Set-Valued Numerical Analysis for Optimal Control and Differential Games

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1 Introduction

We consider the following dynamic of a two players' zero sum differential games:

$$(1)\qquad \begin{cases} x'(t) &= f(x(t),u(t),v(t)), \text{ for almost all } t \geq 0 \\ x(t) &\in X,\ u(t) \in U \text{ and } v(t) \in V \end{cases}$$

where $X := \boldsymbol{R}^N$ is the state space, U the set of controls of the first player whose name is Ursula and V the set of controls of the second player whose name is Victor.

Associated with this dynamic, we can consider two kinds of problems, the *Qualitative* ones and the *Quantitative* ones[1].

Quantitative Problems consist in optimizing some given criterion. This leads to the definition of the Value function. Qualitative Problems consist in studying problems where the "objective may be of some concrete, yes-or-no type". This leads to the definition of Victory domains. Our purpose is to provide a general method for approaching Victory domains and Value functions.

Let us now describe the two control problems and the two differential games problems we discuss further.

- *Qualitative Control Problems.* Many Qualitative Control Problems can be reduced to the following formulation: determine the set of initial points x_0 starting from which there exists a control $v(\cdot)$ such that the associated solution $x(\cdot)$ to the differential system

  $$(2)\qquad x'(t) = f(x(t),v(t)),\ \ v(t) \in V$$

  remains forever in a closed set K. This set is called the *Viability Kernel* of K.

  We give a general method for approaching such a domain. The algorithms for solving just as well control as differential games problems are based on these results.

  As an example, we study the target problem. More precisely we want to determine the set of initial points from which a solution to (2) exists, reaching a target in a finite time while remaining in a set of constraints K.

- *Quantitative Control Problems* We compute the Value function of an optimal control problem with state constraints.

  As an example of Quantitative Control Problem, we show how to determine the Minimal Time function. Namely, it is the function which associates, with any initial condition, the minimal time[2] over solutions to (2) to reach a target while remaining in a set of constraints K. We call this problem the *Minimal Time* problem.

- *Qualitative differential games.* We study differential games, with one target and two players with opposite goals, which dynamic is defined by

  $$(3)\qquad x'(t) = f(x(t),u(t),v(t)),\ \ u(t) \in U,\ \ v(t) \in V.$$

[1] In his book on Differential Games [47], R.Isaacs has distinguished these two questions and called them *game of kind* and *game of degree*.

[2] possibly infinite.

For instance, the problem where Ursula aims at reaching a target while her opponent, Victor, aims at avoiding the target forever, is a *Qualitative problem.*

We provide an algorithm for finding the *victory domain* of each player, namely the set of initial conditions starting from which this player succeeds to reach his goal whatever his opponent plays. The boundary of the victory domains is usually called the *barrier of the game.*

- *Quantitative differential games.* We determine the Value function of a differential game of degree.

  For instance, the problem where Victor aims at reaching a target in a minimal time, whenever Ursula aims at avoiding it as long as possible, is a *Quantitative differential games problem.* This game is a pursuit-evasion game. We call this problem the *Minimal Hitting Time* problem.

We propose to explore how these questions are related and how both can be treated in the framework of Set-Valued Analysis and Viability Theory. In a way, this approach is rather well-adapted to look at these several problems with an unified point of view. Although we shall not insist on Viability Theory, we shall recall, without giving proofs, results on control and differential games relevant to our objective.

We present the same kind of problems for both control *and* differential games. Indeed the introduction and the description of the viability method in the framework of control theory allows a better understanding of the development of this method for studying differential games.

For solving Quantitative problems, the basic idea of our approach is to compute the Value function by determining a Viability Kernel instead of solving a Hamilton-Jacobi-Belmann's equation. In the case of the two-players differential games, we compute the Value function by determining the Discriminating Kernel which is the analogous of Viability Kernels for differential games.

The Qualitative problems in differential games are quite classical. Barriers problems presented here are very similar for instance to those studied and solved by Isaacs [47], Breakwell [21] and Bernhard [17]. Their method, based on the computation of some particular trajectories, amounts to compute explicitly the barrier and the strategies of the players. Let us remark that, unlike the Isaacs-Breakwell-Bernhard's approach, in all results that follow, we do not need to compute any trajectory of the dynamic system to solve Qualitative as well as Quantitative control or differential game problems presented above.

The Quantitative differential game problems, mainly the problem of finding the Optimal Hitting time function, have been tackled through several approaches by many authors.
In his pioneering work [47], Isaacs has proposed a method to compute the solution of these games. This method has been studied and extended by several authors (Breakwell [21], Bernhard [17], [18], [19], [20]).

A second approach of quantitative games is based on the notion of continuous viscosity solutions to Hamilton-Jacobi-Isaacs' equation. It can be found in Crandall & Lions [36], Soner [59], Barles & Perthame [12], Bardi [6], for control problems and Bardi & Soravia [10], Bardi, Falcone & Soravia [8], Subbotin [62] for differential games. For lower semicontinuous Value function, extensions can be found mainly in Barron & Jensen [15], Barles [14], Subbotin [61], Rozyev & Subbotin [56], Bardi & Staicu [11], Bardi, Bottacin & Falcone [7], Soravia [60].

A third approach is due to H. Frankowska. It is based on the notion of contingent solution to Hamilton-Jacobi-Bellman's equations defined thanks to Viability Theory and Set-Valued Analysis. It allows to study lower semicontinuous Value functions of control problems (see [40], [41] and [42]). Some ideas underlying this present work are deeply inspired by her approach.

There is an extensive literature on the approximation of the value function for control problems and some recent papers on differential game problems. The reader may consult Capuzzo-Dolcetta & Falcone [23], Alziary [1], Bardi & Soravia [9], Bardi, Falcone & Soravia [8] and Pourtalier & Tidball [51]. In these papers, the approximation of the Value function is based on a discretization of Partial Differential Equations.

The numerical methods we obtain are based on the numerical approximation method of the Viability Kernel. This is the reason why our numerical schemes differ from those obtained through the discretization of Hamilton-Jacobi-Isaacs' equations.
Concerning the approximation of the Viability Kernel, we refer to the pioneering works of Byrnes & Isidori [22] in the context of so-called *zero dynamics* when K is an affine subspace. The first result of convergence of approximations of the Viability Kernel appeared in [43] but this method is hardly digitizable. In a similar context, it has been developed the so-called *Viability Kernel Algorithm* (see [54], [58]) on which the forthcoming numerical

methods are built. The main ideas of the results presented in this chapter appeared in [25], [27], [54], [58]. However, the present work contains some innovations. First we give general sufficient conditions for the pointwise convergence of a numerical scheme. Second, for all the algorithms, we apply the *Refinement Principle* which allows to avoid redoing computation over all the initial domain at each change of discretization step.

We do not give here any results concerning the rate of convergence. Let us mention the recent paper (Cardaliaguet [32]) where an estimation of the convergence for different discontinuous Value functions is given.

The major advantages of our approach are the following

- it takes into account state constraints without any controllability assumptions on the dynamic, neither at the boundary of targets, nor at the boundary of the constraint set,
- it allows to deal with a large class of systems with minimal regularity and convexity assumptions,
- it gives completely explicit and effective algorithms, adaptable to many situations,
- thanks to the support of Viability Theory, rigorous proofs of the convergence are provided including irregular cases.

We consider state constraints for Quantitative or Qualitative control problems. For differential game problems we do not impose any constraints for Ursula but Victor has to ensure the solution to remain in a constraint set K. The complete problem is much more intricated and its analysis exceeds the scope of this study (see [28] for the general case).

The present work is organized in the following way

- In section 2, we are interested in Qualitative Control Problems and the main concepts are defined. We recall the basic results of Viability Theory and we introduce the numerical scheme to compute the viability kernel. We give the proofs of their convergence. Let us point out that any algorithm of this chapter is an application or an extension of the numerical schemes presented in this first section.
- Section 3 is devoted to Quantitative control problems, and, in particular, to the Minimal Time function.
- In section 4, we study the Target Problem in differential games as an example of Qualitative differential game problem. We give algorithms to compute the Victory Domains of the players.
- The last section deals with the approximation of the Optimal Hitting Time function for differential games.
- In appendix 1, we recall some basic definitions and results of set-valued analysis. In particular, the different definitions of convergence for sets are given.
- In appendix 2, we characterize the Optimal Hitting Time function by the mean of viscosity solutions.

All numerical examples and figures we present have been computed, through the Viability Kernel Algorithm or the Discriminating Kernel Algorithm.

2 Qualitative Control Problems

This section is devoted to the presentation of basic results of viability theory. In particular, we recall the definition and the geometrical characterization of the *Viability Kernel* and of the *Invariance Kernel*.

The approximation of the viability kernel is divided in three steps

- the *semi-discrete algorithm* corresponds to a time discretization, through an Euler scheme. It allows to construct discrete viability kernels approaching the viability kernel,

- the *fully discrete algorithm* corresponds to discretization both in time and in space. As usually for numerical explicit schemes, the space and time discretization steps are linked up. This is done through Theorem 2.19, which is the main result of this section.

- the *Refinement Principle* adjusts the passage to grids more and more thin. This process improves significantly the efficiency of the algorithms.

We complete this section by giving a few applications to Qualitative Control Problems.

2.1 Basics results on Viability Theory

2.1.1 Differential inclusions

Consider the control system:

$$(4) \qquad \begin{cases} x'(t) &=& f(x(t), v(t)), \text{ for almost all } \ t \geq 0 \\ v(t) &\in& V, \ \forall t \geq 0 \end{cases}$$

It is an almost classical result that control system (4) can be represented by the following differential inclusion

$$(5) \qquad x'(t) \in F(x(t)), \text{ for almost all } \ t \geq 0,$$

where $F : X \rightsquigarrow X$ is the set-valued map defined by

$$\forall x \in X, \quad F(x) := \{ f(x, v), \ v \in V \}$$

The systems (4) and (5) have the same absolutely continuous solutions[3].

We shall denote by $S_F(x_0)$, the set of absolutely continuous solutions on $[0, +\infty)$ of (5) starting at $t = 0$ from x_0.

Let us define the *Hamiltonian* associated with the system

$$H(x, p) := \inf_{v \in V} < f(x, v), p >= \inf_{y \in F(x)} < y, p > .$$

The reader can refer to Appendix 1 for more details concerning the following concepts and results. For a complete overview, he may consult [4] and [3].

2.1.2 The Viability kernel

Let us consider a closed nonempty set $K \subset X$. We shall say that a solution $x(\cdot)$ to (4) (or equivalently to (5)) is *viable* in K if and only if $x(t) \in K$ for any $t \geq 0$.

Definition 2.1 *Let K be a closed subset of X. The* Viability kernel *of K for F is the set*

$$\{ x_0 \in K \text{ such that} \, \exists \, x(\cdot) \in S_F(x_0), \ x(t) \in K, \ \forall t \geq 0 \}$$

We denote it by $Viab_F(K)$.

Let us notice that this set is empty if an only if any solution, starting from K, leaves K in a finite time.

For computing $Viab_F(K)$ without computing any trajectory, we need to characterize $Viab_F(K)$ in a geometrical way. For that purpose, we first characterize the closed sets D such that, starting from any point $x_0 \in D$, there exists at least one solution viable in D.

[3] For any measurable $v(\cdot)$, the associated absolutely continuous solution $x(\cdot)$ to (4) starting from x_0 at $t = 0$ is a solution to (5). Conversely, for any absolutely continuous solution $x(\cdot)$ to (5) such that $x(0) = x_0$ there exists a measurable control $v(\cdot)$ for which $x(\cdot)$ is a solution to (4).

Such closed sets D are called *Viability Domains* or *viable* sets.
The Viability Domains are actually the sets D such that $Viab_F(D) = D$.

The following definition specifies the kind of regularity of the set-valued map we usually need

Definition 2.2 *A set-valued map is called a* Marchaud *map if it is upper semicontinuous with convex compact nonempty values and if it has a linear growth.*

In particular, let us consider a map $f : X \times V \to X$ describing a control system. If f continuous, with a linear growth, if V is compact and nonempty, if for any $x \in X$, $F(x) := \bigcup_v f(x, v)$ is convex, then F is a Marchaud map.

We also need to define a geometric tool which allows to handle geometric properties involved within the framework of this approach which is the *proximal normal* at a point x to a closed set K. A vector p is a *proximal normal* if and only if the open ball centered at $x+p$ and of radius $\|p\|$ does not encounter D (See Definition 6.6 in Appendix 1).

The following Theorems provides a characterization of viability domains by using geometrical conditions.

Theorem 2.3 (Viability Theorem) *Let $F : X \rightsquigarrow X$ be a Marchaud map and D a closed subset of X. The following properties are equivalent*

$$(6) \quad \begin{cases} i) & D \text{ is viable} : \ \forall x_0 \in D, \ \exists x(\cdot) \in S_F(x_0), \ x(t) \in D, \ \ \forall t \geq 0 \\ ii) & \forall x \in D, \ \ \forall p \in \mathcal{NP}_D(x), \ \exists v \in V, \ < f(x, v), p > \leq 0 \\ iii) & \forall x \in D, \ \ \forall p \in \mathcal{NP}_D(x), \ \ H(x, p) \leq 0 \end{cases}$$

For other characterizations of the viability domains, in particular for characterizations involving the contingent cone, we refer to [4].

In general, a closed set K is not a viability domain. Then $Viab_F(K)$ is contained in K but not necessarily equal to K.
The viability kernel of K for F can be characterized in the following way

Theorem 2.4 *Let $F : X \rightsquigarrow X$ be a Marchaud map and K a closed subset of X. The viability kernel of K for F is a closed viability domain contained in K. It contains any viability domain contained in K.*
Moreover, any solution viable in K has to remain in $Viab_F(K)$ forever.

For the proof of Theorems 2.3 and 2.4 we refer to [4].

Notations. In the following, $\mathcal{B}_X$ denotes the unit ball of space X. The subscript will be omitted when there cannot be any confusion.

Example 2.1

Let us consider $X = \boldsymbol{R}^2$ and the controlled system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} \ \text{ for almost all } \ t \geq 0$$

where $(v_x(t), v_y(t)) \in \mathcal{B}$. Let us consider the closed set

$$K := \{(x, y) \in \boldsymbol{R}^2 \ \text{ such that } \ \max(|x|, |y|) \leq 1\}$$

For all initial value $(x_0, y_0) \in \mathcal{B}$, the solution $(x(t), y(t)) = (x_0, y_0)$, $\forall t > 0$ is a trivial solution to the system and so (x_0, y_0) belongs to $Viab_F(K)$. The viability kernel of the system is $\mathcal{B}$ and any closed subset contained in $\mathcal{B}$ is a viability domain. □

Example 2.2

Let us consider the system

$$\left(\begin{array}{c} x'(t) \\ y'(t) \end{array}\right) = \left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right) \left(\begin{array}{c} x(t) \\ y(t) \end{array}\right) + \left(\begin{array}{c} v_x(t) \\ v_y(t) \end{array}\right)$$

where $(v_x(t), v_y(t)) \in \mathcal{B}$ and the closed set

$$K := \{z := (x, y) \in \boldsymbol{R}^2 \text{ such that } \max(|x|, |y|) \leq 1\}$$

Using the Proximal Normal characterization, one proves that $\mathcal{B}$ is a viability domain and that K is not a viability domain. Here, to prove that the viability kernel of K is $\mathcal{B}$, it is easy to show that for any solution $z(\cdot) = (x(\cdot), y(\cdot)) \in S_F(z_0)$ starting from a point z_0 which norm is strictly grater than 1 then $\|z(t)\|$ increases to $+\infty$. □

Remark 2.1

For any closed set K' satisfying $Viab_F(K) \subset K' \subset K$ we have $Viab_F(K') = Viab_F(K)$. □

Remark 2.2

Assume that K is convex. If $\text{Graph}(F) := \{(x, y) \in X \times X \mid y \in F(x)\}$ is convex and if F is a Marchaud map, then $Viab_F(K)$ is convex (see [43]). In particular, when the control system f is of the form $f(x, v) := Ax + Bv$ where A and B are matrix and V is convex compact, then $\text{Graph}(F)$ is convex. □

2.1.3 The Invariance kernel

The viability kernel of K for a set-valued map F consists inof the set of initial positions x_0 of K from which at least one solution starts $x(\cdot) \in \mathcal{S}_F(x_0)$ which remains in K. It is quite natural to consider the set of initial conditions x_0 of K such that *any* solution $x(\cdot) \in \mathcal{S}_F(x_0)$ remains in K.

Definition 2.5 *Let K be a closed subset of X. The* Invariance kernel *of K for F is the set*

$$\{x_0 \in K \text{ such that } \forall\, x(\cdot) \in S_F(x) \ : \ x(t) \in K, \ \forall\, t \geq 0\,\}$$

We denote it by $Inv_F(K)$.

As for the viability kernel, it is possible to characterize the invariance kernel by the mean of geometric conditions.

Theorem 2.6 (Invariance Theorem) *Let F be a Lipschitz[4] Marchaud map and D a closed subset of X. The following properties are equivalent*

$$(7) \qquad \left\{\begin{array}{ll} i) & D \text{ is an invariance domain } : \ \forall x_0 \in D, \ \forall x(\cdot) \in S_F(x_0) \ : \ x(t) \in D, \forall t \leq 0 \\ ii) & \forall\, x \in D, \ \forall p \in \mathcal{NP}_D(x), \ \forall v \in V, < f(x, v), p > \leq 0 \\ iii) & \forall x \in D, \ \ \forall p \in \mathcal{NP}_D(x), \ H(x, -p) \geq 0 \end{array}\right.$$

The following property characterizes the invariance kernel:

Proposition 2.7 *Let $F : X \rightsquigarrow X$ be a Lipschitz Marchaud map and K a closed subset of X. The invariance kernel of K for F is the largest closed invariance domain for F contained in K.*

In particular, the invariance kernel is an invariance domain. If an invariance domain is contained in a closed set K, then it is contained in the invariance kernel of K.

The computation of the invariance kernel is actually a particular case of computation of the discriminating kernel as we shall see in the further subsection 4.2, remark 4.1.

[4]For the Definition of Lipschitz set-valued map, see Appendix 1. In particular, if $f : X \times V \to X$ is continuous control system, with V compact and f Lipschitz with respect to x, then the associated set-valued map $F(x) := \bigcup_v f(x, v)$ is Lipschitz.

2.1.4 Target Problems

Let us consider $\mathcal{O}$ an open target and set $K := X \backslash \mathcal{O}$. The controller aims at reaching $\mathcal{O}$. Then we can define two different victory domains:

Definition 2.8 *The* Possible Victory Domain *is the set of initial points in K from which at least one trajectory starts reaching the target $\mathcal{O}$ in finite time.*

The Certain Victory Domain *is the set of initial points in K starting from which every trajectory reaches the target $\mathcal{O}$ in finite time.*

Next proposition states that it is possible to characterize these victory domains with viability and invariance kernels.

Proposition 2.9 *Let F be a Marchaud Lipschitz map and $\mathcal{O}$ an open target. Then*

The Certain Victory domain is the complement of the Viability kernel of K.

The Possible Victory Domain is equal to $X \backslash Inv_F(K)$.

Such an interpretation of target problems in term of viability kernels was provided firstly in [53]. The reader can also refer to ([4], chapter 5). Also the computation of the certain victory domain for target problem is a straightforward application of the computation of the viability kernel. The computation of the possible victory domain is a straightforward application of the computation of the invariance kernel.

The boundary of victory domains is usually called the barrier of the Qualitative control problem. Indeed, it is known that, *if they are smooth*, these barriers contain some trajectories of the system and can be crossed by the other trajectories in only one direction (see for instance in Isaacs [47]). This provided a method for constructing barriers.

However, this approach required an *a priori* regularity which was not satisfied in practice. This difficulty is solved in [53] where it is proved - without regularity assumption - that the boundary of viability kernels have the same property than these barriers.

2.2 Approximation of $Viab_F(K)$

To approach the Viability Kernel, we first replace the initial differential inclusion system by a finite difference inclusion system (semi-discrete scheme). Secondly, we replace the state space X by an integer lattice X_h of X (fully discrete scheme). Finally we apply a Refinement Principle.

2.2.1 Discrete Viability Kernel

Let us consider F_ε some approximation of F and define

$$G_\varepsilon(x) := x + \varepsilon F_\varepsilon(x)$$

The choice of F_ε depends in general on the regularity[5] of the dynamic F.

The discretized dynamic corresponding to the Euler scheme is

$$x_{n+1} \in G_\varepsilon(x_n) := x_n + \varepsilon F_\varepsilon(x_n) \tag{8}$$

which solution $\vec{x} := (x_n)_n$ is a sequence of points of X.

Definition 2.10 *A set closed set D is* discretely viable *(or equivalently a* discrete viability domain*) for G if and only if for any $x_0 \in D$ there exists at least a sequence $\vec{x} := (x_n)_n$ solution to the recursive inclusion $x_{n+1} \in G(x_n)$ starting from x_0 which belongs to D for any $n \geq 0$.*

It is easy to prove that discrete viability domains are also characterized by a geometric condition

Proposition 2.11 *The following propositions are equivalent*

$$\left\{ \begin{array}{ll} i) & D \text{ is discretely viable for } G \\ ii) & \forall\, x \in D,\ G(x) \cap D \neq \emptyset \end{array} \right. \tag{9}$$

When K is not a discrete viability domain, next Proposition states the existence of the discrete viability kernel contained in K

[5] We shall mainly discuss the case when F is $\ell-$Lipschitz and bounded by some constant M. Then we can take $F_\varepsilon(x) := F(x) + \frac{1}{2} M \ell \varepsilon \mathcal{B}$.

Proposition 2.12 *Let $G : X \rightsquigarrow X$ be an upper semicontinuous set-valued map with compact nonempty values.*

a) The largest closed discrete viability domain for G contained in K exists and is called the discrete viability kernel *of K for G. We denote it $\overrightarrow{Viab}_G\,(K)$.*

b) Furthermore, $\overrightarrow{Viab}_G\,(K)$ coincides with the subset of initial values $x_0 \in K$ for which there exists at least one sequence solution viable in K.

Proposition 2.13 provides a constructive proof of existence of the discrete viability kernel. We do not give the proof of b) that can be deduced from Proposition 2.13 below and that can be found in [58]. We shall not need this point in the sequel.

2.2.2 The semi-discrete Viability Kernel Algorithm

Let us consider the decreasing sequence of closed sets K^n defined by

$$\text{(10)} \qquad \begin{cases} K^0 & := & K \\ K^{n+1} & := & \{x \in K^n \mid G(x) \cap K^n \neq \emptyset\} \end{cases}$$

Proposition 2.13 *Let $G : X \rightsquigarrow X$ be an upper semicontinuous set-valued map with compact nonempty values and K a closed set. Then the sequence of sets $\{K^n\}_n$ defined by (10)satisfies*

$$\bigcap_{n=0}^{\infty} K^n \;= \overrightarrow{Viab}_G\,(K)$$

Proof of Proposition 2.13

We denote $K^\infty := \bigcap_{n=0}^\infty K^n$.

Let us prove by induction that the sets K^n are closed. Indeed, K is closed. Assume that K_{n-1} is already closed. Let x_i be a convergent sequence of K^n converging to x. Since $G(x_i) \cap K^{n-1} \neq \emptyset$, there exists $y_i \in G(x_i) \cap K^{n-1}$. Since G is upper semicontinuous at x with nonempty compact values, there exists a subsequence of the sequence $\{y_i\}_i$ converging to some $y \in G(x)$. Since K^{n-1} is closed, $y \in K^{n-1}$. Thus $G(x) \cap K^{n-1} \neq \emptyset$ implies that $x \in K^n$.

So the intersection K^∞ of the set K^n is also closed.

Let us prove that K^∞ is a discrete viability domain. Indeed, let us fix $x \in K^\infty$. Since $K^\infty \subset K^n$ and since the sets $G(x) \cap K^n$ are compact and nonempty for all n, the set $G(x) \cap K^\infty$ is also compact and nonempty. Thanks to Proposition 2.11, K^∞ is a discrete viability domain.

Let D be a closed discrete viability domain contained in K. We have to show that D is contained in K^∞. Since D is a discrete viability domain, $G(x) \cap D$ is nonempty for any $x \in D$. In particular, $G(x) \cap K$ is nonempty for any $x \in D$, so that D is contained in K^1.

In the same way, if D is contained in any K^n then D is contained in K^{n+1}. Thus, by induction, D is contained in K^∞.

This implies that K^∞ is the largest discrete viability domain for G contained in K, and so $K^\infty \;= \overrightarrow{Viab}_G\,(K)$.
Q.E.D.

Remark 2.3

In the same way as in Remark (2.1), for any closed K' satisfying $\overrightarrow{Viab}_G\,(K) \subset K' \subset K$, we have $\overrightarrow{Viab}_G\,(K') = \overrightarrow{Viab}_G\,(K)$. □

2.2.3 The semi-discrete scheme

In this subsection we prove that, under suitable regularity assumptions, we can approach $Viab_F(K)$ by discrete viability kernels $\overrightarrow{Viab}_{G_\varepsilon}\,(K)$.

We assume that the set-valued map F is bounded. Namely

$$\text{(11)} \qquad \exists M \geq 0,\;\; \forall x \in X,\; \forall y \in F(x),\; \|y\| \leq M$$

and we denote $\text{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$ the graph of F.

For any $\varepsilon > 0$, the approximation F_ε of F satisfies the following properties

$$(\mathbf{H0}) \quad F_\varepsilon : X \rightsquigarrow X \text{ is upper semicontinuous with convex compact nonempty values}$$

$$(\mathbf{H1}) \quad \mathrm{Graph}(F_\varepsilon(\cdot)) \subset \mathrm{Graph}(F(\cdot)) + \phi(\varepsilon)\mathcal{B} \text{ where } \lim_{\varepsilon\to 0^+} \phi(\varepsilon) = 0^+$$

$$(\mathbf{H2}) \quad \forall x \in X, \bigcup_{\|y-x\|\le M\varepsilon} F(y) \subset F_\varepsilon(x)$$

Assumption **H0** is the minimal regularity assumption we may do.

Assumptions **H1** and **H2** guaranty the "convergence" of F_ε to F. Even if F admits some discontinuities, the construction of such approximations is also available so long as F_ε "enlarges the discontinuities" of F (see figure 2.2.3).

Assumption **H2** insures the following crucial property : if $x(\cdot) \in S_F(x_0)$ then the sequence $x_n = x(n\varepsilon)$ is a solution to (8). Let us notice that the plain Euler scheme of the form

$$x_{n+1} \in x_n + \varepsilon F(x_n)$$

does not enjoy this property.

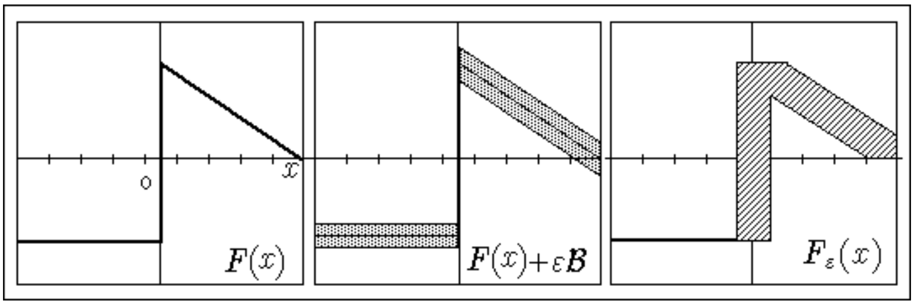

Figure 1: a - Initial set-valued map F,
b - Enlargement of F, **H2** is not satisfied,
c - $Graph(F_\varepsilon)(x) := \bigcup_{\|y-x\|\le z\varepsilon} F(y)$, (**H2**) is satisfied.

In practice, we choose the set-valued map F_ε as small as possible so that it satisfies **H2** and we verify that it satisfies **H0** and **H1**. We give examples of construction of F_ε below.

Theorem 2.14 *Let F be a Marchaud map which is bounded by M and K a closed set. Let - F_ε any approximation of F satisfying* **H0**, **H1**, **H2** *and set $G_\varepsilon(x) := x + \varepsilon F_\varepsilon(x)$.*

Then, for any $\varepsilon > 0$,

$$(12) \qquad Viab_F(K) \subset \overrightarrow{Viab}_{G_\varepsilon}(K)$$

and

$$\operatorname*{Lim}_{\varepsilon\to 0} \overrightarrow{Viab}_{G_\varepsilon}(K) = Viab_F(K) \subset \overrightarrow{Viab}_{G_\varepsilon}(K)$$

where Lim *denotes the Painlevé-Kuratowski limit (See Appendix 1, Definition 6.3).*

Let us point out that assumption **H2** is crucial in order to ensure inclusion (12).

The proof of Theorem 2.14 is given below.

Example in the Lispchitz case : Let us first explain how to construct the discretization F_ε when the set-valued map F is ℓ−Lipschitz and satisfies (11). Let us note that

$$\|y - x\| \le M\varepsilon \ \Rightarrow\ F(y) \subset F(x) + M\ell\varepsilon\mathcal{B}.$$

For guarantying that F_ε satisfies **H2**, a natural choice is

$$(13) \qquad \forall x \in X,\ F_\varepsilon(x) := F(x) + M\ell\varepsilon\mathcal{B}$$

Then F_ε satisfies **H2**. It also clearly satisfies **H0**, **H1** with $\phi(\varepsilon) := M\ell\varepsilon$.

So, from Theorem 2.14, we have:

Corollary 2.15 *Let F be a Marchaud and ℓ-Lipschitz set-valued map, bounded by some constant M (assumption (11)). Let K be a closed subset of X. If $F_\varepsilon(x) := F(x) + M\ell\varepsilon\mathcal{B}$ and $G_\varepsilon(x) := x + \varepsilon F_\varepsilon(x)$, then*

$$\operatorname*{Lim}_{\varepsilon\to 0} \overrightarrow{Viab}_{G_\varepsilon}(K) = Viab_F(K)$$

When F is Hölderian we refer to [58] and [54].

For proving Theorem 2.14, we need the following Propositions

Proposition 2.16 *Let F be a Marchaud map and K a closed subset of X. Let F_ε an approximation of F satisfying assumptions* **H0** *and* **H1** *and $G_\varepsilon(x) := x + \varepsilon F_\varepsilon(x)$.*

If V_ε is a closed viability domain for G_ε, then

$$V^\sharp := \operatorname*{Limsup}_{\varepsilon\to 0^+} V_\varepsilon$$

is a viability domain for F, where Limsup *denotes the Painlevé-Kuratowski upper limit.*

Proposition 2.17 *Let F be a Marchaud map and D a viability domain for F. Let F_ε an approximation of F satisfying assumptions* **H0** *and* **H2** *and $G_\varepsilon(x) := x + \varepsilon F_\varepsilon(x)$.*

Then, for any $\varepsilon > 0$, D is a discrete viability domain for G_ε.

Proof of Theorem 2.14

Let $D^\infty := \operatorname*{Limsup}_{\varepsilon\to 0^+} \overrightarrow{Viab}_{G_\varepsilon}(K)$. From Proposition 2.16, the set D^∞ is a viability domain for F. Moreover, D^∞ is contained in K because, for any $\varepsilon > 0$, $\overrightarrow{Viab}_{G_\varepsilon}(K) \subset K$. Thus D^∞ is contained in $Viab_F(K)$.

Conversely, $Viab_F(K)$ is a discrete viability domain for G_ε from Proposition 2.17. So $Viab_F(K)$ is contained in $\overrightarrow{Viab}_{G_\varepsilon}(K)$. In conclusion

$$\operatorname*{Limsup}_{\varepsilon\to 0^+} \overrightarrow{Viab}_{G_\varepsilon}(K) \subset Viab_F(K) \subset \operatorname*{Liminf}_{\varepsilon\to 0^+} \overrightarrow{Viab}_{G_\varepsilon}(K)$$

where Liminf denotes the Kuratowski lower limit. Since the upper-limit always contains the lower limit, we have finally proved Theorem 2.14. **Q.E.D.**

It remains to prove Propositions 2.16 and 2.17.

Proof of Proposition 2.16

For proving that $V^\sharp$ is a viability domain, we use characterization (ii) in Theorem 2.3. Let us consider $x \in V^\sharp$ and $p \in \mathcal{NP}_{V^\sharp}(x)$. We have to prove that there exists $y \in F(x)$ such that $< y, p > \leq 0$.

For that purpose, we can assume, without loss of generality, that the projection of $x + p$ onto $V^\sharp$ is reduced to the singleton $\{x\}$. Indeed, otherwise setting $p' := \frac{p}{2}$, the projection of $x + p'$ onto $V^\sharp$ is actually reduced to $\{x\}$ (See Proposition 6.7 Appendix 1). The proof below yields the existence of $y \in F(x)$ such that $< y, p' > \leq 0$, which implies $< y, p > \leq 0$.

From the very definition of the upper limit, one can find $\varepsilon_n \to 0$, $x_n \to x$ with $x_n \in V_{\varepsilon_n}$. Let us consider z_n belonging to the projection of $(x + p)$ onto V_{ε_n}.

First let us prove that

$$\lim_{n\to\infty} z_n = x \tag{14}$$

From the very definition of the projection on V_{ε_n} we have

$$\|z_n - (x + p)\| \leq \|x_n - (x + p)\| \tag{15}$$

In particular, the sequence $\{z_n\}_n$ is bounded. Let z be a cluster point of the sequence $\{z_n\}_n$ and consider $\{z_{n_k}\}_k$ a subsequence of $\{z_n\}_n$ converging to z.

Passing to the limit in (15) yields $\|z - (x + p)\| \leq \|p\|$. Since the projection of $(x + p)$ onto $V^\sharp$ is the singleton $\{x\}$, this implies $z = x$ and (14) ensues.

Let us now recall that V_{ε_n} is discretely viable for G_{ε_n}. Thus there exists $y_n \in F_{\varepsilon_n}(z_n)$ such that

$$z_n + \varepsilon_n y_n \in G_{\varepsilon_n}(z_n) \cap V_{\varepsilon_n}$$

Since z_n is a projection of $(x + p)$ onto V_{ε_n}, we have

$$\|z_n - (x+p) + \varepsilon_n y_n\| \geq \|z_n - (x+p)\|$$

Expanding the square of the two terms, subtracting the right one and dividing by ε_n yields

$$\varepsilon_n \|y_n\|^2 + 2 < y_n, z_n - x - p > \geq 0 \tag{16}$$

From assumption **H1**, there exists some subsequence - still denoted y_n - such that $\lim_{n\to\infty} y_n = y$ and $y \in F(x)$. Passing to the limit in (16) provides $< p, y > \leq 0$.

So, $V^\sharp$ is viable for F. **Q.E.D.**

Proof of Proposition 2.17

Let $x \in D$ and consider any solution $x(\cdot) \in S_F(x)$ viable in D. We know that

$$\forall t \geq 0, \;\; x(t) = x + \int_0^t x'(\tau)d\tau.$$

Since $x'(\tau) \in F(x(\tau))$ for almost all $\tau \in [0, \varepsilon]$, we have, thanks to assumption (11)

$$\text{for every } \tau \geq 0, \quad \|x(\tau) - x\| \leq \tau M \leq \epsilon M \tag{17}$$

On the other hand, for any $t \in [0, \varepsilon]$, $x(t) - x = \int_0^t x'(\tau)d\tau$ where

$$x'(\tau) \;\in\; F(x(\tau)) \;\subset\; \bigcup_{\|y-x\|\leq M\epsilon} F(y) \;\subset F_\varepsilon(x)$$

from (17) and assumption **H2**.

We claim that

$$x(\varepsilon) \in x + \varepsilon F_\varepsilon(x) = G_\varepsilon(x) \tag{18}$$

Indeed, since, from assumption **H0**, $F_\varepsilon(x)$ is convex and compact, $G_\varepsilon(x)$ is also convex and compact. So, applying the Separation Theorem, for proving (18) it is sufficient to prove that

$$\forall p \in X, \qquad \sup_{v \in x + \varepsilon F_\varepsilon(x)} < v, p > \geq < x(\varepsilon), p > \tag{19}$$

This is true because, for almost every $\tau \in [0, \varepsilon]$, $x'(\tau) \in F_\varepsilon(x(\tau))$ so that

$$\sup_{v \in F_\varepsilon(x)} < v, p > \;\geq\; < x'(\tau), p > .$$

After integration, we get (19).

So we have proved that

$$x(\varepsilon) \;\in\; G_\varepsilon(x) \cap D \neq \emptyset.$$

Hence D is a discrete viability domain for G_ε. **Q.E.D.**

2.3 The fully discrete viability kernel algorithm

To implement this algorithm we have to associate with G_ε suitable finite set-valued maps defined on finite sets.

For that purpose, we introduce a grid X_h of X, associated with any $h \in \boldsymbol{R}$, satisfying

$$\left\{ \begin{array}{ll} i) & \text{the set } X_h \text{ has a finite intersection} \\ & \text{with any compact of } X \\ ii) & \forall h > 0, \, \forall x \in X, \, \exists x_h \in X_h, \, \|x - x_h\| \leq h \end{array} \right. \tag{20}$$

2.3.1 Approximation of $Viab_F(K)$ by Finite Discrete Viability Kernels

Now we are dealing with systems which are not only discrete but finite.

With any closed set K we associate its "projection onto the grid" defined by

$$K_h := (K + h\mathcal{B}) \cap X_h.$$

The next Proposition is similar to Proposition 2.13 for finite sets

Proposition 2.18 *Let $\Gamma_{\varepsilon,h}$: $X_h \rightsquigarrow X_h$ be a set-valued map with finite nonempty values and K_h a finite subset of X_h.*

Let $\{K^n_{\varepsilon,h}\}_n$ be the sequence of subsets of K_h defined as follows

$$\begin{cases} K^0_{\varepsilon,h} := K_h \\ K^{n+1}_{\varepsilon,h} := \{z_h \in K^n_{\varepsilon,h} \text{ such that: } \Gamma_{\varepsilon,h}(z_h) \cap K^n_{\varepsilon,h} \neq \emptyset\} \end{cases}$$

Then,

(21)
$$\exists n_{\varepsilon,h} \in \mathbb{N} \ : \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}(K_h) = K^{n_{\varepsilon,h}}_{\varepsilon,h}$$

The relevance of this Proposition consists in the fact that it is enough to compute a finite number of stages - which is upper bounded by the number of points of K_h. Furthermore the nonemptiness of an intersection of finite sets is checked instead of an intersection of infinite sets as in Proposition 2.13.

Let $F : X \rightsquigarrow X$ be a Marchaud map and K a closed subset of X. Let us consider the family $(F_\varepsilon)_\varepsilon$ satisfying assumptions **H0**, **H1**, **H2** and set $G_\varepsilon := x + \varepsilon F_\varepsilon(x)$.

A finite set-valued map $\Gamma_{\varepsilon,h} : X_h \rightsquigarrow X_h$ with nonempty finite values, will be a "good discretization" of the set-valued map G_ε if, for any $\varepsilon > 0$ and for any $h > 0$, it satisfies

$$(\mathbf{H3}) \quad \mathrm{Graph}(\Gamma_{\varepsilon,h}(\cdot)) \subset \mathrm{Graph}(G_\varepsilon(\cdot)) + \psi(\varepsilon,h)\mathcal{B} \text{ where } \lim_{\varepsilon\to 0^+,\ \frac{h}{\varepsilon}\to 0^+} \frac{\psi(\varepsilon,h)}{\varepsilon} = 0^+$$

$$(\mathbf{H4}) \quad \forall x_h \in X_h, \quad \bigcup_{\|y-x_h\|\leq h} [G_\varepsilon(y) + h\mathcal{B}] \cap X_h \ \subset \ \Gamma_{\varepsilon,h}(x_h)$$

Assumptions (**H3**) and (**H4**) extend to fully discretization assumptions (**H1**) and (**H2**). Actually assumption (**H4**) plays the same role as assumption (**H2**) for the semi-discretization.

Moreover the condition $\lim_{\varepsilon\to 0^+,\ \frac{h}{\varepsilon}\to 0^+} \frac{\psi(\varepsilon,h)}{\varepsilon} = 0$ expresses the comptability between time and space steps discretization (consistency of the explicit scheme).

We explain below how to construct in practice a discretization $\Gamma_{\varepsilon,h}$ from the set-valued map G_ε. In general, we have to choose $\Gamma_{\varepsilon,h}$ as small as possible satisfying **H3** provided that it verifies **H4**.

Theorem 2.19 *Let F_ε be a "good approximation" of a Marchaud map F satisfying* **H0***,* **H1** *and* **H2** *and $G_\varepsilon(x) := x + \varepsilon F_\varepsilon(x)$.*

Let $\Gamma_{\varepsilon,h} : X_h \rightsquigarrow X_h$ be a fully discrete dynamic and set $K_h := (K + h\mathcal{B}) \cap X_h$. Let $\Gamma_{\varepsilon,h}$ be a "good discretization" of G_ε satisfying assumptions **H3** *and* **H4***.*

Then

(22)
$$Viab_F(K) \ \subset \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}(K_h) + h\mathcal{B}$$

and

(23)
$$Viab_F(K) \ = \ \mathop{\mathrm{Lim}}_{\varepsilon\to 0^+,\ \frac{h}{\varepsilon}\to 0^+} \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}(K_h)$$

In practice, we can often choose $\varepsilon^2 := h$ or $\varepsilon^{\frac{3}{2}} := h$.

Example 2.3 (The Lipschitz case)

When F is a $\ell-$Lipschitz set-valued map, we can construct the set-valued map $\Gamma_{\varepsilon,h}$ as follows. Let us recall that

$$G_\varepsilon(x) := x + \varepsilon(F(x) + M\ell\varepsilon\mathcal{B})$$

so that G_ε is a $(1 + \ell\varepsilon)-$Lipschitz set-valued map. Setting

$$\Gamma_{\varepsilon,h}(x_h) := \left[x_h + \varepsilon F(x_h) + (2h + \ell\varepsilon h + M\ell\varepsilon^2)\mathcal{B}\right] \cap X_h$$

yields a discretization which satisfies **H3** and **H4**. □

Theorem 2.19 is a consequence of the following Propositions 2.20, 2.21 and Lemma 2.22.

Proposition 2.20 *Assume that F_ε satisfies* **H0** *and* **H1** *and that $\Gamma_{\varepsilon,h}$ satisfies* **H3**. *Let $D_{\varepsilon,h}$ a family of finite subsets of X_h.*

If the sets $D_{\varepsilon,h}$ are discrete viability domains for $\Gamma_{\varepsilon,h}$, then

$$D^\sharp := \operatorname*{Limsup}_{\varepsilon\to 0^+,\ \frac{h}{\varepsilon}\to 0^+} D_{\varepsilon,h}$$

is a viability domain for F.

Proof of Proposition 2.20

The proof starts exactly as the proof of Proposition 2.16. Let $\varepsilon_n \to 0^+$ and h_n be such that $\frac{h_n}{\varepsilon_n} \to 0^+$. Let x_n belong to D_{ε_n,h_n} converging to some $x \in D^\sharp$. Let p be a proximal normal to $D^\sharp$ at x. Following Proposition 6.7 given in Annex 1, we can always assume that the projection of $x+p$ onto $D^\sharp$ is reduced to $\{x\}$ so that, if z_n belongs to the projection of $x+p$ onto D_{ε_n,h_n}, then the sequence $\{z_n\}_n$ converges to x.

Since $\{D_{\varepsilon_n,h_n}\}_n$ is a sequence of discrete viability domains for $\Gamma_{\varepsilon_n,h_n}$, there exists y_n such that $z_n+\varepsilon_n y_n$ belongs to $D_{\varepsilon_n,h_n} \cap \Gamma_{\varepsilon_n,h_n}(z_n)$. As in the proof of Proposition 2.16, we have

$$\varepsilon_n\|y_n\|^2 + 2 < y_n, z_n - x - p > \ \geq\ 0 \tag{24}$$

Let us now prove that $\{y_n\}_n$ converges, up to a subsequence, to some $y \in F(x)$. Indeed, from assumption **H3** and from the very definition of G_ε, there exist z'_n and $y'_n \in F_{\varepsilon_n}(z'_n)$ such that

$$\|(z'_n, z'_n + \varepsilon_n y'_n) - (z_n, z_n + \varepsilon_n y_n)\| \leq \psi(\varepsilon_n, h_n)$$

A simple computation yields

$$\|y'_n - y_n\| \ \leq\ \frac{2\psi(\varepsilon_n, h_n)}{\varepsilon_n}$$

which converges to 0. From assumption **H1**, the sequence $\{y'_n\}_n$ converges, up to a subsequence, to some $y \in F(x)$, so that the sequence $\{y_n\}_n$ also converges to y. Moreover, inequality (24) gives

$$< y, p > \leq 0.$$

This implies that $D^\sharp$ is a viability domain for F. **Q.E.D.**

Proposition 2.21 *Assume that F_ε satisfies* **H0** *and* **H2** *and that $\Gamma_{\varepsilon,h}$ satisfies* **H4**. *If a closed set D is a viability domain for F, then $D_h := (D + h\mathcal{B}) \cap X_h$ is a discrete viability domain for $\Gamma_{\varepsilon,h}$.*

Proof of Proposition 2.21 :

Let $x_h \in D_h$. There exists $x \in D$ such that $\|x - x_h\| \leq h$. Since D is viability domain for F, from Proposition 2.17, D is a viability domain for G_ε. So $y \in G_\varepsilon(x) \cap D$ exists. Let $y_h \in X_h$ such that $\|y - y_h\| \leq h$. Let us note that $y_h \in D_h$. Moreover, from assumption **H4**

$$v_h \in [G_\varepsilon(x) + h\mathcal{B}] \cap X_h \ \subset \bigcup_{\|x'-x_h\|\leq h} [G_\varepsilon(x') + h\mathcal{B}] \cap X_h \ \subset\ \Gamma_{\varepsilon,h}(x_h).$$

So y_h belongs to $D_h \cap \Gamma_{\varepsilon,h}(x_h)$ and D_h is a discrete viability domain for $\Gamma_{\varepsilon,h}$. **Q.E.D.**

Lemma 2.22 *Let A be a closed subset of X and define $A_h := (A + h\mathcal{B}) \cap X_h$. Then*

$$A \ =\ \operatorname*{Liminf}_{h\to 0^+} A_h$$

The proof of Lemma 2.22 is given in Appendix

Proof of Theorem 2.19

Let us set $K^\infty := \operatorname*{Limsup}_{\varepsilon\to 0^+,\ h/\varepsilon\to 0^+} \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}(K_h)$. Since K^∞ is the upper limit of discrete viability domains for the set-valued map $\Gamma_{\varepsilon,h}$, Proposition 2.20 states that K^∞ is a viability domain for F. Moreover, K^∞ is contained in K because $\overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}(K_h) \subset K_h \subset K + h\mathcal{B}$. Thus, K^∞ is contained in $Viab_F(K)$.

Since $(Viab_F(K) + h\mathcal{B}) \cap X_h$ is a viability domain for $\Gamma_{\varepsilon,h}$, Proposition 2.20 states that $(Viab_F(K) + h\mathcal{B}) \cap X_h$ is a discrete viability domain for $\Gamma_{\varepsilon,h}$.

From Lemma 2.22, we have

$$Viab_F(K) \;=\; \operatorname*{Liminf}_{h\to 0^+}\ (Viab_F(K)+hB)\cap X_h \;\subset\; \operatorname*{Liminf}_{\varepsilon\to 0^+,\ \frac{h}{\varepsilon}\to 0^+}\ \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}\,(K_h)$$

So we have finally proved that

$$\operatorname*{Limsup}_{\varepsilon\to 0^+,\ \frac{h}{\varepsilon}\to 0^+}\ \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}\,(K_h)\ \subset Viab_F(K)\ \subset\ \operatorname*{Liminf}_{\varepsilon\to 0^+,\ \frac{h}{\varepsilon}\to 0^+}\ \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}\,(K_h)$$

Since the upper-limit is always contained in the lower limit, the proof of Theorem 2.19 is complete. **Q.E.D.**

2.3.2 Refinement Principle

We keep the notations of the previous subsection. Let us consider sequences ε_p and $\frac{h_p}{\varepsilon_p}$ converging to 0^+ and set

$$\Gamma_p := \Gamma_{\varepsilon_p,h_p},\quad K_{h_p} := (K+h_p\mathcal{B})\cap X_{h_p}.$$

Theorem 2.19 states that $\operatorname*{Lim}_{p\to\infty}\ \overrightarrow{Viab}_{\Gamma_p}\,(K_{h_p}) = Viab_F(K)$ and we have already indicated how to compute the discrete viability kernel $\overrightarrow{Viab}_{\Gamma_p}\,(K_{h_p})$. We now show that it is not necessary to restart the calculus of the discrete viability kernel from the whole set K_{h_p} each time the step h_p changes.

In practice, we consider

(25) $$h_p := \frac{1}{2^p},\ \ \varepsilon_p := \sqrt{h_p}\,,\quad \text{and}\ \ X_{h_p} := h_p\mathbb{Z}^N$$

where $\mathbb{Z}^N$ is an integer lattice of $\mathbf{R}^N$ closed under addition and subtraction. For such a choice of steps, it is easy to deduce the grid $X_{h_{p+1}}$ from the grid X_{h_p} when refining.

Theorem 2.23 (Refinement Principle) *Suppose that the assumptions of Theorem 2.19 are fulfilled and define*

$$\begin{cases} \tilde{D}_1 & := (K+h_1\mathcal{B})\cap X_{h_1} := K_{h_1} \\ \tilde{D}_{p+1} & := \overrightarrow{Viab}_{\Gamma_{p+1}}\left[\left(\tilde{D}_p + (h_p+h_{p+1})\mathcal{B}\right)\cap K_{h_{p+1}}\right] \end{cases}$$

Then

$$\operatorname*{Lim}_{p\to\infty}\ \tilde{D}_p \;=\; Viab_F(K).$$

Proof of Theorem 2.23

Let us notice that $\tilde{D}_p \subset Viab_{\Gamma_p}(K_p)$. We first prove by induction that

(26) $$[Viab_F(K)+h_p\mathcal{B}]\cap X_{h_p} \subset \tilde{D}_p.$$

It is clearly true for $p=1$. Assume that this inclusion holds true for some p. Then

$$Viab_F(K) \subset [Viab_F(K)+h_p\mathcal{B}]\cap X_{h_p} + h_p\mathcal{B} \subset \tilde{D}_p + h_p\mathcal{B},$$

so that

$$[Viab_F(K)+h_{p+1}\mathcal{B}]\cap X_{h_{p+1}} \subset \left[\tilde{D}_p + (h_p+h_{p+1})\mathcal{B}\right]\cap K_{h_{p+1}}.$$

From Proposition 2.21, $[Viab_F(K)+h_{p+1}\mathcal{B}]\cap X_{h_{p+1}}$ is a discrete viability domain for Γ_{p+1}. Thus

$$\begin{array}{l}[Viab_F(K)+h_{p+1}\mathcal{B}]\cap X_{h_{p+1}} \\ \quad \subset \overrightarrow{Viab}_{\Gamma_{p+1}}\left(\left[\tilde{D}_p + (h_p+h_{p+1})\mathcal{B}\right]\cap K_{h_{p+1}}\right) \;=\; \tilde{D}_{p+1}.\end{array}$$

By induction, we have proved inclusion (26). From Lemma 2.22,

$$Viab_F(K) = \ \operatorname*{Liminf}_{p}\ [Viab_F(K)+h_p\mathcal{B}]\cap X_{h_p} \subset\ \operatorname*{Liminf}_{p}\ \tilde{D}_p.$$

Since $\tilde{D}_p$ is contained in $\overrightarrow{Viab}_{\Gamma_p}\,(K_p)$, Theorem 2.19 yields that the upper-limit of the $\tilde{D}_p$ is contained in $Viab_F(K)$. So we have finally proved that

$$\operatorname*{Limsup}_{p}\ \tilde{D}_p \subset Viab_F(K) \subset\ \operatorname*{Liminf}_{p}\ \tilde{D}_p.$$

Since the upper-limit always contains the lower-limit, the proof is complete. **Q.E.D.**

2.3.3 Outline of the Algorithm

PROCEDURES OF CONSTRUCTION OF FULLY DISCRETE SETS (parameter p)

Definition of the grid $X_{h_p} := h_p \mathbb{Z}^N$, $h_p = 2^{-p}$, $\varepsilon_p := \sqrt{h_p}$.
Definition of $K_{h_p} := (K + h_p \mathcal{B}) \cap X_{h_p}$.
Definition of F_p, G_p and Γ_p satisfying assumptions **H0 ... H4**.

INITIALIZATION

$p \leftarrow 1$, $D_1^0 \leftarrow K_{h_1}$

MAIN LOOP

For $p := 1$ **to** $\bar{p}$ **do**
  $n := 0$
  Repeat
    $D_p^{n+1} \leftarrow \{x \in D_p^n \mid \Gamma_p(x) \cap D_p^n \neq \emptyset\}$
    $n \leftarrow n + 1$
  until $D_p^{n+1} = D_p^n$
  Set $D_p^\infty \leftarrow D_p^n$
  If $p < \bar{p}$ **then** $D_{p+1}^0 \leftarrow \left[D_p^\infty + 3h_{p+1}\mathcal{B}\right] \cap K_{h_{p+1}}$ { REFINEMENT}

RETURN

$D_{\bar{p}}^\infty$ $\{D_{\bar{p}}^\infty$ is the approached Viability Kernel at step $\bar{p}\}$

2.4 Application to a non-autonomous target problem

Caratheodory has described in [24] many examples of control problems. The problem we study here is derived from the so called Zermelo's problem. Our aim is to illustrate how barriers can be characterized and computed as the boundary of the Viability Kernel.

The problem can be summed up as follows. Let $\varphi(x_1, x_2)$ be the water current of a river at position (x_1, x_2). The current is assume to be decreasing with the distance from the middle axis of the river

$$\varphi(x_1, x_2) := (1 - a x_2^2, 0).$$

A swimmer aims at reaching an island but if he passes over some "waterfall" beyond the island, he completely fails. He is no more allowed to pass over the banks of the river. The swimmer has his own dynamic and he can swim in any direction at a speed $c(t)$. Taking into account that the swimmer is getting tired as time increases, we suppose that the function $c(\cdot)$ is decreasing. Let us take for instance $c(t) := \frac{1}{1+0.25t}$.

Also the global dynamic is given by

$$(x_1'(t), x_2'(t)) \quad = \quad \varphi(x_1(t), x_2(t)) + c(t)v(t) \tag{27}$$

with $v(t) \in \mathcal{B}$. The system is non autonomous and depends explicitly on time t. Let us introduce the variable y which represents the time and consider the following dynamic

$$(x_1'(t), x_2'(t), y'(t)) \in (\varphi(x_1(t), x_2(t)) + c(t)\mathcal{B}, \{1\}) := \Phi(x_1(t), x_2(t), y(t)),$$

Let $C = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1^2 + x_2^2 \leq 0.44\}$ be the (closed) island, $K = [-6, 2] \times [-5, 5] \times \boldsymbol{R}^+$ be the river and $a = \frac{1}{25}$.

These values of parameters are chosen such that the swimmer can go upstream close to the bank for low values of t - when the swimmer is not yet too much tired - since $c(t)$ can be greater than $\varphi(x_1(t), x_2(t))$ whenever $x_2(t) \geq x_2^\star = (\frac{1-c(t)}{a})^{\frac{1}{2}}$.

Our aim is to compute the region of points from which the swimmer can reach the island in a finite time while remaining in K. The following Proposition allows to compute it thanks to the Viability Kernel Algorithm.

Proposition 2.24 *Let Φ be a Marchaud map and K a closed subset of X. Assume that $Viab_\Phi(K) = \emptyset$. Let C be a closed target contained in K and set*

$$F(x) := \begin{cases} \Phi(x) & \text{if } x \notin C \\ \overline{Co}\,[\Phi(x) \cup \{0\}] & \text{if } x \in C \end{cases}$$

The VIability Kernel is precisely the set

$$Viab_\Phi(K \times \boldsymbol{R}^+) = \{x \in K \text{such that } \exists x(\cdot) \in S_\Phi(x),\ \exists \tau < \infty,\ x(t) \in K \forall t \leq \tau,\ x(\tau) \in C\}$$

In this example one can easily show that, if there is no island, since the swimmer is getting more and more tired, any trajectory would reach and pass over the "waterfall". In other words, $Viab_{\Phi}(K) = \emptyset$ and then we can apply Proposition 2.24 with $x = (x_1, x_2)$ and

$$F(x,y) := \begin{cases} \Phi(x_1, x_2, y) & \text{if } x \notin C \\ \overline{Co}(\Phi(x_1, x_2, y), \{0,0,0\}) & \text{if } x \in C \end{cases}$$

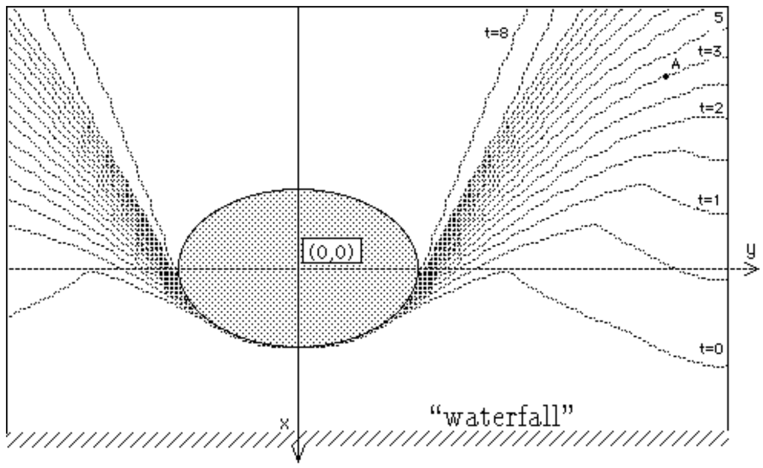

Figure 2: Intersections of the Viability Kernel with equidistant t-constant planes

Figure 2.4 shows the sections of the boundary of the Viability Kernel with equidistant t-constant planes.

Figure 2.4 shows the boundary of the Viability Kernel which is the upper bound of the Viability Kernel. It can be interpreted as follows: any (x_1^0, x_2^0, t^0) belonging to this boundary is such that, standing at position (x_1^0, x_2^0) at a time $t \leq t^0$, the swimmer can reach the target in a finite time. If he is in late and passes at point (x_1^0, x_2^0) at any time $t > t^0$, he will never be able to reach the island and, so, he will disappear in a finite time. If he is at position (x_1^0, x_2^0) exactly at a time t^0, he will be able to reach the island providing that he never deviates from the limit trajectory which follows the boundary of the viability kernel.

This corresponds to a semi permeable property of the boundary of the Viability Kernel (see [53], [57]).

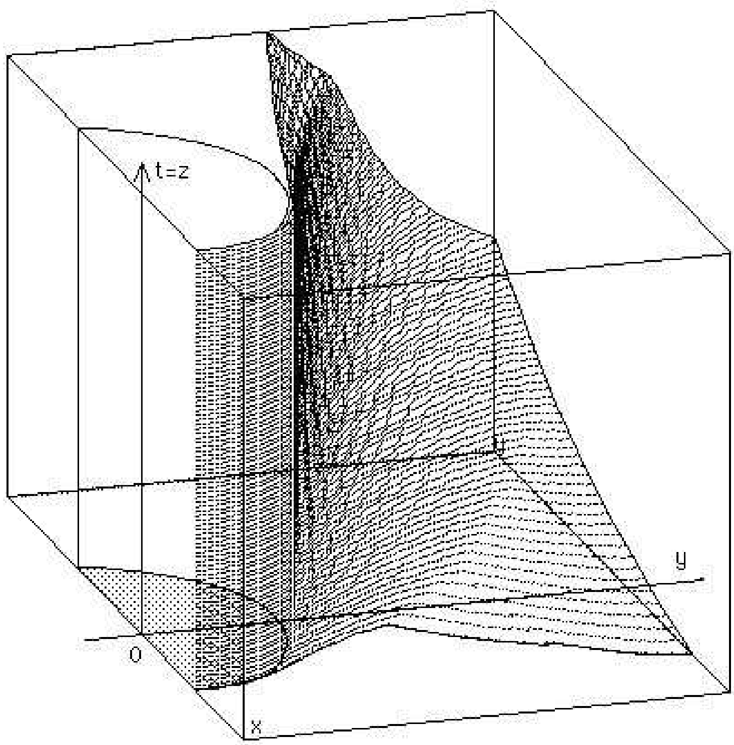

Figure 3: Viability Kernel for a non autonomous target.

On this figure we can observe for instance that, if the swimmer pass through point A at a time $t > 3$ then he will never be able to reach the island because he is already too tired. If he pass at a time $t \leq 3$, he can reach the island.

3 The Minimal Time Function in Control Theory

We consider the following controlled system

$$(28)\qquad \begin{cases} x'(t) = f(x(t), v(t)), \;\; v(t) \in V \\ \text{for almost every } t \geq 0 \end{cases}$$

where the state variable x belongs to a finite dimensional vector space X and V is a compact subset of some finite dimensional vector space.

Let $C \subset X$ be a closed target and $K \subset X$ be a closed set of state constraints.

Our propose is to characterize and to provide numerical schemes for computing the Minimal Time function ϑ_C^K defined, for any initial condition x_0, by:

$$(29)\qquad \vartheta_C^K(x_0) := \inf \left\{ \tau \geq 0 \mid \begin{array}{c} \exists x(\cdot) \text{ solution to (28) with } x(0) = x_0 \\ x(\tau) \in C \text{ and } x(t) \in K \;\; \forall t \in [0, \tau] \end{array} \right\}$$

Conventionnaly we set $\vartheta_C^K(x) = 0$ if $x \in C$.

Roughly speaking, $\vartheta_C^K(x)$ is the first time such that, starting from position x, the state of the system can reach the target C while remaining in the set of state constraints K. Note that ϑ_C^K takes values in $\boldsymbol{R}^+ \cup \{+\infty\}$ and that $\vartheta_C^K(x) = +\infty$ if no solution, starting from x, reaches the target C or if any solution, starting from x, leaves the constraints K before reaching the target.

In the sequel we denote by $dom(\vartheta_C^K)$ the domain of ϑ_C^K: $dom(\vartheta_C^K) := \{x \in X \mid \vartheta_C^K(x) < +\infty\}$.

We recall now a regularity result for ϑ_C^K (see for instance [27]):

Proposition 3.1 *If $f : X \times V \to X$ is continuous, then the Minimal Time function ϑ_C^K satisfies the following properties:*

a - for all $x \in dom(\vartheta_C^K)$, an optimal solution $x(\cdot) \in \mathcal{S}_F(x)$ exists such that

$$\forall t \in [0, \vartheta_C^K(x)), \;\; x(t) \in K \;\text{ and }\; x(\vartheta_C^K(x)) \in C.$$

b - the Minimal Time function $\vartheta_C^K(\cdot) : X \to \boldsymbol{R}^+ \cup \{+\infty\}$ is lower semicontinuous on K.

In particular, the epigraph[6] of ϑ_C^K is closed. We denote it $\mathcal{E}pi(\vartheta_C^K)$. It is a subset of $X \times \boldsymbol{R}^+$. We shall denote by (x, y), with $x \in X$ and $y \in \boldsymbol{R}^+$ any element of $X \times \boldsymbol{R}^+$.

We first characterize the epigraph of the Minimal Time function as a viability kernel of a closed set for an extended dynamic and we deduce from this characterization a numerical scheme for computing ϑ_C^K.

3.1 Characterization of the Minimal Time function

As usually, we set $F(x) := \bigcup_u f(x, u)$ and we replace control system (28) by differential inclusion (5). Then, if f is continuous, F is Marchaud.

Let us define the expanded set-valued map $\Phi : X \times \boldsymbol{R} \rightsquigarrow X \times \boldsymbol{R}$ by:

$$(30)\qquad \Phi(x, y) = \begin{cases} F(x) \times \{-1\} & \text{if } \; x \notin C \\ \overline{Co}((F(x) \times \{-1\}) \cup (\{0\} \times \{0\})) & \text{if } \; x \in C \end{cases}$$

and consider the differential inclusion

$$(31)\qquad (x'(t), y'(t)) \in \Phi(x(t), y(t)), \;\; a.e. \; t \geq 0$$

If F is a Marchaud map, Φ is also a Marchaud map.

Theorem 3.2 *Let $F : X \rightsquigarrow X$ be a Marchaud map, K and C be two closed subsets of X. We set $\mathcal{H} := K \times \boldsymbol{R}^+$.*

Then the epigraph of $\vartheta_C^K(\cdot)$ is the viability kernel of $\mathcal{H}$ for Φ:

$$Viab_\Phi(\mathcal{H}) = \mathcal{E}pi(\vartheta_C^K)$$

[6] Recall that the epigraph of a map $\phi : X \to R^+$ is the set $\{(x, y) \in X \times R \mid \phi(x) \leq y\}$.

Proof of Theorem 3.2

Let $(x,y) \in Viab_\Phi(\mathcal{H})$. We want to prove that $y \geq \vartheta_C^K(x)$. We can assume that $x \notin C$ since if $x \in C$, then $y \geq \vartheta_C^K(x) = 0$.

Let $(x(\cdot), y(\cdot)) \in \mathcal{S}_\Phi(x,y)$ be a solution which forever remains in $\mathcal{H}$. We denote by $\theta_C(x(\cdot)) \in [0,+\infty]$ the first time that the solution $x(\cdot)$ reaches the target C:

$$\theta_C(x(\cdot)) := \inf\{t \geq 0 \mid x(t) \in C\}$$

From the very definition of Φ, on $[0, \theta_C(x(\cdot))]$, $(x(\cdot), y(\cdot))$ is solution to the differential inclusion

$$\begin{cases} x'(t) \in F(x(t)) \\ y'(t) = -1 \\ x(0) = x,\ y(0) = y \end{cases}$$

Let us now recall that $(x(\cdot), y(\cdot))$ remains in $\mathcal{H} := K \times \boldsymbol{R}^+$. Thus $y(t) = y - t$ is not smaller than 0 on $[0, \theta_C(x(\cdot))]$ and one has proved that $y \geq \theta_C(x(\cdot))$. Moreover, the solution $x(\cdot)$ remains in K on $[0, \theta_C(x(\cdot))]$. So finally

$$\vartheta_C^K(x) \leq \theta_C(x(\cdot)) \leq y$$

In particular, $x \in dom(\vartheta_C^K)$ and $(x,y) \in \mathcal{E}pi(\vartheta_C^K)$. So we have proved that

$$Viab_\Phi(\mathcal{H}) \subset \mathcal{E}pi(\vartheta_C^K)$$

Conversely, let $(x,y) \in \mathcal{E}pi(\vartheta_C^K)$. Since $y \geq \vartheta_C^K(x)$, then $\vartheta_C^K(x) < +\infty$ and $x \in dom(\vartheta_C^K)$. If $x \in C$, then $(x,y) \in Viab_\Phi(\mathcal{H})$ because $(0,0) \in \Phi(x,y)$.

Assume now that $x \notin C$. From Proposition 3.1, there is a solution $x(\cdot) \in \mathcal{S}_F(x)$ such that

$$x(t) \in K,\ \ \forall t \in [0, \vartheta_C^K(x)] \ \text{ and } \ x(\vartheta_C^K(x)) \in C$$

Then let us define $x^\star(\cdot)$ and $y^\star(\cdot)$ as follows:

$$(x^\star(t), y^\star(t) = \begin{cases} (x(t), y - t), & \forall t \leq \vartheta_C^K(x) \\ (x(\vartheta_C^K(x)), y - \vartheta_C^K(x)) & \forall t \geq \vartheta_C^K(x) \end{cases}$$

Since $\vartheta_C^K(x) \leq y$, then $\forall t \geq 0,\ \ y(t) \geq 0$. Since $\forall t \geq \vartheta_C^K(x)$, $y'^\star(t) = 0$ and since $x^\star(t) \in C$, then it is clear that the pair $(x^\star(\cdot), y^\star(\cdot))$ is a solution to system (31) starting from (x,y) and viable in $\mathcal{H}$.

Thus we have proved that $(x,y) \in Viab_\Phi(\mathcal{H})$. So

$$Viab_\Phi(\mathcal{H}) \ = \ \mathcal{E}pi(\vartheta_C^K)$$

Q.E.D.

3.2 Numerical approximation of the Minimal Time function

We now explain how the computation of $Viab_\Phi(\mathcal{H})$ leads to a numerical scheme for approximating ϑ_C^K.

3.2.1 Time discretization

Following the approximation method developped in section 2.2, we consider the finite difference inclusion system associated with $\Phi : X \times \boldsymbol{R}^+ \rightsquigarrow X \times \boldsymbol{R}^+$.

We have to define a time discretization $\Phi_\varepsilon : X \times \boldsymbol{R}^+ \rightsquigarrow X \times \boldsymbol{R}^+$ which satisfies **H0**, **H1** and **H2**. Let us define

$$\Phi_\varepsilon(x,y) \ := \ \begin{cases} \{F(x) + (M\ell\varepsilon)\mathcal{B}_X\} \times \{-1\} & \text{if } d_C(x) > M\varepsilon \\ \overline{Co}\,[\{(0,0)\} \ \cup \ \{F(x) + (M\ell\varepsilon)\mathcal{B}_X\} \times \{-1\}] & \text{otherwise} \end{cases}$$

Lemma 3.3 *If F is $\ell-$Lipschitz and bounded by some constant M (satisfying (11)), Φ_ε satisfies assumptions* **H0**, **H1** *and* **H2**.

Proof - Assumption **H0** is clearly fulfilled because F is Marchaud and Lipschitz. Let us show that **H1** is fulfilled with $\phi(\varepsilon) := 2M\ell\varepsilon$.

Let $((x,y),(v_x^\varepsilon, v_y^\varepsilon))$ belong to Graph(Φ_ε). There are two cases

- either $d_C(x) > M\varepsilon$. Then $((x,y),(v_x^\varepsilon, v_y^\varepsilon)) \ \in \ \text{Graph}(\Phi) + M\ell\varepsilon\mathcal{B}_{X\times R}$ because $(v_x^\varepsilon, v_y^\varepsilon) \in \{F(x) + M\ell\varepsilon\mathcal{B}\} \times \{-1\}$,

- or $d_C(x) \le M\varepsilon$. Then there exists some $x' \in C$ such that $\|x' - x\| \le M\varepsilon$. Thus

$$\begin{array}{rl}(v_x^\varepsilon, v_y^\varepsilon) \in & \overline{Co}\,[\{(0,0)\} \,\cup\, \{F(x) + (M\ell\varepsilon)\mathcal{B}_X\} \times \{-1\}] \subset \\ & \overline{Co}\,[\{(0,0)\} \,\cup\, \{F(x') + (2M\ell\varepsilon)\mathcal{B}_X\} \times \{-1\}] \subset \Phi(x', y) + 2M\ell\varepsilon\mathcal{B}_{X\times \boldsymbol{R}}\end{array}$$

so that $((x,y),(v_x^\varepsilon, v_y^\varepsilon))$ belongs to $\text{Graph}(\Phi) + \phi(\varepsilon)\mathcal{B}_{X\times \boldsymbol{R}}$.

For proving **H2**, let us fix $(x_0, y_0) \in X \times \boldsymbol{R}^+$. Let $x \in X$ be such that $\|x - x_0\| \le M\varepsilon$. Let us show that $\Phi(x,y) \subset \Phi_\varepsilon(x_0, y_0), \forall y \in \boldsymbol{R}^+$. Indeed,
- if $x \notin C$, then $\Phi(x,y) \;=\; \{F(x)\} \times \{-1\} \subset \{F(x_0) + M\ell\varepsilon\mathcal{B}_X\} \times \{-1\} \subset \Phi_\varepsilon(x_0, y)$,
- if $x \in C$, then $d_C(x_0) \le M\varepsilon$, and $F(x) \subset F(x_0) + M\ell\varepsilon\mathcal{B}_X$ and

$$\Phi(x,y) := \overline{Co}\,[(F(x) \times \{-1\}) \cup (0,0)\}] \subset \Phi_\varepsilon(x_0, y)$$

Q.E.D.

3.2.2 State discretization

Let Φ_ε be the previous discretization of Φ and set $G_\varepsilon(x,y) := (x,y) + \varepsilon\Phi_\varepsilon(x,y)$.

Let R_h be an integer lattice of $\boldsymbol{R}$ generated by segments of length h and set $R_h^+ := R_h \cap \boldsymbol{R}^+$. Typically, R_h is a grid and is such that

$$(32) \qquad \left\{ \begin{array}{ll} i) & R_h \text{ is stable by addition and substraction} \\ ii) & \forall h, \;\; R_h \subset R_{\frac{h}{2}} \end{array} \right.$$

As in subsection 2.3, we consider a discretization of the state space $X_h \times R_h$ where X_h satisfies (20).

Let us set $\alpha_{\varepsilon h} := \alpha(\varepsilon, h) := 2h + \ell\varepsilon h + M\varepsilon^2$.

We now define $\Gamma_{\varepsilon,h}$ as follows

$$\Gamma_{\varepsilon,h}(x_h, y_h) := \left\{ \begin{array}{l} (\{x_h + \varepsilon F(x_h) + \alpha_{\varepsilon h}\mathcal{B}\} \times \{y_h - \varepsilon + [-h,h]\}) \cap (X_h \times R_h) \\ \hfill \text{if } d_C(x_h) > M\varepsilon + h \\ \overline{Co}[\{x_h + \varepsilon F(x_h) + \alpha_{\varepsilon h}\mathcal{B}\} \times \{y_h - \varepsilon + [-h,h]\} \cup \; \{x_h, y_h\}] \cap \; (X_h \times R_h) \\ \hfill \text{if } d_C(x_h) \le M\varepsilon + h \end{array} \right.$$

Lemma 3.4 *Assume that F is $\ell-$Lipschitz and bounded by some constant M. Then $\Gamma_{\varepsilon,h}$ is a "good discretization" of G_ε satisfying* **H3** *and* **H4**.

The proof is the same as that of Lemma 3.3. Now applying Theorem 2.19 yields

Corollary 3.5 *Suppose that the assumptions of Lemma 3.4 are fulfilled. Let $\mathcal{H}_h := [(K + h\mathcal{B}) \cap X_h] \times [R_h^+]$. Then we have*

$$\mathop{\mathrm{Lim}}_{\varepsilon \to 0^+,\; \frac{h}{\varepsilon} \to 0^+} \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}(\mathcal{H}_h) \;=\; \mathcal{E}pi(\vartheta_C^K)$$

3.2.3 The fully discrete approximation

From Corollary 3.5 we deduce a fully discrete scheme for computing the Minimal Time function $\vartheta_C^K(\cdot)$.

Let $\varepsilon_p \in R_{h_p}$ be a sequence converging to 0^+ when p tends to ∞ such that $\frac{h_p}{\varepsilon_p} \to 0^+$. Moreover, we assume that $\varepsilon_p > h_p$. In practice, we can again choose h_p and ε_p as previously given in (25). We also assume that $\varepsilon_p, \in R_{h_p}$ and $h_p \in R_{h_p}$.

Let us denote $\alpha_p := \alpha(\varepsilon_p, h_p)$ and $K_{h_p} := (K + h_p\mathcal{B}) \cap X_{h_p}$.

Let us suppose that, starting at step $p = 0$ with the initial value $T_0^\infty \equiv 0$, we have computed, from step $k = 1$ to step $k = p - 1$, the functions $T_k^\infty : K_{h_k} \to R_{h_k}$.

At step p, starting from $T_p^0 = T_{p-1}^\infty$, we built recursively the sequence of functions $T_p^n : K_{h_p} \to R_{h_p}$ as follows

$$(33) \qquad T_p^{n+1}(x) := \left\{ \begin{array}{ll} \varepsilon_p - h_p + \displaystyle\min_{v \in V\; b \in \mathcal{B}} T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b), & \text{if } d_C(x) > M\varepsilon_p + h_p \\ T_p^n(x) & \text{otherwise} \end{array} \right.$$

This iterative process corresponds to the algorithm described in Proposition 2.18. The role played by $K_{\varepsilon,h}^n$ in the iterative construction given in Proposition 2.18 is played here by the epigraph of T_p^n. In particular:

Lemma 3.6 *For any $x \in K_{h_p}$, the limit*

$$T_p^\infty(x) := \lim_{n \to +\infty} T_p^n(x)$$

exists and

$$\mathcal{E}pi(T_p^\infty) = \overrightarrow{Viab}_{\Gamma_{\varepsilon,h}}(\mathcal{H}_h)$$

Proof of Lemma 3.6 : Let us set $\Gamma_p := \Gamma_{\varepsilon_p,h_p}$ and let us consider the sequence of (discrete) sets A^n defined as in Proposition 2.18 by $A^0 := \mathcal{E}pi(T_{p-1}^{\infty})$ and

$$A^{n+1} := \{(x,y) \in A^n \mid \Gamma_p(x,y) \cap A^n \neq \emptyset\}$$

We are going to prove by induction (on p and on n) that $A^n = \mathcal{E}pi(T_p^n)$ and that $T_p^{n+1} \geq T_p^n$.

Note that for $n = 0$, the equality $A^0 = \mathcal{E}pi(T_p^0)$ is obvious. Moreover, since $T_p^0 := T_{p-1}^{\infty} \geq 0$ and since $\varepsilon_p > h_p$, this proves that $T_p^1 \geq T_p^0$.

Suppose the result proved up to n. Then, for any $(x,y) \in \mathcal{E}pi(T_p^{n+1})$, one has :

$$y \geq \begin{cases} \varepsilon_p - h_p + \min_{v\in V\ b\in\mathcal{B}} T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b), & \text{if } d_C(x) > M\varepsilon_p + h_p \\ T_p^n(x) & \text{otherwise} \end{cases}$$

Assume that $d_C(x) > M\varepsilon_p + h_p$ and let $v \in V$, $b \in \mathcal{B}$ realize the minimum in the expression. Then $(x + \varepsilon_p f(x,v) + \alpha_p b, y - \varepsilon_p + h_p)$ belongs to $\Gamma_p(x,y)$ (from teh very definition of Γ_p) and to $A^n := \mathcal{E}pi(T_p^n)$ because

$$y \geq \varepsilon_p - h_p + T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b) \geq T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b)$$

Moreover, (x,y) belongs to A^n because $T^{n+1} \geq T^n$. So, if $d_C(x) > M\varepsilon_p + h_p$, then $(x,y) \in A^{n+1}$. Otherwise, (x,y) belongs to $\Gamma_p(x,y)$ and to A^n because $y \geq T_p^n(x)$. So we have proved that $\mathcal{E}pi(T_p^{n+1}) \subset A^{n+1}$.

Conversely, if (x,y) belongs to A^{n+1}, and if $d_C(x) > M\varepsilon_p + h_p$, there are $v \in V$, $b \in B$, $s \in [-h_p, h_p]$ such that $(x + \varepsilon_p f(x,v) + \alpha_p b, y - \varepsilon_p + s)$ belongs to A^n, i.e.,

$$y \geq \varepsilon_p - s + T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b) \geq T_p^{n+1}(x)$$

One another hand, if $d_C(x) \leq M\varepsilon_p + h_p$, then (x,y) also belongs to A^n so that $y \geq T_p^n(x) = T_p^{n+1}(x)$. In both cases, we have proved that (x,y) belongs to $\mathcal{E}pi(T_p^{n+1})$. Thus the equality $A^{n+1} = \mathcal{E}pi(T_p^{n+1})$ is proved.

We have finally to prove that $T_p^{n+2} \geq T_p^{n+1}$. If $d_C(x) \leq M\varepsilon_p + h_p$, this is obvious. Otherwise,

$$\begin{array}{rcl} T_p^{n+2}(x) & = & \varepsilon_p - h_p + T_p^{n+1}(x + \varepsilon_p f(x,v) + \alpha_p b) \\ & \geq & \varepsilon_p - h_p + T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b) \\ & \geq & T_p^{n+1}(x) \end{array}$$

since $T_p^{n+1} \geq T_p^n$. So we have finally proved our claim.

Since $T_p^n(x)$ is non decreasing, it converge to some $T_p^{\infty}(x)$ such that

$$\mathcal{E}pi(T_p^{\infty}) = \bigcap_n A^n = \overrightarrow{Viab}_{\Gamma_p}(\mathcal{H}_p)$$

from Proposition 2.18. **Q.E.D.**

Comments

- For a greater writing convenience, we extend maps T_p^n on the whole set X by setting $T_p^n(x) = +\infty$ whenever $x \notin K_{h_p}$. Also, in this equation, since T_p^n is defined on the whole space X, the term $T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b)$ takes finite values only at points $x + \varepsilon_p f(x,v) + \alpha_p b$ which belong to the grid K_{h_p} for precise values of u and b.
- Let us notice also that, for any $x \in K_{h_p}$ and for any $v \in V$, there exists at least some $b \in \mathcal{B}$ such that $x + \varepsilon_p f(x,v) + \alpha_p b \in K_{h_p}$. So maps T_p^n are well defined on K_{h_p}.
- The convergence can be accelerated when replacing the value of $T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b)$ in (33) by the value of $T_p^{n+1}(x + \varepsilon_p f(x,v) + \alpha_p b)$ as soon as it has been already computed.

Now, exploiting the Refinement Principle stated in Theorem 2.23, we define the initial function T_{p+1}^0 as follows

$$T_{p+1}^0(x) := \begin{cases} -(h_p + h_{p+1}) + \min_{b\in\mathcal{B}} T_p^{\infty}(x + (h_p + h_{p+1})b), & \text{if } x \in K_{h_{p+1}} \\ +\infty & \text{otherwise} \end{cases}$$

Using discretization $\Gamma_p := \Gamma_{\varepsilon_p,h_p}$ defined in the previous subsection, we deduce from Theorem 2.23:

Corollary 3.7 *If the assumptions of Theorem 2.23 are fulfilled, the sequence T_p^∞ converges to ϑ_C^K in the epigraphic sense:*

$$\mathcal{E}pi(\vartheta_C^K) \;=\; \operatorname*{Lim}_{p\to+\infty} \mathcal{E}pi(T_p^\infty)$$

Moreover,

$$\forall x_p \in R_{h_p},\;\; T_p^\infty(x_p) \le \vartheta_C^K((x_p))$$

and T_p^∞ converges pointwisely to ϑ_C^K

$$\forall x \in K,\; \vartheta_C^K(x) \;=\; \lim_{p\to+\infty} \min_{x_p\in(x+h\mathcal{B})\cap X_{h_p}} T_p^\infty(x_p)$$

The epigraphic convergence is a direct consequence of the convergence of fully discrete viability kernels to the continuous viability kernel (Theorem 2.19).

The pointwise convergence is a consequence of the same Theorem where the viability kernel and the fully discrete viability kernels are the epigraphs of functions ϑ_C^K and T_p^∞.

Let us define the set-valued map $\tilde{T}_p^\infty$ which epigraph is precisely $\mathcal{E}pi(T_p^\infty) + h\mathcal{B}_{X\times R}$.

Then, on the one hand, we have

$$\forall x \in K,\;\; \exists x_p \in x + h_p\mathcal{B} \text{ such that }\; \tilde{T}_p^\infty(x) \le T_p^\infty(x_p) \le \tilde{T}_p^\infty(x) + h_p.$$

From (23) stated in Theorem 2.19 we deduce that $\operatorname*{Limsup}_{p\to\infty} \mathcal{E}pi(\tilde{T}_p^\infty) \subset \mathcal{E}pi(\vartheta_C^K)$ and consequently

$$\forall x \in K,\;\; \vartheta_C^K(x) \le \liminf_{p\to\infty, x'\to x} \tilde{T}_p^\infty(x') \le \liminf_{p\to\infty} \min_{\|x_p-x\|\le h_p} T_p^\infty(x_p)$$

On the other hand, from (22) stated in Theorem 2.19, which is written $\mathcal{E}pi(\vartheta_C^K) \subset \mathcal{E}pi(\tilde{T}_p^\infty)$, we have

$$\tilde{T}_p^\infty(x) \le \vartheta_C^K(x).$$

and thus

$$\min_{x_p\in x+h_p\mathcal{B}} T_p^\infty(x_p) \le \vartheta_C^K(x) + h_p$$

so that

$$\limsup_{p\to\infty} \min_{x_p\in x+h_p\mathcal{B}} T_p^\infty(x_p) \le \vartheta_C^K(x)$$

In conclusion we have $\lim_{p\to\infty} \min_{x_p\in x+h_p\mathcal{B}} T_p^\infty(x) = \vartheta_C^K(x)$. **Q.E.D.**

3.2.4 Outline of the Algorithm

PROCEDURE OF CONSTRUCTION OF THE DATA (parameter p)

$R_{h_p} \leftarrow 2^{-p}\mathbb{Z}$, $X_{h_p} \leftarrow 2^{-p}\mathbb{Z}^N$. {Definition of the grids R_{h_p} and X_{h_p}.}
$h_p \leftarrow 2^{-p}$, $\varepsilon_p = \sqrt{h_p/M\ell}$ {Definition of steps h_p and $\varepsilon_p \in h_p\mathbb{Z}$}
$\alpha_p \leftarrow 2h_p + \ell\varepsilon_p(h_p + M\varepsilon_p)$ {Definition of the dilation term.}
$K_{h_p} \leftarrow (K + 2^{-p}\mathcal{B}) \cap X_{h_p}$ {Definition of K_{h_p}}

INITIALIZATION

$p \leftarrow 1$
  if $x \in K_1$
    then $T_1^0(x) \leftarrow 0$
    else $T_1^0(x) \leftarrow +\infty$

MAIN LOOP Minimal Time Problem

For $p := 1$ **to** $\bar{p}$ **do**
  $n := 0$
  Repeat {Beginning of calculus of T_p^∞}
    $x \leftarrow x_{p\downarrow}$ {$x_{p\downarrow}$ is the first point of the grid K_{h_p}}
    Repeat {Scanning of the grid}
      if $d_C(x) > M\varepsilon_p + h_p$
        then $T_p^{n+1}(x) \leftarrow [\varepsilon_p - h_p + \min_{v\in V, b\in\mathcal{B}} T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b)]$

else $T_p^{n+1}(x) \leftarrow T_p^n(x)$
$x \leftarrow Next(x)$ $\{Next(x)$ is the following point of the grid$\}$
until $x \equiv x_{p\uparrow}$ $\{x_{p\uparrow}$ is the last point of the grid$\}$
$n \leftarrow n+1$
until $T_p^{n+1} \equiv T_p^n$ {End of calculus of T_p^∞}
Set $T_p^\infty \leftarrow T_p^n$
if $p < \bar{p}$ **then**
$x \leftarrow x_{(p+1)\downarrow}$
Repeat {Scanning for Refinement Process}
if $x \in K_{h_{p+1}}$
then $T_{p+1}^0(x) \leftarrow -(h_p + h_{p+1}) + \min_{b \in \mathcal{B}} T_p^\infty(x + (h_p + h_{p+1})b)$
else $T_{p+1}^0(x) \leftarrow +\infty$
$x \leftarrow Next(x)$
until $x \equiv x_{(p+1)\uparrow}$ {End Scanning and End of Refinement}

Return

$T_{\bar{p}}^\infty(x)$ $\{T_{\bar{p}}^\infty(\cdot)$ is the approached Minimal Time function at step $\bar{p}$ $\}$

3.3 Minimal Time for a basic target problem with constraints

Let us consider the basic example described by the following dynamic:

$$\left\{ \begin{array}{lcl} x_1'(t) & = & cv_{x_1} \\ x_2'(t) & = & cv_{x_2} \qquad \text{with } v_{x_1}^2 + v_{x_2}^2 \leq 1 \end{array} \right.$$

Let $C = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1^2 + x_2^2 \geq 8\}$ be the target and K the constraint of labyrinth shape as shown on the figure.

Figure 3.3 down represents the minimal time function obtained for a final space discretization step $h = 10^{-3}$.

3.4 Minimal Time for the swimmer problem with obstacles

Let us come back to the previous Zermelo's type problem but, now, in presence of obstacles. The swimmer aims at reaching the island in minimal time. His dynamic now is autonomous. It is described by through the following system:

$$(34) \qquad \begin{array}{l} x_1'(t) = (1 - ax_2^2) + v_{x_1} \\ x_2'(t) = v_{x_2} \quad \text{with } v_{x_1}^2 + v_{x_2}^2 \leq c^2 \end{array}$$

where $C = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1^2 + x_2^2 \leq 0.44\}$ and $K = \{[-6, 2] \times [0, 5]\} \backslash M \times \boldsymbol{R}^+$. The set M is the union of a triangular and a square shapes as viewed on figure 3.4.

Barriers appear corresponding to discontinuities of the Minimal Time function.

4 Qualitative Differential Game Problems: the target problem

We investigate differential games with dynamic described by the differential equation

$$(35) \qquad \left\{ \begin{array}{l} x'(t) = f(x(t), u(t), v(t)), \\ u(t) \in U, \ v(t) \in V \end{array} \right.$$

where $f : X \times U \times V \to X$, U and V being the control sets of the players.

Throughout this section, we study the following game. $\mathcal{O} \subset X$ is an open target (for the first player) and $\mathcal{E} \subset X$ a closed evasion set (for the second player). The first player - Ursula, playing with u - aims at reaching $\mathcal{O}$ in finite time while avoiding $\mathcal{E}$ and the second player - Victor, playing with v - aims at avoiding $\mathcal{O}$ until reaching $\mathcal{E}$. This game is called the target problem.

The aim of the section is to explain how to characterize and to compute the set of initial positions from which a player may win, whatever his adversary plays. This set is called the *victory domains* of the player. The characterization is given by an extension of the Viability approach to the more general framework of differential games.

The compatibility of notations and assumptions between Control and Game problems.

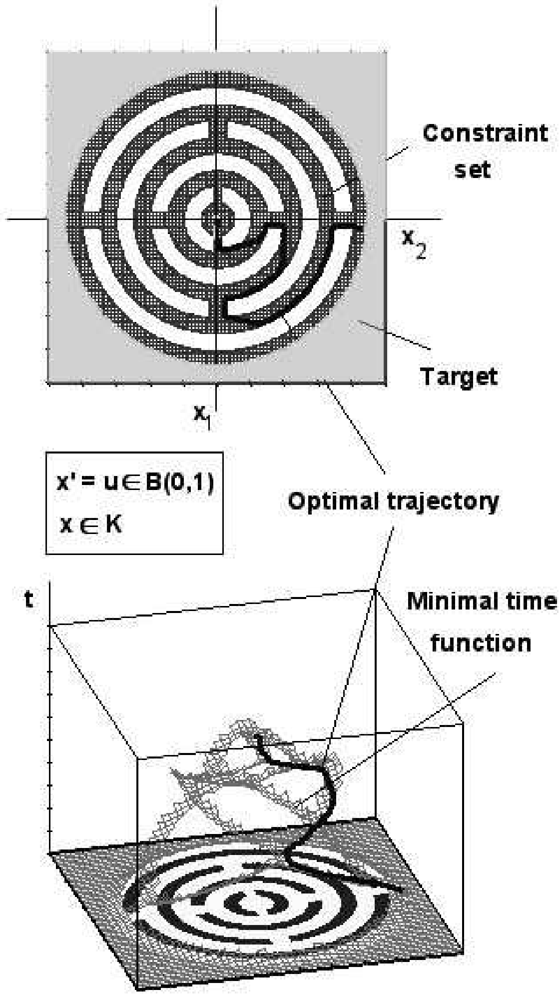

Figure 4: the Minimal Time function with constraints. An optimal solution

Figure 5: Minimal Time function for Zermelo problem

Figure 6: Level Curves of the Minimal Time function for Zermelo problem with obstacles and visualization of the optimal swimming policy

A control problem is a specific differential game problem where one of the players has no action on the system. So, to be in accordance with the notations of Qualitative and Quantitative differential game formulation, we have chosen to denote v the control and V the set of control for Qualitative and Quantitative control problems. The study of a control problem amounts to study a game problem for which the dynamic $f(x, u, v)$ does not depend on u. Moreover, for the initial game problem, Victor seeks after avoiding an open target $\mathcal{O}$ against any action of Ursula. In the same way, in the control problem where Ursula has no action, the objective of the controller is to avoid also an open target $\mathcal{O}$, or equivalently, to remain in the complement of $\mathcal{O}$ denoted K in the former section.

Concerning the notation, the target is denoted C when assumed to be closed and $\mathcal{O}$ when assumed to be open. For control problems we have explained in the former sections that the victory domains - in qualitative case - and the epigraph of the Value function - in quantitative case - are (viability) kernels of suitable problems. This kernel exists only for closed sets. This requires the target denoted $\mathcal{O}$ to be open for the qualitative problem and the target denoted C to be closed for the quantitative problem. For differential games, we shall see that both victory domains - in qualitative case - and epigraph of the Value function - in quantitative case - will be suitable kernels (that will be called *Discriminating Kernel*) of suitable problems. Also, as for control problems, this requires to study problems where the target must be open - and so is denoted $\mathcal{O}$ - for qualitative problems and the target must be closed - and so is denoted $\mathcal{C}$ - for quantitative problems.

4.1 Definition of the game

We study this game in the framework of the *nonanticipative strategies*[7]. Let

$$(36)\qquad \begin{cases} \mathcal{U} = \{u(\cdot) : [0, +\infty[\to U, \text{ measurable function } \} \\ \mathcal{V} = \{v(\cdot) : [0, +\infty[\to V, \text{ measurable function } \} \end{cases}$$

be the sets of time-measurable controls of respectively the first (Ursula) and the second (Victor) player.

Definition 4.1 (Nonanticipative strategies) *A map $\alpha : \mathcal{V} \to \mathcal{U}$ is a nonanticipative strategy (for Ursula) if it satisfies the following condition:*

For any $s \geq 0$, for any $v_1(\cdot)$ and $v_2(\cdot)$ belonging to $\mathcal{V}$ such that $v_1(\cdot)$ and $v_2(\cdot)$ coincide almost everywhere on $[0, s]$, the image $\alpha(v_1(\cdot))$ and $\alpha(v_2(\cdot))$ coincide almost everywhere on $[0, s]$.
Nonanticipative strategies $\beta : \mathcal{U} \to \mathcal{V}$ (for Victor) are defined in the symmetric way.

Assume now that f is continuous and Lipschitz with respect to x. Then, for any $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, for any initial position x_0, there exists only one solution to (35). We denote this solution by $x[x_0, u(\cdot), v(\cdot)]$.

We are now ready to define the victory domains of the game.

Definition 4.2 (Victory domains)
- Victor's victory domain is the set of initial positions $x_0 \notin \mathcal{O}$ for which Victor can find a nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ such that for any time-measurable control $u(\cdot) \in \mathcal{U}$ played by Ursula, the solution $x[x_0, u(\cdot), \beta(u(\cdot))]$ avoids $\mathcal{O}$ until it reaches $\mathcal{E}$ (or forever if it does not reach $\mathcal{E}$). Namely:

$$\begin{array}{l} \exists \tau \in [0, +\infty],\ \forall t \in [0, \tau),\ x[x_0, u(\cdot), \beta(u(\cdot))](t) \notin \mathcal{O} \\ \text{and if } \tau < +\infty, \text{ then } x[x_0, u(\cdot), \beta(u(\cdot))](\tau) \in \mathcal{E}. \end{array}$$

- Ursula's victory domain is the set of initial positions $x_0 \notin \mathcal{O}$ for which Ursula can find a nonanticipative strategy $\alpha : \mathcal{V} \to \mathcal{U}$, positive ε and T such that, for any $v(\cdot) \in \mathcal{V}$ played by Victor, the solution $x[x_0, \alpha(v(\cdot)), v(\cdot)]$ avoids $\mathcal{E} + \varepsilon B$ until it reaches the set $\mathcal{O}_\varepsilon := \{x \mid d_{X \setminus \mathcal{O}}(x) \geq \varepsilon\}$ before T.
(We denote by $d_K(x)$ the distance from a point x to a closed set K.) Namely

$$\begin{array}{l} \exists \tau \leq T,\ d_{X \setminus \mathcal{O}}(x[x_0, \alpha(v(\cdot)), v(\cdot)](\tau)) \geq \varepsilon \\ \text{and } \forall t \in [0, \tau],\ x[x_0, \alpha(v(\cdot)), v(\cdot)](t) \notin \mathcal{E} + \varepsilon \mathcal{B}. \end{array}$$

In the definition of Victor's victory domain, the solution has to avoid the target until reaching the evasion set $\mathcal{E}$.

In the definition of Ursula's victory domain, the solution has not only to reach the target and to avoid the evasion set, but also to remain "sufficiently far" from the evasion set and to enter "sufficiently far" the target (i.e., with a distance larger than ε from the boundary of the target) and in a finite time (say not larger than T). Moreover, both ε and T have to be independent on Victor's response $v(\cdot)$. Let us point out that ε and T are imposed by technical reasons.

[7] These strategies are also called VREK strategies (from Varaiya [63], Roxin [55], Elliot & Kalton [38]).

Example 4.1

In the following example the difficulty of the "ε" and "T" clearly appears. Let $U = V := [-1, 1]$ and $f : \boldsymbol{R} \times U \times V \to \boldsymbol{R}$ defined by $f(x, u, v) := (x + 1)u + v$.

When $\mathcal{E} := \emptyset$ and $\mathcal{O} := \boldsymbol{R}\backslash\{0\}$ then $\mathcal{W}_V = \{0\}$. Indeed $\beta(u(\cdot))(t) = -u(t)$ is a nonanticipative strategy for Victor.

But the following nonanticipative strategy α defined by

$$\forall v(\cdot) \in \mathcal{V},\ \alpha(v(\cdot))(t) = \begin{cases} -1 & \text{if} \quad v(s) \neq -1 \ \text{ for almost all } s \in [0, t] \\ 1 & \text{otherwise,} \end{cases}$$

is such that for any $v(\cdot) \in \mathcal{V}$, the solution $x[0, \alpha(v(\cdot), v(\cdot)]$ reaches $\mathcal{O}$ immediately. □

Assumptions on f.

Let us introduce the set-valued map F defined on $X \times U$ by

$$\forall x \in X,\ \forall v \in V,\ \ F(x, u) := \bigcup_{v \in V} f(x, u, v).$$

In the sequel, we need the following assumptions

$$(37) \qquad \begin{cases} i) & U \text{ and } V \text{ are compact} \\ ii) & f : X \times U \times V \to X \text{ is continuous,} \\ iii) & f \text{ is } \ell\text{-Lipschitz,} \\ iv) & \forall u \in U, \ \ F(x, u) \text{ is convex.} \end{cases}$$

and the Isaacs' condition:

$$(38) \qquad \forall (x, p) \in \boldsymbol{R}^{2N},\ \sup_u \inf_v < f(x, u, v), p >= \inf_v \sup_u < f(x, u, v), p >$$

Let us remark that, in particular, if V is convex and if f is affine with respect to v, assumption (37 iv) is already satisfied. On the other hand, convexity of U is not assumed.

4.2 Characterization of the victory domains

We explain here how to characterize the victory domains of each player. For that purpose, we follow the same method as for control problems. We first define a class of closed sets: the *discriminating domains*. Then we show that any closed set K contains a largest discriminating domain: the *discriminating kernel* of K. The discriminating kernel of the complement of the target is equal to Victor's victory domain - while its complement is equal to Ursula's victory domain. So the discriminating kernel plays the role of the viability kernel in the Two-Players Differential Games Theory.

In this section, we state the results without proof. Indeed, these proofs exceed the scope of this paper (See for instance [29], [30]).

Definition 4.3 (Discriminating Domains) *Let $H : X \times X \to \boldsymbol{R}$. A closed set $D \subset X$ is a discriminating domain for H if:*

$$\forall x \in D,\ \forall p \in \mathcal{NP}_D(x),\ H(x, p) \leq 0$$

We are mainly interested here in the following H:

$$(39) \qquad H(x, p) := \begin{cases} \sup_u \inf_v < f(x, u, v), p > & \text{if } x \notin \mathcal{E} \\ \min\{\sup_u \inf_v < f(x, u, v), p >; 0\} & \text{otherwise} \end{cases}$$

If K is not a discriminating domain, it contains a largest discriminating domain

Theorem 4.4 (Discriminating kernel) *Let $H : X \times X \to \boldsymbol{R}$ be a lower semicontinuous map. Any closed subset K of X contains a largest (closed) discriminating domain for H. This set is called the discriminating kernel of K for H and is denoted by $Disc_H(K)$.*

Any discriminating domain for H contained in a closed K is contained in $Disc_H(K)$. Moreover, $Disc_H(K)$ is itself a discriminating domain for H and $Disc_H(K)$ may be empty if K does not contain any discriminating domain for H.

Let us notice that discriminating domains and kernels are defined by geometric conditions. Thus they do not depend on the strategies chosen to play the game.

The main result of this section is the following characterization of the victory domains

Theorem 4.5 (Characterization of the victory domains) *Assume that f satisfies (37). Recall that the hamiltonian H of the system is defined by (39). Let $\mathcal{O}$ be an open target and set $\mathcal{K} := X \backslash \mathcal{O}$. Then*

- *Victor's victory domain is equal to $Disc_H(\mathcal{K})$.*

and if Isaac's condition (38) holds true,

- *Ursula's victory domain is equal to $\mathcal{K} \backslash Disc_H(\mathcal{K})$.*

The victory domains of the two players form a partition[8] of the closed set $\mathcal{K}$. This characterization Theorem allows to compute numerically the victory domains of each player.

Remark 4.1

We want to underline the relations between viability kernels, invariance kernels and discriminating kernels. In some sense, the notion of discriminating kernels contains the notions of viability and of invariance. Here we assume that $\mathcal{E} = \emptyset$.
a) Assume that $f(x,u,v) := g_1(x,v)$ and that f satisfies assumptions (37). Let us set $G_1(x) := \bigcup_v g_1(x,v)$. Then, for any closed set K, we have $Disc_H(K) = Viab_{G_1}(K)$.
b) Assume now that $f(x,u,v) := g_2(x,u)$ and that f satisfies assumptions (37). Let us set $G_2(x) := \bigcup_u g_2(x,u)$. Then, for any closed set K, we have $Disc_H(K) = Inv_{G_2}(K)$ □

Remark 4.2

In the same way as in Remarks (2.1) and (2.3) it is easy to prove that if $K \subset K'$, then $Disc_H(K) \subset Disc_H(K')$.□

Remark 4.3

Let K be a closed convex subset of X. If the graph of the set-valued maps $x \rightsquigarrow f(x,u,V)$ (for $u \in U$) are convex, then $Disc_H(K)$ is convex (see [25]). □

4.3 Approximation of the discriminating kernel

In this section, we explain how to compute numerically the discriminating kernel of a closed set K.

As for the viability kernel, we first define the discrete discriminating domains and kernel, which are somehow a discrete version of the discriminating domains and kernel. Then we prove that the discrete discriminating kernel provides a good approximation of the (continuous) discriminating kernel.

4.3.1 The discrete discriminating kernel

In the same way as we have defined discrete viability domains and discrete viability kernel for control problems, we now introduce the notion of discrete discriminating domains and of the discrete discriminating kernel. In fact, these sets have an interpretation for discrete differential games.

Definition 4.6 (Discrete discriminating domains) *Let $G : X \times U \rightsquigarrow X$ be a set-valued map. A closed set S is a discrete discriminating domain for G if S enjoys the following property:*

$$\forall x \in S,\ \forall u \in U,\ G(x,u) \cap S \neq \emptyset \tag{40}$$

It is clear that a closed set S is a discrete discriminating domain if and only if it is a discrete viability domain for the set-valued maps $x \rightsquigarrow G(x,u)$ (for any $u \in U$).

If K is not a discrete discriminating domain, it is possible to define the largest discriminating domain contained in K.

Proposition 4.7 (Discrete discriminating kernel) *Let K be a closed subset of X and $G(\cdot,\cdot) : X \times U \rightsquigarrow X$ be an upper semicontinuous set-valued map with compact values. Then, there exists a largest closed discrete discriminating domain contained in K. We call this set the* discrete discriminating kernel *of K, and we denote it $\overrightarrow{\mathrm{Disc}}_G(K)$.*

The proof of Proposition 4.7 is the consequence of the following algorithm to compute the discrete discriminating kernel. This algorithm has a great importance in practice, as we see below.

[8] A similar Alternative Theorem have been obtained by Krasovskii & Subbotin in the framework of the positional strategies (See [48]). In fact, the discriminating domains are very close to Krasovskii & Subbotin's stable bridges, while the discriminating kernel is related with the maximal stable bridges.

4.3.2 The semi-discrete discriminating kernel algorithm

Let us define the following decreasing sequence of closed sets

$$\left\{ \begin{array}{l} K^0 = K, \\ K^{n+1} = \{x \in K^n \mid \forall u \in U,\ G(x,u) \cap K^n \neq \emptyset\}. \end{array} \right. \tag{41}$$

Proposition 4.8 *Let K and G as previously. The decreasing sequence of closed sets $\{K^n\}_n$ defined by (41) converges to $\overrightarrow{\text{Disc}}_G\,(K)$, i.e.,*

$$\bigcap_{n \in N} K^n = \overrightarrow{\text{Disc}}_G\,(K) \tag{42}$$

Proof of Propositions 4.7 and 4.8

We have to prove that the set $K^\infty := \bigcap_{n\in N} K^n$ is the largest discrete discriminating domain of K for G.

Following the beginning of the proof of Theorem 2.13, since G is an upper semicontinuous set-valued map, we prove by induction that the sets K^n and K^∞ are closed.

Let us now show that K^∞ is a discrete discriminating domain. Let x belong to K^∞ and $u \in U$. Since x belongs to K^n for any n, from (41), $G(x,u) \cap K^n$ is nonempty. Since G has compact values, the intersection $G(x,u) \cap K^\infty$ is also nonempty. So, K^∞ is a discrete discriminating domain of K for G.

Let us now prove that K^∞ contains any discrete discriminating domain S contained in K. For that purpose, it is sufficient to prove by induction that such a set S is contained in any K^n. It is clearly true for K^0. Assume that $S \subset K^n$ for some n. Then, since S is a discrete discriminating domain, for any $x \in S$, for any $u \in U$, $G(x,u) \cap S \neq \emptyset$ and consequently $G(x,u) \cap K^n \neq \emptyset$. Thus $S \subset K^{n+1}$. So, by induction, we conclude that S is contained in the intersection of the K^n, i.e., in K^∞.

We have finally proved that K^∞ is the largest discrete discriminating domain contained in K. **Q.E.D.**

4.3.3 Discrete Games

In this subsection we give an interpretation of the discrete discriminating kernel.

We consider the discrete game whose dynamics is

$$x_{n+1} = g\left(x_n, \tilde{u}[x_n], \tilde{v}[x_n, u_n]\right). \tag{43}$$

where $g : X \times U \times V \to X$ is continuous, U and V being compact, and where $\tilde{u}[\cdot] : X \to U$ denotes Ursula's strategy[9] while $\tilde{v}[\cdot,\cdot] : X \times U \to V$ denotes Victor's strategy. Ursula chooses her strategy $\tilde{u}[\cdot]$ in such a way that the solution (x_n) to (43) reaches an open target $\mathcal{O}$ in a finite number of steps, while Victor chooses his strategy $\tilde{v}[\cdot,\cdot]$ in such a way that this solution (x_n) avoid the target $\mathcal{O}$ forever.

Definition 4.9 *Victor's discrete discriminating victory set denoted by $\overrightarrow{\mathcal{W}}_V^d$ is the set of point $x_0 \in X \backslash \mathcal{O}$ for which a strategy $\tilde{v}[\cdot,\cdot] : X \times U \to V$ exists, such that, for any strategy $\tilde{u}[\cdot] : X \to U$, the solution $(x_n)_{n\geq 0}$ of (43) starting from x_0 avoids $\mathcal{O}$.*

Let us set $G(x,u) := \bigcup_{v\in V} g(x,u,v)$.

Theorem 4.10 *Let us posit assumptions of Proposition 4.7. Let us denote by K the closed set $X\backslash\mathcal{O}$ and assume that $G(\cdot,\cdot) : X \times U \rightsquigarrow X$ is an upper semi-continuous set-valued map with compact values.*

Then Victor's discrete discriminating victory set is equal to the discrete discriminating kernel of K.

$$\overrightarrow{\mathcal{W}}_V^d = \overrightarrow{\text{Disc}}_G\,(K)$$

The proof can be found in ([25], Theorem 4.4).

4.3.4 The fully discrete Discriminating Kernel Algorithm

With any $h > 0$, we associate X_h and U_h locally finite subsets of X and U, which span X and U in the sense that

$$\left\{ \begin{array}{l} \forall x \in X,\ \exists x_h \in X_h \text{ such that } \|x - x_h\| \leq h \\ \forall u \in U,\ \exists u_h \in U_h \text{ such that } \|u - u_h\| \leq h \end{array} \right. \tag{44}$$

The discretization X_h shall be always locally finite while the discretization U_h is finite.

[9]For discrete games, a strategy is any application - without regularity. Indeed, there is no existence problem for solution to (43).

Consider a ℓ-Lipschitz map $f : X \times U \times V \to X$ which is bounded by some constant M on the closed set K, i.e.,

$$\forall (x,u,v) \in K \times U \times V, \quad \|f(x,u,v)\| \le M. \tag{45}$$

Let $\mathcal{E}$ be the closed evasion set and let H be the hamiltonian defined by (39). We denote

$$F(x,u) := \begin{cases} \bigcup_v f(x,u,v) & \text{if } x \notin \mathcal{E} \\ \overline{Co}[\{0\} \cup \bigcup_v f(x,u,v)] & \text{otherwise} \end{cases}$$

For any fixed $\varepsilon > 0$, we set:

$$\forall x \in X,\ \forall u \in U,\ \ F_\varepsilon(x,u) := \begin{cases} F(x,u) + M\ell\varepsilon\mathcal{B} & \text{if } d_{\mathcal{E}}(x) > M\varepsilon \\ \overline{Co}[\{0\} \cup (F(x,u) + M\ell\varepsilon\mathcal{B})] & \text{otherwise} \end{cases}$$

and

$$G_\varepsilon(x,u) := x + \varepsilon F_\varepsilon(x,u)$$

We also define the discrete set-valued map $\Gamma_{\varepsilon,h} : X_h \times U_h \rightsquigarrow X_h$

$$\Gamma_{\varepsilon,h}(x_h,u_h) \ := \ \begin{cases} [G_\varepsilon(x_h,u_h) + 2(1+\ell\varepsilon)hB] \cap X_h & \text{if } d_{\mathcal{E}}(x) \ge M\varepsilon + h \\ \overline{Co}\,[\{x_h\} \cup [G_\varepsilon(x_h,u_h) + 2(1+\ell\varepsilon)hB]] \cap X_h & \text{otherwise} \end{cases}$$

The set-valued map $\Gamma_{\varepsilon,h}$ is the discretization of the dynamic system (35) for f. It is rather natural to ask if the discriminating kernel of a closed set K for H can be approached by the discrete discriminating kernel of K for $\Gamma_{\varepsilon,h}$. The answer is positive:

Theorem 4.11 *Assume that $f : X \times U \times V \to X$ satisfies (37) and (45). Let us set $K_h := (K + h\mathcal{B}) \cap X_h$. Then*

$$(Disc_H(K) + h\mathcal{B}) \ \subset \overrightarrow{\mathrm{Disc}}_{\Gamma_{\varepsilon,h}}(K_h)$$

and

$$\mathop{\mathrm{Lim}}_{\varepsilon\to 0,\ \frac{h}{\varepsilon}\to 0^+} \overrightarrow{\mathrm{Disc}}_{\Gamma_{\varepsilon,h}}(K_h) = \mathrm{Disc}_H(K)$$

Remark 4.4

Usually K_h and U_h are finite sets so that $\overrightarrow{\mathrm{Disc}}_{\Gamma_h}(K_h)$ can be computed in a finite number of steps thanks to the algorithm described in Proposition 4.8. □

Proof of Theorem 4.11

For proving the Theorem, we need the following Lemmas.

Lemma 4.12 *If A is a closed discriminating domain for H, then A is a viability domain for $F(\cdot,u)$ for any $u \in U$.*

Lemma 4.13 *For any $u \in U$, the set-valued map $F_\varepsilon(\cdot,u)$ is a good discretization of $F(\cdot,u)$ (i.e., satisfies* **H0**, **H1**, **H2***) while, if $u_h \in U_h$ satisfies $\|u_h - u\| \le h$, then $\Gamma_{\varepsilon,h}(\cdot,u_h)$ is a good discretization of $G_\varepsilon(\cdot,u)$ (i.e., satisfies* **H3**, **H4***)*

The proofs are straightforward. □

Let us set $D^\sharp := \mathop{\mathrm{Limsup}}_{\varepsilon\to 0,\ \frac{h}{\varepsilon}\to 0^+} \overrightarrow{\mathrm{Disc}}_{\Gamma_{\varepsilon,h}}(K_h)$. We are going to prove that $D^\sharp$ is a discriminating domain for H. For that propose, it is sufficient to show that $D^\sharp$ is a viability domain for the set-valued map $x \rightsquigarrow F(x,u)$ for any $u \in U$ (Lemma 4.12). Let us fix $u \in U$ and consider $u_h \in U_h$ such that $\|u_h - u\| \le h$.

Recall now that $\overrightarrow{\mathrm{Disc}}_{\Gamma_{\varepsilon,h}}(K_h)$ is a discrete discriminating domain for $\Gamma_{\varepsilon,h}$, so that it is a discrete viability domain for $\Gamma_{\varepsilon,h}(\cdot,u_h)$. Then Proposition 2.20 state that $D^\sharp$ is a viability domain for $F(\cdot,u)$ because $\Gamma_{\varepsilon,h}(\cdot,u_h)$ is a good approximation of $G_\varepsilon(\cdot,u)$. This holds true for any u. Thus Lemma 4.12 states that $D^\sharp$ is a discriminating domain for H. Since $\overrightarrow{\mathrm{Disc}}_{\Gamma_{\varepsilon,h}}(K_h)$ is contained in K_h, $D^\sharp$ is contained in K. Thus $D^\sharp$ is contained in $Disc_H(K)$.

Fix $u_h \in U_h$. Since $Disc_H(K)$ is a discriminating domain for H, it is a viability domain for $F(\cdot,u_h)$. Since $\Gamma_{\varepsilon,h}(\cdot,u_h)$ is a "good discretization" of $G_\varepsilon(\cdot,u_h)$, Proposition 2.21 states that $[Disc_H(K) + h\mathcal{B}] \cap X_h$ is a discrete viability domain for $\Gamma_{\varepsilon,h}(\cdot,u_h)$ and is contained in K_h. This holds true for any $u_h \in U_h$, so that $[Disc_H(K) + h\mathcal{B}] \cap$

X_h is a discrete discriminating domain for $\Gamma_{\varepsilon,h}$. In particular $[Disc_H(K)+h\mathcal{B}]\cap X_h$ is contained in $\overrightarrow{\text{Disc}}_{\Gamma_{\varepsilon,h}}(K_h)$.

Thanks to Lemma 2.22, we conclude

$$\operatorname*{Limsup}_{\varepsilon\to 0,\ \frac{h}{\varepsilon}\to 0^+} \overrightarrow{\text{Disc}}_{\Gamma_{\varepsilon,h}}(K_h) \subset \text{Disc}_H(K) \subset \operatorname*{Liminf}_{\varepsilon\to 0,\ \frac{h}{\varepsilon}\to 0^+} \overrightarrow{\text{Disc}}_{\Gamma_{\varepsilon,h}}(K_h)$$

Since the upper limit always contains the lower limit, the proof is complete. **Q.E.D.**

4.3.5 The Refinement Principle

We keep the notations of the previous subsection. Let $\varepsilon_p \to 0^+$, $\frac{h_p}{\varepsilon_p} \to 0^+$ and set

$$K_{h_p} := [K + h_p\mathcal{B}] \cap X_{h_p},\ \ \Gamma_p := \Gamma_{\varepsilon_p,h_p}.$$

Theorem 4.11 states that $\text{Lim}_p\ \overrightarrow{\text{Disc}}_{\Gamma_p}(K_{h_p}) = Disc_H(K)$ and we have already indicated the way of computing the discrete discriminating kernels. Now we show that it is not necessary to resume the computation of the discrete discriminating kernel from K_h at each change of step h_p to step h_{p+1}.

For that purpose, let us now define the following sequence of closed sets:

$$\begin{cases} \tilde{D}_1 := K_1 \\ \tilde{D}_{p+1} := \overrightarrow{\text{Disc}}_{\Gamma_{p+1}} \left[\left(\tilde{D}_p + (h_p + h_{p+1})\mathcal{B} \right) \cap K_{h_{p+1}} \right] \end{cases}$$

Theorem 4.14 (Refinement Principle) *Suppose that the assumptions of Theorem 4.11 are fulfilled. Then*

$$\operatorname*{Lim}_{p\to\infty} \tilde{D}_p = Disc_H(K).$$

Remark 4.5

Since X_h is a locally finite set and U_h is a finite set, the sets $\tilde{D}_p$ are computed in a finite number of steps as soon as K is compact, which is reasonable in a numerical point of view. □

Proof of Theorem 4.14

The proof of Theorem 4.14 is quite the same as the proof of Theorem 2.23. We first prove by induction that

$$[Disc_H(K) + h_p\mathcal{B}] \cap X_{h_p} \subset \tilde{D}_p.$$

and we complete the proof in the same way. **Q.E.D.**

4.3.6 Outline of the Algorithm

PROCEDURE OF CONSTRUCTION OF THE DATA (parameter p)

> Definition of the grid $X_{h_p} := h_p\mathbb{Z}^N$, $h_p = 2^{-p}$, $\varepsilon_p := \sqrt{h_p}$.
> Definition of $K_{h_p} := (K + h_p\mathcal{B}) \cap X_{h_p}$.
> Definition of $U_{h_p} \subset U$ {Definition of the grids of U and V.}
> Definition of F_p, G_p and Γ_p.

INITIALIZATION

> $p \leftarrow 1$, $D_1^0 \leftarrow K_{h_1}$

MAIN LOOP Hitting Time Problem

> **For** $p := 1$ **to** $\bar{p}$ **do**
> $n := 0$
>   **Repeat** {Semi-discrete discriminating kernel loop}
> $D_p^{n+1} \leftarrow \left\{ x \in D_p^n \ \mid \forall u_h \in U_{h_p}\ \ \Gamma_p(x, u_h) \cap D_p^n \neq \emptyset \right\}$
> $n \leftarrow n + 1$
>   **until** $D_p^{n+1} = D_p^i$
>   Set $D_p^\infty \leftarrow D_p^n$
>   **if** $p < \bar{p}$ **then** $D_{p+1}^0 \leftarrow \left[D_p^\infty + h_{p-1}\mathcal{B} \right] \cap K_{h_{p+1}}$ {REFINEMENT}

RETURN

> $D_{\bar{p}}^\infty(x)$ {$D_{\bar{p}}^\infty(\cdot)$ is the approached Discriminating Kernel at step $\bar{p}$ }

4.4 Example of Approximation of the Victory Domain

In [46] Rufus Isaacs has described a modelization of the chase of a noisy fugitive by a potentially listening pursuer. He modelized the interception of the fugitive by a straight flying pursuer.

For this problem, P. Bernhard and B. Larrouturou provided an explicit formula characterizing the "barrier" (see [20]).

The key of these approaches is to determine victory domains by computing extremal trajectories starting from a part of the boundary of the target. The surface generated by these extremal trajectories forms the barrier[10]. We have presented in [25] an example where this method fails because the victory domain does not intersect the boundary of the target.

We now study a similar game where one of the victory domains is not connected. It is derived from an acoustic capture problem studied by P. Bernhard where the fugitive must slow down its speed whenever he gets nearer to the pursuer so as to mute its presence. But reducing the speed increases the risk of future capture. The fugitive has to find the domain where he must stand - and the right speed regulation - so as to be able to avoid the capture for ever.

We denote by (x, y) the coordinates of the fugitive in the reduced space where the pursuer's position is centered at the origin and where the x-axe always points in the direction of the pursuer's velocity vector. This allows to rewrite the problem in $\boldsymbol{R}^2$.

Let the target be the open rectangle $\mathcal{O} := \{(x, y) \in \boldsymbol{R}^2,\ -0.2 < x < 0,\ -3.5 < y < 3.5\}$ (in this problem, $\mathcal{E} = \emptyset$) and consider the system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = W_p \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{W_p}{0.8} \begin{pmatrix} -y(t) \\ x(t) \end{pmatrix} u(t) + w(x(t), y(t))\, v(t)$$

where

$$u(t) \in [-1, 1],\ \ v(t) \in \mathcal{B}_{R^2},\ \ w(x, y) = 2\, W_f\ \min(\sqrt{x^2 + y^2}\,, 0.5)$$

The numerical values for the pursuer and fugitive's maximal speed are $W_p = 1$ and $W_f = 1.1$.

K is the complement of the target $\mathcal{O}$.

The victory domain of the pursuer is the hatched region in Figure 3.4. The victory domain of the fugitive, which corresponds to the Discriminating Kernel $Disc_H(K)$, is the complement of the hatched region.

In this case, a connected part of the boundary of the Discriminating Kernel does not intersects the boundary of the target $\mathcal{O}$.

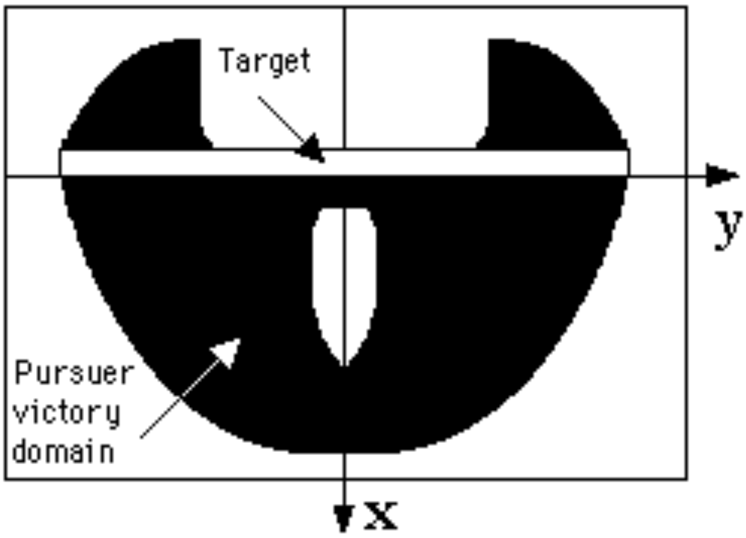

Figure 7: Discriminating Victory Domain.

To have a better understanding of this phenomenon, we propose to make a comparison with the situation of a man who tries to avoid the hits of a boxer. Let us consider a boxer who can move straight ahead with a maximal speed W_p. This speed is smaller than that of the fugitive. The boxer can pivot his arms that we assume for simplicity always outstretched. Also, at the end of his arms, the speed becomes higher than that of the fugitive.

[10]This question has been studied in [53] in the nonsmooth case for control problems. The barrier property has been extended to differential games by considering the boundary of the discriminating kernel (see [30]).

It is easy to understand that the fugitive must run away if he is far enough from the boxer, or, if he is already facing close to the boxer, he can take refuge quite near and always keep one's short distance from him. It is the "hand-to-hand" principle.

Moreover, the speed of the fugitive is assumed to be decreasing with its distance to the pursuer. For this reason an area of non-capture appears which does not encounter the target. In this situation, it is much more difficult to compute the barrier using the BUP of the target as we have mentioned above.

5 The Optimal Hitting Time Problem under State Constraints

We now study the optimal Hitting Time under state constraints. In this game, Ursula aims at avoiding a closed target C as long as possible while Victor aims at reaching this target as soon as possible[11]. Moreover, Victor has to ensure the state of the system to remain in a closed set of constraints K as long as it has not reached the target. Note that we do not impose any constraints for Ursula (see [28] for the general problem).

We assume here that f satisfies assumption (37). We keep the notation $\mathcal{U}$ and $\mathcal{V}$ (see (36)). We shall denote by Δ the set of nonanticipative strategies for Victor $\beta : \mathcal{U} \to \mathcal{V}$. Given $x_0 \in X$, $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, $x[x_0, u(\cdot), v(\cdot)]$ denotes as previously the solution to the differential equation (35).

Now let C be the closed target, K be a closed set of constraints and $x(\cdot)$ be a trajectory. The first Hitting Time $\theta_C^K(x(\cdot))$ of $x(\cdot)$ is:

$$\theta_C^K(x(\cdot)) := \inf\{t \geq 0 \mid x(t) \in C,\ x(s) \in K\ \forall s \in [0,t]\}.$$

If the solution $x(\cdot)$ leaves K before reaching C or if $x(\cdot)$ does not reach C, we set $\theta_C^K(x(\cdot)) := +\infty$.

Here we are interested with the optimal Hitting Time, with state constraints, of the closed set $C \subset X$:

Definition 5.1 (Optimal Hitting Time function) *The optimal Hitting Time function $\vartheta_C^K : K \to \boldsymbol{R}^+ \cup \{+\infty\}$ is the map defined by:*

$$\vartheta_C^K(x_0) := \inf_{\beta(\cdot)\in\Delta} \sup_{u(\cdot)\in\mathcal{U}} \theta_C^K\left(x[x_0, u(\cdot), \beta(u(\cdot))]\right)$$

and we define ϑ_C^K on X by setting $\vartheta_C^K(x) = 0$ if $x \in C$.

Remark: Here we are only interested in one Value function of the game. The other one, namely:

$$\sup_{\alpha} \inf_{v(\cdot)} \theta_C^K(x[x_0, \alpha(v(\cdot)), v(\cdot)])$$

is much more difficult to study for constrained problems[12].

In this section, we characterize the optimal Hitting Time function by the mean of the discriminating kernel. Using this characterization, we provide an algorithm for computing this function and give a proof of its convergence.

5.1 Characterization of the optimal Hitting Time function

We show here that the epigraph of the optimal Hitting Time function is a discriminating kernel of some closed set for a suitable hamiltonian. Recall that the epigraph of ϑ_C^K is a subset of $\boldsymbol{R}^{N+1}$. We shall denote by (x, y) any point of $\boldsymbol{R}^{N+1}$, where $x \in X$ and $y \in \boldsymbol{R}$. A proximal normal to a closed subset of $\boldsymbol{R}^{N+1}$ shall be denoted by (p_x, p_y), where $p_x \in X$ and $p_y \in \boldsymbol{R}$.

Theorem 5.2 *Assume that f satisfies assumption (37). Then*

$$\mathcal{E}pi(\vartheta_C^K) = Disc_H\left(K \times \boldsymbol{R}^+\right)$$

where the hamiltonian $H : \boldsymbol{R}^{N+1} \times \boldsymbol{R}^{N+1} \to \boldsymbol{R}$ is defined for all $(x, y) \in \boldsymbol{R}^{N+1}$ and for all $(p_x, p_y) \in \boldsymbol{R}^{N+1}$ by

$$(46) \qquad H(x, y, p_x, p_y) := \begin{cases} \sup_{u\in U} \inf_{v\in V} < f(x, u, v), p_x > -p_y & \text{if } x \notin C \\ \min\{0, \sup_{u\in U} \inf_{v\in V} < f(x, u, v), p_x > -p_y\} & \text{otherwise} \end{cases}$$

Let us remark that

1. Theorem 5.2 states in particular that the optimal Hitting Time function is lower semicontinuous.

[11] **Warning :** in this problem, the role of Ursula and Victor have been exchanged in comparison with the previous section. Victor is now the pursuer while Ursula is the fugitive.

[12] This difficulty is of the same nature than that of the "ε" and "T" in the definition of Ursula's victory domain in the target problem (see Definition 4.2).

2. Theorem 5.2 can be formulated in terms of viscosity solution (see Appendix 2).

Proof : Let us consider

$$g(x,t,u,v) := \{f(x,u,v)\} \times \{-1\}$$

and let $\mathcal{E} := C \times \boldsymbol{R}$ be the evasion set. Then the hamiltonian H defined by (46) is nothing but the hamiltonian defined from g as in (39).

From Theorem 4.5, if (x_0, ρ_0) belongs to $Disc_H(K \times \boldsymbol{R}^+)$, there is a nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ such that, for any control $u(\cdot) \in \mathcal{U}$, the solution to

$$(47) \qquad \begin{cases} x'(t) = f(x(t), u(t), \beta(u(\cdot))(t)) \\ \rho'(t) = -1 \\ x(0) = x_0,\ \rho(0) = \rho_0 \end{cases}$$

remains in $K \times \boldsymbol{R}^+$ as long as $(x(\cdot), \rho(\cdot))$ does not reach the evasion set $\mathcal{E} = C \times \boldsymbol{R}$. Thus, if we set $\tau := \theta_C^K(x[x_0, u(\cdot), \beta(u(\cdot))])$, then the solution remains in $K \times \boldsymbol{R}^+$ on $[0, \tau]$. Since $\rho(t) = \rho_0 - t$ has to be non negative on $[0, \tau]$, this proves that $\tau \le \rho_0$. In particular,

$$\sup_{u(\cdot)} \theta_C^K(x[x_0, u(\cdot), \beta(u(\cdot))]) \le \rho_0$$

which proves that $\vartheta_C^K(x_0) \le \rho_0$. So

$$\mathcal{E}pi(\vartheta_C^K) \subset Disc_H(K \times \boldsymbol{R}^+)$$

Conversely, let ρ_0 be larger than $\vartheta_C^K(x_0)$ and let us prove that (x_0, ρ_0) belongs to $Disc_H(K \times \boldsymbol{R}^+)$. There is a nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ such that

$$\sup_{u(\cdot)} \theta_C^K(x[x_0, u(\cdot), \beta(u(\cdot))]) \le \rho_0$$

since $\rho_0 > \vartheta_C^K(x_0)$. Thus, for any $u(\cdot) \in \mathcal{U}$, $(x[x_0, u(\cdot), \beta(u(\cdot))], \rho_0 - \cdot)$ is a solution to equation (47) which remains in $K \times \boldsymbol{R}^+$ as long as $x[x_0, u(\cdot), \beta(u(\cdot))]$ does not reach C, i.e., as long as the solution $(x[x_0, u(\cdot), \beta(u(\cdot))], \rho_0 - \cdot)$ does not reach the evasion set $C \times \boldsymbol{R}^+$. From Theorem 4.5 again, this means that (x_0, ρ_0) belongs to $Disc_H(K \times \boldsymbol{R}^+)$ for any $\rho_0 > \vartheta_C^K(x_0)$. Since $Disc_H(K \times \boldsymbol{R}^+)$ is a closed set, this proves the desired equality. **Q.E.D.**

Let us point out that choosing $\rho_0 := \vartheta_C^K(x_0)$ in the first part of the proof yields the existence of strategies

Corollary 5.3 *For any $x_0 \in dom(\vartheta_C^K(x_0))$, there is some optimal nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ for Victor, namely*

$$\vartheta_C^K(x_0) = \sup_{u(\cdot) \in \mathcal{U}} \theta_C^K(x[x_0, u(\cdot), \beta(u(\cdot))])$$

5.2 Approximation of the optimal Hitting Time function

5.2.1 Time and state discretizations

As usually, we set

$$F(x,u) := \bigcup_{v \in V} f(x,u,v)$$

We assume also that F is upper bounded and we denote $M := \sup_{x \in K,\ u \in U,; v \in V} |f(x,u,v)|$ The following Lemma characterizes the discriminating domains for H:

Lemma 5.4 *Let H be the hamiltonian defined by (46), where $f : X \times U \times V \to X$ is $\ell-$Lipschitz. Let us define the set-valued map $\Phi : X \times \boldsymbol{R} \times U \rightsquigarrow X \times \boldsymbol{R}$ by*

$$\begin{array}{rcll} \Phi(x,y,u) & := & F(x,u) \times \{-1\} & \text{if} x \notin C \\ & := & \overline{Co}\,[F(x,u) \times \{-1\} \,\cup\, \{(0,0)\}] & \text{otherwise} \end{array}$$

A closed set $D \subset X \times \boldsymbol{R}$ is a discriminating domain for H if and only if it is a viability domain for $x \rightsquigarrow \Phi(x,u)$ for any $u \in U$.

The proof of Lemma 5.4 is straightforward and is left to the reader.

We now discretize the dynamic Φ. We use the same kind of discretization than that for the Minimal Time function in section 3. Let us define the time discretization Φ_ε of $\Phi(\cdot,\cdot)$

$$\Phi_\varepsilon(x,y,u) := \begin{cases} \{F(x,u) + M\ell\varepsilon\mathcal{B}\} \times \{-1\} & \text{if } d_C(x) > M\varepsilon \\ \overline{Co}\,[\{0,0\} \,\cup\, \{F(x,u) + M\ell\varepsilon\mathcal{B}\} \times \{-1\}] & \text{otherwise} \end{cases}$$

As in subsection 2.3, we consider a discretization $X_h \times R_h$ (R_h is a discretization of $\boldsymbol{R}$ and X_h satisfies (20)) of the state space $X \times \boldsymbol{R}$ and a discretization U_h of the control state U.

We introduce now $Z_h := X_h \times R_h$ and the fully discrete dynamics $\Gamma_{\varepsilon,h} : Z_h \times U_h \rightsquigarrow Z_h$. Let us set $\alpha_{\varepsilon,h} = \alpha(\varepsilon,h) := 2h + \ell\varepsilon h + M\ell\varepsilon^2$. Then

$$\Gamma_{\varepsilon,h}(x_h, y_h, u_h) := \begin{cases} \{x_h + \varepsilon F(x_h,u_h) + \alpha_{\varepsilon,h}\mathcal{B}\} \times \{y_h - \varepsilon + [-h,h]\} \cap Z_h & \text{if } d_C(x_h) > M\varepsilon + h \\ \overline{Co}[\{x_h,y_h\} \cup \{x_h + \varepsilon F(x_h,u_h) + \alpha_{\varepsilon,h}\mathcal{B}\} \times \{y_h - \varepsilon + [-h,h]\}] \cap Z_h & \text{otherwise} \end{cases}$$

Let us point out that, as in Lemma 3.4, such a $\Gamma_{\varepsilon,h}(\cdot,\cdot)$ is a "good" discretization of $\Phi(\cdot,\cdot)$. Thus, from Theorem 4.11,

Proposition 5.5 *Assume that $f : X \times U \times V \to X$ satisfies (37) and is bounded by some constant M (i.e., satisfies (45)). Then*

$$\operatorname*{Lim}_{\varepsilon\to 0^+,\ \frac{h}{\varepsilon}\to 0^+} \overrightarrow{\mathrm{Disc}}_{\Gamma_{\varepsilon,h}}(K \times \boldsymbol{R}^+) \;=\; \mathcal{E}pi(\vartheta_C^K)$$

5.2.2 A fully discrete numerical scheme for the optimal Hitting Time function

Proceeding exactly as in subsection 3.2.3, we deduce a fully discrete scheme for approaching the Hitting Time function $\vartheta_C^K(\cdot)$, applying again the refinement principle.

The only change in the procedure is the expression of $T_p^{n+1}(\cdot)$ in function of $T_p^n(\cdot)$ where is added a *maximizing* operation

$$T_p^{n+1}(x) := \begin{cases} \varepsilon_p - h_p + \max_{u\in U} \min_{v\in V\, b\in\mathcal{B}} T_p^n(x + \varepsilon_p f(x,u,v) + \alpha_p b), & \text{if } d_C(x) > M\varepsilon_p + h_p \\ T_p^n(x) & \text{otherwise} \end{cases}$$

5.2.3 Outline of the Algorithm

PROCEDURE OF CONSTRUCTION OF THE DATA (parameter p)

$R_{h_p} \leftarrow 2^{-p}\mathbb{Z}$, $X_{h_p} \leftarrow 2^{-p}\mathbb{Z}^N$. {Definition of the grids R_{h_p} and X_{h_p}.}
U_p, V_p. {Definition of the grids of U and V.}
$h_p \leftarrow 2^{-p}$, $\varepsilon_p = \sqrt{h_p/M\ell}$ {Definition of steps h_p and $\varepsilon_p \in h_p\mathbb{Z}$}
$\alpha_p \leftarrow 2h_p + \ell\varepsilon_p(h_p + M\varepsilon_p)$ {Definition of the dilation term.}

INITIALIZATION

$p \leftarrow 1$
$T_1^0(x) \leftarrow 0$

MAIN LOOP Hitting Time Problem

For $p := 1$ **to** $\bar{p}$ **do**
  $n := 0$
  Repeat {Beginning of calculus of T_p^∞}
    $x \leftarrow x_{p\downarrow}$ {$x_{p\downarrow}$ is the first point of the grid K_{h_p}}
    Repeat {Scanning of the grid}
     if $d_C(x) > M\varepsilon_p + h_p$
      then $T_p^{n+1}(x) \leftarrow [\varepsilon_p - h_p + \max_{u\in U_p} \min_{v\in V_p, b\in\mathcal{B}} T_p^n(x + \varepsilon_p f(x,v) + \alpha_p b)]$
      else $T_p^{n+1}(x) \leftarrow T_p^n(x)$
     $x \leftarrow Next(x)$ {$Next(x)$ is the following point of the grid}
    until $x \equiv x_{p\uparrow}$ {$x_{p\uparrow}$ is the last point of the grid}
    $n \leftarrow n+1$
    until $T_p^{n+1} \equiv T_p^n$ {End of calculus of T_p^∞}
    Set $T_p^\infty \leftarrow T_p^n$
    if $p < \bar{p}$ **then**
      $x \leftarrow x_{(p+1)\downarrow}$
      Repeat {Scanning for Refinement Process}
       if $x \in K_{h_{p+1}}$
        then $T_{p+1}^0(x) \leftarrow -(h_p + h_{p+1}) + \min_{b\in\mathcal{B}} T_p^\infty(x + (h_p + h_{p+1})b)$

$$\textbf{else} \quad T^0_{p+1}(x) \leftarrow +\infty$$
$$x \leftarrow Next(x)$$
$$\textbf{until } x \equiv x_{(p+1)\uparrow}$$ {End Scanning and End of Refinement}

Return

$T^{\infty}_{\bar{p}}(x)$ $\{T^{\infty}_{\bar{p}}(\cdot)$ is the approached Hitting Time function at step $\bar{p}$ $\}$

Using discretization $\Gamma_p := \Gamma_{\varepsilon_p,h_p}$ defined in the previous subsection, we deduce easily from Theorem 2.23:

Corollary 5.6 *If assumptions of Theorem 2.23 are fulfilled, the sequence T^{∞}_p converges to ϑ^K_C in the epigraphic sense*

$$\mathcal{E}pi(\vartheta^K_C) = \operatorname*{Lim}_{p\to+\infty} \mathcal{E}pi(T^{\infty}_p)$$

Moreover,

$$\forall x_p \in R_{h_p},\ \ T^{\infty}_p(x_p) \leq \vartheta^K_C((x_p))$$

and T^{∞}_p converges pointwisely to ϑ^K_C

$$\forall x \in K,\ \vartheta^K_C(x) = \lim_{p\to+\infty} \left(\min_{x_p \in (x+h_p\mathcal{B})\cap X_{h_p}} T^{\infty}_p(x_p) \right)$$

The proof of this Corollary is very similar to the proof of Corollary 3.7.

5.3 Example of Optimal Hitting Time for differential game

We consider the same problem as presented in Subsection 4.4.

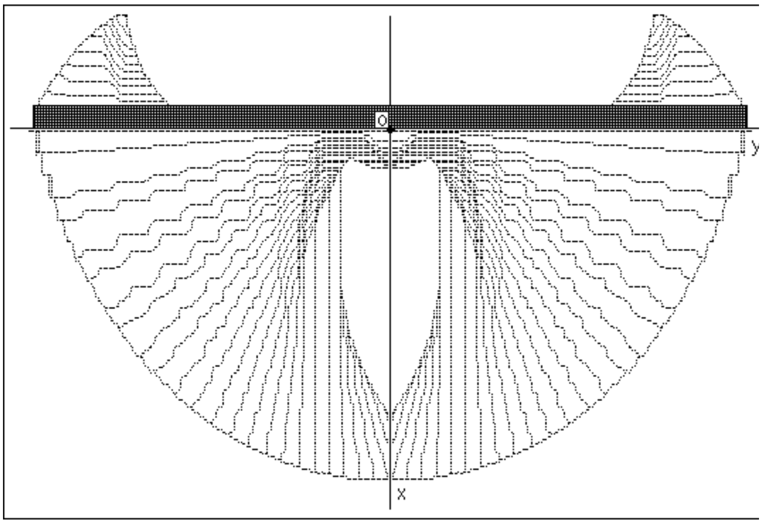

Figure 8: Level curves of minimal Hitting Time of capture for a differential game problem

Let us recall the equations of the system:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = W_p \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{W_p}{0.8} \begin{pmatrix} -y(t) \\ x(t) \end{pmatrix} u(t) + f(x(t), y(t))\, v(t)$$

The closed target is still given by $C = \{(x, y),\ -0.2 \leq x \leq 0,\ -3.5 \leq y \leq 3.5\}$ and the numerical values for the pursuer and fugitive's maximal speed are $W_p = 1$ and $W_f = 1.1$.

On the figure 5.3 are represented the level curves of the minimal Hitting Time function and on the figure 5.3 is represented the graph of this function for positive values of y. One can see the shape of the barrier which borders the "hole of safety".

6 Appendix 1 - Basic notions of Set-Valued Analysis

We present here some basic tools of set-valued analysis. Our propose is not to give an introduction on that (large) subject, but just to state the definitions and results needed all along this paper. For a complete presentation, see monograph [3].

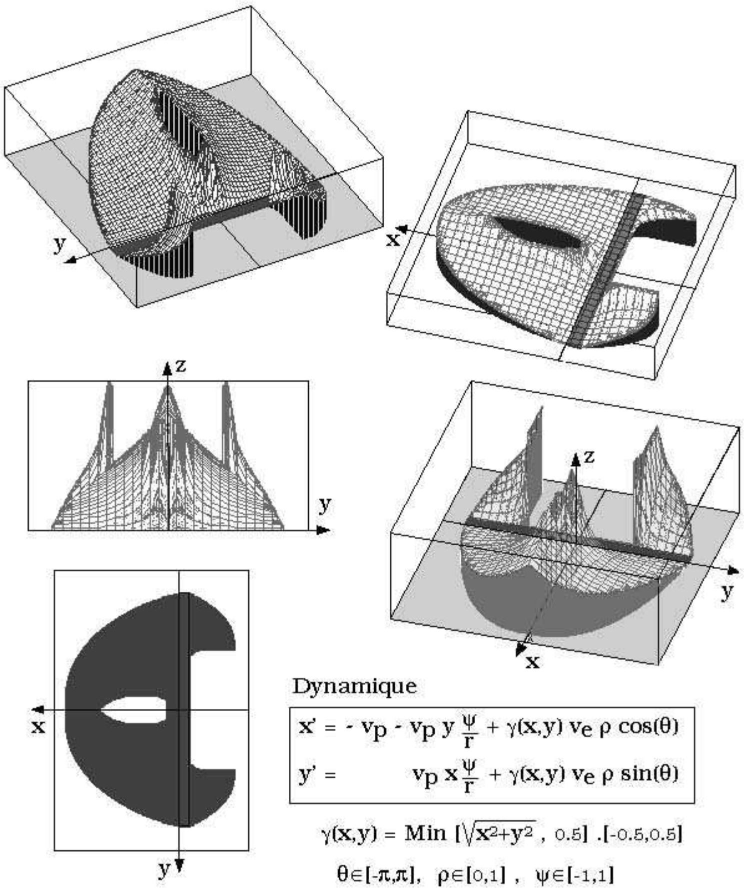

Figure 9: Graph of the minimal Hitting Time function for acoustic capture problem

6.1 Conventions and notations

Let us recall that $\mathcal{B}$ denotes the closed unit ball of the current space. In general - except for the proximal normal below - we do not need to specify the kind of norm we use.
The distance from a point x to the set A is denoted

$$d_A(x) \ := \ \inf_{y \in A} \|x-y\|.$$

For any pair A and B of sets, we define the sum

$$A+B \ := \ \{a+b \mid a \in A \ \& \ b \in B \ \}$$

If A is compact and B is closed, $A+B$ is closed.
If A and B are convex, $A+B$ is convex.
The notation $A+\epsilon\mathcal{B}$ denotes the $\epsilon-$neighborhood of A

$$A+\varepsilon\mathcal{B} \ := \ \{x \in \boldsymbol{R}^N \mid d_A(x) \leq \varepsilon \ \}.$$

and in the same way, for any $x \in \boldsymbol{R}^N$, the notation $x+\varepsilon\mathcal{B}$ denotes the ball centered at x of radius ε

$$x+\varepsilon\mathcal{B} \ := \ \{y \in \boldsymbol{R}^N \mid \|y-x\| \leq \epsilon \ \}.$$

6.2 Convergence of sets

Let $A(s)$ be subsets of $\boldsymbol{R}^N$ parameterized by $s \in S$ where S is a metric space.

Definition 6.1 (Kuratowski Upper Limit) *The upper-limit of $A(s)$ when $s \to \bar{s}$ is the set*

$$\left\{x \in \boldsymbol{R}^N \mid \liminf_{s \to \bar{s}} d_{A(s)}(x) \ = \ 0\right\}$$

We denote the upper-limit of the sets $A(s)$ when $s \to \bar{s}$ by $\operatorname*{Limsup}_{s \to \bar{s}} \ A(s)$.

In particular a point x belongs to the upper limit of the $A(s)$ when $s \to \bar{s}$ if and only if there exists $s_n \to \bar{s}$, $x_n \in A(s_n)$ with $x_n \to x$.

The upper-limit is empty if and only if $d_{A(s)}(0) \to +\infty$ when $s \to \bar{s}$. Thus, if the A_s are contained in a compact set, the upper-limit is not empty.

Definition 6.2 (Kuratowski Lower Limit) *The lower-limit of the $A(s)$ when $s \to \bar{s}$ is the set*

$$\left\{ x \in \boldsymbol{R}^N \mid \lim_{s \to \bar{s}} d_{A(s)}(x) = 0 \right\}$$

We denote the lower-limit of the sets $A(s)$ when $s \to \bar{s}$ by $\operatorname*{Liminf}_{s \to \bar{s}} A(s)$.

A point x belongs to the lower-limit of the sets $A(s)$ when $s \to \bar{s}$ if and only if, for any $s_n \to \bar{s}$, there exists $x_n \in A(s_n)$ with $x_n \to x$.

The lower-limit may be empty. It is always contained in the upper-limit:

$$\operatorname*{Liminf}_{s \to \bar{s}} A(s) \subset \operatorname*{Limsup}_{s \to \bar{s}} A(s)$$

We are now ready to prove the following result (formerly announced as Lemma 2.22): Let A be a closed subset of X and define $A_h := (A + h\mathcal{B}) \cap X_h$, where X_h is a discretization of the space X satisfying (20). Then

$$A = \operatorname*{Liminf}_{h \to 0^+} A_h$$

Proof of 2.22 : Since $A_h \subset A + h\mathcal{B}$, we have

$$\operatorname*{Liminf}_{h \to 0^+} A_h \subset \operatorname*{Limsup}_{h \to 0^+} A_h \subset \operatorname*{Limsup}_{h \to 0^+} (A + h\mathcal{B}) \subset A$$

Conversely, let $x \in A$. From the very definition of X_h, there is some $x_h \in X_h$ such that $\|x - x_h\| \leq h$. In particular, for any h, x_h belongs to A_h. Consequently $x \in \operatorname*{Liminf}_{h \to 0^+} A_h$. So we have proved that $A \subset \operatorname*{Liminf}_{h \to 0^+} A_h$. **Q.E.D.**

Definition 6.3 (Kuratowski Limit) *If the upper-limit of $A(s)$ coincides with the lower-limit of $A(s)$ when $s \to \bar{s}$, then we say that $A(s)$ has a limit when $s \to \bar{s}$.*

We denote the limit of $A(s)$ when $s \to \bar{s}$, when it exists, by $\operatorname*{Lim}_{s \to \bar{s}} A(s)$.

6.3 Continuity of set-valued maps

In this chapter, we are often dealing with *Marchaud* maps and *Lipschitz* set-valued maps. We explain here what it is.

A *set-valued map* is an application F whose values are sets: It associates with any point of a space X a subset of a space Y. We denote it $F : X \rightsquigarrow Y$.

Assume now that X and Y are metric spaces. A set-valued map $F : X \rightsquigarrow Y$ is *upper semicontinuous at $x_0 \in X$* if

$$\operatorname*{Limsup}_{x \to x_0} F(x) \subset F(x_0)$$

The set-valued map F is *upper semicontinuous* if it is upper semicontinuous at each point $x_0 \in X$.

If a set-valued map F is upper semicontinuous, then its graph

$$\operatorname{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$$

is closed. The converse holds true if F is locally bounded.

Definition 6.4 (Marchaud map) *Let X be a finite dimensional space and $F : X \rightsquigarrow X$. The set-valued map F is a Marchaud map if it satisfies the following conditions*

i) F is upper semicontinuous.

ii) for any $x \in X$, the values $F(x)$ are convex compact and nonempty.

iii) F has a linear growth, that is to say that there is a constant $c > 0$ such that

$$\forall x \in X,\ \forall y \in F(x),\ \|y\| \leq c(\|x\| + 1)$$

We mainly consider set-valued maps derived from a control system. Let $f : X \times U \to X$ describe the dynamic of a control system where X is the state space and U the set of controls. The associated set-valued map is then $F(x) := \bigcup_{u \in U} f(x,u)$.
The following conditions on f
i) U is convex compact.
ii) $f : X \times U \to X$ is continuous and affine with respect to u.
iii) f has a linear growth: there exists a constant $c > 0$ such that

$$\forall x \in X, \ \forall u \in U, \ \ \|f(x,u)\| \le c(\|x\| + 1).$$

ensure that the associated set-valued map F is a Marchaud map.

Definition 6.5 (Lipschitz set-valued maps) *Let X be a finite dimensional space and $F : X \rightsquigarrow X$. The set-valued map F is Lipschitz with constant of Lipschitz ℓ (or $\ell-$Lipschitz) if*

$$\forall x \in X, \ \forall y \in X, \ \ F(y) \ \subset F(x) + \ell\|x - y\|\mathcal{B}$$

The following conditions on f
i) U is convex, compact and nonempty.
ii) $f : X \times U \to X$ is continuous, affine with respect to u and Lipschitz with respect to x.
iii) f has a linear growth
ensure that the associated set-valued map F is Marchaud and Lipschitz

6.4 The proximal normals

Definition 6.6 *Let K be a closed subset of a finite dimensional space X. A vector $p \in X$ is a proximal normal to K at $x \in K$ if*

$$d_K(x + p) = \|p\|$$

Here $d_K(y)$ denotes the distance from y to K, $\inf_{z \in K} \|y - z\|$, where $\| \cdot \|$ denotes the euclidean norm.

We denote by $\mathcal{NP}_K(x)$ the set of proximal normals to K at x. The fact that the norm is the euclidian norm is crucial here. A vector p is a proximal normal to K at x if and only if the open ball centered at $x + p$ of radius $\|p\|$ does not intersect K. Note that the closed ball intersects K at least at x. So this ball is somehow tangent to K at x.

If a point y does not belong to the closed set K and if x is a projection of y onto K (i.e., $d_K(y) = \|y - x\|$ and $x \in K$), then the vector $p := y - x$ is a proximal normal to K at x.

Another important property used in this chapter is the following

Proposition 6.7 *Let p be a proximal normal to K at x (with $p \neq 0$). Then, for any $\lambda \in]0, 1[$, x is the unique projection of the point $x + \lambda p$ onto K and λp is also a proximal normal.*

7 Appendix 2 - A Hamilton-Jacobi formulation of the Optimal Hitting Time

Here we show that the optimal Hitting Time function with state constraints is the smallest (viscosity) super-solution of some Hamilton-Jacobi-Isaacs equation[13]. Our purpose is *not* to show that this characterization is suitable for the approximation of Value functions, but is just to precise the link between the viscosity approach that can be found in the literature on control and differential games and the Viability approach.

The following results are mainly inspired by H. Frankowska [42], who first underlined the relations between Viability theory and Hamilton-Jacobi equations (see also [45] for the time-measurable case). The most natural concept of solutions, in the Viability point of view, is the notion of contingent solutions introduced in [42] by H. Frankowska. In fact, the characterization of the optimal Hitting Time function as the smallest (contingent) super-solutions can be deduced from Theorem 5.2.

[13]This characterization is known for problems without state constraints. It has been proved, in addition, that the Value function is the supremum of subsolutions (see [7], [61] and [56]).

7.1 Different formulations for the viscosity solutions

The aim of this subsection is to provide another formulation of the viscosity solutions to

$$H(x, w(x), Dw(x)) = 0 \tag{48}$$

involving the proximal normals. The results of this subsection can be found in [31].

Definition 7.1 (Viscosity super-solutions) *A lower semicontinuous map $w(\cdot) : X \to \boldsymbol{R}$ is a viscosity super-solution to (48) if, for any $x \in X$,*

$$\begin{array}{c}\text{If } \phi \text{ is } \mathcal{C}^1 \text{ and if } w - \phi \text{ has a local minimum at } x, \\ \text{then } H(x, w(x), \nabla\phi(x)) \geq 0\end{array} \tag{49}$$

We intend to provide an equivalent formulation of (49) using the *proximal normals*. If $w(\cdot) : X \to \boldsymbol{R}$ is a lower semicontinuous map, then any proximal normal to $\mathcal{E}pi(w(\cdot))$ at $(x, w(x))$ belongs to $X \times \boldsymbol{R}$. We shall denote it (ν_x, ν_ρ), with $\nu_x \in X$ and $\nu_\rho \in \boldsymbol{R}$. It is not difficult to check that $\nu_\rho \leq 0$.

Theorem 7.2 *Assume that the Hamiltonian $H : X \times \boldsymbol{R} \times X \to \boldsymbol{R}$ is lower semicontinuous. Let $w(\cdot) : X \to \boldsymbol{R} \cup \{+\infty\}$ be an extended lower semicontinuous map.*
Then $w(\cdot)$ is a viscosity super-solution to (48) on X if and only if $w(\cdot)$ satisfies:

$$\begin{array}{c}\forall x \in X,\ \forall (\nu_x, \nu_\rho) \in \mathcal{NP}_{\mathcal{E}pi(w(\cdot))}(x, w(x)), \\ \\ \nu_\rho \neq 0 \implies H(x, w(x), \nu_x/|\nu_\rho|) \geq 0.\end{array} \tag{50}$$

Since this Theorem does not explain what happens for the proximal normals of the form $(\nu_x, 0)$ at $(x, w(x))$ and for the proximal normals at (x, ρ) for $\rho > w(x)$, we shall need the following Lemmas. The first one is a generalization of a Lemma due to Rockafellar. This result is also used in [42].

Lemma 7.3 *Let $w(\cdot) : X \to \boldsymbol{R} \cup \{+\infty\}$ be a lower semicontinuous map and $\bar{x}$ belong to the domain of $w(\cdot)$. If $(\nu_x, 0) \in \mathcal{NP}_{\mathcal{E}pi(w(\cdot))}(\bar{x}, w(\bar{x}))$, there exist sequences $x^n \in Dom(w(\cdot))$ and $(\mu_x^n, \mu_\rho^n) \in \mathcal{NP}_{\mathcal{E}pi(w(\cdot))}(x^n, w(x^n))$ such that*

$$x^n \to x,\ w(x^n) \to w(\bar{x}),\ \mu_\rho^n < 0 \text{ and } \frac{(\mu_x^n, \mu_\rho^n)}{\|(\mu_x^n, \mu_\rho^n)\|} \to \frac{(\nu_x, 0)}{\|\nu_x\|}$$

Lemma 7.4 *Let $w(\cdot) : X \to \boldsymbol{R} \cup \{+\infty\}$ be a lower semicontinuous map and $\bar{x}$ belong to the domain of $w(\cdot)$. If $(\nu_x, \nu_\rho) \in \mathcal{NP}_{\mathcal{E}pi(w(\cdot))}(\bar{x}, \rho)$ (with $\rho > w(\bar{x})$), then $\nu_\rho = 0$ and $(\nu_x, 0) \in \mathcal{NP}_{\mathcal{E}pi(w(\cdot))}(\bar{x}, w(\bar{x}))$.*

7.2 Hamilton-Jacobi formulation of Optimal Hitting Time Problem with Constraints

Our purpose now is to characterize the optimal Hitting Time function ϑ_C^K in differential games (c.f. subsection 5.1) as the smallest super-solution to an Hamilton-Jacobi equation. The hamiltonian of our problem is

$$\forall (x, p) \in X \times X, \quad H_0(x, p) := \inf_{u \in U} \sup_{v \in V} < -f(x, u, v), p > +1.$$

Theorem 7.5 *Suppose that the assumptions of Theorem 5.2 are fulfilled. Then the optimal Hitting Time function ϑ_C^K is the smallest non negative lower semicontinuous super-solution to*

$$\left\{\begin{array}{rl} H_0(x, \nabla u(x)) = 0 & \text{if } x \in K \backslash C \\ \\ u(x) = 0 & \text{if } x \in \partial C \end{array}\right. \tag{51}$$

Similar results are proved, with different definitions of super-solution, in [56]. Recall also that another characterization as contingent solutions can be deduced directly from Theorem 5.2.

Proof of Theorem 7.5

In Theorem 5.2, we have characterized ϑ_C^K as the discriminating kernel of $\mathcal{K} := K \times \boldsymbol{R}^+$ for the hamiltonian H defined by

$$\begin{array}{l}\forall (x, \rho) \in X \times \boldsymbol{R},\ (p_x, p_\rho) \in X \times \boldsymbol{R}, \\ H(x, \rho, p_x, p_\rho) \ := \left\{\begin{array}{ll} \sup_{u \in U} \inf_{v \in V} < f(x, u, v), p_x > -p_\rho & \text{if } x \notin C \\ \min \{ 0 \ ; \ \sup_{u \in U} \inf_{v \in V} < f(x, u, v), p_x > -p_\rho \} & \text{otherwise.} \end{array}\right.\end{array}$$

In particular, for any $x \notin C$, for any $(\nu_x, \nu_\rho) \in \mathcal{NP}_{\mathcal{E}pi(\vartheta_C^K)}(x, \vartheta_C^K)$,

$$\nu_\rho < 0 \;\; \Rightarrow \;\; H(x, \nu_x, \nu_\rho) \leq 0, \text{ i.e., } H_0(x, \frac{\nu_x}{|\nu_\rho|}) \geq 0.$$

from the very definition of H on $X \backslash C$. So Theorem 7.2 states that ϑ_C^K is a lower semicontinuous super-solution to the Hamilton-Jacobi equation (51).

We now prove that, if $w(\cdot)$ is a lower semicontinuous non negative super-solution to Hamilton-Jacobi equation (51), then $w(\cdot)$ is larger than or equal to ϑ_C^K. For that purpose, let us show that $\mathcal{E}pi(w(\cdot))$ is a discriminating domain for H.

Theorem 7.2 states that $\forall x \notin C$, $\forall (\nu_x, \nu_\rho) \in \mathcal{NP}_{\mathcal{E}pi(w(\cdot))}(x, w(x))$,

$$\nu_\rho < 0 \;\Rightarrow\; H_0(x, \frac{\nu_x}{|\nu_\rho|}) \geq 0.$$

Since H is positively homogeneous with respect to (ν_x, ν_ρ)

$$\nu_\rho < 0 \;\Rightarrow\; H(x, \nu_x, \nu_\rho) \leq 0.$$

If $\nu_\rho = 0$, then Lemma 7.3 yields the existence of sequences (x_n), $(\nu_x^n, \nu_\rho^n) \in \mathcal{NP}_{\mathcal{E}pi(u(\cdot))}(x_n, w(x_n))$ such that

$$x_n \to x, \;\nu_\rho^n < 0 \text{ and } \frac{(\nu_x^n, \nu_\rho^n)}{\|(\nu_x^n, \nu_\rho^n)\|} \to \frac{\nu_x}{\|\nu_x\|}$$

Note that $H(x_n, \nu_x^n, \nu_\rho^n) \leq 0$. Since H is lower semicontinuous and positively homogeneous, we conclude that $H(x, \nu_x, 0) \leq 0$.

If $(\nu_x, \nu_\rho) \in \mathcal{NP}_{\mathcal{E}pi(w(\cdot))}(x, \rho)$ with $\rho > u(x)$, then Lemma 7.4 states that $\nu_\rho = 0$ and $(\nu_x, 0) \in \mathcal{NP}_{\mathcal{E}pi(w(\cdot))}(x, w(x))$. Thus $H(x, \nu_x, 0) \leq 0$. So we have finally proved that $\mathcal{E}pi(w(\cdot))$ is a discriminating domain for H.

Since $\mathcal{E}pi(w(\cdot))$ is obviously contained in $\mathcal{K}$, the discriminating domain $\mathcal{E}pi(w(\cdot))$ is contained in the discriminating kernel of $\mathcal{K}$ for H, which is actually the epigraph of ϑ_C^K. So any super-solution $w(\cdot)$ is greater than or equal to the super-solution $\vartheta_C^K(\cdot)$. **Q.E.D.**

References

[1] ALZIARY de ROQUEFORT (1991) *Jeux différentiels et approximation numérique de fonctions valeur*, RAIRO Math. Model. Numer. Anal., 25, 517-560.

[2] AUBIN J.-P. & CELLINA A. (1984) Differential Inclusions Springer-Verlag, Berlin

[3] AUBIN J.-P.& FRANKOWSKA H. (1992) Set-valued analysis. Birkhaüser.

[4] AUBIN J.-P. (1992) Viability Theory. Birkhaüser.

[5] BYRNES C. I. & ISIDORI A. (1988) *Local stabilization of minimum-phase nonlinear systems.* Syst. Contr. Let. Vol. 11, 9-17.

[6] BARDI M. (1989) *A boundary value problem for the minimum time function.* SIAM J. Control and Opti, 26, 776-785.

[7] BARDI M., BOTTACIN & FALCONE M. (1995) *Convergence of discrete schemes for discontinuous value functions of pursuit-evasion games*, New Trends in dynamic games and applications, G.J.Olsder ed., p.273-304, Birkaüser.

[8] BARDI M., FALCONE M & SORAVIA P. (1993) *Fully discrete schemes for the Value function of pursuit-evasion games* in T. Basar e A. Haurie eds., *Annals of Dynamic Games*, Vol. 1, Birkaüser.

[9] BARDI M. & SORAVIA P. (1990) *Approximation of differential games of pursuit-evasion by discrete-time games* Differential games - Developments in modelling and computation, R.P. Hamalainen and H.K. Ethamo eds., Lecture Note Control Inform Sci. 156, pp. 131-143, Springer 1991.

[10] BARDI M. & SORAVIA P. (1991) *Hamilton-Jacobi Equations with singular boundary conditions on a free boundary and applications to differential games* Trans. American Math. Soc., 325, 1, 205-229.

[11] BARDI M. & STAICU V. (1993) *The Bellman Equation for Time-Optimal Control of Noncontrollable, Nonlinear Systems.* Acta Applic. Math. 31, 201-223.

[12] BARLES G. & PERTHAME B. (1988) *Exit time problems in optimal control and vanishing viscosity method* SIAM J. Control and Opti. 26,1133-1148.

[13] BARLES G. & SOUGANIDIS P.E. (1991) *Convergence of approximation schemes for fully non-linear systems*, Asymptotic Anal. 4, 271-283.

[14] BARLES G. (1993) *Discontinuous viscosity solutions of first-order Hamilton-Jacobi Equations: A guided visit.* Nonlinear Analysis, Theory Methods and Appl. Pergamon Press, 9, 1123-1134.

[15] BARRON E. & JENSEN R. (1992) *Optimal control and semicontinuous viscosity solutions.* Proc. American Math. Soc. 113, 397-402.

[16] BERNHARD P. (1976) Commande Optimale, decentralisation, et jeux dynamiques. Dunod.

[17] BERNHARD P. (1979) Contribution à l'étude des jeux différentiels à some nulle et information parfaite. Thèse Université de Paris VI.

[18] BERNHARD P. (1988) *Differential games* in Systems and control Encyclopedia, Theory Technology Application, M, G. Singh Ed, Pergamon Press.

[19] BERNHARD P. (1990) *A simple game with a singular focal line* In J. of Optimization Theory and Appl, vol. 64, 2, 419-428.

[20] BERNHARD P. & LARROUTUROU B. (1989) *Etude de la barrière pour un problème de fuite optimale dans le plan.* preprint Rapport de recherche INRIA.

[21] BREAKWELL J.V. (1977) *Zero-sum differential games with terminal payoff.* In Differential Game and Applications, Hagedorn p., Knobloch H.W. & Olsder G.H. Ed Lecture Notes in Control and Information Sciences Vol.3, Springer Verlag.

[22] BYRNES C.I. & ISIDORI A. (1990) *Régulation asymptotique de Systèmes non Linéaires* Comptes-Rendus de l'Académie des Sciences, Paris, 309, 527-530

[23] CAPUZZO-DOLCETTA I. & FALCONE M. (1989) *Discrete dynamic programming and viscosity solutions of the Bellman Equation.* Ann. I.H.P. Anal. Non Lin. 6, 161-183.

[24] CARATHEODORY C. (1935) Calculus of Variations and Partial Differential Equations of the First Order (1989 Edition) Chelsea Publishing Company, New-York.

[25] CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. (1994) *Some Algorithms for Differential Games with two Players and one Target.* Mathematical Modelling and Numerical Analysis, Vol. 28, N 4, 441-461.

[26] CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. (1994) *Temps optimaux pour des problèmes avec contraintes et sans contrôlabilité locale* - Comptes Rendus de l'Académie des Sciences, T.318, s I, 1994, 607-612 1994

[27] CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. (to appear) *Optimal times for constrained non-linear control problems without local controllability* Applied Math. & Optimization.

[28] CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. (1995) *Differential games with state constraints* Preprint. Cahiers de Mathématiques de la Décision. Université Paris IX Dauphine.

[29] CARDALIAGUET P. (1996) *A differential game with two players and one target.* SIAM J. Contr. Opti., Vol. 34, No. 4, pp. 1441-1460.

[30] CARDALIAGUET P. (to appear) *Non smooth semi-permeable barriers, Isaacs equation and application to a differential game with one target and two players* Applied Math. & Optimization.

[31] CARDALIAGUET P. (in preparation) *Direct Construction of Generalized Motion of a Front Moving along its Normal Direction.*

[32] CARDALIAGUET P. (in preparation) *Regularity Results for Discontinuous Value Functions of Control Problems and Convergence Rates of their Approximations.*

[33] CHENTSOV A.G. (1976) *On a Game Problem of Converging at a Given Instant of Time* Math USSR Sbornik, Vol 20, 3, 353-376.

[34] CHENTSOV A.G. (1978) *An Iterative Program Construction for Differential Games with Fixed Termination Time* Soviet Math Doklady, Vol 19, 3, 559-562.

[35] COLOMBO G. & KRIVAN V. (1993) *A Viability Algorithm*, J. Diff. Equations 102, 236-243

[36] CRANDALL M.G. & LIONS P.L. (1983) *Viscosity solutions of Hamilton-Jacobi Equations.* Trans. Amer. Math. Soc., 277, 1-42.

[37] DOYEN L. & SAINT-PIERRE P. (1995) *Scale of viability and minimal time of crisis*, Chaiers de Mathématiques de la Décision, No. 9513

[38] ELLIOT N.J. & KALTON N.J. (1972) *The existence of Value in Differential Games of Pursuit and Evasion* J. Differential Equations, Vol 12, 504-523.

[39] EVANS L.C. & SOUGANIDIS P.E. (1984) *Differential games and representation formulas for solutions of Hamilton-Jacobi Equations* Indiana Univ. Math. J., 33, 773-797.

[40] FRANKOWSKA H. (1987) *L'équation d'Hamilton-Jacobi Contingente* Comptes-Rendus de l'Académie des Sciences, Paris 303, Série 1, 733-736

[41] FRANKOWSKA H. (1991) *Lower semicontinuous solutions to Hamilton-Jacobi-Bellman equations*, Proceedings of 30th CDC Conference, IEEE, Brighton, England.

[42] FRANKOWSKA H. (1993) *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations*, SIAM J. Control and Optimization, vol.31, N.1, 257-272.

[43] FRANKOWSKA H. & QUINCAMPOIX M. (1991) *Viability kernels of differential inclusions with constraints: Algorithm and applications.* Mathematics of Systems, Estimation and Control. vol.1, No 3, 371-388.

[44] FRANKOWSKA H. & QUINCAMPOIX M. (1991) *Un algorithme déterminant les noyaux de viabilité pour les inclusions différentielles avec contraintes* Comptes-Rendus de l'Académie des Sciences. Série I. PARIS, t. 312, 31-36.

[45] FRANKOWSKA H., PLASCASZ M. & RZEZUCHOWSKI T. (1995) *Measurable Viability Theorem and Hamilton-Jacobi-Bellman Equations*, J. Diff. Eqs., Vol. 116, No. 2, pp. 265-305.

[46] ISIDORI A.(1989) NONLINEAR CONTROL SYSTEMS. 2^{nd} Edition. Springer-Verlag.

[47] ISAACS R. (1965) DIFFERENTIAL GAMES. Wiley, New York

[48] KRASOVSKII N.N. & SUBBOTIN A.I. (1988) GAME-THEORICAL CONTROL PROBLEMS Springer-Verlag, New-York.

[49] PLASKACZ S. *Personal communication.*

[50] PONTRYAGIN N.S. (1968) *Linear Differential Games I and II*, Soviet Math. Doklady, Vol 8, 3 & 4, 769-771 & 910,912.

[51] POURTALLIER O. & TIDBALL M. (1994) *Approximation of the Value Function for a Class of Differential Games with Target*, Preprint Volume to the sixth international symposium on dynamic games and applications, St-Jovite, Québec, Canada.

[52] QUINCAMPOIX M. (1991) *Playable Differential Games.* J. Math. Anal. Appl. vol.1, No 1, 194-211.

[53] QUINCAMPOIX M. (1992) *Differential inclusions and target problems* SIAM J. Control and Optim. Vol. 30, No 2, 324-335.

[54] QUINCAMPOIX M. & SAINT-PIERRE P. (1995) *An Algorithm for Viability Kernels in Holderian Case: Approximation by discrete dynamic systems.* Summary in Journal of Mathematics of Systems, Estimation and Control, vol.5 N.1, 115-118.

[55] ROXIN E. (1969) *The axiomatic approach in differential games*, J. Optim. Theory Appl. 3, 153-163.

[56] ROZYEV I. & SUBBOTIN A.I. (1988) *Semicontinuous solutions of Hamilton-Jacobi Equations.* PMM U.S.S.R., Vol. 52, N. 2, 141-146.

[57] SAINT-PIERRE P. (1991) *Viability of Boundary of the Viability kernel.* Journal of Differential and Integral Equation, Vol. 4, No 3, 1147-1153.

[58] SAINT-PIERRE P. (1994) *Approximation of the Viability Kernel.* Applied Mathematics & Optimisation, 29, 187-209.

[59] SONER M.H. (1986) *Optimal control problems with state space constraints.* SIAM J. on Control and Optimization, 24, 552-562 and 1110-1122.

[60] SORAVIA P. (1993) *Discontinuous viscosity solutions to Dirichlet Problems for Hamilton-Jacobi Equations with convex Hamiltonians.* Commun. P.D.E. 18, 1493-1514.

[61] SUBBOTIN A.I. (1993) *Discontinuous solutions of a Diriclet type boundary value problem for first order partial differential equations*, Russian J. Anal. Math. Modelling, 8, 145-164.

[62] SUBBOTIN A.I. (1991) *Existence and Uniqueness Results for the Hamilton-Jacobi Equations*, Nonlinear Anal. T.M.A., 16, 683-699.

[63] VARAIYA P. (1967) *The existence of solution to a differential game*, SIAM J. Control Optim. 5, 153-162.

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