

Flow of diffeomorphisms for SDEs with unbounded Hölder continuous drift

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Abstract

We consider a SDE with a smooth multiplicative non-degenerate noise and a possibly unbounded Hölder continuous drift term. We prove existence of a global flow of diffeomorphisms by means of a special transformation of the drift of Itô-Tanaka type. The proof requires non-standard elliptic estimates in Hölder spaces. As an application of the stochastic flow, we obtain a Bismut-Elworthy-Li type formula for the first derivatives of the associated diffusion semigroup.

1 Introduction

In this paper we study the existence of a global stochastic flow of diffeomorphisms for the following stochastic differential equation in \mathbb{R}^d

$$dX_t^x = b(X_t^x) dt + \sum_{i=1}^k \sigma_i(X_t^x) dW_t^i, \quad t \geq 0, \quad X_0^x = x, \quad (1)$$

where $W_t = (W_t^1, \dots, W_t^k)$ is a standard Brownian motion in \mathbb{R}^k . We assume that the diffusion coefficients $\sigma_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $i = 1, \dots, k$, are smooth and non-degenerate and we allow the drift term $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ to be unbounded and Hölder continuous.

Following a common language, we say that equation (1) is *weakly complete* if there exists a unique global strong solution for every $x \in \mathbb{R}^d$, and that it is *strongly complete* if there exists a global stochastic flow of homeomorphisms. If the coefficients b and σ_i are globally Lipschitz, then one has strong completeness (see [19] and [20]).

Weak completeness is true under much weaker assumptions: for instance, when the coefficients b and σ_i are locally Lipschitz continuous and have at most linear growth. In dimension one, these assumptions also imply strong completeness (see [19] and [20]) but in dimension larger than one there are counterexamples, from [23], even in the case of smooth bounded coefficients. These examples indicate that some form of global

control at infinity on the increments of the coefficients is necessary. For (at least) locally Lipschitz coefficients, there are indeed positive results of strong completeness (see [7], [22], [24]).

Strong completeness for non-locally Lipschitz coefficients can be established replacing the global Lipschitz condition on the coefficients with global log-Lipschitz type conditions (see [29], [33], [9], [8]). Such log-Lipschitz conditions are stronger than the Hölder continuity.

Many papers prove weak completeness for SDEs with non-locally Lipschitz continuous coefficients assuming a non-degenerate diffusion matrix σ . First papers in this direction were [35] and [31] in which the method of the so called Zvonkin's transformation was introduced. More recent papers dealing with such approach are [13], [17], [32], [34] (see also the references therein). In the case of non-degenerate additive noise and time dependent drift b , the most advanced result (but see also the 1-dimensional results reported in [30]) is [17]; in such paper it is shown that it is sufficient to assume that $b \in L^q(0, T; L^p_{loc}(\mathbb{R}^d))$ with $\frac{d}{p} + \frac{2}{q} < 1$, $p \geq 2$ and $q > 2$, plus a non-explosion condition, to get weak completeness. This result has been generalized in [32] to cover also the case in which σ is variable, time-dependent and non-degenerate. We do not know about strong completeness under such weak assumptions.

The contribution of the present paper is to prove strong completeness for SDEs with "locally uniformly θ -Hölder continuous" drift b , for some $\theta \in (0, 1)$ (see (3)), removing boundedness of b or additional regularity assumed in previous works. Also, we allow non-degenerate, bounded and $C^3_b(\mathbb{R}^d, \mathbb{R}^d)$ -diffusion coefficients $(\sigma_i)_{i=1, \dots, k}$. We point out that our result seems to be new even in the case of constant and non-degenerate $(\sigma_i)_{i=1, \dots, k}$.

In spite of the fact that b is not even differentiable, under the previous assumptions, we construct a stochastic flow of C^1 -diffeomorphisms (see Theorem 7) using the approach of [10] rather than the Zvonkin's transformation method used in the above mentioned works on strong completeness (we compare the two methods in Section 3).

In [10] in order to study a linear stochastic transport equation with a *bounded vector field* $\bar{b}(t, x)$ which is Hölder continuous in x , uniformly in time, we have showed that if in (1) $\sigma = (\sigma_i)$ is constant and non-degenerate and $b = \bar{b}$, then there exists a stochastic flow of C^1 -diffeomorphisms. This result can be extended without difficulties to the case in which σ is not constant, bounded, non-degenerate, and time-dependent (see [34] where this case is investigated by the Zvonkin's transformation or Remark 9 where we show such result following the approach of [10]).

In the present situation, since our b is *unbounded*, we need new global regularity results in Hölder spaces for the solution u of the elliptic equation

$$\lambda u(x) - \frac{1}{2} \text{Tr}(a(x) D^2 u(x)) - b(x) \cdot Du(x) = b(x), \quad x \in \mathbb{R}^d, \quad (2)$$

to be interpreted componentwise, where $\lambda > 0$ is large enough, $a(x) = \sigma(x)\sigma^*(x)$ ($\sigma^*(x)$ denotes the adjoint matrix of $\sigma(x)$). The study of this equation will be the subject of Section 2 of the present paper. The required estimates are not covered by recent papers dealing with elliptic and parabolic equations with unbounded coefficients (compare with [4], [1], [18] and the references therein). To obtain such result we prove a crucial Lemma

4 concerning estimates on the derivatives of the associated diffusion semigroup when it is applied to *unbounded* functions f ; in its proof we also use an argument from the proof of [27, Theorem 3.3]. In Remark 10 we show a possible extension of our Theorem 7 to the case in which b and σ are time-dependent.

We finish the paper by showing that a Bismut-Elworthy-Li formula holds for the diffusion semigroup associated to (1) (see Theorem 11). Under the poor regularity of b assumed here, this result is new. Bismut-Elworthy-Li formula requires a suitable form of differentiability of the solution of (1) with respect to the initial condition x ; we have this result as a byproduct of our Theorem 7 on existence of a differentiable stochastic flow.

Notations and assumption The euclidean norm in any \mathbb{R}^k , $k \geq 1$, will be denoted by $|\cdot|$ and its inner product by \cdot or $\langle \cdot, \cdot \rangle$. For $\theta \in (0, 1)$, we define the set $C^\theta(\mathbb{R}^d; \mathbb{R}^k)$, $k, d \geq 1$, as set of all vector-fields $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$ for which

$$[f]_\theta := \sup_{x \neq y \in \mathbb{R}^d, |x-y| \leq 1} \frac{|f(x) - f(y)|}{|x - y|^\theta} < \infty. \quad (3)$$

These are the “*locally uniformly θ -Hölder continuous*” vector fields mentioned in the introduction. The function $f(x) = |x|^\theta$ is a classical example. We let

$$[f]_{\theta,1} := \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{(|x - y|^\theta \vee |x - y|)} < \infty, \quad (4)$$

where $a \vee b = \max(a, b)$, for $a, b \in \mathbb{R}$. By a simple argument we have $[f]_\theta \leq [f]_{\theta,1} \leq 2[f]_\theta$, so in particular functions in $C^\theta(\mathbb{R}^d; \mathbb{R}^k)$ have *at most linear growth*. The set $C^\theta(\mathbb{R}^d; \mathbb{R}^k)$ becomes a Banach space with respect to the norm

$$\|f\|_\theta = \|(1 + |\cdot|)^{-1} f(\cdot)\|_0 + [f]_\theta,$$

where $\|\cdot\|_0$ denotes the supremum norm over \mathbb{R}^d . We say that $f \in C^{n+\theta}(\mathbb{R}^d; \mathbb{R}^k)$, $n \geq 1$, if $f \in C^\theta(\mathbb{R}^d; \mathbb{R}^k)$ and moreover, for all $i = 1, \dots, n$, the Fréchet derivatives $D^i f$ are *bounded* and θ -Hölder continuous. Define the corresponding norm as

$$\|f\|_{n+\theta} = \|f\|_\theta + \sum_{i=1}^n \|D^i f\|_0 + [D^n f]_\theta. \quad (5)$$

If $\mathbb{R}^k = \mathbb{R}$, we simply write $C^{n+\theta}(\mathbb{R}^d)$ instead of $C^{n+\theta}(\mathbb{R}^d; \mathbb{R})$, $n \geq 0$. $C_b^{n+\theta}(\mathbb{R}^d; \mathbb{R}^k)$ is the subspace of $C^{n+\theta}(\mathbb{R}^d; \mathbb{R}^k)$, consisting of all bounded functions of $C^{n+\theta}(\mathbb{R}^d; \mathbb{R}^k)$. In particular, $C_b^\theta(\mathbb{R}^d)$ is the usual Banach space of all real bounded and θ -Hölder continuous functions on \mathbb{R}^d (cf. [15]). $C_b^n(\mathbb{R}^d; \mathbb{R}^k)$ is the space of all bounded functions from \mathbb{R}^d into \mathbb{R}^k having also bounded derivatives up to the order $n \geq 1$ and we set $C_b^n(\mathbb{R}^d; \mathbb{R}) = C_b^n(\mathbb{R}^d)$. Finally, we say that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is of class $C^{n,\alpha}$, $n \geq 1$, $\alpha \in (0, 1)$, if f is continuous on \mathbb{R}^d , n -times differentiable and the derivatives up to the order n are α -Hölder continuous on each compact set of \mathbb{R}^d .

Throughout the paper we will assume a fixed stochastic basis with a d -dimensional Brownian motion $(\Omega, (\mathcal{F}_t), \mathcal{F}, P, (W_t))$ to be given. Denote by $\mathcal{F}_{s,t}$ the completed σ -algebra generated by $W_u - W_r$, $s \leq r \leq u \leq t$, for each $0 \leq s < t$.

On equation (1), we will consider the following assumptions.

Hypothesis 1 *There exists $\theta \in (0, 1)$ such that $b \in C^\theta(\mathbb{R}^d; \mathbb{R}^d)$.*

Hypothesis 2 *The diffusions coefficients $\sigma_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $i = 1, \dots, k$, are bounded functions of class $C_b^3(\mathbb{R}^d, \mathbb{R}^d)$.*

Hypothesis 3 *Consider the $d \times k$ matrix $\sigma(x) = (\sigma_i(x))$, and its adjoint matrix $\sigma^*(x)$, $x \in \mathbb{R}^d$; we assume that, for any $x \in \mathbb{R}^d$, there exists the inverse of $a(x) = \sigma(x)\sigma^*(x)$ and*

$$\|a^{-1}\|_0 = \sup_{x \in \mathbb{R}^d} \|a^{-1}(x)\| < \infty \quad (6)$$

($\|a^{-1}(x)\|$ denotes the Hilbert-Schmidt norm of the $d \times d$ symmetric matrix $a^{-1}(x)$).

2 Regularity results for the associated elliptic problem

2.1 Estimates on the derivatives of the diffusion semigroup

Here, we consider the SDE (1), assuming that σ satisfies Hypotheses 2 and 3 and imposing in addition that

$$b \in C^3(\mathbb{R}^d; \mathbb{R}^d) \text{ with all bounded derivatives up to the third order.} \quad (7)$$

Clearly this is stronger than Hypothesis 1 but b is not assumed to be bounded.

Let (P_t) be the corresponding diffusion semigroup, i.e., for any $g : \mathbb{R}^d \rightarrow \mathbb{R}$ Borel and bounded,

$$P_t g(x) = \mathbb{E}[g(X_t^x)], \quad x \in \mathbb{R}^d, \quad t \geq 0,$$

where (X_t^x) is the unique strong solution to (1) under (7).

In our next result, we will prove estimates on the spatial derivatives of $P_t f$, $t > 0$, assuming that $f \in C^\theta(\mathbb{R}^d)$. To this purpose, we will use the so-called Bismut-Elworthy-Li formula (see (12)) for the spatial derivatives of $P_t f$ (cf. [6]).

Let us comment on such formula. Probabilistic formulae for the spatial derivatives of Markov semigroups have been much studied for different classes of degenerate and non-degenerate diffusion processes even with jumps (see [3], [21], [6], [5], [4], [12] [26], [27], [34] and the references therein). The martingale approach of [6] mainly works for non-degenerate semigroups (but see also [4, Chapter 3] and [34]); it has been also used for some infinite dimensional diffusion processes (see [5] and [4]). On the other hand, in case of degenerate diffusion semigroups, more complicate formulae for the derivatives can be established by Malliavin Calculus (see [3], [21], [12] and [26]). Some applications to Mathematical Finance are given in [14].

The next lemma is of independent interest since the function f in (8) is not assumed to be bounded (compare with [4, Chapter 1] and [1, Chapter 6]).

Lemma 4 *Assume Hypotheses 2 and 3 and condition (7). There exist constants $c_j > 0$, $M_j > 0$, $j = 1, 2, 3$ (c_j and M_j depends on θ , $\|a^{-1}\|_0$, d , $\|\sigma\|_0$ and on the supremum norms of derivatives of σ and b up to the order j), such that, for any $f \in C^\theta(\mathbb{R}^d)$, $t > 0$, it holds*

$$\|D^j P_t f\|_0 \leq M_j [f]_\theta \frac{e^{c_j t}}{t^{(j-\theta)/2}}, \quad t > 0, \quad \text{for } j = 1, 2, 3. \quad (8)$$

Proof. *I Step.* First note that $\mathbb{E}[\sup_{t \in [0, T]} |X_t^x|^q] \leq C_T(1 + |x|^q)$, for any $T > 0$, $x \in \mathbb{R}^d$, $q \geq 1$ (see, for instance, [19, Chapter II]).

It is also known that, for any $t \geq 0$, the mapping:

$$x \mapsto X_t^x \text{ is three times Fréchet differentiable from } \mathbb{R}^d \text{ into } L^2(\Omega) \quad (9)$$

(see [4, Section 1.3] which contains a more general result). Let us write the Fréchet derivatives:

$$\eta_t(x, h) = D_x(X_t^x)[h], \quad \xi_t(x, h, k) = D_x^2(X_t^x)[h, k], \quad \psi_t(x, h, k, l) = D_x^3(X_t^x)[h, k, l],$$

for any $x, h, k, l \in \mathbb{R}^d$. These derivatives satisfy suitable stochastic variation equations (see [19, Chapter II]). We only write down the variation equation for $\eta_t = \eta_t(x, h)$:

$$d\eta_t = Db(X_t^x)\eta_t + D\sigma(X_t^x)\eta_t dW_t, \quad \eta_0 = h.$$

Using standard estimates, based on the Burkholder inequality, we get that, for any $p \geq 1$, that there exist positive constants C and c (depending on p , $\|Db\|_0$ and $\|D\sigma\|_0$) such that, for any $x \in \mathbb{R}^d$, $h \in \mathbb{R}^d$,

$$\mathbb{E}|\eta_t(x, h)|^p \leq C|h|^p e^{ct}, \quad t \geq 0. \quad (10)$$

In a similar way, using the second and third variation equations, we obtain the estimates:

$$\mathbb{E}|\xi_t(x, h, k)|^p \leq C_2|h|^p|k|^p e^{\hat{c}_2 t}, \quad (11)$$

$$\mathbb{E}|\psi_t(x, h, k, l)|^p \leq C_3|h|^p|k|^p|l|^p e^{\hat{c}_3 t}, \quad t \geq 0,$$

for any $x, h, k, l \in \mathbb{R}^d$ (with positive constants C_i and \hat{c}_i which depend on p and on the supremum norms of the derivatives of b and σ up to the order i , $i = 2, 3$).

II Step. Arguing similarly to [4, Section 1.5] one can prove that, for any $f \in C^\theta(\mathbb{R}^d)$, $t > 0$, the map: $x \mapsto P_t f(x)$ is differentiable on \mathbb{R}^d and, moreover, we have the following Bismut-Elworthy-Li formula:

$$\langle DP_t f(x), h \rangle = \mathbb{E} \left[f(X_t^x) J^1(t, x, h) \right], \quad x, h \in \mathbb{R}^d, \quad t > 0, \quad \text{where} \quad (12)$$

$$J^1(t, x, h) = \frac{1}{t} \int_0^t \langle \sigma^*(X_s^x) a^{-1}(X_s^x) \eta_s(x, h), dW_s \rangle.$$

Note that formula (12) is first proved for bounded $f \in C_b^2(\mathbb{R}^d)$. Then a straightforward approximation argument shows that (12) holds even for (a possibly unbounded) $f \in$

$C^\theta(\mathbb{R}^d)$. However, to be precise, in [4], it is assumed that $\sigma(x)$ is an invertible $d \times d$ matrix and so the expression of J^1 in [4, Section 1.5] contains $\sigma^{-1}(X_s^x)$ instead of our $\sigma^*(X_s^x) a^{-1}(X_s^x)$. We briefly explain why (12) holds following the proof of [25, Theorem 5.1]. We only discuss the crucial point of the argument which is needed to get (12) when $f \in C_b^2(\mathbb{R}^d)$. One has by the Itô formula

$$f(X_t^x) = P_t f(x) + \int_0^t \langle DP_{t-s} f(X_s^x), \sigma(X_s^x) dW_s \rangle.$$

Multiplying both terms of the identity by the martingale

$$K_t = \int_0^t \langle \sigma^*(X_s^x) a^{-1}(X_s^x) \eta_s(x, h), dW_s \rangle,$$

and taking the expectation, one arrives at

$$\mathbb{E}[f(X_t^x) K_t] = \int_0^t \mathbb{E}[\langle DP_{t-s} f(X_s^x), \eta_s(x, h) \rangle] ds = t \langle DP_t f(x), h \rangle.$$

Thus (12) is proved.

Now the problem is to show that, for $f \in C^\theta(\mathbb{R}^d)$, $t > 0$, the map: $x \mapsto \langle DP_t f(x), h \rangle$ is a bounded function (we cannot use as in [4] the boundedness of f).

By using (10), we get easily that there exist $C_1 > 0$ depending on $\|a^{-1}\|_0$, $\|Db\|_0$ and $\|D\sigma\|_0$ such that

$$\mathbb{E}|J^1(t, x, h)|^2 \leq \frac{C_1 e^{C_1 t}}{t} |h|^2, \quad t > 0. \quad (13)$$

Now we prove the crucial estimate of the first derivative in (8). We use an argument from the proof of [27, Theorem 3.3]. Introduce the deterministic process

$$Y_t^x = x + \int_0^t b(Y_s^x) ds, \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

which solves $\dot{Y}_t^x = b(Y_t^x)$, $Y_0^x = x$. Using that σ is bounded and applying the Gronwall lemma, we find, for any $q \geq 1$,

$$\mathbb{E}|X_t^x - Y_t^x|^q \leq M t^{q/2} e^{c_1 t}, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad (14)$$

where M depends on $\|\sigma\|_0$ and q and c_1 on $\|Db\|_0$ and q . Since

$$\mathbb{E}[f(Y_t^x) J^1(t, x, h)] = f(Y_t^x) \langle D(P_t 1)(x), h \rangle = 0, \quad t > 0, \quad h \in \mathbb{R}^d, \quad x \in \mathbb{R}^d,$$

we have (see also (4))

$$\begin{aligned} |\langle DP_t f(x), h \rangle| &= \left| \mathbb{E}[(f(X_t^x) - f(Y_t^x)) J^1(t, x, h)] \right| \\ &\leq 2[f]_\theta \mathbb{E}[(|X_t^x - Y_t^x|^\theta \vee |X_t^x - Y_t^x|) |J^1(t, x, h)|] \end{aligned}$$

$$\leq 2[f]_\theta \left(\mathbb{E}[|X_t^x - Y_t^x|^{2\theta} \vee |X_t^x - Y_t^x|^2] \right)^{1/2} \left(\mathbb{E}|J^1(t, x, h)|^2 \right)^{1/2}, \quad (15)$$

$t > 0$. Using that $a \vee b \leq a + b$, $a, b \geq 0$, and the previous estimates (10) and (14), we find

$$|\langle DP_t f(x), h \rangle| \leq C''[f]_\theta (t^{\theta/2} + t^{1/2}) \frac{e^{c't}}{t^{1/2}} |h| \leq [f]_\theta \frac{C' e^{c't}}{t^{1/2-\theta/2}} |h|, \quad t > 0, x \in \mathbb{R}^d, \quad (16)$$

where C' and c depend on $\|\sigma\|_0$, $\|a^{-1}\|_0$, $\|D\sigma\|_0$, $\|Db\|_0$ and θ .

Let us consider the remaining estimates in (8). We have, using the semigroup law, $P_t f = P_{t/2}(P_{t/2} f)$ and so (cf. [4, formula (1.5.2)]), for any $x, h, k \in \mathbb{R}^d$, $t > 0$,

$$\begin{aligned} \langle D^2(P_t f)(x)k, h \rangle &= D_k \left(\mathbb{E} \left[(P_{t/2} f)(X_{t/2}^{(\cdot)}) J^1(t/2, (\cdot), h) \right] \right) (x) \\ &= \mathbb{E} \left[\langle DP_{t/2} f(X_{t/2}^x), \eta_{t/2}(x, k) \rangle J^1(t/2, x, h) \right] + \mathbb{E} \left[P_{t/2} f(X_{t/2}^x) D_k J^1(t/2, x, h) \right] \\ &= \Gamma_1(t, x) + \Gamma_2(t, x), \end{aligned}$$

where D_k denotes the directional derivative along the vector k (indeed, for any fixed $t > 0$ and $h \in \mathbb{R}^d$, the mapping: $x \mapsto J^1(t/2, x, h)$ is Fréchet differentiable from \mathbb{R}^d into $L^2(\Omega)$; this follows easily, using (9), (14), (10) and (11)). We have

$$\begin{aligned} D_k J^1(t/2, x, h) &= \frac{2}{t} \int_0^{t/2} \langle D\sigma^*(X_s^x)[\eta_s(x, k)] a^{-1}(X_s^x) \eta_s(x, h), dW_s \rangle \\ &\quad - \frac{2}{t} \int_0^{t/2} \langle \sigma^*(X_s^x) a^{-1}(X_s^x) Da(X_s^x)[\eta_s(x, k)] a^{-1}(X_s^x) \eta_s(x, h), dW_s \rangle \\ &\quad + \frac{2}{t} \int_0^{t/2} \langle \sigma^*(X_s^x) a^{-1}(X_s^x) \xi_s(x, h, k), dW_s \rangle. \end{aligned}$$

Using the Schwarz inequality, (13) and

$$\sup_{x \in \mathbb{R}^d} \left(\mathbb{E} |\langle DP_{t/2} f(X_{t/2}^x), \eta_{t/2}(x, k) \rangle|^2 \right)^{1/2} \leq [f]_\theta \frac{C'' e^{c't}}{t^{1/2-\theta/2}} |k|,$$

we get immediately $|\Gamma_1(t, x)| \leq M[f]_\theta \frac{e^{c't}}{t^{(2-\theta)/2}} |h||k|$, $t > 0$, $x \in \mathbb{R}^d$. To estimate Γ_2 , first note that, by taking $f = 1$,

$$0 = \langle D^2(P_t 1)(x)k, h \rangle = 0 + \mathbb{E} [D_k J^1(t/2, x, h)],$$

for any $x, h, k \in \mathbb{R}^d$. We find (arguing similarly to (15))

$$\Gamma_2(t, x) = \mathbb{E} \left[(P_{t/2} f(X_{t/2}^x) - P_{t/2} f(Y_{t/2}^x)) D_k J^1(t/2, x, h) \right].$$

Since

$$|P_s f(x) - P_s f(y)| \leq \mathbb{E} |f(X_s^x) - f(X_s^y)| \leq 2[f]_\theta \mathbb{E} [(|X_s^x - X_s^y|^\theta + |X_s^x - X_s^y|)]$$

$$\leq 2[f]_\theta M(|x-y|^\theta + |x-y|)e^{c_1 s}, \quad s \geq 0, \quad x, y \in \mathbb{R}^d,$$

we find, for any $x \in \mathbb{R}^d$, $t > 0$,

$$\begin{aligned} |\Gamma_2(t, x)| &\leq 2Me^{c_1 t/2} [f]_\theta \mathbb{E}[(|X_{t/2}^x - Y_{t/2}^x|^\theta + |X_{t/2}^x - Y_{t/2}^x|) |D_k J^1(t/2, x, h)|]. \\ &\leq 2Me^{c_1 t/2} [f]_\theta (\mathbb{E}[|X_{t/2}^x - Y_{t/2}^x|^{2\theta} + |X_{t/2}^x - Y_{t/2}^x|^2])^{1/2} (\mathbb{E}|D_k J^1(t/2, x, h)|^2)^{1/2} \\ &\leq [f]_\theta \frac{C_1 e^{c_1 t}}{t^{1/2-\theta/2}} |k| |h|, \end{aligned}$$

where C_1 and c_1 depends on $\|\sigma\|_0$, $\|a^{-1}\|_0$, $\|D\sigma\|_0$, $\|D^2\sigma\|_0$, $\|Db\|_0$, $\|D^2b\|_0$ and θ . We have so obtained estimate in (8) corresponding to $j = 2$.

The estimate for $j = 3$ follows in a similar way. ■

2.2 The main regularity result

With respect to the previous section, here we consider the elliptic operator

$$Lu(x) = \frac{1}{2} \text{Tr}(a(x)D^2u(x)) + b(x) \cdot Du(x), \quad x \in \mathbb{R}^d,$$

with $a(x) = \sigma(x)\sigma^*(x)$, assuming Hypotheses 1, 2 and 3.

The next result provides new estimates for L in Hölder spaces. These estimates are not covered by recent papers dealing with elliptic and parabolic equations with unbounded coefficients, due to the fact that in our case also f can be unbounded (compare with [4], [1], [18] and the references therein).

Theorem 5 *Let $\theta \in (0, 1)$. For any $\theta' \in (0, \theta)$, there exists $\lambda_0 > 0$ (depending on $\theta, \theta', d, [b]_\theta, \|\sigma\|_0, \|a^{-1}\|_0, \|D^k\sigma\|_0, k = 1, 2, 3$) such that, for $\lambda \geq \lambda_0$, for any $f \in C^\theta(\mathbb{R}^d)$, the equation*

$$\lambda u - Lu = f \tag{17}$$

admits a unique classical solution $u = u_\lambda \in C^{2+\theta'}(\mathbb{R}^d)$ for which

$$\|u\|_{2+\theta'} = \|u(\cdot)(1 + |\cdot|)^{-1}\|_0 + \|Du\|_0 + \|D^2u\|_0 + [D^2u]_{\theta'} \leq C(\lambda)\|f\|_\theta \tag{18}$$

with $C(\lambda)$ (independent on u and f) such that $C(\lambda) \rightarrow 0$ as $\lambda \rightarrow +\infty$.

Proof. Uniqueness can be proved by the following argument (cf. [16, page 606]). Consider $\eta(x) = \sqrt{1 + |x|^2}$, $x \in \mathbb{R}^d$.

Defining $u = v\eta$, we obtain an elliptic equation for the bounded function v , i.e.,

$$\begin{aligned} \lambda v(x) - \frac{1}{2} \text{Tr}(a(x)D^2v(x)) - (b(x) + \frac{a(x)D\eta(x)}{\eta(x)}) \cdot Dv(x) \\ - \left(\frac{1}{2} \frac{\text{Tr}(a(x)D^2\eta(x))}{\eta(x)} + b(x) \cdot \frac{D\eta(x)}{\eta(x)} \right) v(x) = \frac{f(x)}{\eta(x)}, \end{aligned} \tag{19}$$

$x \in \mathbb{R}^d$. Note that v has first and second bounded derivatives. For λ large enough (depending on $\|\sigma\|_0$ and $\|\frac{b}{\eta}\|_0$), uniqueness of v follows by the classical maximum principle.

Now we divide the rest of the proof in some steps.

Step I. We assume *in addition* that $b \in C^3(\mathbb{R}^d, \mathbb{R}^d)$ and has all bounded derivatives up to the third order (but it is not necessarily bounded). We prove that, for sufficiently large $\lambda > 0$, there exists a unique solution $u = u_\lambda \in C^{2+\theta}(\mathbb{R}^d)$ to the equation

$$\lambda u - Lu = f \in C^\theta(\mathbb{R}^d).$$

Moreover there exists C (independent on u and f) such that

$$\|u\|_{2+\theta} \leq C\|f\|_\theta. \quad (20)$$

Estimates (20) are new Schauder estimates since f is not assumed to be bounded (compare with [4] and [1])

We consider the function

$$u(x) = \int_0^\infty e^{-\lambda t} \mathbb{E}[f(X_t^x)] dt = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad x \in \mathbb{R}^d, \quad (21)$$

where (X_t^x) is the solution of (1) and show that, for λ large enough, u is a $C^{2+\theta}(\mathbb{R}^d)$ -solution to our PDE.

Using that $\mathbb{E}|X_t^x - X_t^y| \leq C e^{Ct} |x - y|$, $t \geq 0$, $x, y \in \mathbb{R}^d$, we find

$$|u(x) - u(y)| \leq c[f]_{\theta,1} (|x - y|^\theta \vee |x - y|), \quad x, y \in \mathbb{R}^d,$$

and also $\|u(\cdot)(1 + |\cdot|)^{-1}\|_0 \leq C\|f(\cdot)(1 + |\cdot|)^{-1}\|_0$, for λ large enough.

By Lemma 4 we get, for λ large enough,

$$\|Du\|_0 + \|D^2u\|_0 \leq C[f]_\theta.$$

To estimate the second derivatives of u , we proceed as in [27, Theorem 4.2]. We have, for any $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$,

$$\begin{aligned} |D^2u(x) - D^2u(y)| &= \int_0^{|x-y|^2} e^{-\lambda t} |D^2P_t f(x) - D^2P_t f(y)| dt \\ &\quad + \int_{|x-y|^2}^\infty e^{-\lambda t} |D^2P_t f(x) - D^2P_t f(y)| dt \\ &\leq c''|x - y|^\theta [f]_\theta + C|x - y| [f]_\theta \int_{|x-y|^2}^\infty e^{-\lambda t} \frac{e^{ct}}{t^{(3-\theta)/2}} dt \leq c'[f]_\theta |x - y|^\theta. \end{aligned}$$

It remains to check that u is a solution. This is not difficult thanks to Lemma 4 (see, for instance, [4, Chapter 1] or argue as in [27, Theorem 4.1]).

Step II. Under the assumptions of Step I, for any $\alpha \in (0, \theta)$, we have

$$\|u\|_{2+\alpha} \leq C(\lambda)\|f\|_{\theta}, \quad (22)$$

with $C(\lambda) \rightarrow 0$, as $\lambda \rightarrow +\infty$. This is clear if we replace $\|u\|_{2+\alpha}$ with $\|u(\cdot)(1+|\cdot|)^{-1}\|_0 + \|Du\|_0 + \|D^2u\|_0$. Therefore, we only consider $[D^2u]_{\alpha}$.

Combining the interpolatory estimate: $[v]_{\alpha} \leq C\|v\|_0^{1-\alpha} \|Dv\|_0^{\alpha}$, $v \in C_b^1(\mathbb{R}^d)$ (where $C = C(d)$, see [15, Section 3.2]) with estimates of Lemma 4 corresponding to $j = 2, 3$, we find, for any $t > 0$,

$$[D^2P_t f]_{\alpha} \leq C\|D^2P_t f\|_0^{1-\alpha} \|D^3P_t f\|_0^{\alpha} \leq C_4[f]_{\theta} \frac{e^{c_4 t}}{t^{\gamma}},$$

with $\gamma = \frac{2-\theta+\alpha}{2} < 1$ (since $\alpha < \theta$). It follows

$$[D^2u]_{\alpha} \leq C_4[f]_{\theta} \int_0^{+\infty} \frac{e^{(c_4-\lambda)t}}{t^{\gamma}} dt \leq C_5[f]_{\theta}(\lambda - c_4)^{\gamma-1}.$$

The assertion is proved.

Step III. We require that $b \in C^{\theta}(\mathbb{R}^d, \mathbb{R}^d)$ as in Hypothesis 1 and prove the following a-priori estimates: if λ is large enough and $u \in C^{2+\theta'}(\mathbb{R}^d)$, $0 < \theta' < \theta$, is a solution to $\lambda u - Lu = f \in C^{\theta}(\mathbb{R}^d)$, then

$$\|u(\cdot)(1+|\cdot|)^{-1}\|_0 + \|Du\|_0 + \|D^2u\|_{\theta'} \leq K(\lambda)\|f\|_{\theta}, \quad (23)$$

with $K(\lambda) \rightarrow 0$, as $\lambda \rightarrow +\infty$.

To prove the estimate we introduce $\rho \in C_0^{\infty}(\mathbb{R}^d)$, $0 \leq \rho \leq 1$, $\rho(x) = \rho(-x)$, for any $x \in \mathbb{R}^d$, $\int \rho(x) dx = 1$. Moreover, $b * \rho$ indicates b convoluted with ρ .

Write $\lambda u(x) - \frac{1}{2}Tr(a(x)D^2u(x)) - (b * \rho)(x) \cdot Du(x) = f(x) + \langle (b - (b * \rho))(x), Du(x) \rangle$. It is easy to see that $b * \rho$ (even if it can be unbounded) is a C^{∞} -function with all bounded derivatives. Moreover, there exists $C = C(\theta, D\rho, D^2\rho, D^3\rho) > 0$ such that

$$\|D^k(b * \rho)\|_0 \leq C[b]_{\theta}, \quad k = 1, 2, 3. \quad (24)$$

The function $b - (b * \rho)$ is bounded and we have

$$\|b - (b * \rho)\|_0 \leq C[b]_{\theta}.$$

It follows that $b - (b * \rho) \in C_b^{\theta}(\mathbb{R}^d, \mathbb{R}^d)$. Applying Step II, we find that

$$\|u\|_{2+\theta'} \leq C(\lambda)\|f\|_{\theta} + C(\lambda)\|\langle b - (b * \rho), Du \rangle\|_{\theta} \quad (25)$$

with $C(\lambda) \rightarrow 0$. Using that

$$\|\langle b - (b * \rho), Du \rangle\|_{\theta} \leq c[b]_{\theta}\|Du\|_0 + c[b]_{\theta}\|Du\|_{\theta} \leq c[b]_{\theta}\|u\|_{2+\theta'},$$

for some constant c depending on θ , we rewrite (25):

$$\|u\|_{2+\theta'} \leq C(\lambda)\|f\|_{\theta} + C(\lambda)c[b]_{\theta}\|u\|_{2+\theta'}.$$

Choosing $\lambda_0 > 0$ such that $C(\lambda) < \frac{1}{c[b]_{\theta}}$, for $\lambda \geq \lambda_0$, we find, with $u = u_{\lambda}$

$$(1 - C(\lambda)c[b]_{\theta})\|u\|_{2+\theta'} \leq C(\lambda)\|f\|_{\theta}. \quad (26)$$

Defining $K(\lambda) = \frac{C(\lambda)}{1 - C(\lambda)c[b]_{\theta}}$, we get the assertion.

Step IV. We show that for $\lambda \geq \lambda_0$ (see Step III) there exists a classical solution $u = u_{\lambda} \in C^{2+\theta'}(\mathbb{R}^d)$ to (17). This assertion will conclude the proof.

We fix $\lambda \geq \lambda_0$. To prove the result, we will use the continuity method. To this purpose, using the test function ρ of Step III, we consider:

$$\lambda u(x) - \frac{1}{2}Tr(a(x)D^2u(x)) - (1 - \delta)(b * \rho)(x) \cdot Du(x) - \delta b(x) \cdot Du(x) = f(x), \quad (27)$$

$x \in \mathbb{R}^d$, where $\delta \in [0, 1]$ is a parameter. Let us define

$$\Gamma = \{\delta \in [0, 1] : \text{eq. (27) has a unique solution } u = u_{\delta} \in C^{2+\theta'}(\mathbb{R}^d), \text{ for any } f \in C^{\theta}(\mathbb{R}^d)\}.$$

Γ is not empty since $0 \in \Gamma$ by Step I. Let us fix $\delta_0 \in \Gamma$ and rewrite equation (27) corresponding to an arbitrary $\delta \in [0, 1]$ as

$$\begin{aligned} \lambda u(x) - \frac{1}{2}Tr(a(x)D^2u(x)) - (1 - \delta_0)(b * \rho)(x) \cdot Du(x) - \delta_0 b(x) \cdot Du(x) \\ = f(x) + [\delta - \delta_0](b - b * \rho)(x) \cdot Du(x). \end{aligned}$$

Introduce the operator $\mathcal{T} : C^{2+\theta'}(\mathbb{R}^d) \rightarrow C^{2+\theta'}(\mathbb{R}^d)$. For any $v \in C^{2+\theta'}(\mathbb{R}^d)$, $\mathcal{T}v = u$ is the (unique) $C^{2+\theta'}(\mathbb{R}^d)$ -function which solves

$$\begin{aligned} \lambda u(x) - \frac{1}{2}Tr(a(x)D^2u(x)) - (1 - \delta_0)(b * \rho)(x) \cdot Du(x) - \delta_0 b(x) \cdot Du(x) \\ = f(x) + [\delta - \delta_0](b - b * \rho)(x) \cdot Dv(x). \end{aligned}$$

Using the a-priori estimates (26), we get that

$$\|\mathcal{T}v - \mathcal{T}w\|_{2+\theta'} \leq 2K(\lambda)|\delta - \delta_0|[b]_{\theta}\|v - w\|_{2+\theta'}, \quad v, w \in C^{2+\theta'}(\mathbb{R}^d).$$

Choosing $|\delta - \delta_0|$ small enough, the operator \mathcal{T} becomes a contraction on $C^{2+\theta'}(\mathbb{R}^d)$ and it has a unique fixed point which is the solution to (27). Therefore for $|\delta - \delta_0|$ small enough, we have that $\delta \in \Gamma$. A compactness argument shows that $\Gamma = [0, 1]$. The assertion is proved. ■

3 Differentiable stochastic flow

Given $x \in \mathbb{R}^d$, consider the stochastic differential equation in \mathbb{R}^d :

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t, \quad X_s = x, \quad t \geq s \geq 0. \quad (28)$$

As already mentioned our key result is the existence of a *differentiable* stochastic flow $(x, s, t) \mapsto \varphi_{s,t}(x)$ for equation (28). Recall the relevant definition from H. Kunita [19]:

Definition 6 A stochastic flow of diffeomorphisms (*resp.* of class $C^{1,\alpha}$) on the stochastic basis $(\Omega, (\mathcal{F}_t), \mathcal{F}, P, (W_t))$ associated to equation (28) is a map $(s, t, x, \omega) \mapsto \phi_{s,t}(x)(\omega)$, defined for $0 \leq s \leq t$, $x \in \mathbb{R}^d$, $\omega \in \Omega$ with values in \mathbb{R}^d , such that

- (a) given any $s \geq 0$, $x \in \mathbb{R}^d$, the process $X^{s,x} = (X_t^{s,x}(\omega), t \geq s, \omega \in \Omega)$ defined as $X_t^{s,x} = \phi_{s,t}(x)$ is a continuous $\mathcal{F}_{s,t}$ -measurable solution of equation (28);
- (b) *P*-a.s., for all $0 \leq s \leq t$, $\phi_{s,t}$ is a diffeomorphism and the functions $\phi_{s,t}(x)$, $\phi_{s,t}^{-1}(x)$, $D\phi_{s,t}(x)$, $D\phi_{s,t}^{-1}(x)$ are continuous in (s, t, x) (*resp.* of class $C^{1,\alpha}$ in x uniformly in (s, t) , for $0 \leq s \leq t \leq T$, with $T > 0$);
- (c) *P*-a.s., $\phi_{s,t}(x) = \phi_{u,t}(\phi_{s,u}(x))$, for all $0 \leq s \leq u \leq t$, $x \in \mathbb{R}^d$, and $\phi_{s,s}(x) = x$.

Starting from the work of Zvonkin, an important approach to the analysis of SDEs with non-regular drift is based on the transformation $\Psi_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$, solution of the vector valued equation

$$\frac{\partial \Psi_t}{\partial t} + L\Psi_t = 0 \text{ on } [0, T], \quad \Psi_T(x) = x$$

where $\Psi_t(x) = \Psi(t, x)$ and $[0, T]$ is a time interval where the SDE is considered. At time T , the solution is an isomorphism by definition; one has to prove suitable regularity and invertibility of Ψ_t for $t \in [0, T]$. Then $Y_t := \Psi_t(X_t)$ satisfies

$$dY_t = D\Psi_t(\Psi_t^{-1}(Y_t)) \sigma(\Psi_t^{-1}(Y_t)) dW_t.$$

The irregular drift has been removed. This approach, although successful (see [2], [13], [17], [32], [34]), raises two delicate questions: i) one has to deal with unbounded initial conditions; ii) one has to prove some form of invertibility.

We propose a variant, based on the same operator L but on the vector valued equation

$$\lambda\psi - L\psi = b$$

(under other assumptions one can treat also the time-dependent case through the parabolic equation $\lambda\psi_t - \frac{\partial \psi_t}{\partial t} - L\psi_t = b$, see [10]). We find it more tractable than the case of unbounded initial condition; and we translate the difficult invertibility issue in the smallness of the gradient of the solution, obtained by means of a large λ . When the gradient of ψ

is less than one, the function $\Psi(x) = x + \psi(x)$ is invertible and the process $Y_t := \Psi(X_t)$ satisfies

$$dY_t = D\Psi(\Psi^{-1}(Y_t)) \sigma(\Psi^{-1}(Y_t)) dW_t + \lambda\psi(\Psi^{-1}(Y_t)) dt.$$

So, at the end, the transformed equation has the same degree of difficulty as in the case of the Zvonkin's transformation.

Theorem 7 *Assume Hypotheses 1, 2, 3 and fix any $\theta'' \in (0, \theta)$. Then we have the following facts:*

- (i) *(pathwise uniqueness) For every $s \geq 0$, $x \in \mathbb{R}^d$, the stochastic equation (28) has a unique continuous adapted solution $X^{s,x} = (X_t^{s,x}(\omega), t \geq s, \omega \in \Omega)$.*
- (ii) *(differentiable flow) There exists a stochastic flow $\phi = (\phi_{s,t})$ of diffeomorphisms for equation (28). The flow is also of class $C^{1,\theta''}$.*
- (iii) *(stability) Let $(b^n) \subset C^\theta(\mathbb{R}^d, \mathbb{R}^d)$ and let (ϕ^n) be the corresponding stochastic flows. Assume that there exists $b \in C^\theta(\mathbb{R}^d, \mathbb{R}^d)$ such that $b_n - b \in C_b^\theta(\mathbb{R}^d, \mathbb{R}^d)$, $n \geq 1$, and $\|b - b_n\|_{C_b^\theta} \rightarrow 0$ as $n \rightarrow \infty$. If ϕ is the flow associated to b , then, for any $p \geq 1$, $T > 0$,*

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \sup_{0 \leq s \leq T} E \left[\sup_{u \in [s, T]} \frac{|\phi_{s,u}^n(x) - \phi_{s,u}(x)|^p}{(1 + |x|)^p} \right] = 0. \quad (29)$$

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} \sup_{0 \leq s \leq T} E \left[\sup_{u \in [s, T]} \|D\phi_{s,u}^n(x)\|^p \right] < \infty, \quad (30)$$

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \sup_{0 \leq s \leq T} E \left[\sup_{u \in [s, T]} \|D\phi_{s,u}^n(x) - D\phi_{s,u}(x)\|^p \right] = 0. \quad (31)$$

($\|\cdot\|$ denotes the Hilbert-Schmidt norm).

Proof. Step 1 (auxiliary elliptic systems). Let us choose θ' such that $0 < \theta'' < \theta' < \theta$.

For a fixed $\lambda \geq \lambda_0 > 0$ (see Theorem 5) we consider the unique classical solution $\psi = \psi_\lambda \in C^{2+\theta'}(\mathbb{R}^d, \mathbb{R}^d)$ to the elliptic system

$$\lambda\psi_\lambda - L\psi_\lambda = b, \quad (32)$$

where

$$Lu(x) = \frac{1}{2} \text{Tr}(\sigma(x)\sigma^*(x)D^2u(x)) + b(x) \cdot Du(x),$$

for any smooth function $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ (clearly (32) has to be interpreted componentwise).

Define

$$\Psi_\lambda(x) = x + \psi_\lambda(x).$$

Similarly to [10, Lemma 8] we have

Lemma 8 For λ large enough, such that $\|D\psi_\lambda\|_0 < 1$ (see Theorem 5), the following statements hold:

(i) Ψ_λ has bounded first and spatial derivatives and moreover the second (Fréchet) derivative $D_x^2\Psi_\lambda$ is globally θ' -Hölder continuous.

(ii) Ψ_λ is a C^2 -diffeomorphism of \mathbb{R}^d .

(iii) Ψ_λ^{-1} has bounded first and second derivatives and moreover

$$D\Psi_\lambda^{-1}(y) = \sum_{k \geq 0} (-D\psi_\lambda(\Psi_\lambda^{-1}(y)))^k, \quad y \in \mathbb{R}^d. \quad (33)$$

In the sequel we will use a value of λ for which Lemma 8 holds and simply write ψ and Ψ for ψ_λ and Ψ_λ .

Step 2 (conjugated SDE). Define

$$\tilde{b}(y) = \lambda\psi(\Psi^{-1}(y)), \quad \tilde{\sigma}(y) = D\Psi(\Psi^{-1}(y))\sigma(\Psi^{-1}(y))$$

and consider, for every $s \geq 0$ and $y \in \mathbb{R}^d$, the SDE

$$Y_t = y + \int_s^t \tilde{\sigma}(Y_u)dW_u + \int_s^t \tilde{b}(Y_u)du, \quad t \geq s. \quad (34)$$

This equation is equivalent to equation (28), in the following sense. If X_t is a solution to (28), then $Y_t = \Psi(X_t)$ verifies equation (34) with $y = \Psi(x)$: it is sufficient to apply Itô formula to $\Psi(X_t)$ and use equation (32).

Viceversa, given a solution Y_t of equation (34), let $X_t = \Psi^{-1}(Y_t)$, then it is possible to prove by direct application of Itô formula that X_t is a solution of (28) with $x = \Psi^{-1}(y)$. This is not very important since below we will obtain this fact indirectly.

Step 3 (proof of (i) and (ii)). We have clearly \tilde{b} and $\tilde{\sigma} \in C^{1+\theta'}$ (with first order derivatives bounded and in $C_b^{\theta'}$) so that, in particular, they are Lipschitz continuous.

By classical results (see [19, Chapter 2]) this implies existence and uniqueness of a strong solution Y of equation (34) and even the existence of a $C^{1,\theta''}$ stochastic flow of diffeomorphisms $\varphi_{s,t}$ associated to equation (34).

The uniqueness of Y implies the pathwise uniqueness of solutions of the original SDE (1) since two solutions X, \tilde{X} give rise to two processes $Y_t = \Psi(X_t)$ and $\tilde{Y}_t = \Psi(\tilde{X}_t)$ solving (34), then $Y = \tilde{Y}$ and then necessarily $X = \tilde{X}$. By the Yamada-Watanabe theorem pathwise uniqueness together with weak existence (which is a direct consequence of the Girsanov formula) gives the existence of the (unique) solution $(X_t^x)_{t \geq s}$ of eq. (1) starting from x at time s . Moreover setting

$$\phi_{s,t} = \Psi^{-1} \circ \varphi_{s,t} \circ \Psi$$

we realize that $\phi_{s,t}$ is the flow of (1) (in the sense that $X_t^x = \phi_{s,t}(x)$, P -a.s.).

Step 4. (proof of (iii)). Let ψ^n and ψ be the solutions in $C^{2+\theta'}(\mathbb{R}^d; \mathbb{R}^d)$ respectively of the elliptic problem associated to b_n and to $b \in C^\theta(\mathbb{R}^d; \mathbb{R}^d)$. Notice that we can make a choice of λ independent of n . We write

$$\lambda(\psi^n - \psi) - L(\psi^n - \psi) = (b^n - b) + (b^n - b) \cdot D\psi^n, \quad n \geq 1.$$

By Theorem 5 we have $\sup_{n \geq 1} \|\psi^n\|_{C^{2+\theta'}} \leq C < \infty$. Since $b - b_n$ is a bounded function, by the classical maximum principle (see [15]) we infer also that $\psi - \psi_n$ is a bounded function on \mathbb{R}^d and

$$\|\psi - \psi_n\|_0 \leq \frac{C+1}{\lambda} \|b - b_n\|_0, \quad n \geq 1. \quad (35)$$

It follows that $\psi - \psi_n \in C_b^{2+\theta'}(\mathbb{R}^d; \mathbb{R}^d)$ and $\|\psi - \psi_n\|_{C_b^{2+\theta'}} \rightarrow 0$ as $n \rightarrow \infty$.

Fix $p \geq 1$ and consider the flows $\varphi_{s,t}^n = \Psi^n \circ \phi_{s,t}^n \circ (\Psi^n)^{-1}$ which satisfy

$$\varphi_{s,t}^n(y) = y + \int_s^t \tilde{b}^n \circ \varphi_{s,u}^n(y) du + \int_s^t \tilde{\sigma}^n \circ \varphi_{s,u}^n(y) \cdot dW_u, \quad (36)$$

We have $\tilde{\sigma}^n \rightarrow \tilde{\sigma}$ and $\tilde{b}^n \rightarrow \tilde{b}$, as $n \rightarrow \infty$, in $C^{1+\theta'}(\mathbb{R}^d; \mathbb{R}^{d \times k})$ and $C^{1+\theta'}(\mathbb{R}^d; \mathbb{R}^d)$, respectively. By standard arguments, using the Gronwall lemma, the Doob inequality and the Burkholder inequality (compare, for instance, with the proof of [19, Theorem II.2.1]) we obtain the analog of (29) for the auxiliary flows $\varphi_{s,t}^n$ and $\varphi_{s,t}$:

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \sup_{0 \leq s \leq T} E \left[\sup_{u \in [s, T]} \frac{|\varphi_{s,u}^n(x) - \varphi_{s,u}(x)|^p}{(1+|x|)^p} \right] = 0. \quad (37)$$

We can also prove the inequality

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} \sup_{0 \leq s \leq T} E \left[\sup_{u \in [s, T]} \|D\varphi_{s,u}^n(x)\|^p \right] < \infty, \quad (38)$$

for $D\varphi_{s,t}^n(y)$, using the fact that the stochastic equation for $D\varphi_{s,t}^n(y)$ has the identity as initial condition and random coefficients $D\tilde{b}^n(\phi_{s,u}^n)$ and $D\tilde{\sigma}^n(\phi_{s,u}^n)$ which are uniformly bounded functions (since $\|D\tilde{b}^n\|_0 + \|D\tilde{\sigma}^n\|_0 \leq C$, uniformly in n).

To prove (30) is then enough to estimate $D\varphi_{s,u}^n$ using (38), the uniform boundedness of the derivatives of Ψ^n and its inverse (note that the uniform boundedness of the $D(\Psi^n)^{-1}$ can be proved by (33)).

To prove (29) we remark that to estimate the difference $\varphi_{s,t}^n(\Psi^n(x)) - \varphi_{s,t}(\Psi(x))$ we can split it as $\varphi_{s,t}^n(\Psi^n(x)) - \varphi_{s,t}(\Psi^n(x)) + \varphi_{s,t}(\Psi^n(x)) - \varphi_{s,t}(\Psi(x))$. The two differences can then be controlled by

$$\mathbb{E} \left[\sup_{s \leq u \leq T} |\varphi_{s,u}^n(\Psi^n(x)) - \varphi_{s,u}(\Psi^n(x))|^p \right] \leq a_n (1 + |\Psi^n(x)|)^p \leq a_n (1 + |x|)^p,$$

(where $a_n = \sup_{x \in \mathbb{R}^d} \sup_{0 \leq s \leq T} E \left[\sup_{u \in [s, T]} \frac{|\varphi_{s,u}^n(x) - \varphi_{s,u}(x)|^p}{(1+|x|)^p} \right]$ and $\lim_{n \rightarrow \infty} a_n = 0$) and by

$$\mathbb{E} \left[\sup_{s \leq u \leq T} |\varphi_{s,u}(\Psi^n(x)) - \varphi_{s,u}(\Psi(x))|^p \right] \leq \sup_{z \in \mathbb{R}^d} \mathbb{E} \left[\sup_{s \leq u \leq T} \|D\varphi_{s,u}(z)\|^p \right] |\Psi^n(x) - \Psi(x)|^p$$

$$\leq C\|\Psi^n - \Psi\|_0^p,$$

with $\lim_{n \rightarrow \infty} \|\Psi^n - \Psi\|_0 = \lim_{n \rightarrow \infty} \|\psi^n - \psi\|_0 = 0$ (see (35)).

Finally, one has to check that $(\Psi^n)^{-1}$ converges to Ψ^{-1} in the supremum norm. This follows from the inequality

$$\begin{aligned} \sup_{y \in \mathbb{R}^d} |(\Psi^n)^{-1}(y) - \Psi^{-1}(y)| &\leq \sup_{x \in \mathbb{R}^d} |(\Psi^n)^{-1}(\Psi^n(x)) - \Psi^{-1}(\Psi^n(x))| \\ &\leq \sup_{x \in \mathbb{R}^d} |\Psi^{-1}(\Psi^n(x)) - \Psi^{-1}(\Psi(x))| \leq \|D\Psi^{-1}\|_0 \|\Psi - \Psi^n\|_0, \end{aligned}$$

which tends to 0, as $n \rightarrow \infty$.

Arguing as in the proof of [19, Theorem II.3.1], we get the following linear equation for the derivative $D\phi_{s,t}(x)$

$$\begin{aligned} [D\Psi(\phi_{s,t}(x))]D\phi_{s,t}(x) &= D\Psi(x) + \int_s^t [D^2\Psi(\phi_{s,u}(x))]D\phi_{s,u}(x) \sigma(\phi_{s,u}(x))dW_u \\ &+ \int_s^t D\Psi(\phi_{s,u}(x)) [D\sigma(\phi_{s,u}(x))]D\phi_{s,u}(x)dW_u - \lambda \int_s^t [D\psi(\phi_{s,u}(x))]D\phi_{s,u}(x)du, \end{aligned} \quad (39)$$

$0 \leq s \leq t \leq T$, $x \in \mathbb{R}^d$. From the fact that $\lim_{n \rightarrow \infty} \|\psi^n - \psi\|_{C_b^{2+\theta'}} = 0$ together with (30) and (39), we finally obtain

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \sup_{0 \leq s \leq T} E[\sup_{u \in [s, T]} \|D\phi_{s,u}^n(x) - D\phi_{s,u}(x)\|^p] = 0, \quad (40)$$

which concludes the proof. ■

We consider now two possible extensions of Theorem 7 to the case when coefficients b and σ_i are time-dependent continuous functions defined on $[0, T] \times \mathbb{R}^d$, i.e., we are dealing with

$$dX_t^x = b(t, X_t^x) dt + \sum_{i=1}^k \sigma_i(t, X_t^x) dW_t^i, \quad t \in [0, T], \quad X_0 = x. \quad (41)$$

Remark 9 Let us treat the case in which also b is *bounded*. Following [10], an analogous of our Theorem 7 holds for (41) if we require that b and σ_i are continuous and bounded functions such that

$$\sup_{t \in [0, T]} (\|b(t, \cdot)\|_{C_b^\theta} + \|\sigma_i(t, \cdot)\|_{C_b^{1+\theta}}) < \infty, \quad i = 1, \dots, k,$$

and, moreover (as in Hypothesis 3) we assume that $\sigma(t, x)$ is uniformly non-degenerate, i.e., there exists the inverse of $a(t, x) = \sigma(t, x)\sigma^*(t, x)$, for any $t \in [0, T]$, $x \in \mathbb{R}^d$, and

$$\|a^{-1}\|_0 = \sup_{x \in \mathbb{R}^d, t \in [0, T]} \|a^{-1}(t, x)\| < \infty. \quad (42)$$

To prove Theorem 7 under these hypotheses, one can follow the proof of the analogous result proved in [10]. We only give a sketch of the argument.

First note that [10, Theorem 2] remains the same even with the previous non-constant $\sigma = (\sigma_i)$ (indeed it is a special case of a result in [18]). Then [10, Lemma 4] is true with σ in (41) by the following rescaling argument. Consider $\lambda \geq 1$ and

$$\partial_t u_\lambda + Lu_\lambda - \lambda u_\lambda = f \quad \text{in } [0, \infty) \times \mathbb{R}^d,$$

where L is the Kolmogorov operator associated to the SDE, i.e.,

$$L = \frac{1}{2} \text{Tr}[a(t, x) D^2 u(t, x)] + b(t, x) \cdot Du(t, x)$$

(here $(\sigma(t, x) \sigma^*(t, x)) = a(t, x)$ and D and D^2 denote spatial derivatives). Define a function v on $[0, \infty) \times \mathbb{R}^d$ such that $v(\lambda t, \sqrt{\lambda} x) = u_\lambda(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$. It is easy to see that, for any $s \geq 0$, $y \in \mathbb{R}^d$,

$$\partial_s v(s, y) + \text{Tr}\left[a\left(\frac{s}{\lambda}, \frac{y}{\sqrt{\lambda}}\right) D^2 v(s, y)\right] + \frac{1}{\sqrt{\lambda}} b\left(\frac{s}{\lambda}, \frac{y}{\sqrt{\lambda}}\right) \cdot Dv(s, y) - v(s, y) = \frac{1}{\lambda} f\left(\frac{s}{\lambda}, \frac{y}{\sqrt{\lambda}}\right).$$

Now the spatial Hölder seminorms of $(s, y) \mapsto a\left(\frac{s}{\lambda}, \frac{y}{\sqrt{\lambda}}\right)$ and $(s, y) \mapsto b\left(\frac{s}{\lambda}, \frac{y}{\sqrt{\lambda}}\right)$ are clearly independent on $\lambda \geq 1$ and on $s \geq 0$. By [18, Theorem 2.4], we deduce in particular, for any $\lambda \geq 1$,

$$\sup_{s \geq 0} \|Dv(s, \cdot)\|_0 \leq \frac{C}{\lambda} \sup_{s \geq 0} \|f(s, \cdot)\|_\theta,$$

where C is independent of λ . It follows the assertion of [10, Lemma 6] since

$$\sup_{t \geq 0} \|Du_\lambda(t, \cdot)\|_0 = \sqrt{\lambda} \sup_{s \geq 0} \|Dv(s, \cdot)\|_0 \leq \frac{C}{\sqrt{\lambda}} \sup_{s \geq 0} \|f(s, \cdot)\|_\theta.$$

The proof of [10, Theorem 5] (which deals with the stochastic flow) remains true even with σ in (41) by a straightforward modification.

Remark 10 An analogous of Theorem 7 holds for (41) requiring that Hypotheses 1, 2 and 3 are satisfied “uniformly in time”.

One assumes that b and σ_i are continuous functions defined on $[0, T] \times \mathbb{R}^d$, $i = 1, \dots, k$. Moreover, there exists $\theta \in (0, 1)$ such that $b(t, \cdot) \in C^\theta(\mathbb{R}^d; \mathbb{R}^d)$, $t \in [0, T]$, and $\sup_{t \in [0, T]} \|b(t, \cdot)\|_{C^\theta(\mathbb{R}^d, \mathbb{R}^d)} < \infty$. In addition, $\sigma_i(t, \cdot) \in C_b^3(\mathbb{R}^d, \mathbb{R}^d)$, $t \in [0, T]$,

$$\sup_{t \in [0, T]} \|\sigma_i(t, \cdot)\|_{C_b^3(\mathbb{R}^d, \mathbb{R}^d)} < \infty,$$

$i = 1, \dots, k$, and one requires that condition (42) holds. Theorem 7 under these assumptions may be established by adapting the (time-independent) proof given in the present paper. However, the complete argument, even if it does not present special difficulties, is considerably longer (for instance, one has to prove the analogous of the Bismut-Elworthy-Li formula (12) in the time-dependent case).

We close the section by an application of the stochastic flow. We obtain a Bismut-Elworthy-Li type formula for the derivative of the diffusion semigroup (P_t) associated to (1) (compare with [3] and [6]). It seems the first time that such formula is given for diffusion semigroups associated to SDEs with coefficients which are not locally Lipschitz.

Theorem 11 *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be uniformly continuous and bounded. For any $x, h \in \mathbb{R}^d$, we have (cf. (12))*

$$D_h P_t f(x) = \frac{1}{t} \mathbb{E}[f(\phi_t(x)) \int_0^t \langle (\sigma^* a^{-1})(\phi_u(x)) D_h \phi_u(x), dW_u \rangle], \quad t > 0, x \in \mathbb{R}^d,$$

where $\langle DP_t f(x), h \rangle = D_h P_t f(x)$ and $D\phi_u(x)$ solves (39) with $s = 0$ (we set $\phi_u(x) = \phi_{0,u}(x)$).

Proof. We prove the formula when $f \in C_b^\infty(\mathbb{R}^d)$. Indeed, then, by a straightforward uniform approximation of f , one can obtain the formula in the general case.

Let $\vartheta : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth test function such that $0 \leq \vartheta(x) \leq 1$, $x \in \mathbb{R}^d$, $\vartheta(x) = \vartheta(-x)$, $\int_{\mathbb{R}^d} \vartheta(x) dx = 1$, $\text{supp}(\vartheta) \subset B(0, 2)$, $\vartheta(x) = 1$ when $x \in B(0, 1)$. For any $n \geq 1$, let $\vartheta_n(x) = n^d \vartheta(nx)$. Define $b_n = b * \vartheta_n$.

We have that b_n is a C^∞ and Lipschitz vector field such that $b - b_n \in C_b^\theta(\mathbb{R}^d; \mathbb{R}^d)$ and $\|b - b_n\|_{C_b^\theta}$ tends to 0 as $n \rightarrow \infty$. Let (ϕ_t^n) be the associated flow of smooth diffeomorphisms which solves the SDE involving b_n and let (P_t^n) be the corresponding diffusion semigroup. The Bismut-Elworthy-Li formula for (P_t^n) is given by

$$D_h P_t^n f(x) = \frac{1}{t} \mathbb{E}[f(\phi_t^n(x)) \int_0^t \langle (\sigma^* a^{-1})(\phi_u^n(x)) D_h \phi_u^n(x), dW_u \rangle], \quad t > 0, x \in \mathbb{R}^d, n \in \mathbb{N}.$$

Note that $D_h P_t^n f(x) = \mathbb{E}[\langle Df(\phi_t^n(x)), D_h \phi_t^n(x) \rangle]$. Passing to the limit as $n \rightarrow \infty$, using the estimates (29) and (31), we get

$$D_h P_t f(x) = \mathbb{E}[\langle Df(\phi_t(x)), D_h \phi_t(x) \rangle] = \frac{1}{t} \mathbb{E}[f(\phi_t(x)) \int_0^t \langle \sigma^{-1}(\phi_u(x)) D_h \phi_u(x), dW_u \rangle],$$

for any $t > 0$, $x \in \mathbb{R}^d$. ■

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