A uniform Tauberian theorem in optimal control

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Abstract : In an optimal control framework, we consider the value $V_T(x)$ of the problem starting from state x with finite horizon T, as well as the value $W_\lambda(x)$ of the λ-discounted problem starting from x. We prove that uniform convergence (on the set of states) of the values $V_T(\cdot)$ as T tends to infinity is equivalent to uniform convergence of the values $W_\lambda(\cdot)$ as λ tends to 0, and that the limits are identical. An example is also provided to show that the result does not hold for pointwise convergence. This work is an extension, using similar techniques, of a related result by Lehrer and Sorin in a discrete-time framework.

Erratum

We thank very warmly Dmitry Khlopin for the following observation : in section 2.1, in addition to the property of concatenation of trajectories $X \circ_s Y \in \Gamma(x)$, we also need to assume that the following property holds:

If $X \in \Gamma(x)$, then for any $t > 0$, the trajectory $s \to X(t+s)$ is in $\Gamma(X(t))$. That is, any trajectory obtained by cutting the beginning of a feasible trajectory is also feasible.

This property was used implicitly in both parts of the proof of Theorem 6:

- in the proof of Lemma 5 , to get $\int_s^t g(X(r))dr \geq (t-s)V_{t-s}(X(s))$
- in the proof of Proposition 8 , to get $W_\lambda(Y(L)) \leq \lambda \int_0^\infty e^{-\lambda s} g(Y(L+s))ds$.

This property is always satisfied in the framework of optimal control and Theorem 5 thus always holds. However, in the more general framework of section 2.1, one can construct an example that does not satisfy this property and for which Theorem 6 fails [21]. Thus one needs this additional assumption for Theorem 6 to hold in full generality.

1 Introduction

Finite horizon problem of optimal control have been studied intensively since the pioneer work of Stekhov, Pontryagin, Boltyanskii [28], Hestenes [18], Bellman [9] and Isaacs [19,20] during the cold war - see for instance [7,23,24] for major references, or [14] for a short, clear introduction. A classical model considers the following controlled dynamic over $\mathbb{R}+$

$$\begin{cases} y'(s) = f(y(s), u(s)) \\ y(0) = y_0 \end{cases} \tag{1.1}$$

where y is a function from $\mathbb{R}+$ to $\mathbb{R}^n$, y_0 is a point in $\mathbb{R}^n$, u is the control function which belongs to $\mathcal{U}$, the set of Lebesgue-measurable functions from $\mathbb{R}+$ to a metric space U and the function $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies the usual conditions, that is: Lipschitz with respect to the state

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variable, continuous with respect to the control variable and bounded by a linear function of the state variable, for any control u.

Together with the dynamic, an objective function g is given, interpreted as the cost function which is to be minimized and assumed to be Borel-measurable from $\mathbb{R}^n \times U$ to $[0,1]$. For each finite horizon $t \in]0,+\infty[$, the average value of the optimal control problem with horizon t is defined as

$$V_t(y_0) = \inf_{u \in \mathcal{U}} \frac{1}{t} \int_0^t g(y(s,u,y_0),u(s))ds \tag{1.2}$$

It is quite natural to define, whenever the trajectories considered are infinite, for any discount factor $\lambda > 0$, the λ-discounted value of the optimal control problem, as

$$W_\lambda(y_0) = \inf_{u \in \mathcal{U}} \lambda \int_0^{+\infty} e^{-\lambda s} g(y(s,u,y_0),u(s))ds \tag{1.3}$$

In this framework the problem was initially to know whether, for a given finite horizon T and a given starting point y_0, a minimizing control u existed, solution of the optimal control problem (T, y_0). Systems with large, but fixed horizons were considered and, in particular, the class of "ergodic" systems (that is, those in which any starting point in the state space Ω is controllable to any point in Ω) has been thoroughly studied [2, 3, 5, 6, 8, 11, 26]. These systems are asymptotically independent of the starting point as the horizon goes to infinite. When the horizon is infinite, the literature on optimal control has mainly focussed on properties of given trajectories as the time tends to infinity. This approach corresponds to the uniform approach in a game theoretical framework and is often opposed to the asymptotic approach (described below), which we have considered in what follows, and which has received considerably less attention.

In a game-theoretical, discrete time framework, the same kind of problem was considered since [30], but with several differences in the approach: 1) the starting point may be chosen at random, i.e, a probability μ may be given on Ω, which randomly determines the point from which the controller will start the play; 2) the controllability-ergodicity condition is generally not assumed; 3) because of the inherent recursive structure of process played in discrete time, the problem is generally considered for all initial states and time horizons.

For these reasons, what is called the "asymptotic approach" - the behavior of $V_t(\cdot)$ as the horizon t tends to infinity, or of $W_\lambda(\cdot)$ as the discount factor λ tends to zero - has been more studied in this discrete-time setup. Moreover, when it is considered in Optimal Control, in most cases [4, 10] an ergodic assumption is made which not only ensures the convergence of $V_t(y_0)$ to some V, but also forces the limit function V to be independent of the starting point y_0. The general asymptotic case, in which no ergodicity condition is assumed, has been to our knowledge studied for the first time recently. In [11, 29] the authors prove in different frameworks the convergence of $V_t(\cdot)$ and $W_\lambda(\cdot)$ to some non-constant function $V(y_0)$.

Some important, closely related questions are the following : does the convergence of $V_t(\cdot)$ imply the convergence of $W_\lambda(\cdot)$? Or vice versa ? If they both converge, does the limit coincide ? A partial answer to these questions goes back to the beginning of the 20th century, when Hardy and Littlewood proved (see [17]) that for any sequence of bounded real numbers, the convergence of the Cesaro means is equivalent to the convergence of their Abel means, and that the limits are then the same :

Theorem 1 (Hardy-Littlewood 1914) *For any bounded sequence of reals $\{a_n\}_{n\geq 1}$, define $V_n = \frac{1}{n}\sum_{i=1}^n a_i$ and $W_\lambda = \lambda \sum_{i=1}^{+\infty}(1-\lambda)^{i-1}a_i$. Then,*

$$\liminf_{n\to+\infty} V_n \leq \liminf_{\lambda\to 0} W_\lambda \leq \limsup_{\lambda\to 0} W_\lambda \leq \limsup_{n\to+\infty} V_n.$$

Moreover, if the central inequality is an equality, then all inequalities are equalities.

Noticing that $\{a_n\}$ can be viewed as a sequence of costs for some deterministic (uncontrolled) dynamic in discrete-time, this results gives the equivalence between the convergence of V_t and the convergence of W_λ, to the same limit. In 1971, setting $V_t = \frac{1}{t}\int_0^t g(s)ds$ and $W_\lambda = \lambda \int_0^{+\infty} e^{-\lambda s} g(s)ds$, for a given Lebesgue-measurable, bounded, real function g, Feller proved that the same result holds for continuous-time uncontrolled dynamics (particular case of Theorem 2, p.445 in [15]).

Theorem 2 (Feller 1971)

$$\liminf_{n\to+\infty} V_n \leq \liminf_{\lambda\to 0} W_\lambda \leq \limsup_{\lambda\to 0} W_\lambda \leq \limsup_{n\to+\infty} V_n.$$

Moreover, if the central inequality is an equality, then all inequalities are equalities.

In 1992, Lehrer and Sorin [25] considered a discrete-time controlled dynamic, defined by a correspondence $\Gamma : \Omega \rightrightarrows \Omega$, with nonempty values, and by g, a bounded real cost function defined on Ω. A feasible play at $z \in \Omega$ is an infinite sequence $\mathbf{y} = \{y_n\}_{n\geq 1}$ such that $y_1 = z$ and $y_{n+1} \in \Gamma(y_n)$. The value functions are defined by $V_n(z) = \inf \frac{1}{n}\sum_{i=1}^n g(y_i)$ and respectively $W_\lambda(y_0) = \inf \lambda \sum_{i=1}^{+\infty}(1-\lambda)^{i-1} g(y_i)$, where the infima are taken over the feasible plays at z.

Theorem 3 (Lehrer-Sorin 1992)

$$\lim_{n\to+\infty} V_n(z) = V(z) \text{ uniformly on } \Omega \Longleftrightarrow \lim_{\lambda\to 0} W_\lambda(z) = V(z) \text{ uniformly on } \Omega.$$

This result establishes the equivalence between uniform convergence of $W_\lambda(y_0)$ when λ tends to 0 and uniform convergence of $V_n(y_0)$ as n tends to infinity, in the general case where the limit may depend on the starting point y_0. The uniform condition is necessary: in the same article, the authors provide an example where only pointwise convergence holds and the limits differs.

In 1998, Arisawa (see [4]) considered a continuous-time controlled dynamic and proved the equivalence between the uniform convergence of W_λ and the uniform convergence of V_t in the specific case of limits independent of the starting point.

Theorem 4 (Arisawa 1998) *Let $d \in \mathbb{R}$, then*

$$\lim_{t\to+\infty} V_t(z) = d, \text{ uniformly on } \Omega \Longleftrightarrow \lim_{\lambda\to 0+} W_\lambda(z) = d, \text{ uniformly on } \Omega.$$

This does not settle the general case, in which the limit function may depend on the starting point[1]. For a continuous-time controlled dynamic in which $V_t(y_0)$ converges to some function $V(y_0)$, dependent on the state variable y_0, as t goes to infinity, we prove the following

Theorem 5 *$V_t(y_0)$ converges to $V(y_0)$ uniformly on Ω, if and only if $W_\lambda(y_0)$ converges to $V(y_0)$ uniformly on Ω.*

In fact, we will prove this result in a more general framework, as described in section 2. Some basic lemmas which occur to be important tools will also be proven on that section. Section 3 will be devoted to the proof of our main result. Section 4 will conclude by pointing out, via an example, the fact that uniform convergence is a necessary requirement for the Theorem 5

[1] Lemma 6 and Theorem 8 in [4] deal with this general setting, but we believe them to be incorrect since they are stated for pointwise convergence and, consequently, are contradicted by the example in Section 4.

to hold. A very simple dynamic is described, in which the pointwise limits of $V_t(\cdot)$ and $W_\lambda(\cdot)$ exist but differ. It should be noted that our proofs (as well as the counterexample in Section 4) are adaptations in this continuous-time framework of ideas employed in a discrete-time setting in [25]. In the appendix we also point out that an alternative proof of our theorem is obtained using the main theorem in [25] as well as a discrete/continuous equivalence argument.

For completeness, let us mention briefly this other approach, mentioned above as the uniform approach, and which has also been deeply studied, see for exemple [12, 13, 16]. In these models, the optimal average cost value, i.e the V_t, is not taken over a finite period of time $[0, t]$, which is then studied for t growing to infinite, as in [4, 15, 17, 25, 29] or in our framework. On the contrary, only infinite trajectories are considered, among which the value $\overline{V_t}$ is defined as $\inf_{u\in\mathcal{U}} \sup_{\tau\geq t} \frac{1}{\tau}\int_0^\tau g(y(s,u,y_0),u(s))ds$, or some other closely related variation. The asymptotic behavior, as t tends to infinity, of the function $\overline{V_t}$ has also been studied. In [16], both λ-discounted and average evaluations of an infinite trajectory are considered and their limits are compared. However, we stress out that the asymptotic behavior of those quantities is in general[2] not related to the asymptotic behavior of V_t and W_λ.

Finally, let us point out that in the framework of zero-sum differential games, that is when the dynamic is controlled by two players with opposite goals, a Tauberian theorem is given in the ergodic case by Theorem 2.1 in [1]. However, the general, non ergodic case is still an open problem in both the discrete and the continuous settings.

2 Model

2.1 General framework

We consider a deterministic dynamic programming problem in continuous time, defined by a measurable set of states Ω, a subset $\mathcal{T}$ of Borel-measurable functions from $\mathbb{R}_+$ to Ω, and a bounded Borel-measurable real-valued function g defined on Ω. Without loss of generality we assume $g : \Omega \to [0, 1]$. For a given state x, define $\Gamma(x) := \{X \in \mathcal{T},\ X(0) = x\}$ the set of all feasible trajectories starting from x. We assume $\Gamma(x)$ to be non empty, for all $x \in \Omega$. Furthermore, the correspondence Γ is closed under concatenation: given a trajectory $X \in \Gamma(x)$ with $X(s) = y$, and a trajectory $Y \in \Gamma(y)$, the concatenation of X and Y at time s is

$$X \circ_s Y := \begin{cases} X(t) & \text{if } t \leq s \\ Y(t-s) & \text{if } t \geq s \end{cases} \tag{2.1}$$

and we assume that $X \circ_s Y \in \Gamma(x)$.

We are interested in the asymptotic behavior of the average and the discounted values. It is useful to denote the average payoff of a play (or trajectory) $X \in \Gamma(x)$ by:

$$\gamma_t(X) := \frac{1}{t}\int_0^t g(X(s))ds \tag{2.2}$$

$$\nu_\lambda(X) := \lambda\int_0^{+\infty} e^{-\lambda s} g(X(s))ds \tag{2.3}$$

This is defined for $t, \lambda \in (0, \infty)$. Naturally, we define the values as:

$$V_t(x) = \inf_{X\in\Gamma(x)} \gamma_t(X) \tag{2.4}$$

$$W_\lambda(x) = \inf_{X\in\Gamma(x)} \nu_\lambda(X) \tag{2.5}$$

[2]The reader may verify that this is indeed not the case in the example of Section 4.

Our main contribution is the following :

Theorem 6 *(A)* $W_\lambda \underset{\lambda\to 0}{\longrightarrow} V$*, uniformly on* $\Omega \Longleftrightarrow$ *(B)* $V_t \underset{t\to\infty}{\longrightarrow} V$*, uniformly on* Ω*.*

Notice that our model is a natural adaptation to the continuous-time framework of deterministic dynamic programming problems played in discrete time ; as it was pointed out during the introduction, this theorem is an extension to the continuous-time framework of the main result of [25], and our proof use similar technics.

This result can be applied to the model presented in section 1: denote $\widetilde{\Omega} = \mathbb{R}^d \times U$ and for any $(y_0, u_0) \in \widetilde{\Omega}$, define $\widetilde{\Gamma}(y_0, u_0) = \{(y(\cdot), u(\cdot)) \,|u \in \mathcal{U}, u(0) = u_0$ and y is the solution of (1.1).$\}$ Then $\widetilde{\Omega}$, $\widetilde{\Gamma}$ and g satisfy the assumptions of this section. Defining $\widetilde{V_t}$ and $\widetilde{W_\lambda}$ as in (2.4) and (2.5) respectively, since the solution of (1.1) does not depend on $u(0)$ we get that

$$\begin{aligned} \widetilde{V_t}(y_0, u_0) &= V_t(y_0) \\ \widetilde{W_\lambda}(y_0, u_0) &= W_\lambda(y_0). \end{aligned}$$

Theorem 6 applied to $\widetilde{V}$ and $\widetilde{W}$ thus implies that V_t converges uniformly to a function V in Ω if and only if W_λ converges uniformly to V in Ω.

2.2 Preliminary results

We follow the ideas of [25], and start by proving two simple lemmas yet important tools, that will be used in the proof. The first establishes that the value increases along the trajectories. Then, we prove a convexity result linking the finite horizon average payoffs and the discounted evaluations on any given trajectory.

Lemma 1 *Monotonicity (compare with Proposition 1 in [25])*
For all $X \in \mathcal{T}$*, for all* $s \geq 0$*, we have*

$$\liminf_{t\to\infty} V_t(X(0)) \leq \liminf_{t\to\infty} V_t(X(s)) \tag{2.6}$$

$$\liminf_{\lambda\to 0} W_\lambda(X(0)) \leq \liminf_{\lambda\to 0} W_\lambda(X(s)) \tag{2.7}$$

Proof. Set $y := X(s)$ and $x := X(0)$. For $\varepsilon > 0$, take $T \in \mathbb{R}_+$ such that $\frac{s}{s+T} < \varepsilon$. Let $t > T$ and take a ε-optimal trajectory for V_t, i.e. $Y \in \Gamma(y)$ such that $\gamma_t(Y) \leq V_t(y) + \varepsilon$. Define the concatenation of X and Y at time s as in (2.1), where $X \circ_s Y$ is in $\in \Gamma(x)$ by our hypothesis. Hence

$$\begin{aligned} V_{t+s}(x) \leq \gamma_{t+s}(X \circ_s Y) &= \frac{s}{t+s}\gamma_s(X) + \frac{t}{t+s}\gamma_t(Y) \\ &\leq \varepsilon + \gamma_t(Y) \\ &\leq 2\varepsilon + V_t(y). \end{aligned}$$

Since this is true for any $t \geq T$ the result follows.

Similarly, for the discounted case let $\lambda_0 > 0$ be such that $\lambda_0 \int_0^s e^{-\lambda_0 r} dr = 1 - e^{\lambda_0 s} < \varepsilon$. Let $\lambda \in (0, \lambda_0)$ and take $Y \in \Gamma(y)$ a ε-optimal trajectory for $W_\lambda(y)$. Then:

$$\begin{aligned} W_\lambda(x) \leq \nu_\lambda(X \circ_s Y) &= \lambda \int_0^s e^{-\lambda r} g(X(r)) dr + \lambda \int_s^\infty e^{-\lambda r} g(Y(r-s)) dr \\ &\leq \varepsilon + e^{-\lambda s} \nu_\lambda(Y) \\ &\leq 2\varepsilon + W_\lambda(y). \end{aligned}$$

Again, this is true for any $\lambda \in (0, \lambda_0)$, and the result follows.

■

Lemma 2 *Convexity (compare with equation (1) in [25])*
For any play $X \in \mathcal{T}$, for any $\lambda > 0$:

$$\nu_\lambda(X) = \int_0^\infty \gamma_s(X)\mu_\lambda(s)ds, \tag{2.8}$$

where $\mu_\lambda(s)ds := \lambda^2 s e^{-\lambda s} ds$ is a probability density on $[0, +\infty]$.

Proof. It is enough to notice that the following relation holds, by integration by parts :

$$\nu_\lambda(X) = \lambda \int_0^\infty e^{-\lambda s} g(X(s))ds = \lambda^2 \int_0^\infty s e^{-\lambda s} \left(\frac{1}{s} \int_0^s g(X(r))dr \right) ds,$$

and that $\int_0^\infty \lambda^2 s e^{-\lambda s} ds = 1$. ■

The probability measure μ_λ will play an important role in the rest of the paper. Denoting $M(\alpha, \beta; \lambda) := \displaystyle\int_\alpha^\beta \mu_\lambda(s)ds = e^{-\lambda\alpha}(1 + \lambda\alpha) - e^{-\lambda\beta}(1 + \lambda\beta)$, we prove here two estimates that will be helpful in the next section.

Lemma 3 *The two following results hold (compare with Lemma 3 in [25]):*

(i) $\forall t > 0, \exists \epsilon_0$ *such that* $\forall \epsilon \leq \epsilon_0$, $M((1-\epsilon)t, t; 1/t) \geq \epsilon/2e$.

(ii) $\forall \delta > 0, \exists \epsilon_0$ *such that* $\forall \epsilon \leq \epsilon_0$, $\forall t > 0$, $M(\epsilon t, (1-\epsilon)t; 1/t\sqrt{\epsilon}) \geq 1 - \delta$.

Proof. Notice that in these particular cases, M does not depend on t:

(i) $M(t(1-\epsilon), t; 1/t) = (2-\epsilon)e^{-1+\epsilon} - 2e^{-1} = \frac{1}{e}(\epsilon + o(\epsilon)) \geq \frac{\epsilon}{2e}$, for ϵ small enough.

(ii) $M(t\epsilon, t(1-\epsilon); 1/\sqrt{\epsilon}t) = (1+\sqrt{\epsilon})e^{-\sqrt{\epsilon}} - (1 - 1/\sqrt{\epsilon} + \sqrt{\epsilon})e^{-1/\sqrt{\epsilon}+\sqrt{\epsilon}}$. This expression tends to 1 as $\epsilon \to 0$, hence the result.

■

3 Proof of Theorem 6

3.1 From V_t to W_λ

Assume (B) : $V_t(\cdot)$ converges to some $V(\cdot)$ as t goes to infinity, uniformly on Ω. Our proof follows Proposition 4 and Lemmas 8 and 9 in [25].

Proposition 4 *For all $\epsilon > 0$, there exists $\lambda_0 > 0$ such that $W_\lambda(x) \geq V(x) - \varepsilon$ for every $x \in \Omega$ and for all $\lambda \in (0, \lambda_0]$.*

Proof. Let T be such that $\|V_t - V\|_\infty \leq \epsilon/2$ for every $t \geq T$. Choose $\lambda_0 > 0$ such that

$$\lambda^2 \int_T^\infty s e^{-\lambda s} ds = 1 - (1 + \lambda T)e^{-\lambda T} \geq 1 - \epsilon/4$$

for every $\lambda \in]0, \lambda_0[$. Fix $\lambda \in]0, \lambda_0[$ and take a play $Y \in \Gamma(x)$ which is $\epsilon/4$-optimal play for $W_\lambda(x)$. Since $\gamma_s(X) \geq 0$, the convexity formula (2.8) from Lemma 2 gives:

$$\begin{aligned} W_\lambda(x) + \epsilon/4 \geq \nu_\lambda(Y) &\geq 0 + \lambda^2 \int_T^\infty s e^{-\lambda s} \gamma_s(Y) \\ &\geq \lambda^2 \int_T^\infty s e^{-\lambda s} V_s(x) \\ &\geq (1-\epsilon/4)(V(x) - \epsilon/2) \\ &= V(x) - \epsilon/4 V(x) - \epsilon/2 + \epsilon^2/8 \\ &\geq V(x) - 3/4\epsilon \end{aligned}$$

■

Lemma 5 *$\forall \epsilon > 0, \exists M$ such that for all $t \geq M, \forall x \in \Omega$, there is a play $X \in \Gamma(x)$ such that $\gamma_s(X) \leq V(x) + \epsilon$ for all $s \in [\epsilon t, (1-\epsilon)t]$.*

Proof. By (B) there exists M such that $\|V_r - V\| \leq \frac{\epsilon^2}{3}$ for all $r \geq \epsilon M$. Given $t \geq M$ and $x \in \Omega$, let $X \in \Gamma(x)$ be a play (from x) such that $\gamma_t(X) \leq V_t(x) + \frac{\epsilon^2}{3}$. For any $s \leq (1-\epsilon)t$, $t - s \geq \epsilon t \geq \epsilon M$ so Proposition 1 (Monotonicity) imply that

$$V_{t-s}(X(s)) \geq V(X(s)) - \frac{\epsilon^2}{3} \geq V(x) - \frac{\epsilon^2}{3}. \tag{3.1}$$

Since $V(x) + \epsilon^2/3 \geq V_t(x)$, we also have:

$$\begin{aligned} t\left(V(x) + 2\frac{\epsilon^2}{3}\right) &\geq t\left(V_t(x) + \frac{\epsilon^2}{3}\right) \\ &\geq t\gamma_t(X) = \int_0^s g(X(r))dr + \int_s^t g(X(r))dr \\ &\geq s\gamma_s(X) + (t-s)V_{t-s}(X(s)) \\ &\geq s\gamma_s(X) + (t-s)\left(V(x) - \frac{\epsilon^2}{3}\right) \text{ by (3.1).} \end{aligned}$$

Isolating $\gamma_s(X)$ we get:

$$\begin{aligned} \gamma_s(X) &\leq V(x) + \epsilon^2 \frac{t}{s} \\ &\leq V(x) + \epsilon, \quad \text{for } s/\epsilon \geq t, \end{aligned}$$

and we have proved the result for all $s \in [\epsilon t, (1-\epsilon)t]$. ■

Proposition 6 *$\forall \delta > 0, \exists \lambda_0$ such that $\forall x \in \Omega$, for all $\lambda \in (0, \lambda_0]$, we have $W_\lambda(x) \leq V(x) + \delta$.*

Proof. By Lemma 3 (ii), one can choose ϵ small enough such that $M(\epsilon t, (1-\epsilon)t; \frac{1}{t\sqrt{\epsilon}}) \geq 1 - \frac{\delta}{2}$, for any t. In particular, we can take $\epsilon \leq \frac{\delta}{2}$. Using Lemma 5 with $\delta/2$, we get that for $t \geq t_0$ (and thus for $\lambda_t := \frac{1}{t\sqrt{\epsilon}} \leq \frac{1}{t_0\sqrt{\epsilon}}$) and for any $x \in \Omega$, there exists a play $X \in \Gamma(x)$ such that

$$\begin{aligned} \nu_{\lambda_t}(X) &\leq \delta/2 + {\lambda_t}^2 \int_{\epsilon t}^{(1-\epsilon)t} s e^{\lambda_t s} \gamma_s(X) ds \\ &\leq \delta/2 + 1 \cdot (V(x) + \delta/2). \end{aligned}$$

■

Propositions 4 and 6 establish the first part of Theorem 6: $(B) \Rightarrow (A)$.

3.2 From W_λ to V_t

Now assume (A) : $W_\lambda(\cdot)$ converges to some $W(\cdot)$ as λ goes to 0, uniformly on Ω. Our proof follows Proposition 2 and Lemmas 6 and 7 in [25]. Start by a technical Lemma:

Lemma 7 *Let $\epsilon > 0$. For all $x \in \Omega$ and $t > 0$, and for any trajectory $Y \in \Gamma(x)$ which is $\epsilon/2$-optimal for the problem with horizon t, there is a time $L \in [0, t(1-\epsilon/2)]$ such that, for all $T \in]0, t-L]$:*

$$\frac{1}{T}\int_L^{L+T} g(Y(s))ds \leq V_t(x) + \epsilon.$$

Proof. Fix $Y \in \Gamma(x)$ some $\epsilon/2$-optimal play for $V_t(x)$. The function $s \to \gamma_s(Y)$ is continuous on $]0, t]$ and satisfies $\gamma_t(Y) \leq V_t(x) + \epsilon/2$. The bound on g implies that $\gamma_r(Y) \leq V_t(x) + \epsilon$ for all $r \in [t(1-\epsilon/2), t]$.

Consider now the set $\{s \in]0, t] \mid \gamma_s(Y) > V_t(x) + \epsilon\}$. If this set is empty, then take $L = 0$ and observe that $\frac{1}{r}\int_0^r g(Y(s))ds \leq V_t(x) + \epsilon, \forall r \in]0, t]$.

Otherwise, let L be the superior bound of this set. Notice that $L < t(1-\epsilon/2)$ and that by continuity $\gamma_L(Y) = V_t(x) + \epsilon$. Now, for any $T \in [0, t-L]$,

$$\begin{aligned} V_t(x) + \epsilon &\geq \gamma_{L+T}(Y) \\ &= \frac{L}{L+T}\gamma_L(Y) + \frac{T}{L+T}\left(\frac{1}{T}\int_L^{L+T} g(Y(s))ds\right) \\ &= \frac{L}{L+T}\left(V_t(x) + \epsilon\right) + \frac{T}{L+T}\left(\frac{1}{T}\int_L^{L+T} g(Y(s))ds\right) \end{aligned}$$

and the result follows. ■

Proposition 8 *$\forall \epsilon > 0, \exists T$ such that for all $t \geq T$ we have $V_t(x) \geq W(x) - \epsilon$, for all $x \in \Omega$.*

Proof. Let λ be such that $\|W_\lambda - W\| \leq \epsilon/8$, and T such that $\lambda^2 \int_{T\epsilon/4}^{\infty} se^{-\lambda s}ds < \epsilon/8$. Proceed by contradiction and suppose that $\epsilon > 0$ is such that for every T, there exists $t_0 \geq T$ and a state $x_0 \in \Omega$ such that $V_{t_0}(x_0) < W(x_0) - \epsilon$.

Using Lemma 7 with $\epsilon/2$, we get a play $Y \in \Gamma(x_0)$ and a time $L \in [0, t_0(1-\epsilon/4)]$ such that, $\forall s \in [0, t_0 - L]$ (and, in particular, $\forall s \in [0, t_0\epsilon/4]$):

$$\frac{1}{s}\int_L^{L+s} g\left(Y(r)\right)dr \leq V_{t_0}(x_0) + \epsilon/2 < W(x_0) - \epsilon/2.$$

Thus,

$$\begin{aligned} W(Y(L)) - \epsilon/8 &\leq W_\lambda(Y(L)) \\ &\leq \lambda\int_0^{\infty} e^{-\lambda s} g(Y(L+s))ds \\ &\leq \lambda^2\int_0^{t_0\epsilon/4} se^{-\lambda s}\left(\frac{1}{s}\int_L^{L+s} g\left(Y(r)\right)dr\right)ds + \ \epsilon/8 \\ &\leq W(x_0) - \epsilon/2 + \epsilon/8 \\ &= W(x_0) - 3\epsilon/8. \end{aligned}$$

This gives us $W(Y(L)) \leq W(x_0) - \epsilon/4$, contradicting Proposition 1 (Monotonicity). ■

Proposition 9 *$\forall \epsilon > 0, \exists T$ such that for all $t \geq T$ we have $V_t(x) \leq W(x) + \epsilon$, for all $x \in \Omega$.*

Proof. Otherwise, $\exists \epsilon > 0$ such that $\forall T$, $\exists t \geq T$ and $x \in \Omega$ with $V_t(x) > W(x) + \epsilon$. For any $X \in \Gamma(x)$ consider the (continuous in s) payoff function $\gamma_s(X) = \frac{1}{s}\int_0^s g(X(r))dr$. Of course, $\gamma_t(X) \geq V_t(x) > W(x) + \epsilon$. Furthermore, because of the bound on g,

$$\gamma_r(X) \geq W(x) + \epsilon/2, \ \forall r \in [t(1-\epsilon/2), t].$$

By Lemma 3, we can take ϵ small enough, so that for all t, $M(t(1-\epsilon/2), t; 1/t) \geq \epsilon/4e$ holds. We set $\delta := \epsilon/4e$.

By Proposition 8, there is a K such that $V_t \geq W(x) - \delta\epsilon/8$, for all $t \geq K$.

For K fixed, we consider $M(0, K; 1/t) = 1 - e^{-K/t}(1 + K/t)$ as a function of t. Clearly, it tends to 0 as t tends to infinity, so let t be such that this quantity is smaller than $\delta\epsilon/16$. Also, let t be big enough so that $\|W_{1/t} - W\| < \delta\epsilon/5$, which is a consequence of assumption (A).

On the following, we set $\tilde{\lambda} := 1/t$ and consider the $\tilde{\lambda}$-payoff of some play $X \in \Gamma(x)$. We'll split the integral over $[0, +\infty]$ in three parts : $\mathcal{K} = [0, K], \mathcal{R} = [t(1-\epsilon/2), t]$, and $(\mathcal{K} \cup \mathcal{R})^c$. The three parts are clearly disjoint since $t >> K$. We have seen that $\mu_\lambda(s)ds = \lambda^2 s e^{-\lambda s} ds$ is a probability measure on $[0, +\infty]$, for any $\lambda > 0$. Then by the Convexity formula (2.8), we can write:

$$\nu_{\tilde{\lambda}}(X) = \left(\int_{\mathcal{K}} \gamma_s(X)\mu_{\tilde{\lambda}}(ds) + \int_{\mathcal{R}} \gamma_s(X)\mu_{\tilde{\lambda}}(ds) + \int_{(\mathcal{K}\cup\mathcal{R})^c} \gamma_s(X)\mu_{\tilde{\lambda}}(ds) \right).$$

Recall that

$$\begin{aligned} \gamma_s(X)_{|\mathcal{K}} &\geq 0 \\ \gamma_s(X)_{|(\mathcal{K}\cup\mathcal{R})^c} &\geq W(x) - \delta\epsilon/8 \\ \gamma_s(X)_{|\mathcal{R}} &\geq W(x) + \epsilon/2 \end{aligned}$$

It is straightforward that

$$\begin{aligned} \nu_{\tilde{\lambda}}(X) &\geq 0 + \delta \cdot (W(x) + \epsilon/2) + (1 - \delta - \delta\epsilon/16) \cdot (W(x) - \delta\epsilon/8) \\ &= W(x) + \delta\epsilon \left(\frac{1}{2} - \frac{1}{16} - \frac{1}{8} - \frac{\delta}{8} + \frac{\delta\epsilon}{64} \right) \\ &\geq W(x) + \delta\epsilon/4. \end{aligned}$$

This is true for any play, so its infimum also satisfies $W_{\tilde{\lambda}}(x) \geq W(x) + \delta\epsilon/4$, which is a contradiction, for we assumed that $W_{\tilde{\lambda}} < W(x) + \delta\epsilon/5$. ■

Propositions 8 and 9 establish the second half of Theorem 6 : $(A) \Rightarrow (B)$.

4 A counter example for pointwise convergence

In this section we give an example of an optimal control problem in which both $V_t(\cdot)$ and $W_\lambda(\cdot)$ converge pointwise on the state space, but to two different limits. As implied by Theorem 6, the convergence is not uniform on the state space.

Lehrer and Sorin were the first to construct such an example [25], in the discrete-time framework. We consider here one of its adaptation in continuous time, which was studied as Example 5 in [29][3], where the notations are the same that in Section 1:

[3]We thank Marc Quincampoix for pointing out this example to us, which is simpler that our original one.

- The state space is $\Omega = \mathbb{R}_+^2$.
- The payoff function is given by $g(x, y) = 0$ if $x \in [1, 2]$, 1 otherwise.
- The set of control is $U = [0, 1]$.
- The dynamic is given by $f(x, y, u) = (y, u)$ (thus Ω is forward invariant.)

An interpretation is that the couple $(x(t), y(t))$ represents the position and the speed of some mobile moving along an axis, and whose acceleration $u(t)$ is controlled. Observe that since $U = [0, 1]$, the speed $y(t)$ increases during a play. We claim that for any $(x_0, y_0) \in \mathbb{R}_+^2$, $V_t(x_0, y_0)$ (resp $W_\lambda(x_0, y_0)$) converges to $V(x_0, y_0)$ as t goes to infinity (resp. converges to $W(x_0, y_0)$ as λ tends to 0, where:

$$V(x_0, y_0) = \begin{cases} 1 & \text{if } y_0 > 0 \text{ or } x_0 > 2 \\ 0 & \text{if } y_0 = 0 \text{ and } 1 \leq x_0 \leq 2 \\ \frac{1-x_0}{2-x_0} & \text{if } y_0 = 0 \text{ and } x_0 < 1 \end{cases}$$

$$W(x_0, y_0) = \begin{cases} 1 & \text{if } y_0 > 0 \text{ or } x_0 > 2 \\ 0 & \text{if } y_0 = 0 \text{ and } 1 \leq x_0 \leq 2 \\ 1 - \frac{(1-x_0)^{1-x_0}}{(2-x_0)^{2-x_0}} & \text{if } y_0 = 0 \text{ and } x_0 < 1 \end{cases}$$

Here we only prove that $V(0, 0) = \frac{1}{2}$ and $W(0, 0) = \frac{3}{4}$; the proof for $y_0 = 0$ and $0 < x_0 < 1$ is similar and the other cases are easy.

First of all we prove that for any t or λ and any admissible trajectory (that is, any function $X(t) = (x(t), y(t))$ compatible with a control $u(t)$), starting from $(0, 0)$, $\gamma_t(X) \geq \frac{1}{2}$ and $\nu_\lambda(X) \geq \frac{3}{4}$. This is clear if $x(t)$ is identically 0, so consider this is not the case. Since the speed $y(t)$ is increasing, we can define t_1 and t_2 as the time at which $x(t_1) = 1$ and $x(t_2) = 2$ respectively, and moreover we have $t_2 \leq 2t_1$. Then,

$$\begin{aligned} \gamma_t(X) &= \frac{1}{t}\left(\int_0^{min(t,t_1)} ds + \int_{min(t,t_2)}^{t} ds\right) \\ &= 1 + \min\left(1, \frac{t_1}{t}\right) - \min\left(1, \frac{t_2}{t}\right) \\ &\geq 1 + \min\left(1, \frac{t_2}{2t}\right) - \min\left(1, \frac{t_2}{t}\right) \\ &\geq \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \nu_\lambda(X) &= \int_0^{t_1} \lambda e^{-\lambda s} ds + \int_{t_2}^{+\infty} \lambda e^{-\lambda s} ds \\ &= 1 - e^{-\lambda t_1} + e^{-\lambda t_2} \\ &\geq 1 - e^{-\lambda t_1} + e^{-2\lambda t_1} \\ &\geq \min_{a>0}\{1 - a + a^2\} \\ &\geq \frac{3}{4}. \end{aligned}$$

On the other hand, one can prove [29] that $\limsup V_t(0,0) \leq \frac{1}{2}$: in the problem with horizon t, consider the control "$u(s) = 1$ until $s = \frac{2}{t}$ and then 0". Similarly one proves that $\limsup W_\lambda(0,0) \leq \frac{3}{4}$: in the λ-discounted problem, consider the control "$u(s) = 1$ until $s = \frac{\lambda}{\ln 2}$ and then 0".

So the functions V_t and W_λ converge pointwise on Ω, but their limits V and W are different, since we have just shown $V(0,0) \neq W(0,0)$. One can verify that neither convergence is uniform on Ω by considering $V_t(1,\varepsilon)$ and $W_\lambda(1,\varepsilon)$ for small positive ε.

Remark 1 *One may object that this example is not very regular since the payoff g is not continuous and the state space is not compact. However a related, smoother example can easily be constructed:*

1. *The set of controls is still* $[0,1]$.
2. *The continuous cost $g(x)$ is equal to 1 outside the segment [0.9,2.1], to 0 on [1,2], and linear on the two remainings intervals.*
3. *The compact state space is* $\Omega = \{(x,y) | 0 \leq y \leq \sqrt{2x} \leq 2\sqrt{2}\}$.
4. *The dynamic is the same that in the original example for $x \in [0,3]$, and $f(x,y,u) = ((4-x)y, (4-x)u)$ for $3 \leq x \leq 4$. The inequality $y(t)y'(t) \leq x'(t)$ is thus satisfied on any trajectory, which implies that Ω is forward invariant under this dynamic.*

With these changes the values $V_t(\cdot)$ and $W_\lambda(\cdot)$ still both converge pointwise on Ω to some $\widetilde{V}(\cdot)$ and $\widetilde{W}(\cdot)$ respectively, and $\widetilde{V}(0,0) \neq \widetilde{W}(0,0)$.

5 Possible extensions

- We considered the finite horizon problem and the discounted one, but it should be possible to establish similar Tauberian theorems for other, more complex, evaluations of the payoff. This was settled in the discrete time case in [27].
- It would be very fruitful to establish necessary or sufficient conditions for uniform convergence to hold. In this direction we mention [29] in which sufficient conditions for the stronger notion of Uniform Value (meaning that there are controls that are nearly optimal no matter the horizon, provided it is large enough) are given in a general setting.
- In the discrete case an example is constructed in [27] in which there is no uniform value despite uniform convergence of the families V_t and W_λ. It would be of interest to construct such an example in continuous time, in particular in the framework of section 1.
- It would be very interesting to study Tauberian theorems for dynamic systems that are controlled by two conflicting controllers. In the framework of differential games this has been done recently (Theorem 2.1 in [1]): an extension of Theorem 4 has been accomplished for two player games in which the limit of V_T or W_λ is assumed to be independent of the starting point. The similar result in the discrete time framework is a consequence of Theorems 1.1 and 3.5 in [22]. Existence of Tauberian theorems in the general setup of two-persons zero-sum games with no ergodicity condition remains open in both the discrete and the continuous settings.

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Appendix

We give here another proof[4] of Theorem 6 by using the analoguous result in discrete time [25] as well as an argument of equivalence between discrete and continuous dynamic.

Consider a deterministic dynamic programming problem in continuous time as defined in section 2.1, with a state space Ω, a payoff g and a dynamic Γ. Recall that, for any $\omega \in \Omega$, $\Gamma(\omega)$ is the non empty set of feasible trajectories, starting from ω. We construct an associated deterministic dynamic programming problem in *discrete* time as follows.

Let $\widetilde{\Omega} = \Omega \times [0,1]$ be the new state space and let $\widetilde{g}$ be the new cost function, given by $\widetilde{g}(\omega, x) = x$. We define a multivalued-function with nonempty values $\widetilde{\Gamma} : \widetilde{\Omega} \rightrightarrows \widetilde{\Omega}$ by

$$(\omega, x) \in \widetilde{\Gamma}(\omega', x') \Longleftrightarrow \exists X \in \Gamma(\omega'), \text{ with } X(1) = \omega \text{ and } \int_0^1 g(X(t))dt = x.$$

Following [25], we define, for any initial state $\widetilde{\omega} = (\omega, x)$

$$v_n(\widetilde{\omega}) = \inf \frac{1}{n} \sum_{i=1}^{n} \widetilde{g}(\widetilde{\omega}_i) \tag{5.1}$$

$$w_\lambda(\widetilde{\omega}) = \inf \lambda \sum_{i=1}^{+\infty} (1-\lambda)^{i-1} \widetilde{g}(\widetilde{\omega}_i) \tag{5.2}$$

where the infima are taken over the set of sequences $\{\widetilde{\omega}_i\}_{i\in\mathbb{N}}$ such that $\widetilde{\omega}_0 = \widetilde{\omega}$ and $\widetilde{\omega}_{i+1} \in \widetilde{\Gamma}(\widetilde{\omega}_i)$ for every $i \geq 0$.

Theorem 6 is then the consequence of the following three facts. Firstly, the main theorem of Lehrer and Sorin in [25], which states that uniform convergence (on $\widetilde{\Omega}$) of v_n to some v is equivalent to uniform convergence of w_λ to the same v.

Secondly, the concatenation hypothesis (2.1) on Γ implies that for any $(\omega, x) \in \widetilde{\Omega}$

$$v_n(\omega, x) = V_n(\omega)$$

where $V_t(\omega) = \inf_{X\in\Gamma(\omega)} \frac{1}{t} \int_0^n g(X(s))ds$, as defined in equation (2.4). Consequently, because of the bound on g, for any $t \in \mathbb{R}+$ we have

$$|V_t(\omega) - v_{\lfloor t \rfloor}(\omega, x)| \leq \frac{2}{\lfloor t \rfloor}$$

where $\lfloor t \rfloor$ stands for the integral part of t.

Finally, again because of hypothesis (2.1), for any $\lambda \in]0,1]$,

$$w_\lambda(\omega, x) = \inf_{X\in\Gamma(\omega)} \lambda \int_0^{+\infty} (1-\lambda)^{\lfloor t \rfloor} g(X(t))dt.$$

[4] We thank Frédéric Bonnans for the idea of this proof

Hence, by equation (2.5) and the bound on the cost function, for any $\lambda \in]0,1]$,

$$|W_\lambda(\omega) - w_\lambda(\omega, x)| \leq \lambda \int_0^{+\infty} \left|(1-\lambda)^{\lfloor t \rfloor} - e^{-\lambda t}\right| dt$$

which tends uniformly (with respect to x and ω) to 0 as λ goes to 0 by virtue of the following lemma.

Lemma 10 *The function*

$$\lambda \longrightarrow \lambda \int_0^{+\infty} \left|(1-\lambda)^{\lfloor t \rfloor} - e^{-\lambda t}\right| dt$$

converges to 0 as λ tends to 0

Proof. Since $\lambda \int_0^{+\infty} (1-\lambda)^{\lfloor t \rfloor} = \lambda \int_0^{+\infty} e^{-\lambda t} dt = 1$, for any $\lambda > 0$, the lemma is equivalent to the convergence to 0 of

$$E(\lambda) := \lambda \int_0^{+\infty} \left[(1-\lambda)^{\lfloor t \rfloor} - e^{-\lambda t}\right]_+ dt$$

where $[x]_+$ denotes the positive part of x. Now, from the relation $1 - \lambda \leq e^{-\lambda}$, true for any λ, one can easily deduce that, for any $\lambda > 0,\ t \geq 0$, the relation $(1-\lambda)^{\lfloor t \rfloor} e^{\lambda t} \leq e^{\lambda}$ holds. Hence,

$$\begin{aligned} E(\lambda) &= \lambda \int_0^{+\infty} e^{-\lambda t} \left[(1-\lambda)^{\lfloor t \rfloor} e^{\lambda t} - 1\right]_+ dt \\ &\leq \lambda \int_0^{+\infty} e^{-\lambda t} (e^{\lambda} - 1) dt \\ &= e^{\lambda} - 1 \end{aligned}$$

which converges to 0 as λ tends to 0. ■

References

[1] Alvarez, O.; Bardi, M. (2007) Ergodic Problems in Differential Games. *Advances in dynamic game theory*, 131-152, Ann. Int'l. Soc. Dynam. Games, **9**, Birkhäuser Boston, MA.

[2] Alvarez, O.; Bardi, M. (2010) Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations, Memoirs of the Amer. Math. Soc. 960 **204**, 1-90.

[3] Arisawa, M. (1997) Ergodic problem for the Hamilton-Jacobi-Bellman equation I. *Ann. Inst. Henri Poincare*, **14**, 415-438.

[4] Arisawa, M. (1998) Ergodic problem for the Hamilton-Jacobi-Bellman equation II. *Ann. Inst. Henri Poincare*, **15**, 1-24.

[5] Arisawa, M.; Lions, P.-L. (1998) On ergodic stochastic control. *Comm. Partial Differential Equations 23*, **11-12**, 2187-2217.

[6] Artstein Z.; Gaitsgory V. (2000) The value function of singularly perturbed control systems. *Appl. Math. Optim. 41* **3**, 425-445.

[7] Bardi, M.; Capuzzo-Dolcetta, I. (1997) *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser.

[8] Barles, G. (2007) Some homogenization results for non-coercive Hamilton-Jacobi equations. *Calculus of Variations and Partial Differential Equations*, **30**, 4, 449-466.

[9] Bellman, R. (1952) On the theory of dynamic programming - Proc. Nat. Ac. of Sci., U.S.A.

[10] Bettiol, P. (2005) On ergodic problem for Hamilton-Jacobi-Isaacs equations, *ESAIM: COCV*, **11**, 522-541.

[11] Cardaliaguet, P. (2010) Ergodicity of Hamilton-Jacobi equations with a non coercive non convex Hamiltonian in $\mathbb{R}^2/\mathbb{Z}^2$. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis* **27**, 837-856.

[12] Carlson, D.A.; Haurie, A.B.; Leizarowitz, A. (1991) *Optimal Control on Infinite Time Horizon*, Springer-Verlag, Berlin.

[13] Colonius, F.; Kliemann, W. (1989) Infinite Time Optimal Control and Periodicity, *Appl. Math. Optim.*, **20**, 113-130.

[14] Evans, L.C. An Introduction to Mathematical Optimal Control Theory. Unpublished lecture notes, U.C. Berkeley.

[15] Feller, W. (1971), *An Introduction to Probability Theory and its Applications Vol. II*. Second edition. New York: John Wiley & Sons.

[16] Grune, L. (1998) On the Relation between Discounted and Average Optimal Value Functions. *Journal of Differential Equations* **148**, 65-99.

[17] Hardy, G.H.; Littlewood, J.E. (1914) Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive. *Proc. London Math. Soc.* **13**, 174-191.

[18] Hestenes, M. (1950) A General Problem in the Calculus of Variations with Applications to the Paths of Least Time. *RAND Corporation*, Research Memorandum **100**.

[19] Isaacs, R. (1951) Games of Pursuit. Paper P-**257**, *RAND Corporation*, Santa Monica, California.

[20] Isaacs, R. (1965) *Differential Games. A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization.* John Wiley and Sons, New York.

[21] Khlopin, D.(2016), On an example for the Uniform Tauberian theorem in abstract control systems. https://arxiv.org/pdf/1604.07111v1

[22] Kohlberg, E.; Neyman, A.(1981), Asymptotic behavior of nonexpansive mappings in normed linear spaces. *Israel Journal of Mathematics* **38**, 269-275.

[23] Kirk, D.E. (1970), *Optimal Control Theory: an Introduction*. Prentice Hall.

[24] Lee, E.B; Markus, L.(1967) *Foundations of Optimal Control Theory*. SIAM, Philadelphia.

[25] Lehrer, E.; S. Sorin (1992), A uniform Tauberian theorem in dynamic programming. *Mathematics of Operation Research* **17**, 303-307.

[26] Lions, P.-L.; Papanicolaou G.; Varadhan S.R.S. (1986) Homogenization of Hamilton-Jacobi Equations. Unpublished.

[27] Monderer, M., Sorin, S. (1993), Asymptotic Properties in Dynamic Programming. *International Journal of Game Theory* **22**, 1-11.

[28] Pontryiagin, L.S., Boltyanskii, V.G., Gamkrelidge (1962). *The mathematical theory of optimal processes.* Nauka, Moskow. Engl. Trans. Wiley.

[29] Quincampoix, M., Renault, J. (2009), On the existence of a limit value in some non expansive optimal control problems. Preprint.

[30] Shapley, L. S. (1953) Stochastic games. *Proc. Nat. Acad. Sciences* **39**, 1095-1100.