13], the main motivations are applications in physics, chemistry, and biology (for instance small transverse vibrations in a grid of strings, vibration of a grid of beams, drainage system, electrical equation with Kirchhoff law, wave equation, heat equation,...). Linear diffusions of the form (1), with a Kirchhoff law, are also naturally associated with stochastic processes living on graphs. These processes were introduced in the seminal papers [3] and [4]. Another motivation for studying (1) is the analysis of associated stochastic optimal control problems with a control at the junction. The result of this paper will allow us in a future work to characterize the value function of such problems.

There has been several works on linear and quasilinear parabolic equations of the form (1). For linear equations, von Below [15] shows that, under natural smoothness and compatibility conditions, linear boundary value problems on a network with a linear Kirchhoff condition and an additional Dirichlet boundary condition at the vertex point, are well-posed. The proof consists mainly in showing that the initial boundary value problem on a junction is equivalent to a well-posed initial boundary value problem for a parabolic system, where the boundary conditions are such that the classical results on linear parabolic equations [7] can be applied. The same author investigates in [16] the strong maximum principle for semi linear parabolic operators with Kirchhoff condition, while in [17] he studies the classical global solvability for a class of semilinear parabolic equations on ramified networks, where a dynamical node condition is prescribed: Namely the Neumann condition at the junction point $x = 0$ in (1), is replaced by the dynamic one

$$\partial_t u(t,0) + F(t, u(t,0), \partial_x u(t,0)) \quad = \quad 0.$$

In this way the application of classical estimates for domains established in [7] becomes possible. The author then establish the classical solvability in the class $\mathcal{C}^{1+\alpha,2+\alpha}$, with the aid of the Leray-Schauder-principle and the maximum principle of [16]. Let us note that this kind of proof fails for equation (1) because in this case one cannot expect an uniform bound for the term $|\partial_t u(t,0)|$ (the proof of Lemma 3.1 of [7] VI.3 fails). Still in the linear setting, another approach, yielding similar existence results, was developed by Fijavz, Mugnolo and Sikolya in [2]: the idea is to use semi-group theory as well as variational methods to understand how the spectrum of the operator is related to the structure of the network.

Equations of the form (1) can also be analyzed in terms of viscosity solutions. The first results on viscosity solutions for Hamilton-Jacobi equations on networks have been obtained by Schieborn in [14] for the Eikonal equations and later discussed in many contributions on first order problems [1, 6, 8], elliptic equations [9] and second order problems with vanishing diffusion at the vertex [10].

In contrast second order Hamilton-Jacobi equations with a non vanishing viscosity at the boundary have seldom been studied in the literature and our aim is to show the well-posedness of classical solutions for (1) in suitable Höder spaces: see Theorem 2.2 for the existence and Theorem 2.4 for the comparison, and thus the uniqueness. Our main assumptions are that the equation is uniformly parabolic with smooth coefficients and that the term $F = F(u,p)$ at the junction is either decreasing with respect to u or increasing with respect to p.

The main idea of the proof is to use a time discretization, exploiting at each step the solvability in $C^{2+\alpha}$ of the elliptic problem

$$\begin{cases} -\sigma_i(x,\partial_x u_i(x))\partial^2_{x,x}u_i(x) + H_i(x,u_i(x),\partial_x u_i(x)) = 0, \\ F(u(0),\partial_x u(0)) = 0. \end{cases} \tag{2}$$

The paper is organized as follows. In section 2, we introduce the notations and state our main results. In Section 3, we review the mains results of existence and uniqueness of the elliptic problem (2). Finally Section 4, is dedicated to the proof of our main results.

2. MAIN RESULTS

In this section we state our main result Theorem 2.2, on the solvability of the parabolic problem with Neumann boundary condition at the vertex, on a bounded junction

$$\begin{cases} \partial_t u_i(t,x) - \sigma_i(x,\partial_x u_i(t,x))\partial^2_{x,x}u_i(t,x) + \\ H_i(x,u_i(t,x),\partial_x u_i(t,x)) = 0, \quad \text{if } (t,x) \in (0,T)\times(0,a_i), \\ F(u(t,0),\partial_x u(t,0)) = 0, \quad \text{if } t \in [0,T), \\ \forall i \in \{1\dots I\}, \quad u_i(t,a_i) = \phi_i(t), \quad \text{if } t \in [0,T], \\ \forall i \in \{1\dots I\}, \quad u_i(0,x) = g_i(x), \quad \text{if } x \in [0,a_i]. \end{cases} \tag{3}$$

There will be two typical assumptions for $F = F(u,p)$: either F is decreasing with respect to u or F is increasing with respect to p (Kirchhoff conditions).

2.1. Notations and preliminary results.

Let us start by introducing the main notation used in this paper as well as an interpolation result.

Let $I \in \mathbb{N}^*$ be the number of edges, and $a = (a_1,\dots a_I) \in (0,\infty)^I$ be the length of each edge.

The bounded junction is defined by

$$\mathcal{J}^a \quad = \quad \Big\{ \quad X = (x, i), \quad x \in [0, a_i] \text{ and } i \in \{1, \dots, I\} \quad \Big\},$$

where all the points $(0, i)$, $i = 1, \dots, I$, are identified to the vertex denoted by 0. We can then write

$$\mathcal{J}^a \quad = \quad \bigcup_{i=1}^{I} J_i^{a_i},$$

with $J_i^{a_i} := [0, a_i] \times \{i\}$, $J_i^{a_i} \cap J_j^{a_j} = \{0\}$. For $T > 0$, the time-space domain $\mathcal{J}_T^a$ is defined by

$$\mathcal{J}_T^a \quad = \quad [0, T] \times \mathcal{J}^a.$$

The interior of $\mathcal{J}_T^a$ set minus the junction point 0 is denoted by $\overset{\circ}{\mathcal{J}_T^a}$, and is defined by

$$\overset{\circ}{\mathcal{J}_T^a} \quad = \quad (0, T) \times \Big(\bigcup_{i=1}^{I} \overset{\circ}{J_i^{a_i}} \Big).$$

For the functionnal spaces that will be used in the sequel, we use here the notations of Chapter 1.1 of [7]. For the convenience of the reader, we recall these notations in Appendix A.

In addition we introduce the parabolic Hölder space on the junction $\Big(\mathcal{C}^{\frac{l}{2}, l}(\mathcal{J}_T^a), \|.\|_{\mathcal{C}^{\frac{l}{2}, l}(\mathcal{J}_T^a)} \Big)$ and the space $\mathcal{C}_b^{\frac{l}{2}, l}(\overset{\circ}{\mathcal{J}_T^a})$, defined by (where $l > 0$, see Annexe A for more details)

$$\mathcal{C}^{\frac{l}{2}, l}(\mathcal{J}_T^a) \quad := \Big\{ \quad f : \mathcal{J}_T^a \to \mathbb{R}, \quad (t, (x, i)) \mapsto f_i(t, x), \quad \forall (i, j) \in \{1 \dots I\}^2, \quad \forall t \in (0, T),$$

$$f_i(t, 0) = f_j(t, 0) = f(t, 0), \quad \forall i \in \{1 \dots I\}, \quad (t, x) \mapsto f_i(t, x) \in \mathcal{C}^{\frac{l}{2}, l}([0, T] \times [0, a_i]) \quad \Big\},$$

$$\mathcal{C}_b^{\frac{l}{2}, l}(\overset{\circ}{\mathcal{J}_T^a}) \quad := \Big\{ \quad f : \mathcal{J}_T^a \to \mathbb{R}, \quad (t, (x, i)) \mapsto f_i(t, x),$$

$$\forall i \in \{1 \dots I\}, \quad (t, x) \mapsto f_i(t, x) \in \mathcal{C}_b^{\frac{l}{2}, l}((0, T) \times (0, a_i)) \quad \Big\},$$

with

$$\|u\|_{\mathcal{C}^{\frac{l}{2}, l}(\mathcal{J}_T^a)} \quad = \quad \sum_{1 \le i \le I} \|u_i\|_{\mathcal{C}^{\frac{l}{2}, l}([0, T] \times [0, a_i])}.$$

We will use the same notations, when the domain does not depend on time, namely $T = 0$, $\Omega_T = \Omega$, removing the dependence on the time variable.
We continue with the definition of a nondecreasing maps $F : \mathbb{R}^I \to \mathbb{R}$.
Let $(x = (x_1, \dots x_I), y = (y_1 \dots y_I)) \in \mathbb{R}^{2I}$, we say that

$$x \leq y, \text{ if } \quad \forall i \in \{1 \dots I\}, \quad x_i \leq y_i,$$

and

$$x < y, \text{ if } \quad x \leq y, \text{ and there exists } j \in \{1 \dots I\}, \quad x_j < y_j.$$

We say that $F \in \mathcal{C}(\mathbb{R}^I, \mathbb{R})$ is nondecreasing if

$$\forall (x, y) \in \mathbb{R}^I, \text{ if } x \leq y, \text{ then } F(x) \leq F(y),$$

increasing if

$$\forall (x, y) \in \mathbb{R}^I, \text{ if } x < y, \text{ then } F(x) < F(y).$$

Next we recall an interpolation inequality, which will be useful in the sequel.

Lemma 2.1. *Suppose that $u \in \mathcal{C}^{0,1}([0, T] \times [0, R])$ satisfies an Hölder condition in t in $[0, T] \times [0, R]$, with exponent $\alpha \in (0, 1]$, constant ν_1, and has derivative $\partial_x u$, which for any $t \in [0, T]$ are Hölder continuous in the variable x, with exponent $\gamma \in (0, 1]$, and constant ν_2. Then the derivative $\partial_x u$ satisfies in $[0, T] \times [0, R]$, an Hölder condition in t, with exponent $\frac{\alpha\gamma}{1+\gamma}$, and constant depending only on ν_1, ν_2, γ. More precisely*

$$\forall (t, s) \in [0, T]^2, \quad |t - s| \leq 1, \quad \forall x \in [0, R],$$
$$|\partial_x u(t, x) - \partial_x u(s, x)| \quad \leq \quad \left(2\nu_2 \left(\frac{\nu_1}{\gamma\nu_2}\right)^{\frac{\gamma}{1+\gamma}} \quad + \quad 2\nu_1 \left(\frac{\gamma\nu_2}{\nu_1}\right)^{-\frac{1}{1+\gamma}}\right) |t - s|^{\frac{\alpha\gamma}{1+\gamma}}.$$

This is a special case of Lemma II.3.1, in [7], (see also [13]). The main difference is that we are able to get global Hölder regularity in $[0, T] \times [0, R]$ for $\partial_x u$ in its first variable. Let us recall that this kind of result fails in higher dimensions.

Proof. Let $(t,s) \in [0,T]^2$, with $|t-s| \leq 1$, and $x \in [0,R]$. Suppose first that $x \in [0,\frac{R}{2}]$. Let $y \in [0,R]$, with $y \neq x$, we write

$$\partial_x u(t,x) - \partial_x u(s,x) = \frac{1}{y-x}\int_x^y \left(\partial_x u(t,x) - \partial_x u(t,z)\right) + \left(\partial_x u(t,z) - \partial_x u(s,z)\right) + \left(\partial_x u(s,z) - \partial_x u(s,x)\right)\, dz.$$

Using the Hölder condition in time satisfied by u, we have

$$\Big|\frac{1}{y-x}\int_x^y \left(\partial_x u(t,z) - \partial_x u(s,z)\right)dz\Big| \leq \frac{2\nu_1|t-s|^\alpha}{|y-x|}.$$

On the other hand, using the Hölder regularity of $\partial_x u$ in space satisfied, we have

$$\Big|\frac{1}{y-x}\int_x^y \left(\partial_x u(t,x) - \partial_x u(t,z)\right) + \left(\partial_x u(s,z) - \partial_x u(s,x)\right)dz\Big| \leq 2\nu_2|y-x|^\gamma.$$

It follows

$$|\partial_x u(t,x) - \partial_x u(s,x)| \leq 2\nu_2|y-x|^\gamma + \frac{2\nu_1|t-s|^\alpha}{|y-x|}.$$

Assuming that $|t-s| \leq \Big((\frac{3R}{2})^{1+\gamma}\frac{\gamma\nu_2}{\nu_1}\Big)^{\frac{1}{\alpha}} \wedge 1$, minimizing in $y \in [0,R]$, for $y > x$, the right side of the last equation, we get that the infimum is reached for

$$y^* = x + \Big(\frac{\nu_1|t-s|^\alpha}{\gamma\nu_2}\Big)^{\frac{1}{1+\gamma}},$$

and then

$$|\partial_x u(t,x) - \partial_x u(s,x)| \leq C(\nu_1,\nu_2,\gamma)|t-s|^{\frac{\alpha\gamma}{1+\gamma}},$$

where the constant $C(\nu_1,\nu_2,\gamma)$, depends only on the data (ν_1,ν_2,γ), and is given by

$$C(\nu_1,\nu_2,\gamma) = 2\nu_2\Big(\frac{\nu_1}{\gamma\nu_2}\Big)^{\frac{\gamma}{1+\gamma}} + 2\nu_1\Big(\frac{\gamma\nu_2}{\nu_1}\Big)^{-\frac{1}{1+\gamma}}.$$

For the cases $y < x$, and $x \in [\frac{R}{2},R]$, we argue similarly, which completes the proof. □

2.2. **Assumptions and main results.** We state in this subsection the central Theorem of this note, namely the solvability and uniqueness of (1) in the class $\mathcal{C}^{\frac{\alpha}{2},1+\alpha}(\mathcal{J}_T^a) \cap \mathcal{C}_b^{1+\frac{\alpha}{2},2+\alpha}(\overset{\circ}{\mathcal{J}_T^a})$. **In the rest of these notes, we fix** $\alpha \in (0,1)$.

Let us state the assumptions we will work on.

Assumption $(\mathcal{P})$

We introduce the following data

$$\begin{cases} F \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^I, \mathbb{R}) \\ g \in \mathcal{C}^1(\mathcal{J}^a) \cap \mathcal{C}^2_b(\mathring{\mathcal{J}}^a) \end{cases},$$

and for each $i \in \{1 \dots I\}$

$$\begin{cases} \sigma_i \in \mathcal{C}^1([0, a_i] \times \mathbb{R}, \mathbb{R}) \\ H_i \in \mathcal{C}^1([0, a_i] \times \mathbb{R}^2, \mathbb{R}) \\ \phi_i \in \mathcal{C}^1([0, T], \mathbb{R}) \end{cases}.$$

We suppose furthermore that the data satisfy

(i) Assumption on F

$$\begin{cases} a) \quad F \text{ is decreasing with respect to its first variable,} \\ b) \quad F \text{ is nondecreasing with respect to its second variable,} \\ c) \quad \exists (b, B) \in \mathbb{R} \times \mathbb{R}^I, \quad F(b, B) = 0, \end{cases}$$

or satisfies the Kirchhoff condition

$$\begin{cases} a) \quad F \text{ is nonincreasing with respect to its first variable,} \\ b) \quad F \text{ is increasing with respect to its second variable,} \\ c) \quad \exists (b, B) \in \mathbb{R} \times \mathbb{R}^I, \quad F(b, B) = 0. \end{cases}$$

We suppose moreover that there exist a parameter $m \in \mathbb{R}$, $m \geq 2$ such that we have

(ii) The (uniform) ellipticity condition on the $(\sigma_i)_{i \in \{1 \dots I\}}$: there exists $\underline{\nu}, \overline{\nu}$, strictly positive constants such that

$$\forall i \in \{1 \dots I\}, \quad \forall (x, p) \in [0, a_i] \times \mathbb{R},$$
$$\underline{\nu}(1 + |p|)^{m-2} \quad \leq \quad \sigma_i(x, p) \quad \leq \quad \overline{\nu}(1 + |p|)^{m-2}.$$

(iii) The growth of the $(H_i)_{i \in \{1 \dots I\}}$ with respect to p exceed the growth of the σ_i with respect to p by no more than two, namely there exists μ an increasing real continuous function such that

$$\forall i \in \{1 \dots I\}, \quad \forall (x,u,p) \in [0,a_i] \times \mathbb{R}^2, \quad |H_i(x,u,p)| \quad \leq \quad \mu(|u|)(1+|p|)^m.$$

(iv) We impose the following restrictions on the growth with respect to p of the derivatives for the coefficients $(\sigma_i, H_i)_{i \in \{1\dots I\}}$, which are for all $i \in \{1 \dots I\}$,

$$\begin{aligned} a) \quad |\partial_p \sigma_i|_{[0,a_i]\times\mathbb{R}^2}(1+|p|)^2 + |\partial_p H_i|_{[0,a_i]\times\mathbb{R}^2} \quad &\leq \quad \gamma(|u|)(1+|p|)^{m-1}, \\ b) \quad |\partial_x \sigma_i|_{[0,a_i]\times\mathbb{R}^2}(1+|p|)^2 + |\partial_x H_i|_{[0,a_i]\times\mathbb{R}^2} \quad &\leq \quad \Big(\varepsilon(|u|) + P(|u|,|p|)\Big)(1+|p|)^{m+1}, \\ c) \quad \forall (x,u,p) \in [0,a_i] \times \mathbb{R}^3, \quad -C_H \quad \leq \quad \partial_u H_i(x,u,p) \quad &\leq \quad \Big(\varepsilon(|u|) + P(|u|,|p|)\Big)(1+|p|)^m, \end{aligned}$$

where γ and ε are continuous non negative increasing functions. P is a continuous function, increasing with respect to its first variable, and tends to 0 for $p \to +\infty$, uniformly with respect to its first variable, from $[0,u_1]$ with $u_1 \in R$, and $C_H > 0$ is real strictly positive number. We assume that $(\gamma, \varepsilon, P, C_H)$ are independent of $i \in \{1 \dots I\}$.
(v) A compatibility conditions for g and $(\phi_i)_{\{1\dots I\}}$

$$F(g(0), \partial_x g(0)) \quad = \quad 0 \quad ; \quad \forall i \in \{1 \dots I\}, \quad g_i(a_i) = \phi_i(0).$$

Theorem 2.2. *Assume* $(\mathcal{P})$. *Then system* (3) *is uniquely solvable in the class* $\mathcal{C}^{\frac{\alpha}{2},1+\alpha}(\mathcal{J}_T^a) \cap \mathcal{C}_b^{1+\frac{\alpha}{2},2+\alpha}(\overset{\circ}{\mathcal{J}_T^a})$. *There exist constants* (M_1, M_2, M_3), *depending only the data introduced in assumption* $(\mathcal{P})$,

$$\begin{aligned} M_1 = M_1\Big(\max_{i \in \{1\dots I\}} \Big\{ \quad & \sup_{x \in (0,a_i)} | - \sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x))| + \\ & |\partial_t \phi_i|_{(0,T)} \quad \Big\}, \max_{i \in \{1\dots I\}} |g_i|_{(0,a_i)}, C_H \Big), \\ M_2 = M_2\Big(& \overline{\nu}, \underline{\nu}, \mu(M_1), \gamma(M_1), \varepsilon(M_1), \sup_{|p|\geq 0} P(M_1,|p|), |\partial_x g_i|_{(0,a_i)}, M_1 \Big), \\ M_3 = M_3\Big(& M_1, \underline{\nu}(1+|p|)^{m-2}, \mu(|u|)(1+|p|)^m, \quad |u| \leq M_1, \quad |p| \leq M_2 \Big), \end{aligned}$$

such that

$$||u||_{\mathcal{C}(\mathcal{J}_T^a)} \leq M_1, \quad ||\partial_x u||_{\mathcal{C}(\mathcal{J}_T^a)} \leq M_2, \quad ||\partial_t u||_{\mathcal{C}(\mathcal{J}_T^a)} \leq M_1, \quad ||\partial_{x,x} u||_{\mathcal{C}(\mathcal{J}_T^a)} \leq M_3.$$

Moreover, there exists a constant $M(\alpha)$ depending on $\Big(\alpha, M_1, M_2, M_3\Big)$ such that

$$||u||_{\mathcal{C}^{\frac{\alpha}{2},1+\alpha}(\mathcal{J}_T^a)} \leq M(\alpha).$$

We continue this Section by giving the definitions of super and sub solution, and stating a comparison Theorem for our problem.

Definition 2.3. *We say that $u \in \mathcal{C}^{0,1}(\mathcal{J}_T^a) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}_T^a})$, is a super solution (resp. sub solution) of*

$$\begin{cases} \partial_t u_i(t,x) - \sigma_i(x, \partial_x u_i(t,x))\partial_{x,x}^2 u_i(t,x) + \\ H_i(x, u_i(t,x), \partial_x u_i(t,x)) \quad = \quad 0, \quad \text{if } (t,x) \in (0,T) \times (0,a_i), \\ F(u(t,0), \partial_x u(t,0)) \quad = \quad 0, \quad \text{if } t \in (0,T), \end{cases} \tag{4}$$

if

$$\begin{cases} \partial_t u_i(t,x) - \sigma_i(x, \partial_x u_i(t,x))\partial_{x,x}^2 u_i(t,x) + \\ H_i(x, u_i(t,x), \partial_x u_i(t,x)) \quad \geq \quad 0, \quad (\text{resp.} \leq 0), \quad \forall (t,x) \in (0,T) \times (0,a_i), \\ F(u(t,0), \partial_x u(t,0)) \quad \leq \quad 0, \quad (\text{resp.} \geq 0), \quad \forall t \in (0,T) \end{cases}$$

Theorem 2.4. *Parabolic comparison.*
Assume $(\mathcal{P})$. Let $u \in \mathcal{C}^{0,1}(\mathcal{J}_T^a) \cap \mathcal{C}_b^{1,2}(\overset{\circ}{\mathcal{J}_T^a})$ (resp. $v \in \mathcal{C}^{0,1}(\mathcal{J}_T^a) \cap \mathcal{C}_b^{1,2}(\overset{\circ}{\mathcal{J}_T^a})$) a super solution (resp. a sub solution) of (4), satisfying for all $i \in \{1 \dots I\}$, $u_i(t,a_i) \geq v_i(t,a_i)$, for all $t \in [0,T]$, and $u_i(0,x) \geq v_i(0,x)$, for all $x \in [0,a_i]$.
Then for each $(t,(x,i)) \in \mathcal{J}_T^a : u_i(t,x) \geq v_i(t,x)$.

Proof. We start by showing that for each $0 \leq s < T$, for all $(t,(x,i)) \in \mathcal{J}_s^a$, $u_i(t,x) \geq v_i(t,x)$.
Let $\lambda > 0$. Suppose that $\lambda > C_1 + C_2$, where the expression of the constants (C_1, C_2) are given in the sequel (see (5), and (6)). We argue by contradiction assuming that

$$\sup_{(t,(x,i)) \in \mathcal{J}_s^a} \exp(-\lambda t + x)\Big(v_i(t,x) - u_i(t,x)\Big) > 0.$$

Using the boundary conditions satisfied by u and v, the supremum above is reached at a point $(t_0, (x_0, j_0)) \in (0, s] \times \mathcal{J}$, with $0 \leq x_0 < a_{j_0}$.

Suppose first that $x_0 > 0$, the optimality conditions imply that

$$\begin{aligned}
\exp(-\lambda t_0 + x_0)\Big(-\lambda(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)) &+ \partial_t v_{j_0}(t_0,x_0) - \partial_t u_{j_0}(t_0,x_0)\Big) \geq 0,\\
\exp(-\lambda t_0 + x_0))\Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0) &+ \partial_x v_{j_0}(t_0,x_0) - \partial_x u_{j_0}(t_0,x_0)\Big) = 0,\\
\exp(-\lambda t_0 + x_0)\Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0) &+ 2\Big(\partial_x v_{j_0}(t_0,x_0) - \partial_x u_{j_0}(t_0,x_0)\Big)\\
&+ \Big(\partial^2_{x,x} v_{j_0}(t_0,x_0) - \partial^2_{x,x} u_{j_0}(t_0,x_0)\Big)\Big) =\\
\exp(-\lambda t_0 + x_0)\Big(- \Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)\Big) &+ \partial^2_{x,x} v_{j_0}(t_0,x_0) - \partial^2_{x,x} u_{j_0}(t_0,x_0)\Big) \leq 0.
\end{aligned}$$

Using assumptions $(\mathcal{P})$ (iv) a), (iv) c) and the optimality conditions above we have

$$\begin{aligned}
H_{j_0}(x_0, u_i(t_0,x_0), \partial_x u_{j_0}(t_0,x_0)) - H_{j_0}(x_0, v_{j_0}(t_0,x_0), \partial_x v_{j_0}(t_0,x_0)) &\leq\\
\Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)\Big)\Big(C_H + \gamma(|\partial_x v_{j_0}(t_0,x_0)|)\Big)\Big((1 + |\partial_x u_{j_0}(t_0,x_0))| \vee |\partial_x v_{j_0}(t_0,x_0))|)^{m-1}\Big)&\\
\leq C_1\Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)\Big),&
\end{aligned}$$

where

$$\begin{aligned}
C_1 := \max_{i\in\{1\ldots I\}} \Big\{ \sup_{(t,x)\in[0,T]\times[0,a_i]}\Big\{ \Big(C_H + \gamma(|\partial_x v_i(t,x)|\Big)\Big(1 + |\partial_x u_i(t,x))|&\\
\vee|\partial_x v_i(t,x))|\Big)^{m-1} \Big\}\Big\}. \qquad (5)
\end{aligned}$$

On the other hand we have using assumption $(\mathcal{P})$ (ii), (iv) a), (iv) c), and the optimality conditions

$$\begin{aligned}
\sigma_{j_0}(x_0, \partial_x v_{j_0}(t_0,x_0))\partial^2_{x,x} v_{j_0}(t_0,x_0) - \sigma_{j_0}(x_0, \partial_x u_{j_0}(t_0,x_0))\partial^2_{x,x} u_{j_0}(t_0,x_0) &\leq\\
\Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)\Big)\Big(\overline{\nu}(1 + |\partial_x v_{j_0}(t_0,x_0)|)^{m-2} + \Big|\partial^2_{x,x} u_{j_0}(t_0,x_0)\Big|&\\
+ \gamma(|\partial_x u_{j_0}(t_0,x_0)|)(1 + |\partial_x u_{j_0}(t_0,x_0))| \vee |\partial_x v_{j_0}(t_0,x_0))|)^{m-1} \Big)&\\
\leq C_2\Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)\Big),&
\end{aligned}$$

where

$$\begin{aligned}
C_2 := \max_{i\in\{1\ldots I\}} \Big\{ \sup_{(t,x)\in[0,T]\times[0,a_i]}\Big\{ \overline{\nu}(1 + |\partial_x v_i(t,x)|)^{m-2} + \Big|\partial^2_{x,x} u_i(t,x)\Big|&\\
+ \gamma(|\partial_x u_i(t,x)|)(1 + |\partial_x u_i(t,x))| + |\partial_x v_i(t,x))|)^{m-1} \Big\}\Big\}. \qquad (6)
\end{aligned}$$

Using now the fact that v is a sub-solution while u is a super-solution, we get

$$\begin{aligned}
&0 \leq \\
&\partial_t u_{j_0}(t_0,x_0) - \sigma_{j_0}(x_0,\partial_x u_{j_0}(t_0,x_0))\partial^2_{x,x}u_{j_0}(t_0,x_0) + H_{j_0}(x_0,u_i(t_0,x_0),\partial_x u_{j_0}(t_0,x_0)) \\
&-\partial_t v_{j_0}(t_0,x_0) + \sigma_{j_0}(x_0,\partial_x v_{j_0}(t_0,x_0))\partial^2_{x,x}v_{j_0}(t_0,x_0) - H_{j_0}(x_0,v_{j_0}(t_0,x_0),\partial_x v_{j_0}(t_0,x_0)) \\
&\leq -(\lambda - (C_1+C_2))(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)) \quad < \quad 0,
\end{aligned}$$

which is a contradiction. Therefore the supremum is reached at $(t_0,0)$, with $t_0 \in (0,s]$. We apply a first order Taylor expansion in space, in the neighborhood of the junction point 0. Since for all $(i,j) \in \{1\ldots I\}$, $u_i(t_0,0) = u_j(t_0,0) = u(t_0,0)$, and $v_i(t_0,0) = v_j(t_0,0) = v(t_0,0)$, we get from

$$\begin{aligned}
&\forall (i,j) \in \{1,\ldots I\}^2, \quad \forall h \in (0, \min_{i\in\{1\ldots I\}} a_i] \\
&v_j(t_0,0) - u_j(t_0,0) \quad \geq \quad \exp(h)\Big(v_i(t_0,h) - u_i(t_0,h)\Big),
\end{aligned}$$

that

$$\begin{aligned}
&\forall (i,j) \in \{1,\ldots I\}^2, \quad \forall h \in (0, \min_{i\in\{1\ldots I\}} a_i] \\
&v_j(t_0,0) - u_j(t_0,0) \quad \geq \quad v_i(t_0,0) - u_i(t_0,0) \quad + \\
&h\Big(\quad v_i(t_0,0) - u_i(t_0,0) \quad + \quad \partial_x v_i(t_0,0) - \partial_x u_i(t_0,0) \quad \Big) \quad + \quad h\varepsilon_i(h),
\end{aligned}$$

where

$$\forall i \in \{1,\ldots I\}, \quad \lim_{h\to 0}\varepsilon_i(h) = 0.$$

We get then

$$\forall i \in \{1,\ldots I\}, \quad \partial_x v_i(t_0,0) \quad \leq \quad \partial_x u_i(t_0,0) - \Big(v_i(t_0,0) - u_i(t_0,0)\Big) \quad < \quad \partial_x u_i(t_0,0).$$

Using the growth assumptions on F (assumption $(\mathcal{P})$(i)), and the fact that v is a sub-solution while u is a super-solution, we get

$$0 \quad \leq \quad F(t_0,v(t_0,0),\partial_x v(t_0,0)) \quad < \quad F(t_0,u(t_0,0),\partial_x u(t_0,0)) \quad \leq \quad 0,$$

and then a contradiction.

We deduce then for all $0 \leq s < T$, for all $(t,(x,i)) \in [0,s] \times \mathcal{J}^a$,

$$\exp(-\lambda t + x)\Big(v_i(t,x) - u_i(t,x)\Big) \quad \leq \quad 0.$$

Using the continuity of u and v, we deduce finally that for all $(t,(x,i)) \in [0,T] \times \mathcal{J}^a$,

$$v_i(t,x) \quad \leq \quad u_i(t,x).$$

□

3. The elliptic problem

As explained in the introduction, the construction of a solution for our parabolic problem (3) relies on a time discretization and on the solvability of the associated elliptic problem. We review in this section the well-posedness of the elliptic problem (2), which is formulated for regular maps $(x,i) \mapsto u_i(x)$, continuous at the junction point, namely each $i \neq j \in \{1 \dots I\}$, $u_i(0) = u_j(0) = u(0)$, that follows at each edge

$$-\sigma_i(x, \partial_x u_i(x))\partial^2_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) \quad = \quad 0,$$

and u_i satisfy the following non linear Neumann boundary condition at the vertex

$$F(u(0), \partial_x u(0)) \quad = \quad 0, \quad \text{where} \quad \partial_x u(0) \quad = \quad (\partial_x u_1(0), \dots, \partial_x u_I(0)).$$

We introduce the following data for $i \in \{1 \dots I\}$

$$\begin{cases} F \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^I, \mathbb{R}), \\ \sigma_i \in \mathcal{C}^1([0,a_i] \times \mathbb{R}, \mathbb{R}) \\ H_i \in \mathcal{C}^1([0,a_i] \times \mathbb{R}^2, \mathbb{R}) \\ \phi_i \in \mathbb{R} \end{cases},$$

satisfying the following assumptions

Assumption $(\mathcal{E})$

(i) Assumption on F

$$\begin{cases} a) \;\; F \text{ is decreasing with respect to its first variable,} \\ b) \;\; F \text{ is nondecreasing with respect to its second variable,} \\ c) \;\; \exists (b,B) \in \mathbb{R}\times\mathbb{R}^I, \text{ such that : } F(b,B) \;\; = \;\; 0, \end{cases}$$

or F satisfy the Kirchhoff condition

$$\begin{cases} a) \;\; F \text{ is nonincreasing with respect to its first variable,} \\ b) \;\; F \text{ is increasing with respect to its second variable,} \\ c) \;\; \exists (b,B) \in \mathbb{R}\times\mathbb{R}^I, \text{ such that : } F(b,B) \;\; = \;\; 0. \end{cases}$$

(ii) The ellipticity condition on the σ_i

$$\exists c > 0, \quad \forall i \in \{1 \dots I\}, \quad \forall (x,p) \in [0,a_i]\times\mathbb{R}, \quad \sigma_i(x,p) \geq c.$$

(iii) For the Hamiltonians H_i, we suppose

$$\exists C_H > 0, \;\; \forall i \in \{1 \dots I\}, \;\; \forall (x,u,v,p) \in (0,a_i)\times\mathbb{R}^3,$$
$$\text{if } u \leq v, \;\; C_H(u-v) \leq H_i(x,u,p) - H_i(x,v,p).$$

For each $i \in \{1 \dots I\}$, we define the following differential operators $(\delta_i, \overline{\delta}_i)_{i\in\{1\dots I\}}$ acting on $\mathcal{C}^1([0,a_i]\times\mathbb{R}^2,\mathbb{R})$, for $f = f(x,u,p)$ by

$$\delta_i \;\; := \;\; \partial_u + \frac{1}{p}\partial_x; \;\; \overline{\delta_i} \;\; := \;\; p\partial_p.$$

(iv) We impose the following restrictions on the growth with respect to p for the coefficients $(\sigma_i, H_i)_{i\in\{1\dots I\}} = (\sigma_i(x,p), H_i(x,u,p))_{i\in\{1\dots I\}}$, which are for all $i \in \{1 \dots I\}$

$$\begin{aligned} \delta_i\sigma_i &= o(\sigma_i), \\ \overline{\delta_i}\sigma_i &= \mathcal{O}(\sigma_i), \\ H_i &= \mathcal{O}(\sigma_i p^2), \\ \delta_i H_i &\leq o(\sigma_i p^2), \\ \overline{\delta_i} H_i &\leq \mathcal{O}(\sigma_i p^2), \end{aligned}$$

where the limits behind are understood as $p \to +\infty$, uniformly in x, for bounded u.
The main result of this section is the following Theorem, for the solvability and uniqueness of the elliptic problem at the junction, with non linear Neumann condition at the junction point.

Theorem 3.1. *Assume $(\mathcal{E})$. The following elliptic problem at the junction, with Neumann boundary condition at the vertex*

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x))\partial^2_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) & = & 0, \ \ \textit{if } x \in (0, a_i), \\ F(u(0), \partial_x u(0)) & = & 0, \\ \forall i \in \{1 \dots I\}, \quad u_i(a_i) & = & \phi_i, \end{cases} \tag{7}$$

is uniquely solvable in the class $\mathcal{C}^{2+\alpha}(\mathcal{J}^a)$.

Theorem 3.1 is stated without proof in [9]. For the convenience of the reader, we sketch its proof in the Appendix.
The uniqueness of the solution of (7), is a consequence of the elliptic comparison Theorem for smooth solutions, for the Neumann problem, stated in this Section, and whose proof uses the same arguments of the proof of the parabolic comparison Theorem 2.4. We complete this section by recalling the definition of super and sub solution for the elliptic problem (7), and the corresponding elliptic comparison Theorem.

Definition 3.2. *Let $u \in \mathcal{C}^2(\mathcal{J}^a)$. We say that u is a super solution (resp. sub solution) of*

$$\begin{cases} -\sigma_i(x, \partial_x f_i(x))\partial^2_{x,x} f_i(x) + H_i(x, f_i(x), \partial_x f_i(x)) & = & 0, \ \ \textit{if } x \in (0, a_i), \\ F(f(0), \partial_x f(0)) & = & 0, \end{cases} \tag{8}$$

if

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x))\partial^2_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) & \geq & 0, \quad (\textit{resp.} \leq 0), \quad \textit{if } x \in (0, a_i), \\ F(u(0), \partial_x u(0)) & \leq & 0, \quad (\textit{resp.} \geq 0). \end{cases}$$

Theorem 3.3. *Elliptic comparison Theorem, see for instance Theorem 2.1 of* [9].
Assume $(\mathcal{E})$. Let $u \in \mathcal{C}^2(\mathcal{J}^a)$ (resp. $v \in \mathcal{C}^2(\mathcal{J}^a)$) a super solution (resp. a sub solution)

of (8), satisfying for all $i \in \{1 \dots I\}$, $u_i(a_i) \geq v_i(a_i)$. Then for each $(x,i) \in \mathcal{J}^a$: $u_i(x) \geq v_i(x)$.

4. The parabolic problem

In this Section, we prove Theorem 2.2. The construction of the solution is based on the results obtained in Section 3 for the elliptic problem, and is done by considering a sequence $u^n \in \mathcal{C}^2(\mathcal{J}^a)$, solving on a time grid an elliptic scheme defined by induction. We will prove that the solution u^n converges to the required solution.

4.1. **Estimates on the discretized scheme.** Let $n \in \mathbb{N}^*$, we consider the following time grid, $(t_k^n = \frac{kT}{n})_{0 \leq k \leq n}$ of $[0,T]$, and the following sequence $(u_k)_{0\leq k \leq n}$ of $\mathcal{C}^{2+\alpha}(\mathcal{J}^a)$, defined recursively by

$$\text{for} \quad k = 0, \quad u_0 = g,$$

and for $1 \leq k \leq n$, u_k is the unique solution of the following elliptic problem

$$\begin{cases} n(u_{i,k}(x) - u_{i,k-1}(x)) - \sigma_i(x, \partial_x u_{i,k}(x))\partial_{x,x}^2 u_{i,k}(x)) + \\ H_i(x, u_{i,k}(x), \partial_x u_{i,k}(x)) \quad = \quad 0, \quad \text{if } x \in (0, a_i), \\ F(u_k(0), \partial_x u_k(0)) \quad = \quad 0, \\ \forall i \in \{1 \dots I\}, \quad u_{i,k}(a_i) \quad = \quad \phi_i(t_k^n). \end{cases} \tag{9}$$

The solvability of the elliptic scheme (9) can be proved by induction, using the same arguments as for Theorem 3.1. The next step consists in obtaining uniform estimates of $(u_k)_{0 \leq k \leq n}$. We start first by getting uniform bounds for $n|u_k - u_{k-1}|_{(0,a_i)}$ using the comparison Theorem 3.3.

Lemma 4.1. *Assume $(\mathcal{P})$. There exists a constant $C > 0$, independent of n, depending only the data $C = C\Big(\max_{i\in\{1\dots I\}}\Big\{\ \sup_{x\in(0,a_i)} |-\sigma_i(x, \partial_x g_i(x))\partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x))| + |\partial_t \phi_i|_{(0,T)}\ \Big\}, C_H\Big)$, such that*

$$\sup_{n\geq 0} \ \max_{k\in\{1\dots n\}} \ \max_{i\in\{1\dots I\}} \Big\{ \ n|u_{i,k} - u_{i,k-1}|_{(0,a_i)} \ \Big\} \ \leq \ C,$$

and then

$$\sup_{n\geq 0} \max_{k\in\{0\dots n\}} \max_{i\in\{1\dots I\}} \Big\{ \;\; |u_{i,k}|_{(0,a_i)} \;\; \Big\} \;\; \leq \;\; C + \max_{i\in\{1\dots I\}} \Big\{ \;\; |g_i|_{(0,a_i)} \;\; \Big\}.$$

Proof. Let $n > \lfloor C_H \rfloor$, where C_H is defined in assumption $(\mathcal{P})$ (iv) c). Let $k \in \{1 \dots n\}$, we define the following sequence

$$\begin{cases} M_0 & = & \max_{i\in\{1\dots I\}} \Big\{ \;\; \sup_{x\in(0,a_i)} | -\sigma_i(x,\partial_x g_i(x))\partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x))| + |\partial_t \phi_i|_{(0,T)} \;\; \Big\}, \\ M_{k,n} & = & \dfrac{n}{n - C_H} M_{k-1,n}, \quad k \in \{1 \dots n\}. \end{cases}$$

We claim that for each $k \in \{1 \dots n\}$

$$\max_{i\in\{1\dots I\}} \Big\{ \;\; n|u_{i,k} - u_{i,k-1}|_{(0,a_i)} \Big\} \;\; \leq \;\; M_{k,n}.$$

We give a proof by induction. For this, if $k = 1$, let us show that the map h defined on the junction by

$$h := \begin{cases} \mathcal{J}^a \to \mathbb{R} \\ (x,i) \mapsto \frac{M_{1,n}}{n} + g_i(x), \end{cases}$$

is a super solution of (9), for $k = 1$. For this we will use the Elliptic Comparison Theorem 3.3.

Using the compatibility conditions satisfied by g, namely assumption $(\mathcal{P})$ (v), and the assumptions of growth on F, assumption $(\mathcal{P})$ (i), we get for the boundary conditions

$$\begin{aligned} F(h(0), \partial_x h(0)) \;\; &= \;\; F(\frac{M_{1,n}}{n} + g(0), \partial_x g(0)) \;\; \leq \;\; F(g(0), \partial_x g(0)) \;\; = \;\; 0, \\ h(a_i) \;\; &= \;\; \frac{M_{1,n}}{n} + g_i(a_i) \;\; \geq \;\; \frac{M_{0,n}}{n} + g_i(a_i) \;\; \geq \;\; \phi_i(t_1^n). \end{aligned}$$

For all $i \in \{1 \dots I\}$, and $x \in (0, a_i)$, we get using assumption $(\mathcal{P})$ (iii)

$$\begin{aligned} n(h_i(x) - g_i(x)) - \sigma_i(x, \partial_x h_i(x))\partial_x^2 h_i(x) + H_i(x, h_i(x), \partial_x h_i(x)) \;\; &= \\ M_{1,n} - \sigma_i(x, \partial_x g_i(x))\partial_x^2 g_i(x) + H_i(x, \frac{M_{1,n}}{n} + g_i(x), \partial_x g_i(x)) \;\; &\geq \\ M_{1,n} - \sigma_i(x, \partial_x g_i(x))\partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x)) - \frac{M_{1,n} C_H}{n} \;\; &\geq \;\; 0. \end{aligned}$$

It follows from the comparison Theorem 3.3, that for all $i \in \{1 \dots I\}$, and $x \in [0, a_i]$

$$u_{1,i}(x) \quad \leq \quad \frac{M_{1,n}}{n} + g_i(x).$$

Using the same arguments, we show that

$$h := \begin{cases} \mathcal{J}^a \to \mathbb{R} \\ (x,i) \mapsto -\frac{M_{1,n}}{n} + g_i(x), \end{cases}$$

is a sub solution of (9) for $k = 1$, and we then get

$$\max_{i \in \{1 \dots I\}} \Big\{ \sup_{x \in (0,a)} n|u_{1,i}(x) - g_i(x)| \Big\} \quad \leq \quad M_{1,n}.$$

Let $2 \leq k \leq n$, suppose that the assumption of induction holds true. Let us show that the following map

$$h := \begin{cases} \mathcal{J}^a \to \mathbb{R} \\ (x,i) \mapsto \frac{M_{k,n}}{n} + u_{i,k-1}(x), \end{cases}$$

is a super solution of (9). For the boundary conditions, using assumption $(\mathcal{P})$ (i), we get

$$\begin{aligned} F(h(0), \partial_x h(0)) \quad &= \quad F(\frac{M_{k,n}}{n} + u_{k-1}(0), \partial_x u_{k-1}(0)) \quad \leq \quad F(u_{k-1}(0), \partial_x u_{k-1}(0)) \quad \leq \quad 0, \\ h(a_i) \quad &= \quad \frac{M_{k,n}}{n} + u_{i,k-1}(a_i) \quad \geq \quad \frac{M_{0,n}}{n} + u_{i,k-1}(a_i) \quad \geq \quad \phi_i(t_k^n). \end{aligned}$$

For all $i \in \{1 \dots I\}$, and $x \in (0, a_i)$

$$\begin{aligned} n(h_i(x) - u_{i,k-1}(x)) - \sigma_i(x, \partial_x h(x))\partial_x^2 h(x) + H_i(x, h(x), \partial_x h(x)) \quad &= \\ M_{k,n} - \sigma_i(x, \partial_x u_{i,k-1}(x))\partial_x^2 u_{i,k-1}(x) + H_i(x, \frac{M_{k,n}}{n} + u_{i,k-1}(x), \partial_x u_{k-1}(x)) \quad &\geq \\ M_{k,n} - \sigma_i(x, \partial_x u_{i,k-1}(x))\partial_x^2 u_{i,k-1}(x) + H_i(x, u_{i,k-1}(x), \partial_x u_{k-1}(x)) - \frac{C_H M_{k,n}}{n}&. \end{aligned}$$

Since we have for all $x \in (0, a_i)$

$$-\sigma_i(x, \partial_x u_{i,k-1}(x))\partial_x^2 u_{i,k-1}(x) + H_i(x, u_{i,k-1}(x), \partial_x u_{i,k-1}(x)) \quad = \quad -n(u_{i,k-1}(x) - u_{i,k-2}(x)),$$

using the induction assumption we get

$$\begin{aligned} n(h_i(x)-u_{i,k-1}(x))-\sigma_i(x,\partial_x h(x))\partial_x^2 h(x)+H_i(x,\partial_x h(x),\partial_x h(x)) &\geq \\ M_{k,n}-n(u_{i,k-1}(x)-u_{i,k-2}(x))-\frac{C_H M_{k,n}}{n} \quad \geq \quad M_{k,n}\frac{n-C_H}{n}-M_{k-1,n} &\geq 0. \end{aligned}$$

It follows from the comparison Theorem 3.3, that for all $(x,i)\in\mathcal{J}^a$

$$u_{i,k}(x) \quad \leq \quad \frac{M_{k,n}}{n}+u_{i,k-1}(x).$$

Using the same arguments, we show that

$$h := \begin{cases} \mathcal{J}^a \to \mathbb{R} \\ (x,i) \mapsto -\frac{M_{k,n}}{n}+u_{i,k-1}(x), \end{cases}$$

is a sub solution of (9), and we get

$$\max_{i\in\{1\ldots I\}}\Big\{ \ n|u_{i,k}(x)-u_{i,k-1}(x)|_{(0,a_i)} \ \Big\} \quad \leq \quad M_{k,n}.$$

We obtain finally using that for all $k\in\{1\ldots n\}$

$$\begin{cases} M_{k,n} & \leq \quad M_{n,n}, \\ M_{k,n} & = \quad \Big(\dfrac{n}{n-C_H}\Big)^k M_0, \end{cases}$$

and

$$\begin{aligned} M_{n,n} \xrightarrow{n\to+\infty} \quad M \quad := \quad \exp(C_H)\max_{i\in\{1\ldots I\}}\Big\{ \ & \sup_{x\in(0,a_i)}|-\sigma_i(x,\partial_x g_i(x))\partial_x^2 g_i(x)+ \\ & H_i(x,g_i(x),\partial_x g_i(x))|+|\partial_t\phi_i|_{(0,T)} \ \Big\}, \end{aligned}$$

that

$$\begin{aligned} \sup_{n\geq 0} \quad \max_{k\in\{1\ldots n\}} \quad \max_{i\in\{1\ldots I\}}\Big\{ \ n|u_{i,k}-u_{i,k-1}|_{(0,a_i)} \ \Big\} \quad &\leq \quad C, \\ \sup_{n\geq 0} \quad \max_{k\in\{0\ldots n\}} \quad \max_{i\in\{1\ldots I\}}\Big\{ \ |u_{i,k}|_{(0,a_i)} \ \Big\} \quad \leq \quad C+\max_{i\in\{1\ldots I\}}\Big\{ \ |g_i|_{(0,a_i)} \ \Big\}&. \end{aligned}$$

That completes the proof. □

The next step consists in obtaining uniform estimates for $|\partial_x u_k|_{(0,a_i)}$, in terms of $n|u_k-u_{k-1}|_{(0,a_i)}$ and the quantities $(\underline{\nu},\overline{\nu},\mu,\gamma,\varepsilon,P)$ introduced in assumption $(\mathcal{P})$ (ii), (iii) and (iv). More precisely, we use similar arguments as for the proof of Theorem 14.1 of [5], using a classical argument of upper and lower barrier functions at the boundary. The assumption of growth $(\mathcal{P})$ (ii) and (iii) are used in a key way to get an uniform bound on the gradient at the boundary. Finally to conclude, we appeal to a gradient maximum principle, using the growth assumption $(\mathcal{P})$ (iv), adapting Theorem 15.2 of [5] to our elliptic scheme.

Lemma 4.2. *Assume $(\mathcal{P})$. There exists a constant $C > 0$, independent of n, depending only the data*

$$\Big(\overline{\nu}, \underline{\nu}, \mu(|u|), \gamma(|u|), \varepsilon(|u|), \sup_{|p|\geq 0} P(|u|,|p|), |\partial_x g_i|_{(0,a_i)},$$
$$|u| \leq \sup_{n\geq 0} \max_{k\in\{0\ldots n\}} \max_{i\in\{1\ldots I\}} \Big\{ \ |u_{i,k}|_{(0,a_i)} \ \Big\},$$
$$\sup_{n\geq 0} \max_{k\in\{1\ldots n\}} \max_{i\in\{1\ldots I\}} \Big\{ \ n|u_{i,k} - u_{i,k-1}|_{(0,a_i)} \ \Big\}\Big),$$

such that

$$\sup_{n\geq 0} \max_{k\in\{0\ldots n\}} \max_{i\in\{1\ldots I\}} \Big\{ \ |\partial_x u_{i,k}|_{(0,a_i)} \ \Big\} \ \leq \ C.$$

Proof. **Step 1 :** We claim that, for each $k \in \{1 \ldots n\}$, $\max_{i\in\{1\ldots I\}}\Big\{ \ |\partial_x u_{i,k}|_{\partial(0,a_i)} \ \Big\}$ is bounded by the data, uniformly in n.

It follows from Lemma 4.1, that there exists $M > 0$ such that

$$\sup_{n\geq 0} \max_{k\in\{0\ldots n\}} \max_{i\in\{1\ldots I\}} \Big\{ \ |u_{i,k}|_{(0,a_i)} + n|u_{i,k} - u_{i,k-1}|_{(0,a_i)} \ \Big\} \ \leq \ M.$$

We fix $i \in \{1 \ldots I\}$. We apply a barrier method consisting in building two functions $w^+_{i,k}, w^-_{i,k}$ satisfying in a neighborhood of 0, for example $[0, \kappa]$, with $\kappa \leq a_i$

$$Q_i(x, w^+_{i,k}(x), \partial_x w^+_{i,k}(x), \partial^2_x w^+_{i,k}(x)) \geq 0, \quad \forall x \in [0,\kappa], \quad w^+_{i,k}(0) = u_{i,k}(0), \quad w^+_{i,k}(\kappa) \geq M,$$
$$Q_i(x, w^-_{i,k}(x), \partial_x w^-_{i,k}(x), \partial^2_x w^-_{i,k}(x)) \leq 0, \quad \forall x \in [0,\kappa], \quad w^-_{i,k}(0) = u_{i,k}(0), \quad w^-_{i,k}(\kappa) \leq -M,$$

where we recall that for each $(x, u, p, S) \in [0, a_i] \times R^3$

$$Q_i(x,u,p,S) \ = \ n(u - u_{i,k-1}(x)) - \sigma_i(x,p)S + H_i(x,u,p).$$

For $n > \lfloor C_H \rfloor$, where C_H is defined in assumption $\mathcal{P}$ (iv) c), it follows then from the comparison principle that

$$w^-_{i,k}(x) \leq u_{i,k}(x) \leq w^+_{i,k}(x), \quad \forall x \in [0, \kappa],$$

and then

$$\partial_x w^-_{i,k}(0) \leq \partial_x u_{i,k}(0) \leq \partial_x w^+_{i,k}(0).$$

We look for $w^+_{i,k}$ defined on $[0, \kappa]$ of the form

$$w^+_{i,0} = g_i(x)$$
$$w^+_{i,k} : x \mapsto u_{i,k}(0) + \frac{1}{\beta} \ln(1 + \theta x),$$

where the constants (β, θ, κ) will be chosen in the sequel independent of k.
Remark first that for all $x \in [0, \kappa]$, $\partial^2_x w^+_{i,k}(x) = -\beta \partial_x w^+_{i,k}(x)^2$, and $w^+_{i,k}(0) = u_{i,k}(0)$. Let we choose (θ, κ), such that

$$\forall k \in \{1 \dots n\}, \quad 0 < \kappa \leq \min_{i \in \{1 \dots I\}} a_i, \quad w^+_{i,k}(\kappa) \geq M, \quad \partial_x w^+_{i,k}(\kappa) \geq \beta. \tag{10}$$

We choose for instance

$$\theta = \beta^2 \exp(2\beta M) \quad + \quad \frac{1}{\min_{i \in \{1 \dots I\}} a_i} \exp(2\beta M)$$
$$\kappa = \frac{1}{\theta}\Big(\exp(2\beta M) - 1 \Big). \tag{11}$$

The constant β will be chosen in order to get

$$\beta \quad \geq \quad \sup_{k \in \{1 \dots n\}} \sup_{x \in [0,\kappa]} \frac{\mu(w^+_{i,k}(x))(1 + \partial_x w^+_{i,k}(x))^m + M}{\underline{\nu}(1 + \partial_x w^+_{i,k}(x))^{m-2} \partial_x w^+_{i,k}(x)^2}, \tag{12}$$

where $(\mu(.), \underline{\nu}, m)$ are defined in assumption $(\mathcal{P})$ (ii) and (iii). Since we have

$$\forall x \in [0, \kappa], \quad w^+_{i,k}(x) \leq w^+_{i,k}(\kappa) = 2M,$$
$$\beta \leq \partial_x w^+_{i,k}(\kappa) \leq \partial_x w^+_{i,k}(x) \leq \partial_x w^+_{i,k}(0).$$

We can then choose β large enough to get (12), for instance

$$\beta \quad \geq \quad \frac{\mu(2M)}{\underline{\nu}} \Big(1 + \frac{1}{\beta^2} \Big) + \frac{M}{\underline{\nu} \beta^2}.$$

It is easy to show by induction that $w_{i,k}^+$ is lower barrier of $u_{i,k}$ in the neighborhood $[0,\kappa]$. More precisely, since $w_{i,0}^+ = u_{i,0}$, and for all $k \in \{1\dots n\}$

$$w_{i,k}^+(0) = u_{i,k}(0), \quad w_{i,k}^+(\kappa) \geq u_{i,k}(\kappa),$$
$$w_{i,k}^+(x) = w_{i,k-1}^+(x) + u_{i,k}(0) - u_{i,k-1}(0) \geq w_{i,k-1}^+(x) - \frac{M}{n},$$

we get using the assumption of induction, assumption $(\mathcal{P})$ (ii) and (iii), and (12) that for all $x \in (0,\kappa)$

$$n(w_{i,k}^+(x) - u_{i,k-1}(x)) - \sigma_i(x, \partial_x w_{i,k}^+(x))\partial_{x,x} w_{i,k}^+(x) + H_i(x, w_{i,k}^+(x), \partial_x w_{i,k}^+(x)) \geq$$
$$-M + \beta\sigma_i(x, \partial_x w_{i,k}^+(x))\partial_x w_{i,k}^+(x)^2 + H_i(x, w_{i,k}^+(x), \partial_x w_{i,k}^+(x)) \geq$$
$$-M + \beta\underline{\nu}(1 + \partial_x w_{i,k}^+(x))^{m-2}\partial_x w_{i,k}^+(x)^2 + \mu(w_{i,k}^+(x))(1 + \partial_x w_{i,k}^+(x))^m \geq 0.$$

We obtain therefore

$$\sup_{n\geq 0} \max_{k\in\{0\dots n\}} \max_{i\in\{1\dots I\}} \partial_x u_{i,k}(0) \quad \leq \quad \frac{\theta}{\beta} \vee \partial_x g_i(0).$$

With the same arguments we can show that

$$w_{i,0}^- = g_i(x)$$
$$w_{i,k}^- : x \mapsto u_{i,k}(0) - \frac{1}{\beta}\ln(1 + \theta x),$$

is a lower barrier in the neighborhood of 0. Using the same method, we can show that $\partial_x u_{i,k}(a_i)$ is uniformly bounded by the same upper bounds, which completes the proof of **Step 1**.

Step 2 : For the convenience of the reader, we do not detail all the computations of this Step, since they can be found in the proof of Theorem 15.2 of [5]. It follows from Lemma 4.1 that there exists $M > 0$ such that

$$\sup_{n\geq 0} \quad \max_{k\in\{0\dots n\}} \quad \max_{i\in\{1\dots I\}} \quad \Big\{ \ |u_{i,k}|_{(0,a_i)} \ \Big\} \quad \leq \quad M.$$

We set furthermore

$$\forall (x,u,p) \in [0,a_i] \times \mathbb{R}^2, \quad H_{i,k}^n(x,u,p) = n(u - u_{i,k-1}(x)) + H_i(x,u,p).$$

Let u be a solution of the elliptic equation, for $x \in (0, a_i)$

$$\sigma_i(x, \partial_x u(x))\partial_{x,x} u(x) - H^n_{i,k}(x, u(x), \partial_x u(x)) = 0,$$

and assume that $|u|_{(0,a_i)} \leq M$. The main key of the proof will be in the use of the following equalities

$$\delta_i H^n_{i,k}(x, u, p) = \delta_i H_i(x, u, p) + \frac{n(p - \partial_x u_{i,k-1}(x))}{p}, \quad \overline{\delta}_i H^n_{i,k}(x, u, p) = \delta_i H_i(x, u, p), \quad (13)$$

where we recall that the operators δ_i and $\bar{\delta}_i$ are defined in assumption $(\mathcal{E})$ (iii). We follow the proof of Theorem 15.2 in [5]. We set $u = \psi(\overline{u})$, where $\psi \in \mathcal{C}^3[\overline{m}, \overline{M}]$, is increasing and $\overline{m} = \phi(-M)$, $\overline{M} = \phi(M)$. In the sequel, we will set $v = \partial_x u^2$ and $\overline{v} = \partial_x \overline{u}^2$. To simplify the notations, we will omit the variables $(x, u(x), \partial_x u(x))$ in the functions σ_i and $H^n_{i,k}$, and the variable $\overline{u}$ for ψ. We assume first that the solution $u \in \mathcal{C}^3([-M, M])$, and we follow exactly all the computations that lead to equation of (15.25) of [5] to get the following inequality

$$\sigma_i \partial_{x,x} \overline{v} + B_i \partial_x \overline{v} + G^n_{i,k} \geq 0, \quad (14)$$

where B_i and $G^n_{i,k}$ have the same expression in (15.26) of [5] with $(\sigma_i = \sigma_i^*, c_i = 0)$. We choose $(r = 0, s = 0)$, since we will see in the sequel (15), that condition (15.32) of [5] holds under assumption assumption $(\mathcal{P})$. We have more precisely

$$B_i = \psi' \partial_p \sigma_i \partial_{x,x} \overline{u} - \partial_p H_i + \omega \partial_p(\sigma_i p^2),$$

$$G^n_{i,k} = \frac{\omega'}{\psi'} + \kappa_i \omega^2 + \beta_i \omega + \theta^n_{i,k},$$

$$\omega = \frac{\psi''}{\psi'^2} \in \mathcal{C}^1([\overline{m}, \overline{M}]),$$

$$\kappa_i = \frac{1}{\sigma_i p^2}\Big(\overline{\delta}_i(\sigma_i p^2) + \frac{p^2}{4\sigma_i}|(\overline{\delta}_i + 1)\sigma_i|^2\Big),$$

$$\beta_i = \frac{1}{\sigma_i p^2}\Big(\delta_i(\sigma_i p^2) - \overline{\delta}_i H_i + \frac{p^2}{2\sigma_i}((\overline{\delta}_i + 1)\sigma_i)(\delta_i \sigma_i)\Big),$$

$$\theta^n_{i,k} = \frac{1}{\sigma_i p^2}\Big(\frac{p^2}{4\sigma_i}|\delta_i \sigma_i|^2 - \delta_i H^n_{i,k}\Big) = \theta_i - \frac{1}{\sigma_i p^2}\Big(\frac{n(p - \partial_x u_{i,k-1}(x))}{p}\Big),$$

$$\theta_i = \frac{1}{\sigma_i p^2}\Big(\frac{p^2}{4\sigma_i}|\delta_i \sigma_i|^2 - \delta_i H_i\Big).$$

We set in the sequel

$$G_i = \frac{\partial_x \omega}{\partial_x \psi} + \kappa_i \omega^2 + \beta_i \omega + \theta_i, \text{ in order to get } G^n_{i,k} = G_i - \frac{1}{\sigma_i p^2}\Big(\frac{n(p - \partial_x u_{i,k-1}(x))}{p}\Big).$$

More precisely, we see from (13) that all the coefficients $(B_i, \kappa_i, \beta_i, \theta_i)$ can be chosen independent of n and $u_{i,k-1}$. The main argument then to get a bound of $\partial_x u$ is to apply a maximum principle for $\overline{v}$ in (14), and this will be done as soon as we ensure

$$G^n_{i,k} \leq 0, \text{ for } |\partial_x u| \geq L^n_k.$$

On the other hand, using assumption $(\mathcal{P})$ (ii) (iii) and (iv), it is easy to check that there exists a constants (a, b, c), depending only on the data

$$\Big(\overline{\nu}, \underline{\nu}, \mu(M), \gamma(M), \varepsilon(M), \sup_{|p|\geq 0} P(M, |p|)\Big),$$

such that

$$\begin{array}{llll} \sup_{x\in[0,a_i],|u|\leq M} & \limsup_{|p|\to+\infty} & \kappa_i(x,u,p) & \leq & a, \\ \sup_{x\in[0,a_i],|u|\leq M} & \limsup_{|p|\to+\infty} & \beta_i(x,u,p) & \leq & b, \\ \sup_{x\in[0,a_i],|u|\leq M} & \limsup_{|p|\to+\infty} & \theta_i(x,u,p) & \leq & c, \end{array}$$

where

$$a = \frac{1}{\underline{\nu}}(\gamma(M) + \overline{\nu}) + \frac{1}{2} + \frac{\gamma(M)^2}{\underline{\nu}^2},$$

$$b = \frac{\varepsilon(M) + \sup_{|p|\geq 0} P(M,|p|) + \gamma(M)}{\underline{\nu}} + \frac{(\varepsilon(M) + \sup_{|p|\geq 0} P(M,|p|))(\overline{\nu} + \gamma(M))}{\underline{\nu}^2},$$

$$c = \frac{(\varepsilon(M) + \sup_{|p|\geq 0} P(M,|p|))^2}{4\underline{\nu}^2} + \frac{2(\varepsilon(M) + \sup_{|p|\geq 0} P(M,|p|))}{\underline{\nu}}.$$

As it has been on the proof of Theorem 15.2 of [5], we choose then $L = L(a, b, c)$, and $\psi(\cdot) = \psi(a, b, c)(\cdot)$ such that we have

$$G_i \leq 0, \quad \text{if} \quad |\partial_x u(x)| \geq L(a, b, c).$$

We see then from the expression of $\theta^n_{i,k}$ that we get

$$G^n_{i,k} \leq 0, \quad \text{if} \quad |\partial_x u(x)| \geq L(a, b, c) \vee |\partial_x u_{i,k-1}(x)|.$$

Therefore applying the maximum principle to $\overline{v}$ in (14), and from the relation $u = \psi(\overline{u})$, $\overline{v} = \partial_x \overline{u}^2$ we get finally

$$|\partial_x u|_{(0,a_i)} \leq \max\Big(\frac{\max \psi'(a,b,c)(\cdot)}{\min \psi'(a,b,c)(\cdot)}, |\partial_x u|_{\partial(0,a_i)}, L(a,b,c), |\partial_x u_{i,k-1}|_{(0,a_i)}\Big).$$

This upper bound still holds if $u \in \mathcal{C}^2([0,a_i])$, (cf. (15.30) and (15.31) of the proof of Theorem 15.2 in [5]). Finally applying the upper bound above to the solution u_k, we get by induction that

$$\begin{aligned} &\sup_{n\geq 0} \quad \max_{k\in\{0\ldots n\}} \quad \max_{i\in\{1\ldots I\}} \quad \Big\{ \quad |\partial_x u_{i,k}|_{(0,a_i)} \quad \Big\} \\ \leq & \quad \max\Big(\frac{\max \psi'(a,b,c)(\cdot)}{\min \psi'(a,b,c)(\cdot)}, |\partial_x u_{i,k}|_{\partial(0,a_i)}, L(a,b,c), |\partial_x g_i|_{(0,a_i)}\Big). \end{aligned}$$

This completes the proof. □

The following Proposition follows from Lemmas 4.1 and 4.2, assumption $(\mathcal{P})$ (ii) (iii), and from the relation

$$\begin{aligned} \forall x \in [0,a_i], \quad |\partial^2_{x,x} u_{i,k}(x))| &\leq \frac{|n(u_{i,k}(x) - u_{i,k-1}(x))| + |H_i(x, u_{i,k}(x), \partial_x u_{i,k}(x))|}{\sigma_i(x, \partial_x u_{i,k}(x))} \\ &\leq \frac{|n(u_{i,k}(x) - u_{i,k-1}(x))| + \mu(|u_{i,k}(x)|)(1 + |\partial_x u_{i,k}(x)|^m)}{\underline{\nu}(1 + |\partial_x u_{i,k}(x)|^{m-2})}. \end{aligned}$$

Proposition 4.3. *Assume $(\mathcal{P})$. There exist constants (M_1, M_2, M_3), depending only the data introduced in assumption $(\mathcal{P})$*

$$\begin{aligned} M_1 = M_1\Big(&\max_{i\in\{1\ldots I\}}\Big\{ \quad \sup_{x\in(0,a_i)} |-\sigma_i(x, \partial_x g_i(x))\partial^2_x g_i(x) + H_i(x, g_i(x), \partial_x g_i(x))| + \\ &|\partial_t \phi_i|_{(0,T)} \quad \Big\}, \max_{i\in\{1\ldots I\}} |g_i|_{(0,a_i)}, C_H\Big), \\ M_2 = M_2\Big(&\overline{\nu}, \underline{\nu}, \mu(M_1), \gamma(M_1), \varepsilon(M_1), \sup_{|p|\geq 0} P(M_1, |p|), |\partial_x g_i|_{(0,a_i)}, M_1\Big), \\ M_3 = M_3\Big(&M_1, \underline{\nu}(1+|p|)^{m-2}, \mu(|u|)(1+|p|)^m, \quad |u| \leq M_1, \quad |p| \leq M_2\Big), \end{aligned}$$

such that

$$\begin{aligned}\sup_{n\geq 0}\ \max_{k\in\{0\dots n\}}\ \max_{i\in\{1\dots I\}}\ \Big\{\ |u_{i,k}|_{(0,a_i)}\ \Big\}\ &\leq\ M_1,\\ \sup_{n\geq 0}\ \max_{k\in\{0\dots n\}}\ \max_{i\in\{1\dots I\}}\ \Big\{\ |\partial_x u_{i,k}|_{(0,a_i)}\ \Big\}\ &\leq\ M_2,\\ \sup_{n\geq 0}\ \max_{k\in\{1\dots n\}}\ \max_{i\in\{1\dots I\}}\ \Big\{\ |n(u_{i,k}-u_{i,k-1})|_{(0,a_i)}\ \Big\}\ &\leq\ M_1,\\ \sup_{n\geq 0}\ \max_{k\in\{0\dots n\}}\ \max_{i\in\{1\dots I\}}\ \Big\{\ |\partial_{x,x} u_{i,k}|_{(0,a_i)}\ \Big\}\ &\leq\ M_3.\end{aligned}$$

Unfortunately, we are unable to give an upper bound of the modulus of continuity of $\partial^2_{x,x}u_{i,k}$ in $\mathcal{C}^\alpha([0,a])$ independent of n. However, we are able to formulate in the weak sense a limit solution. From the regularity of the coefficients, using some tools introduced in Section 1, Lemma 2.1, we get interior regularity, and a smooth limit solution.

4.2. **Proof of Theorem 2.2.**

Proof. The uniqueness is a result of the comparison Theorem 2.4. To simplify the notations, we set for each $i\in\{1\dots I\}$, and for each $(x,q,u,p,S)\in[0,a_i]\times R^4$

$$Q_i(x,u,q,p,S)\ =\ q-\sigma_i(x,p)S+H_i(x,u,p).$$

Let $n\geq 0$. Consider the subdivision $(t^n_k=\frac{kT}{n})_{0\leq k\leq n}$ of $[0,T]$, and $(u_k)_{0\leq k\leq n}$ the solution of (9).

From estimates of Proposition 4.3, there exists a constant $M>0$ independent of n, such that

$$\begin{aligned}\sup\nolimits_{n\geq 0}\ \max\nolimits_{k\in\{1\dots n\}}\ \max\nolimits_{i\in\{1\dots I\}}\Big\{\ |u_{i,k}|_{(0,a_i)}\ +\ |n(u_{i,k}-u_{i,k-1})|_{(0,a_i)}\ +\\ |\partial_x u_{i,k}|_{(0,a_i)}\ +\ |\partial_{x,x}u_{i,k}|_{(0,a_i)}\ \Big\}\ \leq\ M.\end{aligned}\tag{15}$$

We define the following sequence $(v_n)_{n\geq 0}$ in $\mathcal{C}^{0,2}(\mathcal{J}^a_T)$, piecewise differentiable with respect to its first variable by

$$\begin{aligned}\forall i\in\{1\dots I\},\quad v_{i,0}(0,x)=g_i(x)\quad \text{if } x\in[0,a_i],\\ v_{i,n}(t,x)\ =\ u_{i,k}(x)+n(t-t^n_k)(u_{i,k+1}(x)-u_{i,k}(x))\quad \text{if } (t,x)\in[t^n_k,t^n_{k+1})\times[0,a_i].\end{aligned}$$

We deduce then from (15), that there exists a constant M_1 independent of n, depending only on the data of the system, such that for all $i \in \{1 \dots I\}$

$$|v_{i,n}|^{\alpha}_{[0,T]\times[0,a_i]} \quad + \quad |\partial_x v_{i,n}|^{\alpha}_{x,[0,T]\times[0,a_i]} \quad \leq \quad M_1.$$

Using Lemma 2.1, we deduce that there exists a constant $M_2(\alpha) > 0$, independent of n, such that for all $i \in \{1 \dots I\}$, we have the following global Hölder condition

$$|\partial_x v_{i,n}|^{\frac{\alpha}{2}}_{t,[0,T]\times[0,a_i]} \quad + \quad |\partial_x v_{i,n}|^{\alpha}_{x,[0,T]\times[0,a_i]} \quad \leq \quad M_2(\alpha).$$

We deduce then from Ascoli's Theorem, that up to a sub sequence n, $(v_{i,n})_{n\geq 0}$ converge in $\mathcal{C}^{0,1}([0,T]\times[0,a_i])$ to v_i, and then $v_i \in \mathcal{C}^{\frac{\alpha}{2},1+\alpha}([0,T]\times[0,a_i])$.
Since v_n satisfies the following continuity condition at the junction point

$$\forall (i,j) \in \{1\dots I\}^2, \quad \forall n \geq 0, \quad \forall t \in [0,T], \quad v_{i,n}(t,0) \quad = \quad v_{j,n}(t,0) \quad = \quad v_n(t,0),$$

we deduce then $v \in \mathcal{C}^{\frac{\alpha}{2},1+\alpha}(\mathcal{J}_T^a)$.
We now focus on the regularity of v in $\overset{\circ}{\mathcal{J}_T^a}$, and we will prove that $v \in \mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(\overset{\circ}{\mathcal{J}_T^a})$, and satisfies on each edge

$$Q_i(x, v_i(t,x), \partial_t v_i(t,x), \partial_x v_i(t,x), \partial^2_{x,x} v_i(t,x)) \quad = \quad 0, \quad \text{if } (t,x) \in (0,T)\times(0,a_i).$$

Using once again (15), there exists a constant M_3 independent of n, such that for each $i \in \{1 \dots I\}$

$$\|\partial_t v_{i,n}\|_{L_2((0,T)\times(0,a_i))} \quad \leq \quad M_3, \quad \|\partial^2_{x,x} v_{i,n}\|_{L_2((0,T)\times(0,a_i))} \quad \leq \quad M_3.$$

Hence we get up to a sub sequence, that

$$\partial_t v_{i,n} \ \rightharpoonup \ \partial_t v_i, \quad \partial^2_{x,x} v_{i,n} \ \rightharpoonup \ \partial^2_{x,x} v_i,$$

weakly in $L_2((0,T)\times(0,a_i))$.
The continuity of the coefficients $(\sigma_i, H_i)_{i\in\{1\dots I\}}$, Lebesgue Theorem, the linearity of Q_i in

the variable ∂_t and $\partial^2_{x,x}$, allows us to get for each $i \in \{1 \dots I\}$, up to a subsequence n_p

$$\int_0^T \int_0^{a_i} \Big(Q_i(x, v_{i,n_p}(t,x), \partial_t v_{i,n_p}(t,x), \partial_x v_{i,n_p}(t,x), \partial^2_{x,x} v_{i,n_p}(t,x)) \Big) \psi(t,x) dx dt$$
$$\xrightarrow{p \to +\infty} \int_0^T \int_0^{a_i} \Big(Q_i(x, v_i(t,x), \partial_t v_i(t,x), \partial_x v_i(t,x), \partial^2_{x,x} v_i(t,x)) \Big) \psi(t,x) dx dt,$$
$$\forall \psi \in \mathcal{C}^\infty_c((0,T) \times (0,a_i)).$$

We now prove that for any $\psi \in \mathcal{C}^\infty_c((0,T) \times (0,a_i))$

$$\int_0^T \int_0^{a_i} \Big(Q_i(x, v_{i,n_p}(t,x), \partial_t v_{i,n_p}(t,x), \partial_x v_{i,n_p}(t,x), \partial^2_{x,x} v_{i,n_p}(t,x))) \Big) \psi(t,x) dx dt \quad \xrightarrow{p \to +\infty} \quad 0.$$

Using that $(u_k)_{0 \le k \le n}$ is the solution of (9), we get for any $\psi \in \mathcal{C}^\infty_c((0,T) \times (0,a_i))$

$$\begin{aligned} &\int_0^T \int_0^{a_i} \Big(Q_i(x, v_{i,n}(t,x), \partial_t v_{i,n}(t,x), \partial_x v_{i,n}(t,x), \partial^2_{x,x} v_{i,n}(t,x)) \Big) \psi(t,x) dx dt = \\ &\sum_{k=0}^{n-1} \int_{t^n_k}^{t^n_{k+1}} \int_0^{a_i} \Big(\sigma_i(x, \partial_x u_{i,k+1}(x)) \partial^2_{x,x} u_{i,k+1}(x) - \sigma_i(x, \partial_x v_{i,n}(t,x)) \partial^2_{x,x} v_{i,n}(t,x) \\ &\quad + H_i(x, v_{i,n}(t,x), \partial_x v_{i,n}(t,x)) - H_i(x, u_{i,k+1}(x), \partial_x u_{i,k+1}(x)) \Big) \psi(t,x) dx dt. \end{aligned} \tag{16}$$

Using assumption $(\mathcal{P})$ more precisely the Lipschitz continuity of the Hamiltonians H_i, the Hölder equicontinuity in time of $(v_{i,n}, \partial_x v_{i,n})$, there exists a constant $M_4(\alpha)$ independent of n, such that for each $i \in \{1 \dots I\}$, for each $(t,x) \in [t^n_k, t^n_{k+1}] \times [0, a_i]$

$$|H_i(x, u_{i,k+1}(x), \partial_x u_{i,k+1}(x)) - H_i(x, v_{i,n}(t,x), \partial_x v_{i,n}(t,x))| \le M_4(\alpha)(t - t^n_k)^{\frac{\alpha}{2}},$$

and therefore for any $\psi \in \mathcal{C}^\infty_c((0,T) \times (0,a_i))$

$$\Big| \sum_{k=0}^{n-1} \int_{t^n_k}^{t^n_{k+1}} \int_0^{a_i} \Big(H_i(x, u_{i,k+1}(x), \partial_x u_{i,k+1}(x)) - H_i(x, v_{i,n}(t,x), \partial_x v_{i,n}(t,x)) \Big) \psi(t,x) dx dt \Big| \le$$
$$a_i M_4(\alpha) |\psi|_{(0,T) \times (0,a_i)} n^{-\frac{\alpha}{2}} \quad \xrightarrow{n \to +\infty} \quad 0.$$

For the last term in (16), we write for each $i \in \{1 \dots I\}$, for each $(t,x) \in (t^n_k, t^n_{k+1}) \times (0, a_i)$

$$\sigma_i(x, \partial_x u_{i,k+1}(x)) \partial^2_{x,x} u_{i,k+1}(x) - \sigma_i(x, \partial_x v_{i,n}(t,x)) \partial^2_{x,x} v_{i,n}(t,x) =$$
$$\Big(\sigma_i(x, \partial_x u_{i,k+1}(x)) - \sigma_i(x, \partial_x v_{i,n}(t,x)) \Big) \partial^2_{x,x} u_{i,k}(x) + \tag{17}$$
$$\Big(\sigma_i(x, \partial_x u_{i,k+1}(x)) - n(t - t^n_k) \sigma_i(x, \partial_x v_{i,n}(t,x)) \Big) \Big(\partial^2_{x,x} u_{i,k+1}(x) - \partial^2_{x,x} u_{i,k}(x) \Big). \tag{18}$$

Using again the Hölder equicontinuity in time of $(v_{i,n}, \partial_x v_{i,n})$ as well as the uniform bound on $|\partial^2_{x,x} u_{i,k}|_{[0,a_i]}$ (15), we can show that for (17), for any $\psi \in \mathcal{C}^\infty_c((0,T)\times(0,a_i))$,

$$\Big|\sum_{k=0}^{n-1}\int_{t_k^n}^{t_{k+1}^n}\int_0^{a_i}\Big(\sigma_i(x,\partial_x u_{i,k+1}(x)) - \sigma_i(x,\partial_x v_{i,n}(t,x))\Big)\partial^2_{x,x}u_{i,k}(x)\psi(t,x)dxdt\Big| \xrightarrow{n\to+\infty} 0.$$

Finally, from assumptions $(\mathcal{P})$, for all $i \in \{1\dots I\}$, σ_i is differentiable with respect to all its variable, integrating by part we get for (18)

$$\Big|\sum_{k=0}^{n-1}\int_{t_k^n}^{t_{k+1}^n}\int_0^{a_i}\Big(\sigma_i(x,\partial_x u_{i,k+1}(x)) - n(t-t_k^n)\sigma_i(x,\partial_x v_{i,n}(t,x))\Big)$$
$$\Big(\partial^2_{x,x}u_{i,k+1}(x) - \partial^2_{x,x}u_{i,k}(x)\Big)\psi(t,x)dxdt\Big| =$$
$$\Big|\sum_{k=0}^{n-1}\int_{t_k^n}^{t_{k+1}^n}\int_0^{a_i}\Big(\partial_x\Big(\sigma_i(x,\partial_x u_{i,k+1}(t,x))\psi(t,x)\Big) - n(t-t_k^n)\partial_x\Big(\sigma_i(x,\partial_x v_{i,n}(t,x))\psi(t,x)\Big)\Big)$$
$$\Big(\partial_x u_{i,k+1}(x) - \partial_x u_{i,k}(x)\Big)dxdt\Big| \xrightarrow{n\to+\infty} 0.$$

We conclude that for any $\psi \in \mathcal{C}^\infty_c((0,T)\times(0,a_i))$

$$\int_0^T\int_0^{a_i}\Big(Q_i(x,v_i(t,x),\partial_t v_i(t,x),\partial_x v_i(t,x),\partial^2_{x,x}v_i(t,x)))\Big)\psi(t,x)dxdt = 0.$$

It is then possible to consider the last equation as a linear one, with coefficients $\tilde{\sigma}_i(t,x) = \sigma_i(x,\partial_x v_i(t,x))$, $\tilde{H}_i(t,x) = H_i(x,v_i(t,x),\partial_x v_i(t,x))$ belonging to the class $\mathcal{C}^{\frac{\alpha}{2},\alpha}((0,T)\times(0,a_i))$, and using Theorem III.12.2 of [7], we get finally that for all $i \in \{1\dots I\}$, $v_i \in \mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}((0,T)\times(0,a_i))$, which means that $v \in \mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(\overset{\circ}{\mathcal{J}^a_T})$.

We deduce that v_i satisfies on each edge

$$Q_i(x,v_i(t,x),\partial_t v_i(t,x),\partial_x v_i(t,x),\partial^2_{x,x}v_i(t,x))) = 0, \quad \text{if } (t,x)\in(0,T)\times(0,a_i).$$

From the estimates (15), we know that $\partial_t v_{i,n}$ and $\partial^2_{x,x}v_{i,n}$ are uniformly bounded by n. We deduce finally that $v \in \mathcal{C}_b^{1+\frac{\alpha}{2},2+\alpha}(\overset{\circ}{\mathcal{J}^a_T})$.

We conclude by proving that v satisfies the non linear Neumann boundary condition at the vertex. For this, let $t\in(0,T)$; we have up to a sub sequence n_p

$$F(v_{n_p}(t,0),\partial_x v_{n_p}(t,0)) \xrightarrow[p\to+\infty]{} F(v(t,0),\partial_x v(t,0)).$$

On the other hand, using that $F(u_k(0), \partial_0 u_k(x)) = 0$, we know from the continuity of F (assumption $(\mathcal{P})$), the Hölder equicontinuity in time of $t \mapsto v_n(t,0)$, and $t \mapsto \partial_x v(t,0)$, that there exists a constant $M_5(\alpha)$ independent of n, such that if $t \in [t_k^n, t_{k+1}^n)$

$$|F(v_n(t,0), \partial_x v_n(t,0))| = |F(v_n(t,0), \partial_x v_n(t,0)) - F(u_k(0), \partial_x u_k(0))| \leq$$
$$\sup \Big\{ |F(u,x) - F(v,y)|, \quad |u-v| + \|x-y\|_{\mathbb{R}^I} \leq M_5(\alpha) n^{-\frac{\alpha}{2}} \Big\} \xrightarrow{n \to +\infty} 0.$$

Therefore, we conclude once more from the continuity of F (assumption $(\mathcal{P})$), the compatibility condition (assumption $(\mathcal{P})$ (v)), that for each $t \in [0,T)$

$$F(v(t,0), \partial_x v(t,0)) \quad = \quad 0.$$

On the other hand, it is easy to get

$$\forall i \in \{1 \dots I\}, \quad \forall x \in [0, a_i], \quad v_i(0,x) = g_i(x), \quad \forall t \in [0,T], \quad v_i(t, a_i) = \phi_i(t).$$

Finally, the expression of the upper bounds of the solution given in Theorem 2.2, are a consequence of Proposition 4.3, and Lemma 2.1, which completes the proof. □

4.3. **On the existence for unbounded junction.** We give in this subsection a result on the existence and the uniqueness of the solution for the parabolic problem (1), in a unbounded junction $\mathcal{J}$ defined for $I \in \mathbb{N}^*$ edges by

$$\mathcal{J} \quad = \quad \Big\{ X = (x,i), \ x \in \mathbb{R}_+ \text{ and } i \in \{1, \dots, I\} \Big\}.$$

In the sequel, $\mathcal{C}^{0,1}(\mathcal{J}_T) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}_T})$ is the class of function with regularity $\mathcal{C}^{0,1}([0,T] \times [0,+\infty)) \cap \mathcal{C}^{1,2}((0,T) \times (0,+\infty))$ on each edge, and $L^\infty(\mathcal{J}_T)$ is the set of measurable real bounded maps defined on $\mathcal{J}_T$.

We introduce the following data

$$\begin{cases} F \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^I, \mathbb{R}) \\ g \in \mathcal{C}^1_b(\mathcal{J}) \cap \mathcal{C}^2_b(\overset{\circ}{\mathcal{J}}) \end{cases},$$

and for each $i \in \{1 \dots I\}$

$$\begin{cases} \sigma_i \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}) \\ H_i \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R}) \\ \phi_i \in \mathcal{C}^1([0,T], \mathbb{R}) \end{cases}.$$

We suppose furthermore that the data satisfy the following assumption

Assumption $(\mathcal{P}_\infty)$

(i) Assumption on F

$$\begin{cases} a) \quad F \text{ is decreasing with respect to its first variable,} \\ b) \quad F \text{ is nondecreasing with respect to its second variable,} \\ c) \quad \exists (b,B) \in \mathbb{R} \times \mathbb{R}^I, \quad F(b,B) = 0, \end{cases}$$

or the Kirchhoff condition

$$\begin{cases} a) \quad F \text{ is nonincreasing with respect to its first variable,} \\ b) \quad F \text{ is increasing with respect to its second variable,} \\ c) \quad \exists (b,B) \in \mathbb{R} \times \mathbb{R}^I, \quad F(b,B) = 0. \end{cases}$$

We suppose moreover that there exist a parameter $m \in \mathbb{R}$, $m \geq 2$ such that we have
(ii) The (uniform) ellipticity condition on the $(\sigma_i)_{i \in \{1 \dots I\}}$: there exists $\underline{\nu}, \overline{\nu}$, strictly positive constants such that

$$\forall i \in \{1 \dots I\}, \quad \forall (x,p) \in \mathbb{R}_+ \times \mathbb{R},$$
$$\underline{\nu}(1+|p|)^{m-2} \quad \leq \quad \sigma_i(x,p) \quad \leq \quad \overline{\nu}(1+|p|)^{m-2}.$$

(iii) The growth of the $(H_i)_{i \in \{1 \dots I\}}$ with respect to p exceed the growth of the σ_i with respect to p by no more than two, namely there exists μ an increasing real continuous function such that

$$\forall i \in \{1 \dots I\}, \quad \forall (x,u,p) \in \mathbb{R}_+ \times \mathbb{R}^2, \quad |H_i(x,u,p)| \quad \leq \quad \mu(|u|)(1+|p|)^m.$$

(iv) We impose the following restrictions on the growth with respect to p of the derivatives for the coefficients $(\sigma_i, H_i)_{i\in\{1\ldots I\}}$, which are for all $i \in \{1 \ldots I\}$,

$$\begin{aligned} a)\ \ |\partial_p\sigma_i|_{\mathbb{R}_+\times\mathbb{R}^2}(1+|p|)^2 + |\partial_p H_i|_{\mathbb{R}_+\times\mathbb{R}^2} &\leq \gamma(|u|)(1+|p|)^{m-1}, \\ b)\ \ |\partial_x\sigma_i|_{\mathbb{R}_+\times\mathbb{R}^2}(1+|p|)^2 + |\partial_x H_i|_{\mathbb{R}_+\times\mathbb{R}^2} &\leq \Big(\varepsilon(|u|) + P(|u|,|p|)\Big)(1+|p|)^{m+1}, \\ c)\ \ \forall (x,u,p)\in\mathbb{R}_+\times\mathbb{R}^2,\ \ -C_H \leq \partial_u H_i(x,u,p) &\leq \Big(\varepsilon(|u|) + P(|u|,|p|)\Big)(1+|p|)^{m}, \end{aligned}$$

where γ and ε are continuous non negative increasing functions. P is a continuous function, increasing with respect to its first variable, and tends to 0 for $p \to +\infty$, uniformly with respect to its first variable, from $[0, u_1]$ with $u_1 \in R$, and $C_H > 0$ is real strictly positive number. We assume that $(\gamma, \varepsilon, P, C_H)$ are independent of $i \in \{1 \ldots I\}$.

(v) A compatibility conditions for g

$$F(g(0), \partial_x g(0)) = 0.$$

We state here a comparison Theorem for the problem 1, in a unbounded junction.

Theorem 4.4. *Assume $(\mathcal{P}_\infty)$. Let $u \in \mathcal{C}^{0,1}(\mathcal{J}_T)\cap\mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}_T})\cap L^\infty(\mathcal{J}_T)$ (resp. $v \in \mathcal{C}^{0,1}(\mathcal{J}_T)\cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}_T})\cap L^\infty(\mathcal{J}_T)$) be a super solution (resp. a sub solution) of* (4) *(where $a_i = +\infty$), satisfying for all $i \in \{1 \ldots I\}$ for all $x \in [0, +\infty)$, $u_i(0,x) \geq v_i(0,x)$. Then for each $(t,(x,i)) \in \mathcal{J}_T : u_i(t,x) \geq v_i(t,x)$.*

Proof. Let $s \in [0,T)$, $K = (K \ldots K) > (1, \ldots 1)$ in $\mathbb{R}^I$, and $\lambda = \lambda(K) > 0$, that will be chosen in the sequel. We argue as in the proof of Theorem 2.4, assuming

$$\sup_{(t,(x,i))\in\mathcal{J}_s^K} \exp(-\lambda t - \frac{(x-1)^2}{2})\Big(v_i(t,x) - u_i(t,x)\Big) > 0.$$

Using the boundary conditions satisfied by u and v, the above supremum is reached at a point $(t_0,(x_0,j_0)) \in (0,s]\times\mathcal{J}$, with $0 \leq x_0 \leq K$.

If $x_0 \in [0,K)$, the optimality conditions are given for $x_0 \neq 0$ by

$$\begin{aligned} -\lambda(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)) &+ \partial_t v_{j_0}(t_0,x_0) - \partial_t u_{j_0}(t_0,x_0) \geq 0, \\ -(x_0-1)\Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)\Big) &+ \partial_x v_{j_0}(t_0,x_0) - \partial_x u_{j_0}(t_0,x_0) = 0, \\ \Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)\Big) &- 2(x_0-1)^2\Big(v_{j_0}(t_0,x_0) - u_{j_0}(t_0,x_0)\Big) \\ &+ \Big(\partial^2_{x,x} v_{j_0}(t_0,x_0) - \partial^2_{x,x} u_{j_0}(t_0,x_0)\Big) \leq 0, \end{aligned}$$

and if $x_0 = 0$,

$$\forall i \in \{1,\dots I\}, \quad \partial_x v_i(t_0,0) \leq \partial_x u_i(t_0,0) - \Big(v_i(t_0,0) - u_i(t_0,0)\Big) < \partial_x u_i(t_0,0).$$

If $x_0 = 0$, we obtain a contradiction exactly as in the proof of Theorem 2.4. On the other hand if $x_0 \in (0,K)$, using assumptions $(\mathcal{P})$ (iv) a), (iv) c) and the optimality conditions, we can choose $\lambda(K)$ of the form $\lambda(K) = C(1+K^2)$, (see (5) and (6)), where $C > 0$ is a constant independent of K, to get again a contradiction. We deduce that, if

$$\sup_{(t,(x,i))\in\mathcal{J}^K_s} \exp(-\lambda(K)t - \frac{(x-1)^2}{2})\Big(v_i(t,x) - u_i(t,x)\Big) > 0,$$

then for all $(t,(x,i)) \in [0,T]\times\mathcal{J}^K$

$$\exp(-\lambda(K)t - \frac{(x-1)^2}{2})\Big(v_i(t,x) - u_i(t,x)\Big) \leq \exp(-\lambda(K)t - \frac{(K-1)^2}{2})\Big(v_i(t,K) - u_i(t,K)\Big).$$

Hence for all $(t,(x,i)) \in [0,T]\times\mathcal{J}^K$

$$\exp(-\frac{(x-1)^2}{2})\Big(v_i(t,x) - u_i(t,x)\Big) \leq \exp(-\frac{(K-1)^2}{2})\Big(v_i(t,K) - u_i(t,K)\Big).$$

On the other hand, if

$$\sup_{(t,(x,i))\in\mathcal{J}^K_s} \exp(-\lambda(K)t - \frac{(x-1)^2}{2})\Big(v_i(t,x) - u_i(t,x)\Big) \leq 0,$$

then for all $(t,(x,i)) \in [0,T]\times\mathcal{J}^K$

$$\exp(-\lambda(K)t - \frac{(x-1)^2}{2})\Big(v_i(t,x) - u_i(t,x)\Big) \leq 0.$$

So

$$\exp(-\frac{(x-1)^2}{2})\Big(v_i(t,x) - u_i(t,x)\Big) \leq 0.$$

Finally we have, for all $(t,(x,i)) \in [0,T] \times \mathcal{J}^K$

$$\max\Big(0, \exp(-\frac{(x-1)^2}{2})\Big(v_i(t,x) - u_i(t,x)\Big)\Big) \leq \exp(-\frac{(K-1)^2}{2})\Big(||u||_{L^\infty(\mathcal{J}_T)} + ||v||_{L^\infty(\mathcal{J}_T)}\Big).$$

Sending $K \to \infty$ and using the boundedness of u and v, we deduce the inequality $v \leq u$ in $[0,T] \times \mathcal{J}$. □

Theorem 4.5. *Assume $(\mathcal{P}_\infty)$. The following parabolic problem with Neumann boundary condition at the vertex*

$$\begin{cases} \partial_t u_i(t,x) - \sigma_i(x, \partial_x u_i(t,x))\partial^2_{x,x} u_i(t,x) + \\ H_i(x, u_i(t,x), \partial_x u_i(t,x)) \quad = \quad 0, \quad \text{if } (t,x) \in (0,T) \times (0,+\infty), \\ F(u(t,0), \partial_x u(t,0)) \quad = \quad 0, \quad \text{if } t \in [0,T), \\ \forall i \in \{1\dots I\}, \quad u_i(0,x) \quad = \quad g_i(x), \quad \text{if } x \in [0,+\infty), \end{cases} \tag{19}$$

is uniquely solvable in the class $\mathcal{C}^{\frac{\alpha}{2},1+\alpha}(\mathcal{J}_T) \cap \mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(\overset{\circ}{\mathcal{J}_T})$. There exist constants (M_1, M_2, M_3), depending only the data introduced in assumption $(\mathcal{P}_\infty)$

$$M_1 = M_1\Big(\max_{i\in\{1\dots I\}}\Big\{\ \sup_{x\in(0,+\infty)} | - \sigma_i(x, \partial_x g_i(x))\partial^2_x g_i(x) + H_i(x, g_i(x), \partial_x g_i(x))|\Big\},$$
$$\max_{i\in\{1\dots I\}} |g_i|_{(0,+\infty)}, C_H\Big),$$
$$M_2 = M_2\Big(\overline{\nu}, \underline{\nu}, \mu(M_1), \gamma(M_1), \varepsilon(M_1), \sup_{|p|\geq 0} P(M_1, |p|), |\partial_x g_i|_{(0,+\infty)}, M_1\Big),$$
$$M_3 = M_3\Big(M_1, \underline{\nu}(1+|p|)^{m-2}, \mu(|u|)(1+|p|)^m, \quad |u| \leq M_1, \quad |p| \leq M_2\Big),$$

such that

$$||u||_{\mathcal{C}(\mathcal{J}_T)} \leq M_1, \quad ||\partial_x u||_{\mathcal{C}(\mathcal{J}_T)} \leq M_2, \quad ||\partial_t u||_{\mathcal{C}(\mathcal{J}_T)} \leq M_1, \quad ||\partial_{x,x} u||_{\mathcal{C}(\mathcal{J}_T)} \leq M_3.$$

Moreover, there exists a constant $M(\alpha)$ depending on $\Big(\alpha, M_1, M_2, M_3\Big)$ such that for any $a \in (0,+\infty)^I$

$$||u||_{\mathcal{C}^{\frac{\alpha}{2},1+\alpha}(\mathcal{J}_T^a)} \leq M(\alpha).$$

Proof. Assume $(\mathcal{P}_\infty)$ and let $a = (a, \ldots, a) \in (0, +\infty)^I$. Applying Theorem 2.2, we can define $u^a \in \mathcal{C}^{0,1}(\mathcal{J}_T^a) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}_T^a})$ as the unique solution of

$$
\begin{cases}
\partial_t u_i(t,x) - \sigma_i(x, \partial_x u_i(t,x))\partial^2_{x,x} u_i(t,x) + \\
H_i(x, u_i(t,x), \partial_x u_i(t,x)) \quad = \quad 0, \quad \text{if } (t,x) \in (0,T) \times (0,a), \\
F(u(t,0), \partial_x u(t,0)) \quad = \quad 0, \quad \text{if } t \in [0,T), \\
\forall i \in \{1 \ldots I\}, \quad u_i(t,a) \quad = \quad g_i(a), \quad \text{if } t \in [0,T], \\
\forall i \in \{1 \ldots I\}, \quad u_i(0,x) \quad = \quad g_i(x), \quad \text{if } x \in [0,a].
\end{cases}
\tag{20}
$$

Using assumption $(\mathcal{P}_\infty)$ and Theorem 2.2, we get that there exists a constant $C > 0$ independent of a such that

$$\sup_{a \geq 0} \quad ||u^a||_{\mathcal{C}^{1,2}(\mathcal{J}_T^a)} \quad \leq \quad C.$$

We are going to send a to $+\infty$ in (20).
Following the same argument as for the proof of Theorem 2.2, we get that, up to a sub sequence, u^a converges locally uniformly to some map u which solves (19). On the other hand, uniqueness of u is a direct consequence of the comparison Theorem 4.4, since $u \in L^\infty(\mathcal{J}_T)$. Finally the expression of the upper bounds of the derivatives of u given in Theorem 4.5, are a consequence of Theorem 2.2 and assumption $(\mathcal{P}_\infty)$. □

Appendix A. Functionnal spaces

In this section, we recall several classical notations from [7]. Let $l, T \in (0, +\infty)$ and Ω be an open and bounded subset of $\mathbb{R}^n$ with smooth boundary $(n > 0)$. We set $\Omega_T = (0,T) \times \Omega$, and we introduce the following spaces :
-if $l \in 2\mathbb{N}^*$, $\left(\mathcal{C}^{\frac{l}{2},l}(\overline{\Omega_T}), \| \cdot \|_{\mathcal{C}^{\frac{l}{2},l}(\Omega_T)}\right)$ is the Banach space whose elements are continuous functions $(t,x) \mapsto u(t,x)$ in $\overline{\Omega_T}$, together with all its derivatives of the form $\partial_t^r \partial_x^s u$, with $2r + s < l$. The norm $\| \cdot \|_{\mathcal{C}^{\frac{l}{2},l}(\Omega_T)}$ is defined for all $u \in \mathcal{C}^{\frac{l}{2},l}(\overline{\Omega_T})$ by

$$\|u\|_{\mathcal{C}^{\frac{l}{2},l}(\Omega_T)} \quad = \quad \sum_{2r+s=j} \sup_{(t,x) \in \Omega_T} |\partial_t^r \partial_x^s u(t,x)|.$$

-if $l \notin \mathbb{N}^*$, $\left(\mathcal{C}^{\frac{l}{2},l}(\overline{\Omega_T}), \|.\|_{\mathcal{C}^{\frac{l}{2},l}(\Omega_T)}\right)$ is the Banach space whose elements are continuous functions $(t,x) \mapsto u(t,x)$ in $\overline{\Omega_T}$, together with all its derivatives of the form $\partial_t^r \partial_x^s u$, with $2r+s<l$, and satisfying an Hölder condition with exponent $\frac{l-2r-s}{2}$ in their first variable, and with exponent $(l-\lfloor l \rfloor)$ in their second variable, over all the connected components of Ω_T whose radius is smaller than 1.

The norm $\|\cdot\|_{\mathcal{C}^{\frac{l}{2},l}(\Omega_T)}$ is defined for all $u \in \mathcal{C}^{\frac{l}{2},l}(\overline{\Omega_T})$ by

$$\|u\|_{\mathcal{C}^{\frac{l}{2},l}(\Omega_T)} = |u|^l_{\Omega_T} + \sum_{j=0}^{\lfloor l \rfloor} |u|^j_{\Omega_T},$$

with

$$\forall j \in \{0,\dots,l\}, \quad |u|^j_{\Omega_T} = \sum_{2r+s=j} \sup_{(t,x)\in\Omega_T} |\partial_t^r \partial_x^s u(t,x)|,$$

$$|u|^l_{\Omega_T} = |u|^l_{x,\Omega_T} + |u|^{\frac{l}{2}}_{t,\Omega_T},$$

$$|u|^l_{x,\Omega_T} = \sum_{2r+s=\lfloor l \rfloor} |\partial_t^r \partial_x^s u(t,x)|^{l-\lfloor l \rfloor}_{x,\Omega_T},$$

$$|u|^l_{t,\Omega_T} = \sum_{0<l-2r-s<2} |\partial_t^r \partial_x^s u(t,x)|^{\frac{l-2r-s}{2}}_{t,\Omega_T},$$

$$|u|^\alpha_{x,\Omega_T} = \sup_{t\in(0,T)} \sup_{x,y\in\Omega, x\neq y, |x-y|\leq 1} \frac{|u(t,x)-u(t,y)|}{|x-y|^\alpha}, \quad 0<\alpha<1,$$

$$|u|^\alpha_{t,\Omega_T} = \sup_{x\in\Omega} \sup_{t,s\in(0,T), t\neq s, |t-s|\leq 1} \frac{|u(t,x)-u(s,x)|}{|t-s|^\alpha}, \quad 0<\alpha<1.$$

- $\mathcal{C}^{\frac{l}{2},l}(\Omega_T)$ is the set whose elements f belong to $\mathcal{C}^{\frac{l}{2},l}(\overline{O_T})$ for any open set O_T separated from the boundary of Ω_T by a strictly positive distance, namely

$$\inf_{y\in\partial\Omega_T, x\in\overline{O_T}} ||x-y||_{\mathbb{R}^n} > 0.$$

- $\mathcal{C}_b^{\frac{l}{2},l}(\Omega_T)$ is the subset of $\mathcal{C}^{\frac{l}{2},l}(\Omega_T)$ consisting in maps u such that the derivatives of the form $\partial_t^r \partial_x^s u$, (with $2r+s<l$) are bounded, namely $\sup_{(t,x)\in\Omega_T} |\partial_t^r \partial_x^s u(t,x)| < +\infty$.

We use the same notations when the domain does not depend on time, namely $T=0$, $\Omega_T = \Omega$, just removing the dependence on the time variable.

For $R>0$, we denote by $L_2((0,T)\times(0,R))$ the usual space of square integrable maps and by $\mathcal{C}_c^\infty((0,T)\times(0,R))$ the set of infinite continuous differentiable functions on $(0,T)\times(0,R)$, with compact support.

Appendix B. The Elliptic problem

Proposition B.1. *Let $\theta \in \mathbb{R}$, $i \in \{1, \dots, I\}$ and assume ($\mathcal{E}$) holds. Let $u_i^\theta \in \mathcal{C}^2([0, a_i])$ be the solution to*

$$\begin{cases} -\sigma_i(x, \partial_x u_i^\theta(x))\partial_{x,x}^2 u_i^\theta(x) + H_i(x, u_i^\theta(x), \partial_x u_i^\theta(x)) = 0, \text{ if } x \in (0, a_i) \\ u_i^\theta(0) = u^\theta(0) = \theta, \\ u_i^\theta(a_i) = \phi_i. \end{cases} \tag{21}$$

Then the following map

$$\Psi := \begin{cases} \mathbb{R} \to \mathcal{C}^2([0, a_i]) \\ \theta \mapsto u_i^\theta \end{cases}$$

is continuous.

Proof. Let θ_n a sequence converging to θ. Using the Schauder estimates Theorem 6.6 of [5], we get that there exists a constant $M > 0$ independent of n, depending only the data, such that for all $\alpha \in (0, 1)$

$$\|u_i^{\theta_n}\|_{\mathcal{C}^{2+\alpha}([0,a_i])} \leq M.$$

From Ascoli's Theorem, $u_i^{\theta_n}$ converges up to a subsequence to v in $\mathcal{C}^2([0, a_i])$ solution of (21). By uniqueness of the solution of (21), $u_i^{\theta_n}$ converges necessary to the solution u_i^θ of (21) in $\mathcal{C}^2([0, a_i])$, which completes the proof. □

Proof of Theorem 3.1.

Proof. The uniqueness of (7) results from the elliptic comparison Theorem 3.3.
We turn to the solvalbility, and for this let $\theta \in \mathbb{R}$. We consider the elliptic Dirichlet problem at the junction

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x))\partial_{x,x}^2 u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) = 0, \quad \text{if } x \in (0, a_i), \\ \forall i \in \{1 \dots I\}, \quad u_i(0) = u(0) = \theta, \\ u_i(a_i) = \phi_i. \end{cases} \tag{22}$$

For all $i \in \{1 \dots I\}$, each elliptic problem is uniquely solvable on each edge in $\mathcal{C}^{2+\alpha}([0, a_i])$, then (22) is uniquely solvable in the class $\mathcal{C}^{2+\alpha}(\mathcal{J}^a)$, and we denote by u^θ its solution. We turn to the Neumann boundary condition at the vertex. Let us recall assumption $(\mathcal{E})$(i)

$$\begin{cases} F \text{ is decreasing in its first variable, nondecreasing in its second variable,} \\ \text{or } F \text{ is nonincreasing in its first variable, increasing in its second variable,} \\ \exists (b, B) \in \mathbb{R} \times \mathbb{R}^I, \text{ such that : } F(b, B) \quad = \quad 0. \end{cases}$$

Fix now

$$\begin{aligned} K_i &= \sup_{(x,u) \in (0,a_i) \times (-a_i B_i, a_i B_i)} |H_i(x, u, B_i)|, \\ \theta &\geq |b| + \max_{i \in \{1 \dots I\}} \Big\{ |\phi_i| + |a_i B_i| + \frac{K_i}{C_H} \Big\}, \end{aligned}$$

and let us show that $f : x \mapsto \theta + B_i x$, is a super solution on each edge $J_i^{a_i}$ of (22). We have the boundary conditions

$$f(0) \quad = \quad \theta, \quad f(a_i) = \theta + a_i B_i \quad \geq \quad |\phi_i| + |a_i B_i| + a_i B_i \quad \geq \quad \phi_i,$$

and using assumption $(\mathcal{E})$ (iii), we have for all $x \in (0, a_i)$

$$\begin{aligned} -\sigma_i(x, \partial_x f(x)) \partial^2_{x,x} f(x) + H_i(x, f(x), \partial_x f(x)) \quad &= \quad H_i(x, \theta + B_i x, B_i) \quad \geq \quad H_i(x, B_i x, B_i) \\ &+ \; C_H \theta \quad \geq \quad H_i(x, B_i x, B_i) + K_i \quad \geq \quad 0. \end{aligned}$$

We then get that for each $i \in \{1 \dots I\}$, $x \in [0, a_i]$, $u_i^\theta(x) \leq \theta + B_i x$, and a Taylor expansion in the neighborhood of the junction point gives that for each $i \in \{1 \dots I\}$, $\partial_x u_i^\theta(0) \leq B_i$. Since $u^\theta(0) = \theta \geq b$, we then get from assumption $(\mathcal{E})$ (i)

$$F(u^\theta(0), \partial_x u^\theta(0)) \quad \leq \quad F(b, B) \quad \leq \quad 0.$$

Similarly, fixing

$$\theta \quad \leq \; - \; |b| \; - \; \min_{i \in \{1 \dots I\}} \Big\{ - |\phi_i| - |a_i B_i| - \frac{K_i}{C_H} \Big\},$$

the map $f : x \mapsto \theta + xB_i$ is a sub solution on each vertex $J_i^{a_i}$ of (22), then for each $i \in \{1 \dots I\}$, $\partial_x u_i^\theta(0) \geq B_i$, which means

$$F(u^\theta(0), \partial_x u^\theta(0)) \;\geq\; 0.$$

From Proposition B.1, we know that the real maps $\theta \mapsto u^\theta(0)$ and $\theta \mapsto \partial_x u^\theta(0)$ are continuous. Using the continuity of F (assumption $(\mathcal{E})$), we get that $\theta \mapsto F(u^\theta(0), \partial_x u^\theta(0))$ is continuous, and therefore there exists $\theta^* \in \mathbb{R}$ such that

$$F(u^{\theta^*}(0), \partial_x u^{\theta^*}(0)) \;=\; 0.$$

We remark that θ^* is bounded by the data, namely θ^* belongs to the following interval

$$\Big[\; -|b| - \max_{i\in\{1\dots I\}} \Big\{ |\phi_i| + |a_i B_i| + \frac{\sup_{(x,u)\in(0,a_i)} |H_i(x, B_i x, B_i)|}{C_H} \Big\},$$
$$|b| + \max_{i\in\{1\dots I\}} \Big\{ |\phi_i| + |a_i B_i| + \frac{\sup_{(x,u)\in(0,a_i)} |H_i(x, B_i x, B_i)|}{C_H} \Big\} \; \Big].$$

This completes the proof. Finally, since the solution u^{θ^*} of (7) is unique, we get the uniqueness of θ^*. □

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